POISSON AND SYMPLECTIC STRUCTURES ON LIE ALGEBRAS. I

D.V. Alekseevsky* and A.M. Perelomov**

**
Departamento de Fisica Teórica
University de Valencia
46100 Burjassot (Valencia)
Spain

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-33

POISSON AND SYMPLECTIC STRUCTURES ON LIE ALGEBRAS. I .

D.V. Alekseevsky *

Max-Planck-Institute für Mathematik,
Gottfried-Claren-Strasse 26,
D-53225 Bonn, Germany

and

A.M. Perelomov **

Departamento de Fiśica Teórica,
Univ. de Valencia
46100 Burjassot (Valencia), Spain

* E-mail: dalekseе@mpim-bonn.mpg.de

** On leave from Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia; E-mail: perelomo@evalvx.ific.uv.es

Contents

Introduction

The purpose of this paper is to describe a new class of Poisson brackets on simple Lie algebras and symplectic structures on some solvable Lie algebras. This gives a new class of solutions of the classical Yang–Baxter equation.

Let us first recall some basic facts on the classical Yang–Baxter equation, referring the reader for more details to the well-known paper by Belavin and Drinfel'd [BD 1982].

The classical Yang–Baxter equation (CYBE) is the functional equation

$$[X_{12}(\lambda_1,\lambda_2),\, X_{13}(\lambda_1,\lambda_3)] + [X_{12}(\lambda_1,\lambda_2),\, X_{23}(\lambda_2,\lambda_3)] + [X_{13}(\lambda_1,\lambda_3),\, X_{23}(\lambda_2,\lambda_3)] = 0 \tag{0.1}$$

for the function $X(\lambda,\,\mu)$ taking the values in $\mathcal{G}\otimes\mathcal{G}$, where $\mathcal{G}$ is the Lie algebra. In order to define the quantity $X_{12}(\lambda_1,\lambda_2)$, following [BD 1982], we fix an associative algebra $\mathcal{A}$ with unit, which contains $\mathcal{G}$ and the linear maps $\varphi_{12},\varphi_{23}$ and φ_{13}, so that

$$\varphi_{12}\colon \mathcal{G}\otimes\mathcal{G}\to\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A},\quad \varphi_{12}(a\otimes b)=a\otimes b\otimes 1 \tag{0.2}$$

and analogously for maps φ_{23} and φ_{13}.

Note that if $X(\lambda,\,\mu)$ is a solution of eq. (0.1) and $\varphi(u)$ is a function with values in $\mathcal{G}$, then $\tilde X(\lambda,\mu) = (\varphi(\lambda)\otimes\varphi(\mu))\,X(\lambda,\mu)$ is also a solution of (0.1) and we will consider the solutions X and $\tilde X$ as equivalent. Let us introduce the following definition.

Definition 0.1. *The function $X(\lambda,\mu)$ is invariant relative to $g\in\operatorname{Aut}\mathcal{G}$, if*

$$(g\otimes g)\,X(\lambda,\mu) = X(\lambda,\mu).$$

The set of all such g forms the group that is called the invariance group of $X(\lambda,\mu)$. The function $X(\lambda,\mu)$ is said to be invariant with respect to $h\in\mathcal{G}$ if

$$[h\otimes 1 + 1\otimes h,\; X(\lambda,\mu)] = 0,$$

i.e. if it is invariant relative to $\exp\{\operatorname{ad} h\}$ for any t.

Note that if $X(\lambda,\mu)$ is a solution of (0.1), which is invariant relative to the subalgebra $\mathcal{H}\subset\mathcal{G}$, and if a tensor r from $\mathcal{H}\otimes\mathcal{H}$ satisfies the following Yang-Baxter equation

$$[r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] = 0, \tag{0.3}$$

$$r_{21} = -r_{12}, \tag{0.4}$$

then the function $\tilde X(\lambda,\mu) = X(\lambda,\mu) + r$ is also a solution of (0.1). Note also that if the algebra $\mathcal{H}$ is Abelian, then (0.3) is satisfied automatically.

It is usually supposed that the Lie algebra $\mathcal{G}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$. In [BD 1982] the solutions of (0.1) have been studied in details, such that:

(i) $X(\lambda,\mu)$ is a meromorphic function; $\lambda,\mu\subset\mathcal{D}$, $\mathcal{D}$ is a domain in $\mathbb{C}$;

(ii) the determinant of the matrix formed by the coordinates of the tensor $X(\lambda,\mu)$ is not identically zero;

(iii) $X(\lambda,\mu)$ depends only on the difference $(\lambda-\mu)$.

In fact, as was shown in [BD 1983], the condition (iii) indeed follows from (i) and (ii).

