On the distribution of the order and index of $g(\text{mod } p)$ over residue classes II

Pieter Moree

Abstract

For a fixed rational number $g \notin \{-1, 0, 1\}$ and integers a and d we consider the set $N_g(a, d)$ of primes p for which the order of $g(\text{mod } p)$ is congruent to $a(\text{mod } d)$. It is shown, assuming the Generalized Riemann Hypothesis (GRH), that this set has a natural density $\delta_g(a, d)$. Moreover, $\delta_g(a, d)$ is computed in terms of degrees of certain Kummer extensions. Several properties of $\delta_g(a, d)$ are established in case d is a power of an odd prime.

1 Introduction

Let $g \notin \{-1, 0, 1\}$ be a rational number (this assumption on g will be maintained throughout this paper). For u a rational number, let $v_p(u)$ denote the exponent of p in the canonical factorisation of u (throughout the letter p will be used to indicate prime numbers). If $v_p(g) = 0$, then there exists a smallest positive integer k such that $g^k \equiv 1(\text{mod } p)$. We put $\text{ord}_p(g) = k$. This number is the *(residual) order* of $g(\text{mod } p)$. The index of the subgroup generated by g mod p inside the multiplicative group of residues mod p, $|(\mathbb{Z}/p\mathbb{Z})^\times : \langle g(\text{mod } p)\rangle|$, is denoted by $r_g(p)$ and called the *(residual) index* mod p of g. Note that $\text{ord}_g(p)r_g(p) = p - 1$.

We let $N_g(a_1, d_1; a_2, d_2)(x)$ count the number of primes $p \le x$ with $p \equiv a_1(\text{mod } d_1)$ and $\text{ord}_g(p) \equiv a_2(\text{mod } d_2)$. For convenience denote $N_g(0, 1; a, d)(x)$ by $N_g(a, d)(x)$. Although our main interest is in the behaviour of $N_g(a, d)(x)$ it turns out that sometimes it is fruitful to partition $N_g(a, d)(x)$ in sets of the form $N_g(a_1, d_1; a_2, d_2)(x)$ with a well-chosen d_1. Let $r|s$ be positive integers. By $K_{s,r}$ we denote the number field $\mathbb{Q}(\zeta_s, g^{1/r})$, where $\zeta_s = \exp(2\pi i/s)$. The main result of this paper is as folows (for notational convenience (a, b), $[a, b]$ will be written for the greatest common divisor, respectively lowest common multiple of a and b and by (GRH) we indicate that GRH is assumed).

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Deutschland, E-mail: moree@mpim-bonn.mpg.de

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Theorem 1 (GRH). *Let* $(a_1, d_1) = 1$. *Then*

$$N_g(a_1, d_1; a_2, d_2)(x) = \delta_g(a_1, d_1; a_2, d_2)\frac{x}{\log x} + O_{d_1,d_2,g}(\frac{x}{\log^{3/2} x}),$$

for a number $\delta_g(a_1, d_1; a_2, d_2)$ *that is given by* (13).

Specialising to $a_1 = 0$ and $d_1 = 1$ the following result is obtained.

Theorem 2 (GRH). *We have*

$$N_g(a, d)(x) = \delta_g(a, d)\frac{x}{\log x} + O_{d,g}(\frac{x}{\log^{3/2} x}),$$

with

$$\delta_g(a,d) = \sum_{\substack{t=1 \\ (1+ta,d)=1}}^{\infty} \sum_{\substack{n=1 \\ (n,d)|a}}^{\infty} \frac{\mu(n)c_g(1+ta, dt, nt)}{[K_{[d,n]t,nt} : \mathbb{Q}]}, \tag{1}$$

where, for $(b, f) = 1$,

$$c_g(b, f, v) = \begin{cases} 1 & \text{if } \sigma_b|_{\mathbb{Q}(\zeta_f)\cap K_{v,v}} = \mathrm{id}; \\ 0 & \text{otherwise} \end{cases},$$

where σ_b *is the automorphism of* $\mathbb{Q}(\zeta_f)$ *that sends* ζ_f *to* ζ_f^b.

In part I [5] this result was established, by a slightly different method, in case $3 \le d \le 4$ and explicit formulae for $\delta_g(a, 3)$ and $\delta_g(a, 4)$ were derived. By a different method some of the results for $d = 4$ with weaker error term and restricted values of g are established in [1] and [6], see also Corollary 1 of [5].

It turns out that the numbers c_g appearing in (1) have a strong tendency to equal one. This motivates the following definition:

$$\delta_g^{(0)}(a,d) = \sum_{\substack{t=1 \\ (1+ta,d)=1}}^{\infty} \sum_{\substack{n=1 \\ (n,d)|a}}^{\infty} \frac{\mu(n)}{[K_{[d,n]t,nt} : \mathbb{Q}]}. \tag{2}$$

For example, if $d|a$, then σ_{1+ta} is the identity element of the Galois group of $\mathbb{Q}(\zeta_{dt})$ over $\mathbb{Q}$ and then trivially $\delta_g(a, d) = \delta_g^{(0)}(a, d)$.

Generically the degree $[K_{[d,n]t,nt} : \mathbb{Q}]$ appearing in (2) equals $\varphi([d, n]t)nt$. On substituting this value in (2) a number $\delta(a, d)$ is obtained that no longer depends on g:

$$\delta(a,d) = \sum_{\substack{t=1 \\ (1+ta,d)=1}}^{\infty} \sum_{\substack{n=1 \\ (n,d)|a}}^{\infty} \frac{\mu(n)}{\varphi([d,n]t)nt}. \tag{3}$$

In [3] it is shown that $\delta(a, d)$ is the average density of elements in a finite field having order $\equiv a (\mathrm{mod}\ d)$. The number $\delta(a, d)$ can be a regarded as a naïve heuristic for $\delta_g(a, d)$.

In case d is a power of an odd prime q the coefficients c_g are easily evaluated. This case is considered in extenso in Section 4. There it is shown, for example, that $\delta_g(a, q^s) = \delta(a, q^s)$ for almost all integers g with $|g| \le x$.

This paper is part of a sequal of three papers, each one considering a larger class of moduli d. Due to the inherit arithmetic complexity, results in later papers are less explicit than those in the earlier ones. Thus only part of the results in the earlier papers follows from those of the later ones.

2 Preliminaries

2.1 Some notation

We recall some notation from our earlier papers [5, 3] on this topic. For any Dirichlet character χ of $(\mathbb{Z}/d\mathbb{Z})^*$, we let h_χ denote the Dirichlet convolution of χ and μ. As usual we let $L(s,\chi)$ denote the Dirichlet L-series for χ. For properties of h_χ the reader is referred to [5]. From [5] we furthermore recall that

$$C_\chi(h,r,s) = \sum_{(r,v)=1,\ s|v}^{\infty} \frac{h_\chi(v)(h,v)}{v\varphi(v)} \text{ and } A_\chi = \prod_{p:\chi(p)\neq 0}\left(1+\frac{[\chi(p)-1]p}{[p^2-\chi(p)](p-1)}\right).$$

The constants A_χ turn out to be the basic constants for this problem. In many cases $A_\chi \in \mathbb{C}\backslash\mathbb{R}$, see [3, Table 3]. It can be shown that

$$A_\chi \prod_{p|d}\left(1-\frac{1}{p(p-1)}\right) = A\frac{L(2,\chi)L(3,\chi)}{L(6,\chi^2)}\prod_{r=1}^{\infty}\prod_{k=3r+1}^{\infty} L(k,\chi^r)^{\lambda(k,r)},$$

where A denotes the Artin constant and the numbers $\lambda(k,r)$ are non-zero integers that can be related to Fibonacci numbers [4]. The latter expansion of A_χ can be used to approximate A_χ with high numerical accuracy (see [3, Section 6]).