In the paper [BD 1982] it was shown that solutions of (0.1) are of three types:

a) Elliptic solutions,
b) Trigonometric solutions,
c) Rational solutions,

and all elliptic and trigonometric solutions were found. As for rational solutions, only few of them were found, and the main purpose of this paper is to extend this class.

The paper is organized as follows. In section 1 we recall some standard definitions and facts about Poisson structures. In the section 2 a decomposition of a Lie algebra $\mathcal{G}$ into sum of two subalgebras is considered and relations between Poisson and symplectic structures on $\mathcal{G}$ and its subalgebras are studied.

These results are used in the section 3 to describe explicitely closed 2-forms and Poisson structures on the elementary Lie algebra $\mathcal{E}_{n+1}$ which is the Iwasawa subalgebra of $su(2,n)$. The main result of this section is Theorem 3.7. It reduces the description of Poisson and symplectic structures of a Lie algebra $\mathcal{G}$, which is semi-direct sum of a subalgebra $\mathcal{F}$ and the ideal $\mathcal{E}_{n+1}$ to the description of such structures on $\mathcal{F}$. This gives an algorithm for description of all closed 2-forms and of symplectic structures on any Lie algebra which is decomposed into semi-direct sum of elementary subalgebras. In the section 4 we construct canonical decomposition of the Borel subalgebra $\mathcal{B}(\mathcal{G})$ of a semisimple Lie algebra $\mathcal{G}$ into semi-derect sum of elementary subalgebras (plus, may be, a commutative subalgebra of the Cartan subalgebra). Applying the results of the section 3, we obtain a description of closed 2-forms and symplectic forms (if they exist) on the Borel subalgebra $\mathcal{B}(\mathcal{G})$ of a semisimple Lie algebra $\mathcal{G}$. As a biproduct, we get description of the second cohomology group $H^2(\mathcal{B}(\mathcal{G}))$.

Acknowledgments

Both authors thank the Erwin Schrödinger Institute for Mathematical Physics where the work has been began, D.A. thanks Max-Planck-Institute für Mathematik, Bonn and A.P. thanks Depart. de Fisica Teorica of University of Valencia, Spain for hospitality and support.

1. POISSON BRACKETS ON A LIE ALGEBRA

1.1. The Schouten bracket in the space of polyvectors and Poisson bivectors

Let $\mathcal{G}$ be a Lie algebra and $\wedge\mathcal{G} = \sum \wedge^i \mathcal{G}$ be the exterior algebra over $\mathcal{G}$. The Lie bracket on $\mathcal{G}$ determines naturally the bracket on $\wedge\mathcal{G}$:

$$[x_1 \wedge \ldots \wedge x_p, \qquad y_1 \wedge \ldots \wedge y_q] =$$

$$= \Sigma(-1)^{p-i+j-1} x_1 \wedge \ldots \hat{x}_i \ldots \wedge x_p \wedge [x_i, y_j] \wedge y_1 \wedge \ldots \hat{y}_j \ldots \wedge y_q;$$

$$x_1, \ldots, x_p, y_1, \ldots, y_q \in \mathcal{G}.$$

This bracket is called the Schouten bracket and it turns the space $\wedge\mathcal{G}$ into a graded Lie superalgebra.

Definition 1.1.

1. *A bivector $\Lambda \in \wedge^2\mathcal{G}$ is called a Poisson bivector (or a Poisson bracket in a Lie algebra $\mathcal{G}$) if it commutes with itself*

$$[\Lambda, \Lambda] = 0 \tag{1.1}$$

(this is equivalent to the classical Yang–Baxter equation 0.3.).

2. *Two Poisson bivectors Λ_1, Λ_2 are called compatible if they commute:*

$$[\Lambda_1, \Lambda_2] = 0. \tag{1.2}$$

Note that this is equivalent to the fact that any linear combination $\lambda\Lambda_1 + \mu\Lambda_2$ is a Poisson bivector.

If $\{e_i\}$ is a basis in $\mathcal{G}$, then Λ may be written as

$$\Lambda = \sum \Lambda^{ij} e_i \wedge e_j$$

and

$$[\Lambda, \Lambda] = \sum \Lambda^{ij} \Lambda^{kl} [e_i \wedge e_j, e_k \wedge e_l],$$

where the bracket of two simple bivectors is given by

$$[x \wedge y, u \wedge v] = [x, u] \wedge y \wedge v + x \wedge [y, u] \wedge v + y \wedge [x, v] \wedge u + x \wedge u \wedge [y, v].$$

1.2. Poisson bracket induced by a Poisson bivector on a $\mathcal{G}$-manifold

Let M be a $\mathcal{G}$-manifold, i.e. a manifold with fixed homomorphism

$$\varphi : \mathcal{G} \to \mathcal{X}(M)$$

of a Lie algebra $\mathcal{G}$ into the Lie algebra $\mathcal{X}(M)$ of vector fields on M.

The homomorphism φ may be extended to a homomorphism

$$\varphi : \wedge\mathcal{G} \to \wedge(M)$$

of the Lie superalgebra of polyvectors on $\mathcal{G}$ into the Lie superalgebra of polyvector fields on M (relative to the Schouten bracket).

In particular, the Poisson bivector $\Lambda = \Lambda^{ij} e_i \wedge e_j \in \wedge^2\mathcal{G}$ determines the bivector $\mathcal{G}(\Lambda) = \Lambda^{ij}\varphi(e_i) \wedge \varphi(e_j)$ on the manifold, such that $[\varphi(\Lambda), \varphi(\Lambda)] = 0$. This bivector defines a Poisson bracket in the space of functions on the manifold according to the formula

$$\{f, g\} = \varphi(\Lambda)(df, dg) = \Lambda^{ij}(X_i \cdot f)\,(X_j \cdot g), \quad X_i = \varphi(e_i).$$

In particular, because the Lie algebra $\mathcal{G}$ acts naturally in $\mathcal{G}^*$ and also in $\mathcal{G}$, the Poisson bivector Λ determines Poisson brackets in the spaces of functions on $\mathcal{G}^*$ and $\mathcal{G}$. These Poisson brackets are defined by

$$\{e_i, e_j\} = \Lambda^{ij}, \tag{1.3}$$

where $\{e_j\}$ is a basis of the space $\mathcal{G}$, which is considered as the space of linear functions on $\mathcal{G}^*$ and respectively by

$$\{\xi, \eta\}_\Lambda = \sum \Lambda^{ij}(\mathrm{ad}^*_{e_i}\xi)(\mathrm{ad}^*_{e_j}\eta), \quad \xi, \eta \in \mathcal{G}^*.$$

Note that bivector fields, corresponding to brackets, have the form

$$\Lambda_{\mathcal{G}^*} = \Lambda^{ij} e_i \wedge e_j, \quad \Lambda_{\mathcal{G}} = \Lambda^{ij} C^a_{ik} C^b_{jl} x^k x^l e^*_a \wedge e^*_b,$$

where e^*_a is the basis of $\mathcal{G}^*$ dual to the basis e_i of $\mathcal{G}$, and C^a_{ik} are structure constants of $\mathcal{G}$.

1.3. Support of a Poisson bracket and symplectic structures on Lie algebras

Let Λ be a bivector on a Lie algebra $\mathcal{G}$. Then Λ determines the linear mapping

$$\Lambda : \mathcal{G}^* \to \mathcal{G}, \quad \xi \to \Lambda \cdot \xi = i_\xi \Lambda.$$

Definition 1.2. *The subspace* supp$\Lambda = \Lambda(\mathcal{G}^*)$*, which is the image of this linear mapping, is called the support of the bivector* $\Lambda \in \wedge^2 \mathcal{G}_\Lambda$.

Lemma 1.3. *The support* suppΛ *of the Poisson bivector* Λ *of the Lie algebra* $\mathcal{G}$ *is the Lie subalgebra of* $\mathcal{G}$.

Recall that a symplectic form (or a symplectic structure) on a Lie algebra $\mathcal{G}$ is a closed non-degenerate two-form $\omega \in \wedge^2 \mathcal{G}^*$.

The closeness condition means that

$$0 = d\omega(x, y, z) = \sigma_{x,y,z}\omega([x, y], z), \quad x, y, z \in \mathcal{G},$$

where $\sigma_{x,y,z}$ denotes the sum of cyclic permutations of x, y, z. If ω is a symplectic form, then the tensor $\Lambda = \omega^{-1}$ is a Poisson bivector. More generally, we have

Proposition 1.4. *Let* $\mathcal{A}$ *be a subalgebra of a Lie algebra* $\mathcal{G}$ *and* ω *be a symplectic form on* $\mathcal{A}$. *Then the inverse tensor*

$$\Lambda = \omega^{-1} \in \wedge^2 \mathcal{A} \subset \wedge^2 \mathcal{G}$$

is a Poisson bivector with support $\mathcal{A}$, *and any Poisson bivector may be obtained using this construction.*

Hence the classification problem for Poisson bivectors on a Lie algebra $\mathcal{G}$ reduces to the classification of Lie subalgebras $\mathcal{A} \subset \mathcal{G}$ with a symplectic form.