In [5, Lemma 10] it is shown that $C_\chi(h,r,s) = cA_\chi$, where c can be explicitly written down and is in $\mathbb{Q}(\zeta_{o_\chi})$. Here o_χ is the order of the character χ, i.e. the smallest positive integer k such that $\chi^k = \chi_0$, the trivial character.

We recall from [5] that, if $(b,f)=1$,

$$\sum_{\substack{t\equiv b(\mathrm{mod}\ f)\\ t|v}} \mu(\frac{v}{t}) = \frac{1}{\varphi(f)}\sum_{\chi\in G_f}\overline{\chi(b)}h_\chi(v), \tag{4}$$

where the sum is over the characters in the character group G_f of $(\mathbb{Z}/\mathbb{Z}f)^*$.

2.2 Preliminaries on algebraic number theory

We first review some properties of the Kronecker symbol (a rarely discussed symbol in books on number theory). To this end we first recall the definition of the Legendre and the Jacobi symbol. By definition the Legendre symbol $(\frac{n}{p})$, where $p \geq 3$ is a prime number and $n \in \mathbb{Z}$, $p \nmid n$, is equal to 1 if n is a quadratic residue mod p, and to -1 otherwise.

Let $m > 0$ be an odd integer relatively prime to n. The Jacobi symbol $(\frac{n}{m})$ is defined as the product of the Legendre symbols $(\frac{n}{m}) = (\frac{n}{p_1})\cdots(\frac{n}{p_s})$, where $m = p_1\cdots p_s$ and each p_i is a prime.

The Kronecker symbol $(\frac{c}{d})$ is defined for $c \in \mathbb{Z}$, $c \equiv 0(\mathrm{mod}\ 4)$ or $c \equiv 1(\mathrm{mod}\ 4)$, c not a square, and $d \geq 1$ an integer; if $b = p_1p_2\cdots p_s$ is the decomposition of b as a product of primes, we put $(\frac{a}{-b}) = (\frac{a}{b}) = (\frac{a}{p_1})(\frac{a}{p_2})\cdots(\frac{a}{p_s})$. If p is an odd prime $(\frac{a}{p}) = 0$ when p divides a, while $(\frac{a}{p})$ is the Legendre symbol $(\frac{a}{p})$ when p does not divide a and, finally, $(\frac{a}{2}) = 1$ when $a \equiv 1(\mathrm{mod}\ 8)$, while $(\frac{a}{2}) = -1$ when $a \equiv 5(\mathrm{mod}\ 8)$. Then if a and b are such that both the Jacobi and Kronecker symbols are defined, then these symbols coincide. If a is odd, then $(\frac{a}{2})$ = Jacobi symbol $(\frac{2}{|a|})$. If $b>0$, $(a,b)=1$, a is odd, then $(\frac{a}{b})=(\frac{b}{|a|})$, where the symbol on the right hand side is the Jacobi symbol. Most importantly, if $b>0$, $(a,b)=1$ and $a=2^r\tilde{a}$ with $\tilde{a}$ odd, then

$$(\frac{a}{b})=(\frac{2}{b})^r(-1)^{\frac{\tilde{a}-1}{2}\frac{b-1}{2}}(\frac{b}{|a|}),$$

where the symbols on the right hand side are Jacobi symbols.

Let $K:\mathbb{Q}$ be an abelian number field. By the Kronecker-Weber theorem there exists an integer f such that $K\subseteq\mathbb{Q}(\zeta_f)$. The smallest such integer is called the conductor of K. Note that $K\subseteq\mathbb{Q}(\zeta_n)$ iff n is divisible by the conductor. Note also that the conductor of a cyclotomic field is never congruent to 2(mod 4). The following lemma allows one to determine all quadratic subfields of a given cyclotomic field (for a proof see e.g. [10, p. 263]).

Lemma 1 *The conductor of a quadratic number field is equal to the absolute value of its discriminant.*

Consider the cyclotomic extension $\mathbb{Q}(\zeta_f)$ of the rationals. There are $\varphi(f)$ distinct automorphisms each determined uniquely by $\sigma_a(\zeta_f)=\zeta_f^a$, with $1\le a\le f$ and $(a,f)=1$. We need to know when the restriction of such an automorphism to a given quadratic subfield of $\mathbb{Q}(\zeta_f)$ is the identity. In this direction we have:

Lemma 2 *Let $\mathbb{Q}(\sqrt{d})\subseteq\mathbb{Q}(\zeta_f)$ be a quadratic field of discriminant Δ_d and b be an integer with $(b,f)=1$. We have $\sigma_b|_{\mathbb{Q}(\sqrt{d})}=\mathrm{id}$ iff $(\frac{\Delta_d}{b})=1$, with $(\frac{\cdot}{\cdot})$ the Kronecker symbol.*

Proof. Using Lemma 1 we see that we can restrict to the case where $f=|\Delta_d|$. Define χ by $\chi(b)=\sigma_b(\sqrt{d})/\sqrt{d}$, $1\le b\le|\Delta_d|$, $(b,\Delta_d)=1$. Then χ is the unique non-trivial character of the character group of $\mathbb{Q}(\sqrt{d})$. As is well-known (see e.g. [7, p. 437]), the primitive character induced by this is $(\frac{\Delta_d}{b})$. Using Lemma 1, we see that χ is a primitive character mod $|\Delta_d|$. Thus $\chi(b)=(\frac{\Delta_d}{b})$. Now $\sigma_b|_{\mathbb{Q}(\sqrt{d})}=\mathrm{id}$ iff $\chi(b)=(\frac{\Delta_d}{b})=1$. □

`Remark 1`. Another way to prove Lemma 2 is to note that $\sigma_b|_{\mathbb{Q}(\sqrt{d})}=\mathrm{id}$ iff there exists a prime $p\equiv b(\mathrm{mod}\ f)$ that splits completely in $\mathbb{Q}(\sqrt{d})$. It is well-known that there exists a prime $p\equiv b(\mathrm{mod}\ f)$ that splits completely in the field $\mathbb{Q}(\sqrt{d})$ iff $(\frac{\Delta_d}{p})=1$ (see e.g. [10, p. 236]). Since $(\frac{\Delta_d}{p})=(\frac{\Delta_d}{b})$, the result follows.

`Remark 2`. The action of σ on $\sqrt{d}$ can also be determined by relating $\sqrt{d}$ to a Gauss sum 'living' in $\mathbb{Q}(\zeta_f)$. It is straigthforward to determine the action of σ on such a Gauss sum.

2.3 Preliminaries on field degrees

In order to explicitly evaluate certain densities in this paper, the following result will play a crucial role. Let $g_1\neq0$ be a rational number. By $D(g_1)$ we denote the discriminant of the field $\mathbb{Q}(\sqrt{g_1})$. The notation $D(g_1)$ along with the notation

g_0, h and n_r introduced in the next lemma will reappear again and again in the sequel.