2. DECOMPOSITIONS OF A LIE ALGEBRA WITH A POISSON BIVECTOR OR A CLOSED 2-FORM

Proposition 2.1. *Let $\mathcal{G}$ be a Lie algebra with a non-degenerate Poisson bivector Λ and $\omega = \Lambda^{-1}$ the associated symplectic form.*

Let $\Lambda = \Lambda_1 + \Lambda_2$ be a decomposition of Λ into a sum of two bivectors Λ_i and $\mathcal{A}_i = \mathrm{supp}\Lambda_i$. Assume that $\mathcal{A}_1 \cap \mathcal{A}_2 = 0$. This means that $\mathcal{G} = \mathcal{A}_1 + \mathcal{A}_2$ is a decomposition of $\mathcal{G}$ into a sum of ω-non-degenerate subspaces and $\Lambda_i = (\omega|_{\mathcal{A}_i})^{-1}$.

Then

i) $\mathcal{A}_1$ is a subalgebra $\leftrightarrow$ Λ_1 is a Poisson bivector,
ii) $\mathcal{A}_1, \mathcal{A}_2$ are subalgebras $\leftrightarrow$ Λ_1 and Λ_2 are commuting Poisson bivectors,
iii) assertion (ii) holds if $[\Lambda_1, \Lambda_2] = 0$ or $\mathcal{A}_1$ is an ideal.

Proof. It follows immediately from the remarks that $\Lambda_i \in \wedge^2\mathcal{A}_i$ and

$$[\wedge^2\mathcal{A}_i, \wedge^2\mathcal{A}_i] \subset \wedge^2\mathcal{A}_i \wedge [\mathcal{A}_i, \mathcal{A}_i],$$

$$[\wedge^2\mathcal{A}_1, \wedge^2\mathcal{A}_2] \subset \mathcal{A}_1 \wedge \mathcal{A}_2 \wedge [\mathcal{A}_1, \mathcal{A}_2].$$

The assertion (iii) implies the following

Corollary 2.2. *Let $\mathcal{G}$ be a Lie algebra with a symplectic form ω and $\mathcal{A}$ is a non-degenerate ideal of $\mathcal{A}$. Then ω-orthogonal complement $\mathcal{A}^\perp$ to $\mathcal{A}$ in $\mathcal{G}$ is a subalgebra.*

Proposition 2.3. *Let $\mathcal{G} = \mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_1 \cap \mathcal{A}_2 = 0$ be a decomposition of a Lie algebra $\mathcal{G}$ into a direct sum of two ideals, and $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda'$ be the corresponding decomposition of a Poisson bivector $\Lambda, \Lambda_i \in \wedge^2\mathcal{A}_i, \Lambda' \in \mathcal{A}_1 \wedge \mathcal{A}_2$. Then Λ_1, Λ_2 are commuting Poisson bivectors.*

Proof. This follows from relations

$$[\wedge^2\mathcal{A}_i, \wedge^2\mathcal{A}_i] \subset \wedge^3\mathcal{A}_i$$

$$[\wedge^2\mathcal{A}_1, \mathcal{A}_1 \wedge \mathcal{A}_2] \subset \wedge^2\mathcal{A}_1 \wedge \mathcal{A}_2,$$

$$[\mathcal{A}_1 \wedge \mathcal{A}_2, \mathcal{A}_1 \wedge \mathcal{A}_2] \subset \mathcal{A}_1 \wedge (\wedge^2\mathcal{A}_2) + \wedge^2\mathcal{A}_1 \wedge \mathcal{A}_2.$$

Proposition 2.4. *Let $\mathcal{G} = \mathcal{A} + \mathcal{V}$ be a semi-direct sum of a Lie subalgebra $\mathcal{A}$ and a commutative ideal $\mathcal{V}$.*

Let Λ be a Poisson bivector and $\Lambda = \Lambda_{\mathcal{A}} + \Lambda_{\mathcal{V}} + \Lambda'$ be its decomposition.

Then

i) $\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{V}}$ are Poisson bivectors,
ii) $[\Lambda_{\mathcal{A}}, \Lambda'] = [\Lambda_{\mathcal{V}}, \Lambda'] = 0$,
iii) $[\Lambda', \Lambda'] + 2[\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{V}}] = 0$.

In particular, $\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{V}}$ are commuting bivectors iff Λ' is a Poisson bivector.