Lemma 3 [5]. *Write $g = \pm g_0^h$, where g_0 is positive and not an exact power of a rational. Let $D(g_0)$ denote the discriminant of the field $\mathbb{Q}(\sqrt{g_0})$. Put $m = D(g_0)/2$ if $\nu_2(h) = 0$ and $D(g_0) \equiv 4 (\text{mod } 8)$ or $\nu_2(h) = 1$ and $D(g_0) \equiv 0 (\text{mod } 8)$, and $m = [2^{\nu_2(h)+2}, D(g_0)]$ otherwise. Put*

$$n_r = \begin{cases} m & \text{if } g < 0 \text{ and } r \text{ is odd;} \\ [2^{\nu_2(hr)+1}, D(g_0)] & \text{otherwise.} \end{cases}$$

We have

$$[K_{kr,k} : \mathbb{Q}] = [\mathbb{Q}(\zeta_{kr}, g^{1/k}) : \mathbb{Q}] = \frac{\varphi(kr)k}{\epsilon(kr, k)(k, h)},$$

where, for $g > 0$ or $g < 0$ and r even we have

$$\epsilon(kr, k) = \begin{cases} 2 & \text{if } n_r | kr; \\ 1 & \text{if } n_r \nmid kr, \end{cases}$$

and for $g < 0$ and r odd we have

$$\epsilon(kr, k) = \begin{cases} 2 & \text{if } n_r | kr; \\ \frac{1}{2} & \text{if } 2|k \text{ and } 2^{\nu_2(h)+1} \nmid k; \\ 1 & \text{otherwise.} \end{cases}$$

`Remark`. Note that if h is odd, then $n_r = [2^{\nu_2(r)+1}, D(g)]$. Note that $n_r = n_{\nu_2(r)}$.

From the latter lemma many consequences can be deduced.

Lemma 4 *Let v, w and z be natural numbers with $v|w$ and with z an odd divisor of w. Then $[K_{zw,v} : \mathbb{Q}] = z[K_{w,v} : \mathbb{Q}]$.*

Proof. The proof easily follows from Lemma 3 on observing that the odd part of n_r is squarefree and that $\varphi(zw) = z\varphi(w)$. □

Lemma 5 *The intersection $\mathbb{Q}(\zeta_f) \cap K_{v,v}$ is equal to $\mathbb{Q}(\zeta_{(f,v)})$ or a quadratic extension thereof. More precisely,*

$$[\mathbb{Q}(\zeta_f) \cap K_{v,v} : \mathbb{Q}(\zeta_{(f,v)})] = \frac{\epsilon([f, v], v)}{\epsilon(v, v)}.$$

Proof. Clearly this intersection field is abelian and contains $\mathbb{Q}(\zeta_{(f,v)})$. We have

$$[K_{[f,v],v} : \mathbb{Q}] = \frac{\varphi(f)[K_{v,v} : \mathbb{Q}]}{[\mathbb{Q}(\zeta_f) \cap K_{v,v} : \mathbb{Q}]}. \tag{5}$$

On noting that $\varphi((f, v))\varphi([f, v]) = \varphi(f)\varphi(v)$, it follows from Lemma 3 and (5) that

$$[\mathbb{Q}(\zeta_f) \cap K_{v,v} : \mathbb{Q}(\zeta_{(f,v)})] = \frac{[\mathbb{Q}(\zeta_f) \cap K_{v,v} : \mathbb{Q}]}{\varphi((f, v))} = \frac{\epsilon([f, v], v)}{\epsilon(v, v)}. \tag{6}$$

It is not difficult to infer from Lemma 3 that the latter quotient is either 1 or 2 (so the apparent possibility 4 does never arise). We conclude that $\mathbb{Q}(\zeta_f) \cap K_{v,v}$ is equal to $\mathbb{Q}(\zeta_{(f,v)})$ or a quadratic extension thereof. □

Lemma 6 *Let q be an odd prime, $s \geq 0$ and suppose $q \nmid v$. Put $q^* = (\frac{-1}{q})q$. Consider the following conditions*
i) $q|D(g_0)$, $s = 0$ and $\frac{n_1}{q}|v$.
ii) If $g < 0$ and $2|v$, then $2^{\nu_2(h)+1}|v$.
We have

$$\mathbb{Q}(\zeta_{q^{s+1}}) \cap K_{q^s v, q^s v} = \begin{cases} \mathbb{Q}(\sqrt{q^*}) & \text{if both i and ii are satisfied;} \\ \mathbb{Q}(\zeta_{q^s}) & \text{otherwise.} \end{cases}$$

Proof. By Lemma 5 we have $[\mathbb{Q}(\zeta_{q^{s+1}}) \cap K_{q^s v, q^s v} : \mathbb{Q}(\zeta_{q^s})] = \epsilon(q^{s+1}v, q^s v)/\epsilon(q^s v, q^s v)$ and hence, by Lemma 3, $\mathbb{Q}(\zeta_{q^{s+1}}) \cap K_{q^s v, q^s v}$ is a quadratic extension of $\mathbb{Q}(\zeta_{q^0}) = \mathbb{Q}$ if conditions i and ii are satisfied and $\mathbb{Q}(\zeta_{q^{s+1}}) \cap K_{q^s v, q^s v} = \mathbb{Q}(\zeta_{q^s})$ otherwise. Since $\mathbb{Q}(\sqrt{q^*})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_q)$, the result now follows. □

`Remark 3`. If $g > 0$ or $g < 0$ and v is odd, then condition ii is vacuously satisfied.

Lemma 7 *Let n be squarefree. Put $t_d = \prod_{p|(t,d)} p^{\nu_p(t)}$. The density of primes p such that $p \equiv 1 + ta (\text{mod } dt)$ and p splits completely in $K_{nt,nt}$ equals zero if $(d, n) \nmid a$ or $(1 + ta, d) > 1$, otherwise it equals*

$$\frac{c_g(1 + ta, dt, nt)}{[K_{[d,n]t,nt} : \mathbb{Q}]} = \frac{c_g(1 + ta, dt_d, nt)}{[K_{[d,n]t,nt} : \mathbb{Q}]}. \tag{7}$$

Proof. This follows from Chebotarev's density theorem together with the observation that the two systems of congruences

$$\begin{cases} x \equiv 1 + ta (\text{mod } dt) \\ x \equiv 1 (\text{mod } nt) \end{cases} \text{ and } \begin{cases} x \equiv 1 + ta (\text{mod } dt_d) \\ x \equiv 1 (\text{mod } nt) \end{cases}$$

are equivalent. □

Lemma 8 *Assume that $(b, f_1 f_2) = 1$, $f_1|f_2$ and*

$$\frac{[K_{[f_1,v],v} : \mathbb{Q}]}{\varphi(f_1)} = \frac{[K_{[f_2,v],v} : \mathbb{Q}]}{\varphi(f_2)}. \tag{8}$$

Then $c_g(b, f_1, v) = c_g(b, f_2, v)$.

Proof. By (5) the assumption (8) implies that

$$[\mathbb{Q}(\zeta_{f_1}) \cap K_{v,v} : \mathbb{Q}] = [\mathbb{Q}(\zeta_{f_2}) \cap K_{v,v} : \mathbb{Q}].$$

This, together with the assumption that $f_1|f_2$ ensures that $\mathbb{Q}(\zeta_{f_1}) \cap K_{v,v} = \mathbb{Q}(\zeta_{f_2}) \cap K_{v,v} = L$, say, whence $L = \mathbb{Q}(\zeta_{(f_1,f_2)}) \cap K_{v,v}$. Since the map $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{f_1})/\mathbb{Q})$ that sends ζ_{f_1} to $\zeta_{f_1}^b$ and the map $\sigma' \in \mathrm{Gal}(\mathbb{Q}(\zeta_{f_2})/\mathbb{Q})$ that sends ζ_{f_2} to $\zeta_{f_2}^b$ act in the same way when restricted to $\mathbb{Q}(\zeta_{(f_1,f_2)})$, it follows that $\sigma_b|_{\mathbb{Q}(\zeta_{f_1}) \cap K_{v,v}} = \sigma'_b|_{\mathbb{Q}(\zeta_{f_2}) \cap K_{v,v}}$ and hence $c_g(b, f_1, v) = c_g(b, f_2, v)$. □

2.4 Remaining preliminaries

The following result is due to Wirsing [13].

Lemma 9 [13]. *Suppose $f(n)$ is a multiplicative function such that $f(n) \geq 0$, for $n \geq 1$, and such that there are constants γ_1 and γ_2, with $\gamma_2 < 2$, such that for every prime p and every $\nu \geq 2$, $f(p^\nu) \leq \gamma_1 \gamma_2^\nu$. Assume that as $x \to \infty$,*

$$\sum_{p \leq x} f(p) \sim \tau \frac{x}{\log x},$$

where $\tau > 0$ is a constant. Then, as $x \to \infty$,

$$\sum_{n \leq x} f(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \frac{f(p^3)}{p^3} + \cdots\right),$$

where γ is Euler's constant and $\Gamma(\tau)$ denotes the gamma-function.

We use it to establish the following lemma.

Lemma 10 *Let $d \geq 3$. The number of integers $1 \leq g \leq x$ such that $D(g)$ has no prime divisor p with $p \equiv 1 (\mathrm{mod}\ d)$ is $O_d(x \log^{-1/\varphi(d)} x)$. The same assertion holds with $D(g)$ replaced by $D(-g)$.*

Proof. Denote the number of integers counted in the formulation of the lemma by $T_d(x)$. We define a multiplicative function $f_d(n)$ as follows:

$$f_d(p^\alpha) = \begin{cases} 0 & \text{if } 2 \nmid \alpha \text{ and } p \equiv 1 (\mathrm{mod}\ d); \\ 1 & \text{otherwise.} \end{cases}$$

Note that $T_d(x) = \sum_{g \leq x} f_d(g)$. Using Lemma 9, it then follows that

$$T_d(x) = O_d\left(\frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{f_d(p)}{p} + \frac{f_d(p^2)}{p^2} + \cdots\right)\right) = O_d\left(\frac{x}{\log x} \prod_{\substack{p \leq x \\ p \not\equiv 1 (\mathrm{mod}\ d)}} (1 + \frac{1}{p})\right).$$

Since by Mertens' theorem for arithmetic progressions (cf. [12]) we have

$$\prod_{\substack{p \leq x \\ p \equiv a (\mathrm{mod}\ d)}} (1 + \frac{1}{p}) \sim c_{a,d} \log^{\frac{1}{\varphi(d)}} x, \ x \to \infty,$$

for some $c_{a,d} > 0$, the result follows. □

3 The proofs of Theorems 1 and 2

In this section we analyse the growth behaviour of the counting functions $N_g(a, d)(x)$ and $N_g(a_1, d_1; a_2, d_2)(x)$. Throughout we restrict to those primes p with $\nu_p(g) = 0$. Let $\omega(d) = \sum_{p|d} 1$ denote the number of distinct prime divisors of d.

Let

$$V_g(a, d; t)(x) = \#\{p \leq x : r_g(p) = t, \ p \equiv 1 + ta (\mathrm{mod}\ dt)\}.$$

Note that $N_g(a,d)(x) = \sum_{t=1}^{\infty} V_g(a,d;t)(x)$. If $(1+ta,d) > 1$ then there is at most one prime counted by $V_g(a,d;t)(x)$ and this prime has to divide d. It follows that $N_g(a,d)(x) = \sum_{\substack{t=1 \\ (1+ta,d)=1}}^{\infty} V_g(a,d;t)(x) + O(\omega(d))$. Let $x_1 = \sqrt{\log x}$. Assume GRH. By [5, Lemma 7] it follows that $= \#\{p \le x : r_g(p) > x_1\} = O_g(x \log^{-3/2} x)$. We thus infer that

$$N_g(a,d)(x) = \sum_{\substack{t \le x_1 \\ (1+ta,d)=1}} V_g(a,d;t)(x) + O_{g,d}(\frac{x}{\log^{3/2} x}). \tag{9}$$

For fixed t the term $V_g(a,d;t)$ can be estimated by a variation of Hooley's classical argument [2]. However, we need to carry this out with a certain uniformity. This requires one to merely keep track of the dependence on t of the various estimates. This results in the following lemma.

Lemma 11 (GRH). *For $t \le x^{1/3}$ we have*

$$V_g(a,d;t)(x) = \frac{x}{\log x} \sum_{\substack{n=1 \\ (n,d)|a}}^{\infty} \frac{\mu(n) c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt} : \mathbb{Q}]} + O\left(\frac{x \log\log x}{\varphi(t) \log^2 x} + \frac{x}{\log^2 x}\right),$$

where the implied constant depends at most on d and g.

Proof. Let

$$M_g'(x,y) = \#\{p \le x : p \equiv 1 + ta (\mathrm{mod}\ dt),\ t | r_g(p),\ qt \nmid r_g(p),\ q \le y\} \text{ and}$$

$$M_g(x,y,z) = \#\{p \le x : qt | r_g(p),\ y \le q \le z\},$$

where q denotes a prime number. Note that

$$V_g(a,d;t)(x) = M_g'(x,\xi_1) + O(M_g(x,\xi_1,\xi_2)) + O(M_g(x,\xi_2,\xi_3)) + O(M_g(x,\xi_3,\frac{x-1}{t})).$$

We take $\xi_1 = \log x/6$, $\xi_2 = \sqrt{x} \log^{-2} x$ and $\xi_3 = \sqrt{x} \log x$. The three error terms were estimated in [5, Theorem 4]. Taking them together it is found that

$$V_g(a,d;t)(x) = M_g'(x,\xi_1) + O_{d,g}\left(\frac{x \log\log x}{\varphi(t) \log^2 x} + \frac{x}{\log^2 x}\right), \tag{10}$$

By inclusion and exclusion it follows that

$$M_g'(x,\xi_1) = \sum_{P(n) \le \xi_1} \mu(n) \#\{p \le x : p \equiv 1 + ta (\mathrm{mod}\ dt),\ nt | r_g(p)\},$$

where $P(n)$ denotes the greatest prime factor of n. The integers n counted in the latter sum are all less than $x^{1/3}$ (cf. (6) of [2]). The counting functions in the latter sum can be estimated by an effective form of Chebotarev's density theorem, cf. Theorem 3 of [5] and the discussion immediately following this theorem. This yields that

$$\#\{p \le x : p \equiv 1 + ta (\mathrm{mod}\ dt),\ nt | r_g(p)\} = \frac{c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt} : \mathbb{Q}]} \mathrm{Li}(x) + O(\sqrt{x} \log x),$$

where the error term depends at most d and g. Indeed, applying applying Theorem 3 of [5] results in an error term of $O(\sqrt{x}\log(d_L x^{[L:\mathbb{Q}]})/[L:\mathbb{Q}])$, with an absolute implied constant and $L = K_{[d,n]t,nt}$. On invoking Lemma 3 and [5, Lemma 2] it follows that this is $O_{d,g}(\sqrt{x}\log x)$. Proceeding as in section 6 of [2] it is then inferred that

$$M'_g(x,\xi_1) = \frac{x}{\log x}\sum_{n=1}^{\infty}\frac{\mu(n)c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt}:\mathbb{Q}]} + O_{d,g}(\frac{x}{\log^2 x}). \tag{11}$$

If $nt|r_g(p)$, then $p \equiv 1(\text{mod } nt)$. By the chinese remainder theorem it now follows that if $(n,d)|a$ then $\#\{p \le x : p \equiv 1+ta(\text{mod } dt),\ nt|r_g(p)\}$ is finite for every x and hence $c_g(1+ta,dt,nt) = 0$ (an alternative way to see this is to note that in this case σ_{1+ta} does not act like the identity on $\mathbb{Q}(\zeta_{nt})$). It follows that in the sum in (11) we can restrict to those n satisfying $(n,d)|a$. On taking this into account and combining (10) and (11), the result follows. □