Corollary 2.5. *Under the notation of Proposition 2.4, assume moreover that the sum is direct, i.e. $\mathcal{V}$ is a central subalgebra. Then a bivector Λ with the decomposition $\Lambda = \Lambda_{\mathcal{A}} + \Lambda_{\mathcal{V}} + \Lambda'$ is a Poisson bivector if and only if*

i) $\Lambda_{\mathcal{A}}$ is a Poisson bivector,
ii) $\Lambda' \in C_{\mathcal{A}}(\Lambda_{\mathcal{A}}) \wedge \mathcal{V}$, where

$$C_{\mathcal{A}}(\Lambda_{\mathcal{A}}) = \{a \in \mathcal{A}, \mathrm{ad}\, a\, \Lambda_{\mathcal{A}} = 0\}$$

is the stability subalgebra of $\Lambda_{\mathcal{A}}$, *and*

iii) $\mathrm{supp}\Lambda' \cap \mathcal{A}$ *is a commutative Lie algebra. In this case* $\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{V}}, \Lambda'$ *are mutually commuting Poisson bivectors.*

Proof. Calculating the bracket, we obtain

$$[\Lambda, \Lambda] = [\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{A}},] + [\Lambda_{\mathcal{A}}, \Lambda'] + [\Lambda', \Lambda'].$$

Since the summands belong to the different homogeneous components, the left-hand side vanishes iff all summands of the right-hand side are equal to zero. It is easy to check that the condition $[\Lambda_{\mathcal{A}}, \Lambda'] = 0$ is equivalent to the condition (ii) and the condition $[\Lambda', \Lambda'] = 0$ gives (iii).

Proposition 2.6. *Let* $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_1 \cap \mathcal{A}_2 = 0$ *be a decomposition of a Lie algebra* $\mathcal{G}$ *into a sum of two subalgebras and* ω_i *be a symplectic form on* $\mathcal{A}_i, i = 1, 2$. *Then* $\omega = \omega_1 + \omega_2$ *is a symplectic form on* $\mathcal{G}$ *iff the natural representation* $\mathrm{ad}\mathcal{A}_i$ *of* $\mathcal{A}_i$ $(i = 1, 2)$ *into the space* $\mathcal{G}/\mathcal{A}_i \approx \mathcal{A}_{i'}$, $\{i, i'\} = \{1, 2\}$ *is symplectic, i.e. it preserves the symplectic form* $\omega_{i'}$.

Corollary 2.7. *Let* $(\mathcal{A}_i, \omega_i)$, $i = 1, 2$ *be two Lie algebras with symplectic forms and* $\varphi : \mathcal{A}_1 \to \mathrm{Der}(\mathcal{A}_2)$ *a representation of* $\mathcal{A}_1$ *by means of derivations of the Lie algebra* $\mathcal{A}_2$. *If the linear Lie algebra* $\varphi(\mathcal{A}_1)$ *is symplectic, i.e. if it preserves* ω_2, *then the semi-direct sum* $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ *has the symplectic form* $\omega = \omega_1 + \omega_2$.

Proof of Proposition 2.6. Let $a_i, b_i, c_i \in \mathcal{A}_i$, $i = 1, 2$. Then we have

$$d\omega(a_1, b_1, c_1) = d\omega_1(a_1, b_1, c_1) = 0,$$

$$d\omega(a_1, b_1, c_2) = \omega([a_1, b_1], c_2) + \omega([b_1, c_2], a_1) + \omega([c_2, a_1], b_1) =$$

$$\omega(ad_{c_2}a_1, b_1) + \omega(a_1, ad_{c_2}b_1) = (\mathrm{ad}c_2^*\omega_1)(a_1, b_1),$$

$$d\omega(a_2, b_2, c_1) = (ad^*_{c_1}\omega_2)(a_2, b_2).$$

Hence, $d\omega = 0 \Leftrightarrow \mathrm{ad}^*_{\mathcal{A}_1}\omega_2 = \mathrm{ad}^*_{\mathcal{A}_2}\omega_1 = 0$.

The following Lemma gives a description of closed two-forms on a semi-direct sum of two Lie algebras.

Lemma 2.8. *Let* $\mathcal{G} = \mathcal{A} + \mathcal{B}$ *be a semi-direct decomposition of a Lie algebra into a sum of a subalgebra* $\mathcal{A}$ *and an ideal* $\mathcal{B}$. *Then*

i) the $\wedge^2\mathcal{A}$*-component* $\Lambda_{\mathcal{A}}$ *of any Poisson bivector* Λ *on* $\mathcal{G}$ *is a Poisson bivector,*

ii) any closed 2-form ω *on* $\mathcal{G}$ *has the canonical decomposition*

$$\omega = \omega_{\mathcal{A}} + \omega_{\mathcal{B}} + \omega', \tag{2.1}$$

where $\omega_{\mathcal{A}} = \omega \mid \mathcal{A}, \omega_{\mathcal{B}} = \omega \mid \mathcal{B}$ *are closed forms on* $\mathcal{A}$ *and* $\mathcal{B}$ *(considered in the natural way as forms on* $\mathcal{G}$*) and* $\omega' \in \mathcal{A}^* \wedge \mathcal{B}^* \subset \wedge^2\mathcal{G}^*$ *is a 2-form that satisfies conditions:*

$$\omega'(a, [b, b']) = (\mathrm{ad}^*_a\omega_{\mathcal{B}})(b, b') = \omega_{\mathcal{B}}([a, b], b') + \omega_{\mathcal{B}}(b, [a, b']), \tag{2.2}$$

$$\omega'([a, a'], b) + \omega'([a', b], a) + \omega'([b, a], a') = 0 \tag{2.3}$$

for $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$.

Conversely, for any closed 2-forms $\omega_{\mathcal{A}}, \omega_{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$ and a 2-form $\omega' \in \mathcal{A}^* \wedge \mathcal{B}^*$ which satisfies (2.2) and (2.3), formula (2.1) defines a closed 2-form on $\mathcal{G}$.

The proof is straightforward.

We shall denote by $z^i(\mathcal{A})$ the space of closed i-forms on a Lie algebra $\mathcal{A}$.

Corollary 2.9. *Let $\mathcal{G} = \mathcal{A} + \mathcal{B}$ be the direct sum of two ideals . Then*

$$z^2(\mathcal{G}) = z^2(\mathcal{A}) + z^2(\mathcal{B}) + z^1(\mathcal{A}) \wedge z^1(\mathcal{B}).$$

In particular, if $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$ then

$$z^2(\mathcal{G}) = z^2(\mathcal{A}) + z^2(\mathcal{B}).$$

As another corollary of Lemma, we have

Proposition 2.10. *Let $\mathcal{G} = \mathcal{A} + \mathcal{B}$ be a semi-direct decomposition of a Lie algebra $\mathcal{G}$ as in Lemma 2.8.*

i) Assume that $[\mathcal{B}, \mathcal{B}] = 0$. Then the space $z^2(\mathcal{G})$ of the closed 2-forms on $\mathcal{G}$ is given by

$$z^2(\mathcal{G}) = z^2(\mathcal{A}) + z^2(\mathcal{B})^{\mathcal{A}} + z^2_{\mathcal{AB}},$$

where $z^2(\mathcal{B})^{\mathcal{A}}$ is the space of $\mathrm{ad}\mathcal{A}$-invariant 2-forms on $\mathcal{B}$ and $z^2_{\mathcal{AB}}$ is the space of 2-forms from $\mathcal{A}^* \wedge \mathcal{B}^*$ that satisfy (2.3).

ii) Assume that $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$, $\quad [\mathcal{A}, \mathcal{B}] \subset [\mathcal{B}, \mathcal{B}]$ and let ω be a closed 2-form with the canonical decomposition (2.1). If $\omega_{\mathcal{B}}$ is $\mathrm{ad}\,\mathcal{A}$-invariant form, then $\omega' = 0$ and the decomposition $\mathcal{G} = \mathcal{A} + \mathcal{B}$ is ω-orthogonal: $\omega(\mathcal{A}, \mathcal{B}) = 0$.

Applying this proposition to the Levi–Malcev decomposition $\mathcal{G} = \mathcal{S} + \mathcal{R}$ of a Lie algebra $\mathcal{G}$, we obtain

Theorem 2.11. *Let $\mathcal{G} = \mathcal{S} + \mathcal{R}$ be the Levi–Malcev decomposition of a Lie algebra $\mathcal{G}$, where $\mathcal{S}$ is a semi-simple subalgebra and $\mathcal{R}$ is the radical. Let ω be a closed 2-form on $\mathcal{G}$. Assume that its restriction of $\omega_{\mathcal{R}}$ to $\mathcal{R}$ is $\mathrm{ad}\,\mathcal{S}$-invariant. Then $\omega(\mathcal{S}, \mathcal{R}) = 0$ and $\omega = \omega_{\mathcal{S}} + \omega_{\mathcal{R}}$, where $\omega_{\mathcal{S}}$ in the restriction of ω to $\mathcal{S}$. In particular, the form ω is degenerate.*

Note that if the semi-simple part $\mathcal{S}$ is compact, then any closed 2-form on $\mathcal{G}$ is cohomologic to a closed $\mathrm{ad}\,\mathcal{S}$-invariant 2-form.

Proposition 2.12. *Let $\mathcal{A} \subset gl(V)$ be a linear Lie algebra and $\mathcal{G} = \mathcal{A} + V$ the associated inhomogeneous Lie algebra, which is the semi-direct sum of the subalgebra $\mathcal{A}$ and the vector ideal V.*

Then the space $z^2(\mathcal{G})$ of closed 2-forms on $\mathcal{G}$ is the direct sum of three subspaces:

$$z^2(\mathcal{G}) = z^2(\mathcal{A}) \oplus \wedge^2(V^*)^{\mathcal{A}} \oplus z^1(\mathcal{A}, V^*),$$

where $\wedge^2(V^*)^{\mathcal{A}}$ is the space of $\mathrm{ad}^*\mathcal{A}$-invariant 2-forms on V and

$$z^1(\mathcal{A}, V^*) = \{\omega \in \mathcal{A}^* \wedge V^* | \, \omega([A, B], x) = \omega(A, Bx) - \omega(B, Ax); \; A, B \in \mathcal{A}, x \in V\}.$$