It is now straigthforward to establish Theorem 2.
Proof of Theorem 2. Recall that $x_1 = \sqrt{\log x}$. Combination of (9) and Lemma 11 yields

$$N_g(a,d)(x) = \frac{x}{\log x}\sum_{\substack{t \le x_1 \\ (1+ta,d)=1}}\sum_{\substack{n=1 \\ (n,d)|a}}^{\infty}\frac{\mu(n)c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt}:\mathbb{Q}]} + O_{g,d}\left(\frac{x}{\log^{3/2} x}\right). \tag{12}$$

Denote the latter double sum by $D(x)$. By Lemma 3 and [5, Lemma 5] we find

$$D(x) = \delta_g(a,d) + O(\sum_{t>x_1}\frac{h}{t\varphi(t)}) = \delta_g(a,d) + O(\frac{h}{\sqrt{\log x}}).$$

On inserting the latter estimate in (12) the proof is then completed. □

A variation of the above (but notationally rather more awkward and hence we only sketch it) gives Theorem 1 with

$$\delta_g(a_1,d_1;a_2,d_2) = \sum_{\substack{t=1,\ (1+ta_2,d_2)=1 \\ 1+ta_2\equiv a_1(\text{mod }(d_1,d_2t))}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)c_g(a_1,d_1,1+ta_2,d_2t,nt)}{[K_{nt,nt}(\zeta_{d_1},\zeta_{d_2t}):\mathbb{Q}]}, \tag{13}$$

where, for $(b_1,f_1) = (b_2,f_2) = 1$ and $b_1 \equiv b_2(\text{mod } (f_1,f_2))$, we define

$$c_g(b_1,f_1,b_2,f_2,v) = \begin{cases} 1 & \text{if } \tau|_{\mathbb{Q}(\zeta_{[f_1,f_2]})\cap K_{v,v}} = \text{id}; \\ 0 & \text{otherwise} \end{cases},$$

where τ is the (unique) automorphism of $\mathbb{Q}(\zeta_{[f_1,f_2]})$ determined by $\tau(\zeta_{f_1}) = \zeta_{f_1}^{b_1}$ and $\tau(\zeta_{f_2}) = \zeta_{f_2}^{b_2}$.

Proof of Theorem 1. This is a variation of the proof of Theorem 2. Most error terms can be estimated as before on dropping the condition that $p \equiv a_1(\text{mod } d_2)$, which brings us in the situation of Theorem 2.

We first generalise Lemma 11. For that we only have to replace $M'_g(x,\xi_1)$

$M''_g(x,\xi_1)$ say, where $M''_g(x,\xi_1)$ is defined as $M'_g(x,\xi_1)$ with $a=a_2$ and $d=d_2$, but where now furthermore the primes p are required to satisfy $p\equiv a_1(\text{mod } d_1)$. The estimation of $M''_g(x,\xi_1)$ can be carried out completely similarly to that of $M'_g(x,\xi_1)$.

The analogue of (9) is easily derived to be

$$N_g(a_1,d_1;a_2,d_2)(x)=\sum_{\substack{t\le x_1,\ (1+ta_2,d_2)=1\\ 1+ta_2\equiv a_1(\text{mod } (d_1,d_2t))}} V_g(a_1,d_1;a_2,d_2;t)(x)+O_{g,d}(\frac{x}{\log^{3/2}x}).$$

From here on the proof is completed as before. □

4 The case where d is an odd prime

Let $d=q$ be an odd prime. In this case it turns out to be fruitful to consider separately the primes p with $p\equiv 1(\text{mod } q)$ and those with $p\not\equiv 1(\text{mod } q)$.

4.1 The case where $q|a$

Trivially $N_g(a,q)=N_g(0,q)$ and w.l.o.g, we may assume that $a=0$. Note that the primes counted by $N_g(0,q)(x)$ must satisfy $p\equiv 1(\text{mod } q)$. Let $j=\nu_q(p-1)$. Note that $p\not\in N_g(0,q)$ iff $q^j|r_g(p)$. Thus we infer that

$$N_g(0,q)(x)=\#\{p\le x: p\equiv 1(\text{mod } q)\}-$$

$$\sum_{j=1}^{\infty}\#\{p\le x: p\equiv 1(\text{mod } q^j),\ p\not\equiv 1(\text{mod } q^{j+1}),\ q^j|r_g(p)\}.$$

The density of primes p satisfying $p\equiv 1(\text{mod } q^j)$, $p\not\equiv 1(\text{mod } q^{j+1})$ and $q^j|r_g(p)$ can be computed by Chebotarev's density theorem and equals

$$\frac{1}{[K_{q^j,q^j}:\mathbb{Q}]}-\frac{1}{[K_{q^{j+1},q^j}:\mathbb{Q}]}.$$

A more refined analysis [11], cf. [8, 9] (with weaker error term), shows that

$$N_g(0,q)(x)=\delta_g(0,q)\text{Li}(x)+O_{g,q}\left(\frac{x(\log\log x)^4}{\log^3 x}\right),$$

with

$$\delta_g(0,q)=\frac{1}{q-1}-\sum_{j=1}^{\infty}\left(\frac{1}{[K_{q^j,q^j}:\mathbb{Q}]}-\frac{1}{[K_{q^{j+1},q^j}:\mathbb{Q}]}\right). \tag{14}$$

Note that $N_g(1,q;0,q)(x)=N_g(0,q)(x)$ and hence $\delta_g(1,q;0,q)=\delta_g(0,q)$. The density $\delta_g(0,q)$ can be explicitly evaluated using Lemma 3:

$$\delta_g(1,q;0,q)=\delta_g(0,q)=\frac{q^{1-\nu_q(h)}}{q^2-1}. \tag{15}$$

Let us now assume GRH. Using Theorem 2 it is inferred that

$$\delta_g(0,q)=\delta_g^{(0)}(0,q)=\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{[K_{[q,n]t,nt}:\mathbb{Q}]}=S_1+S_q,$$

say, where in S_1 we take together those n with $q\nmid n$ and in S_q those with $q|n$. We have

$$S_1=\sum_{v=1}^{\infty}\frac{\sum_{n|v,\ q\nmid n}\mu(n)}{[K_{qv,v}:\mathbb{Q}]}=\sum_{j=0}^{\infty}\frac{1}{[K_{q^{j+1},q^j}:\mathbb{Q}]},$$

and similarly

$$S_q=\sum_{t=1}^{\infty}\sum_{\substack{n=1\\ q|n}}^{\infty}\frac{\mu(n)}{[K_{nt,nt}:\mathbb{Q}]}=-\sum_{t=1}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)}{[K_{qnt,qnt}:\mathbb{Q}]}=-\sum_{j=1}^{\infty}\frac{1}{[K_{q^j,q^j}:\mathbb{Q}]},$$

On adding S_q to S_1 we find (14) on noting that $[K_{q,1}:\mathbb{Q}]=q-1$.

4.2 The case where $q\nmid a$

In the remainder of this section we assume that $q\nmid a$. Recall that $q^*=(\frac{-1}{q})q$.

Proposition 1 (GRH). *Let q be an odd prime and $q\nmid a$. Then*

$$\delta_g(1,q;a,q)=\frac{1}{(q-1)^2}\left(1-\frac{q^{1-\nu_q(h)}}{q+1}\right).$$

In particular $\delta_g(1,q;a,q)\in\mathbb{Q}_{>0}$ and does not depend on a.

Proof. The density $\delta_g(1,q;a,q)$ equals, using (13) and Lemma 7,

$$\sum_{\substack{t=1,\ q|t\\ (1+ta,q)=1}}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)c_g(1+ta,qt,nt)}{[K_{qnt,nt}:\mathbb{Q}]}=\sum_{\substack{t=1\\ q|t}}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)c_g(1+ta,q^{1+\nu_q(t)},nt)}{[K_{qnt,nt}:\mathbb{Q}]}.$$

Suppose that $q\nmid n$. By Lemma 6 it follows that $\mathbb{Q}(\zeta_{q^{1+\nu_q(t)}})\cap K_{nt,nt}=\mathbb{Q}(\zeta_{q^{\nu_q(t)}})$. Since $1+ta\equiv 1(\mathrm{mod}\ q^{\nu_q(t)})$, the automorphism σ_{1+ta} in Theorem 2 acts like the identity on the latter field intersection and hence $c_g(1+ta,q^{1+\nu_q(t)},nt)=1$. We thus infer that

$$\delta_g(1,q;a,q)=\sum_{\substack{t=1\\ q|t}}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)}{[K_{qnt,nt}:\mathbb{Q}]}. \tag{16}$$

In particular, it follows that $\delta_g(1,q;a,q)$ does not depend on a. We present two ways to complete the proof from this point onwards.
First way. From (16) we infer that

$$\delta_g(1,q;a,q)=\sum_{t=1}^{\infty}\sum_{n=1,\ q\nmid n}^{\infty}\frac{\mu(n)}{[K_{q^2nt,qnt}:\mathbb{Q}]}=\sum_{v=1}^{\infty}\frac{\sum_{n|v,\ q\nmid n}\mu(n)}{[K_{q^2v,v}:\mathbb{Q}]}=\sum_{j=1}^{\infty}\frac{1}{[K_{q^{j+1},q^j}:\mathbb{Q}]}.$$

Using Lemma 3 the latter sum is easily evaluated.
Second way. Note that $\sum_{0\le a\le q-1}\delta_g(1,q;a,q)$ equals the density of primes p with $p\equiv 1(\mathrm{mod}\ q)$ and hence

$$\sum_{0\le a\le q-1}\delta_g(1,q;a,q)=\frac{1}{q-1}. \tag{17}$$

Since, provided that $q\nmid a$, $\delta_g(1,q;a,q)$ does not depend on a, we conclude from (17) that

$$\delta_g(1,q;a,q)=\frac{1}{q-1}\left(\frac{1}{q-1}-\delta_g(1,q;0,q)\right).$$

Now invoke (15). □

`Remark 4`. Using a different method in [5, Theorem 10] the values of $\delta_g(1,3^s;a,3)$ for $s\ge 1$ were calculated.

In order to determine $\delta_g(a,q)$ it turns out to be convenient to determine $\delta_g(a,d)-\delta_g(1,q;a,q)$ first.

Lemma 12 (GRH). *Let q be an odd prime and $q\nmid a$. Then*

$$\delta_g(a,q)-\delta_g(1,q;a,q)=$$

$$\frac{1}{q-1}-\sum_{\substack{v=1\\ q\nmid v}}^{\infty}\frac{\sum_{t\equiv -\frac{1}{a}(\mathrm{mod}\ q),\ t|v}\mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]}-\sum_{\substack{v=1,\ q\nmid v\\ \sqrt{q^*}\in K_{v,v}}}^{\infty}\frac{\sum_{(\frac{ta+1}{q})=-1,\ t|v}\mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]}.$$

Proof. By Theorem 1, Theorem 2 and Lemma 7 we infer that

$$\delta_g(a,q)-\delta_g(1,q;a,q)=\sum_{\substack{t=1,\ q\nmid t\\ (1+ta,q)=1}}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)c_g(1+ta,q,nt)}{[K_{qnt,nt}:\mathbb{Q}]}.$$

Let us restrict now to values of n and t that occur in the latter double sum. We have, by Lemma 5,

$$\mathbb{Q}(\zeta_q)\cap K_{nt,nt}=\begin{cases}\mathbb{Q}(\sqrt{q^*}) & \text{if } \sqrt{q^*}\in K_{nt,nt};\\ \mathbb{Q} & \text{otherwise.}\end{cases}$$

Using Lemma 2 it then follows that

$$c_g(1+ta,q,nt)=\begin{cases}(1+(\frac{q^*}{1+ta}))/2 & \text{if } \sqrt{q^*}\in K_{nt,nt};\\ 1 & \text{otherwise.}\end{cases}$$

By the properties of the Kronecker symbol we have $(q^*/1+ta)=(1+ta/q)$, where the symbol $(1+ta/q)$ is just the Legendre symbol. We can thus write $\delta_g(a,q)-\delta_g(1,q;a,q)=J_1-J_2$, where

$$J_1=\sum_{\substack{t=1,\ q\nmid t\\ (1+ta,q)=1}}^{\infty}\sum_{\substack{n=1\\ q\nmid n}}^{\infty}\frac{\mu(n)}{[K_{qnt,nt}:\mathbb{Q}]}\ \text{and}\ J_2=\sum_{\substack{t=1,\ q\nmid t\\ (\frac{1+ta}{q})=-1}}^{\infty}\sum_{\substack{n=1,\ q\nmid n\\ \sqrt{q^*}\in K_{nt,nt}}}^{\infty}\frac{\mu(n)}{[K_{qnt,nt}:\mathbb{Q}]}. \tag{18}$$

On writing $nt = v$ we obtain

$$
\begin{aligned}
J_1 &= \sum_{\substack{v=1 \\ q\nmid v}}^{\infty} \frac{\sum_{t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]} - \sum_{\substack{v=1 \\ q\nmid v}}^{\infty} \frac{\sum_{t\equiv -\frac{1}{a} (\mathrm{mod}\ q),\ t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]} \\
&= \frac{1}{q-1} - \sum_{\substack{v=1 \\ q\nmid v}}^{\infty} \frac{\sum_{t\equiv -\frac{1}{a} (\mathrm{mod}\ q),\ t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]} \qquad (19)
\end{aligned}
$$

and

$$J_2 = \sum_{\substack{v=1,\ q\nmid v \\ \sqrt{q^*}\in K_{v,v}}}^{\infty} \frac{\sum_{(\frac{1+ta}{q})=-1,\ t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]}.$$

On combining these expressions with $\delta_g(a,q) - \delta_g(1,q;a,q) = J_1 - J_2$, the result follows. □

`Example`. For $q = 3$ and $3 \nmid a$ we obtain, on GRH, that

$$
\begin{aligned}
\delta_g(2,3;1,3) &= \frac{1}{2} - \sum_{\substack{v=1 \\ 3\nmid v}}^{\infty} \frac{\sum_{t\equiv 2(\mathrm{mod}\ 3),\ t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]} - \sum_{\substack{v=1,\ 3\nmid v \\ \sqrt{-3}\in K_{v,v}}}^{\infty} \frac{\sum_{t\equiv 1(\mathrm{mod}\ 3),\ t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]}. \\
&= \frac{1}{2} - \sum_{\substack{v=1 \\ 3\nmid v}}^{\infty} \frac{\sum_{t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]} + \sum_{\substack{v=1,\ 3\nmid v \\ \sqrt{-3}\notin K_{v,v}}}^{\infty} \frac{\sum_{t\equiv 1(\mathrm{mod}\ 3),\ t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]}. \\
&= \sum_{\substack{v=1,\ 3\nmid v \\ \sqrt{-3}\notin K_{v,v}}}^{\infty} \frac{\sum_{t\equiv 1(\mathrm{mod}\ 3),\ t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]}.
\end{aligned}
$$

More generally, we have

$$\delta_g(2,3;a,3) = \sum_{\substack{v=1,\ 3\nmid v \\ \sqrt{-3}\notin K_{v,v}}}^{\infty} \frac{\sum_{t\equiv a(\mathrm{mod}\ 3),\ t|v} \mu(\frac{v}{t})}{[K_{3v,v}:\mathbb{Q}]}.$$

Rewriting this expression in terms of $h'_\chi s$ we obtain Theorem 11 of [5].