We note that we may consider $z^1(\mathcal{A}, V^*)$ as the space of closed V^*-valued 1-forms on $\mathcal{A}$, where the differential $d\omega$ of a 1-form $\omega : \mathcal{G} \to V^*$ is given by

$$d\omega(A, B) = \omega([A, B]) - \omega(B)A + \omega(A)B.$$

(Here we denote by $\xi \mapsto \xi A$ the action of $A \in \mathcal{A}$ on the 1-form $\xi \in V^*$, $(\xi A)(x) = \xi(Ax)$ for $x \in V$.)

Remark that any 1-form $\xi \in V^*$ may be considered as a 0-form on $\mathcal{A}$ with values in V^* and, hence, it defines the exact 1-form $\omega^\xi = d\xi \in dz^0(\mathcal{A}, V^*) \subset z^1(\mathcal{A}, V^*)$:

$$\omega^\xi(A, x) = \xi(Ax).$$

Corollary 2.13. *Assume that $H^1(\mathcal{A}, V^*) = 0$. Then*

$$z^2(\mathcal{G}) = z^2(\mathcal{A}) \oplus \wedge^2(V^*)^{\mathcal{A}} \oplus dz^0(\mathcal{A}, V^*).$$

Corollary 2.14. *Assume that the action of $\mathcal{A}$ on V preserves no non-zero 2-form on V, i.e. $\Lambda^2(V^*)^{\mathcal{A}} = 0$ and $\dim\mathcal{A} < \dim V$. Then any closed 2-form ω on $\mathcal{G}$ is degenerate. In particular $\mathcal{A}$ does not admit a symplectic form.*

Proof. Since $\wedge^2(V^*)^{\mathcal{A}} = 0$, any closed 2-form may be written as $\omega = \omega_{\mathcal{A}} + \omega'$, where $\omega_{\mathcal{A}} \in z^2(\mathcal{A}), \omega' \in z^1(\mathcal{A}, V^*)$. Since $\dim \omega'\mathcal{A} < \dim V^*$, there exists $v \in V$ such that $\omega'(\mathcal{A}, v) = 0$. It belongs to the kernel of ω.

3. CLASSIFICATION OF POISSON BIVECTORS ON SOME LIE ALGEBRAS

Let $\mathcal{A}$ be a commutative subalgebra of a Lie algebra $\mathcal{G}$. Then any bivector $\Lambda \in \wedge^2\mathcal{A} \subset \wedge^2\mathcal{G}$ is a Poisson bivector. It has commutative subalgebra $\mathrm{supp}\Lambda \subset \mathcal{A}$ as the support.

The following simple proposition gives the complete description of all Poisson bivectors in a compact Lie algebra.

Proposition 3.1. *Any Poisson bivector Λ on a compact Lie algebra $\mathcal{G}$ has a commutative support.*

Proof. Since any subalgebra of a compact Lie algebra is a compact Lie algebra, i.e. the Lie algebra of a compact Lie group, the support $\mathcal{A} = \mathrm{supp}\,\Lambda$ of a Poisson bivector Λ is a compact Lie algebra with a non-degenerate Poisson bracket Λ.

A compact Lie algebra $\mathcal{A}$ is the direct sum of a compact semi-simple Lie algebra $\mathcal{A}'$ and a commutative subalgebra $\mathcal{B}$. By Corollary 2.9 the symplectic form $\omega = \Lambda^{-1}$ on $\mathcal{A}$ is the sum of symplectic form ω' of $\mathcal{A}'$ and a symplectic form $\omega_{\mathcal{B}}$ of $\mathcal{A}'$. To finish the Proof, we must show that $\mathcal{A}' = 0$. This follows from the well-known

Lemma 3.2. *Any closed 2-form ω on a semi-simple Lie algebra $\mathcal{G}$ is exact, i.e. it has the form*

$$\omega = \mathrm{d}\xi$$

for some 1-form $\xi \in \mathcal{G}^*$.

Its kernel $\mathrm{Ker}\,\omega \neq 0$ and it coincides with the centralizer in $\mathcal{G}$ of the vector $X = B^{-1}\xi \in \mathcal{G}$ associated with ξ by means of the Killing–Cartan form B of $\mathcal{G}$. In particular, there is no symplectic form on $\mathcal{G}$. This shows that $\mathcal{A}' = 0$ and proves Proposition 3.1.