We now have the ingredients to establish the following result.

Theorem 3 (GRH). *Let q be an odd prime and $q \nmid a$. Then*

$$\delta_g^{(0)}(a,q) = \frac{1}{q-1} + \frac{1}{(q-1)^2}\left(1 - \frac{q^{1-\nu_q(h)}}{q+1}\right) - \sum_{\substack{v=1 \\ q\nmid v}}^{\infty} \frac{\sum_{t\equiv -\frac{1}{a} (\mathrm{mod}\ q),\ t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]} \qquad (20)$$

and

$$\delta_g(a,q) = \delta_g^{(0)}(a,q) - \sum_{\substack{v=1,\ q\nmid v \\ \sqrt{q^*}\in K_{v,v}}}^{\infty} \frac{\sum_{(\frac{ta+1}{q})=-1,\ t|v} \mu(\frac{v}{t})}{[K_{qv,v}:\mathbb{Q}]}. \qquad (21)$$

If $q \nmid D(g_0)$, then $\delta_g(a,q) = \delta_g^{(0)}(a,q)$.

Proof. Considering the terms with $q|t$ and $q\nmid t$ in the double sum for $\delta_g(a,q)$ separately we have, using (16) and (18) that $\delta_g^{(0)}(a,q)=\delta_q(1,q;a,q)+J_1$. On using Proposition 1, (19) and Lemma 12 the first two assertions are established.

If $q\nmid D(g_0)$, then by Lemma 6 there is no integer v such that $q\nmid v$ and $\sqrt{q^*}\in K_{v,v}$ and hence the double sum in (21) equals zero. □

`Remark 5`. The double sum (21) can be rewritten, using Lemma 6, as

$$\sum_{v=1,\ q\nmid v}^{\infty}\left(\frac{2}{[K_{qv,v}:\mathbb{Q}]}-\frac{2}{(q-1)[K_{v,v}:\mathbb{Q}]}\right)\sum_{(\frac{ta+1}{q})=-1,\ t|v}\mu(\frac{v}{t}).$$

`Remark 6`. Note that the proof of Theorem 3 makes essential use of the law of quadratic reciprocity (this law enters in the proof of Lemma 12).

Let G_q be the character group associated to $(\mathbb{Z}/\mathbb{Z}q)^*$. Since the latter group is abelian it follows that $G_q\cong(\mathbb{Z}/\mathbb{Z}q)^*$. The principal character mod q will be denoted by χ_0. Recall that o_χ denotes the order of χ considered as an element of G_q.

Using (4) and the following lemma, $\delta_g(a,q)$ and $\delta_g^{(0)}(a,q)$ can be expressed as simple linear combinations of the constants C_χ introduced in Section 2.1. Each such constant can be explicitly evaluated and is of the form cA_χ with $c\in\mathbb{Q}(\zeta_{o_\chi})$. This allows one to explicitly evaluate $\delta_g(a,q)$ and $\delta_g^{(0)}(a,q)$. For reasons of space we only will work this out in the case $g\in\mathcal{G}$, where $\mathcal{G}$ is the set of rational numbers g that can not be written as $-g_0^h$ or g_0^h with $h>1$ an integer and g_0 a rational number.

Lemma 13 [5, Lemma 11]. *Let r,s be integers with $s|r$. Let χ be a Dirichlet character. Then, if $g>0$ or $g<0$ and s is even,*

$$\sum_{(r,v)=1}\frac{h_\chi(v)}{[K_{sv,v}:\mathbb{Q}]}=\frac{1}{\varphi(s)}\left(C_\chi(h,r,1)+C_\chi(h,r,\frac{n_s}{(n_s,s)})\right).$$

When $g<0$ and s is odd, the latter sum equals

$$\frac{1}{\varphi(s)}\left(C_\chi(h,r,1)-\frac{1}{2}C_\chi(h,r,2)+\frac{1}{2}C_\chi(h,r,2^{\nu_2(h)+1})+C_\chi(h,r,\frac{n_s}{(n_s,s)})\right).$$

Now we can formulate one of our main results.

Theorem 4 (GRH). *Let q be an odd prime and $q\nmid a$. We have*

$$\delta_g(a,q)=\sum_{\chi\in G_q}\chi(-a)c_\chi A_\chi,$$

where $c_\chi\in\mathbb{Q}(\zeta_{o_\chi})$ may depend on q and g (but not a) and can be explicitly evaluated.

Proof. We only deal with the case where $g > 0$ or $g < 0$ and h is odd (the remaining more space consuming case being left to the reader).

Using the identity (4) and Lemma 13 we can rewrite (20) as

$$\delta_g^{(0)}(a,q) = \frac{1}{q-1}+$$

$$\frac{1}{(q-1)^2}\left(1-\frac{q^{1-\nu_q(h)}}{q+1}\right)-\frac{1}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)\left(C_\chi(h,q,1)+C_\chi(h,q,\frac{n_q}{(n_q,q)})\right).$$

Similarly, using Remark 5, we rewrite (21) as

$$\delta_g(a,q) = \delta_g^{(0)}(a,q)-\frac{2}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)\left(C_\chi(h,q,\frac{n_q}{(n_q,q)})-C_\chi(h,q,n_1)\right)\sum_b\overline{\chi(b)},$$

where the sum is over the integers $1 \le b \le q-1$ for which $(\frac{1-b}{q}) = -1$. Note that $n_1 = n_q$. If $q \nmid D(g_0)$, the latter double sum equals zero and we infer (as before) that $\delta_g(a,q) = \delta_g^{(0)}(a,q)$. If $q|D(g_0)$, then $C_\chi(h,q,n_1) = 0$ and we infer that

$$\delta_g(a,q) = \delta_g^{(0)}(a,q) - \frac{2}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)C_\chi(h,q,\frac{n_1}{q})\sum_b\overline{\chi(b)}.$$

By [5, Lemma 10] we can write $C_\chi(h,r,s)$ as cA_χ with $c \in \mathbb{Q}(\zeta_{o_\chi})$, where c can be explicitly given. Using this the proof is easily completed. □

The following result is an easy consequence of the latter proof.

Proposition 2 (GRH). *If h is odd and $8|D(g)$, then $\delta_g(a,q) = \delta_{-g}(a,q)$.*

Proof. The assumptions imply that $|n_1| = |n_q| = |D(\pm g)|$ and on noting that $C_\chi(h,r,s) = C_\chi(h,r,-s)$ the character sum expression for $\delta_g(a,q)$ given in the proof of Theorem 4 is seen to equal that of $\delta_{-g}(a,q)$ in case $q \nmid a$. If $q|a$, then by (15) we have $\delta_{\pm g}(0,q) = q/(q^2-1)$. □

The following result demonstates Theorem 4 in the special (but important) case where $g \in \mathcal{G}$. Since almost all integers are in $\mathcal{G}$ it can be thought of as the set of 'generic' integers g.

Proposition 3 (GRH). *Let q be an odd prime and $q \nmid a$. Suppose that $g \in G$. Put*

$$\epsilon_g(\chi) = \begin{cases} 1 & \text{if } 2 \nmid D(g); \\ \frac{\chi(2)}{4} & \text{if } 4||D(g); \\ \frac{\chi(2)^2}{16} & \text{if } 8|D(g). \end{cases}$$

If $q \nmid D(g)$, then $\delta_g(a,q)$ equals $\delta_q^0(a,q)$ which on its turn equals

$$\frac{q^2}{(q-1)(q^2-1)} - \frac{1}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)A_\chi\left(1+\epsilon_g(\chi)\prod_{p|2D(g)}\frac{p(\chi(p)-1)}{p^3-p^2-p+\chi(p)}\right).$$

If $q|D(g)$, then

$$\delta_g(a,q)=\frac{q^2}{(q-1)(q^2-1)}$$

$$-\frac{1}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)A_\chi\left(1+\epsilon_g(\chi)\Big\{1+2\sum_b\overline{\chi(b)}\Big\}\prod_{p|2D(g)/q}\frac{p(\chi(p)-1)}{p^3-p^2-p+\chi(p)}\right),$$

where the sum is over all $1\le b\le q-1$ for which $(\frac{1-b}{q})=-1$.

Proof. By Lemma 3 we have

$$n_1=n_q=\begin{cases}[2,D(g_0)] & \text{if } g>0;\\ D(g_0)/2 & \text{if } g<0 \text{ and } D(g)\equiv 4(\text{mod } 8);\\ [4,D(g_0)] & \text{if } g<0 \text{ and } D(g)\not\equiv 4(\text{mod } 8).\end{cases}$$

Now note that $n_1=n_q=[2,D(g)]$ and that $q|D(g_0)$ iff $q|D(g)$. Working out the formulae involving the C_χ's in the proof of Theorem 4 using [5, Lemma 10], the proof is then completed. □

In case $d=q$ is an odd prime it can be shown [3, Proposition 7] that

$$\delta(a,q)=\frac{q^2}{(q-1)(q^2-1)}-\frac{1}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)A_\chi.$$

Assume GRH. Proposition 3 shows that if $g\in G$ and $|g|$ tends to infinity, then $\delta_g(a,q)$ tends to $\delta(a,q)$. Proposition 3 also shows that if $g\in G$ and $D(g)$ contains a prime divisor p with $p\equiv 1(\text{mod } q)$, then $\delta_g(a,q)=\delta(a,q)$. It is not difficult to show that almost all integers g with $|g|\le x$ satisfy $g\in G$ and have $D(g)$ with at least one prime divisor p with $p\equiv 1(\text{mod } q)$. Consequently we infer that for almost all integers g with $|g|\le x$, $\delta_g(a,q)=\delta(a,q)$. This shows that for a 'generic' g, $\delta_g(a,q)=\delta(a,q)$. Furthermore, if $\delta_g(a,q)$ is not equal to $\delta(a,q)$ then usually it will be quite close to it. The results in the next section allow one to also draw these conclusions in the case where d is an odd prime power, which will be done in the final section.

5 The case where d is an odd prime power

The case where $d=q^s$ with q an odd prime is easily reduced to the case $d=q$ by the following result.

Theorem 5 (GRH). *Suppose that $d|d_1$, the quotient d_1/d is odd and $\omega(d_1)=\omega(d)$. Then $\delta_g(a,d_1)=\frac{d}{d_1}\delta_g(a,d)$.*

Corollary 1 *Let q be an odd prime and $j\ge 1$, then $\delta_g(a,q^j)=q^{1-j}\delta_g(a,q)$.*

`Remark 7`. From formula (3) for $\delta(a,d)$ it is easily inferred that if $d|d_1$ and $\omega(d_1)=\omega(d)$, then

$$\delta(a,d_1)=\frac{d}{d_1}\delta(a,d). \tag{22}$$

Proof of Theorem 5. If $d|d_1$ and $\omega(d)=\omega(d_1)$ and for all n and t with $(1+ta,d)=1$, $(n,d)|a$ and n is squarefree, we have $[K_{[d_1,n]t,nt}:\mathbb{Q}]/\varphi(d_1)=[K_{[d,n]t,nt}:\mathbb{Q}]/\varphi(d)$, then using Lemma 8 we infer that

$$\begin{aligned}\delta_g(a,d_1) &= \sum_{\substack{t=1\\(1+ta,d)=1}}^{\infty}\sum_{\substack{n=1\\(n,d)|a}}^{\infty}\frac{\mu(n)c_g(1+ta,d_1t,nt)}{[K_{[d_1,n]t,nt}:\mathbb{Q}]}\\ &= \frac{d}{d_1}\sum_{\substack{t=1\\(1+ta,d)=1}}^{\infty}\sum_{\substack{n=1\\(n,d)|a}}^{\infty}\frac{\mu(n)c_g(1+ta,dt,nt)}{[K_{[d,n]t,nt}:\mathbb{Q}]}=\frac{d}{d_1}\delta_g(a,d),\end{aligned}$$

where we used that $\varphi(d_1)/\varphi(d)=d_1/d$. Now invoke Lemma 4. □

6 Connection between $\delta_g(a,q^s)$ and $\delta(a,q^s)$

Define $\overline{\delta}_g(d)=(\delta_g(0,d),\delta_g(1,d),\ldots,\delta_g(d-1,d))$ (if this exists) and

$$\overline{\delta}(d)=(\delta(0,d),\delta(1,d),\ldots,\delta(d-1,d)).$$

The next result implies that, under GRH, for almost all integers g we have $\overline{\delta}_g(q^s)=\overline{\delta}(q^s)$.

Theorem 6 (GRH). *Let $s\geq 1$ and q an odd prime. Then there are at most $O_q(x\log^{-1/(q-1)}x)$ integers g with $|g|\leq x$ such that $\overline{\delta}_g(q^s)\neq\overline{\delta}(q^s)$.*

Proof. By the results of the previous section it is enough to prove this in case $s=1$. By [3, Proposition 7] we have, for $1\leq a\leq q-1$,

$$\delta(a,q)=\frac{q^2}{(q-1)(q^2-1)}-\frac{1}{(q-1)^2}\sum_{\chi\in G_q}\chi(-a)A_\chi,$$

and $\delta(0,q)=q/(q^2-1)$ [3, Theorem 1]. Since there are at most $O(\sqrt{x}\log x)$ integers $|g|\leq x$ that are not in $\mathcal{G}$, we can restrict ourselves to the case where $g\in\mathcal{G}$. For such a g we then have, using (15), that $\delta_g(0,q)=\delta(0,q)$. Now by Proposition 3 we infer that if $D(g)$ has a prime divisor p with $p\equiv 1(\mathrm{mod}\ q)$, then $\overline{\delta}_g(q)=\overline{\delta}(q)$ (since $\chi(p)=1$ for every $\chi\in G_q$). It follows that the number of $g\in\mathcal{G}$ with $|g|\leq x$ such that $\overline{\delta}_g(q)\neq\overline{\delta}(q)$ is bounded above by the number of g with $|g|\leq x$ such that $D(g)$ has no prime factor p satisfying $p\equiv 1(\mathrm{mod}\ q)$. By Lemma 10 the proof is then completed. □

If it is not true that $\overline{\delta}_g(q^s)=\overline{\delta}(q^s)$, then our final result shows that $\overline{\delta}_g(q^s)$ will be close to $\overline{\delta}(q^s)$,

Proposition 4 (GRH). *Suppose that $g\in\mathcal{G}$. As $|g|$ tends to infinity, $\overline{\delta}_g(q^s)$ tends to $\overline{\delta}(q^s)$.*

Proof. A simple consequence of the results in the previous section, the formula given for $\delta(a,q)$ in the proof of Theorem 6, (15) and Proposition 3. □

For a numerical illustration of the latter result (with $g = -19$) see Table 1 of [3].

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