Now we associate with a symplectic vector space (V, σ) over field $k = \mathbb{R}, \mathbb{C}$ some $(2n+2)$-dimensional Lie algebra $\mathcal{E}_{n+1}$ with the canonical symplectic form ω_{can} and the canonical Poisson bivector $\Lambda_{can} = \omega_{can}^{-1}$. It is defined as follows:

$$\mathcal{E}_{n+1} = ke_0 + ke_1 + V^{2n} = kh + kr + k\{p_j, q_k\},$$

$$[e_1, V^n] = 0, \quad [x, y] = \sigma(x, y)e_1, \quad x, y \in V^n,$$

$$[e_0, e_1] = 2e_1, \quad \mathrm{ad}\, e_0 | V^n = 1.$$

$$\Lambda_{can} = \frac{1}{2} e_0 \wedge e_1 + \sum (p_i \wedge q_i),$$

$$\omega_{can} = de_1^* = 2e_0^* \wedge e_1^* + \sigma.$$

Here $\{p_j, q_k\}$ denotes a standard symplectic base of V^n:

$$\omega(p_i, p_j) = \omega(q_i, q_j) = 0, \quad \omega(p_i, q_k) = \delta_{jk}.$$

The basis $\{h = e_0, r = e_1, p_i, q_j\}$ will be called a standard basis of $\mathcal{E}_{n+1}$. The dual basis of the dual space is denoted by $\{e_0^* = h^*, \quad e_1^* = r^*, \quad p_i^*, \quad q_j^*\}$.

Following [PS 1962] we will call $\mathcal{E}_{n+1}$ an elementary Kähler algebra. It is the Lie algebra of the Iwasawa subgroup AN of the Lie group $G = SU(1, n+1) = KAN$. The proof of the following lemma is straightforward.

Lemma 3.3. *Any closed 2-form ρ on $\mathcal{E}_{n+1}$ is exact and is a linear combination of the form ω_{can} and a form of the type*

$$dv^* = e_0^* \wedge v^*, \quad v^* \in (V^n)^*. \tag{3.1}$$

The form ρ is degenerate if and only if it is given by (3.1).

Corollary 3.4. *Any non-degenerate Poisson bivector on $\mathcal{E}_{n+1}$ may be written as*

$$\Lambda = \lambda\Lambda_{can} + e_1 \wedge v, \quad v \in V^n, \quad 0 \neq \lambda \in k.$$

Lemma 3.5. *The stabilizer*

$$C_{\mathcal{E}_{n+1}}(\Lambda) = \{x \in \mathcal{E}_{n+1}, \quad (\mathrm{ad}\, x)\Lambda = 0\}$$

of any non-degenerate Poisson bivector $\Lambda = \Lambda_{can} + e_1 \wedge v$ is equal to

$$C_{\mathcal{E}_{n+1}}(\Lambda) = ke_1.$$

We say that a bivector Λ is homogeneous of weight k if

$$(\mathrm{ad}\, e_0)\Lambda = k\Lambda.$$

Note that any symplectic subspace U^{2m} of the dimension $2m$ of the symplectic space V^{2n} defines a subalgebra

$$\mathcal{E}_{m+1} = ke_0 + U^{2m} + ke_1$$

of $\mathcal{E}_{n+1}$. It will be called a standard subalgebra of $\mathcal{E}_{n+1}$.

Lemma 3.6. *Let Λ be a homogeneous Poisson bivector on $\mathcal{E}_{n+1}$ of weight 2. Then either $\mathcal{A} = \mathrm{supp}\,(\Lambda)$ is a standard subalgebra of $\mathcal{E}_{n+1}$ and $\Lambda = \Lambda$ isproportional to the canonical Poisson bivector of the elementary algebra $\mathcal{A}$, or $\mathcal{A}$ is a commutative subalgebra.*

Proof. We may write Λ as

$$\Lambda = \lambda e_0 \wedge e_1 + \Lambda_V,$$

where $\Lambda_V \in \wedge^2 V$.

If $\lambda = 0$, then $\mathcal{C} = \mathrm{supp}\,\Lambda$ is a commutative subalgebra of V.

Assume now that $\lambda \neq 0$. Then

$$\mathcal{C} = \mathrm{supp}\,\Lambda = k\,\{e_0,\, e_1\} + W,$$

where W is a subspace of V. We can write $\mathcal{C}$ as a semi-direct sum $\mathcal{C} = \mathcal{A} + \mathcal{B}$, where $\mathcal{B}$ is the kernel of the canonical symplectic form σ on W and $\mathcal{A} = \mathbb{C}\{e_0,\, e_1\} + U$ is a standard subalgebra. Since $\mathcal{B}$ is a commutative ideal, we can apply Proposition 2.10.

Note that

$$\mathrm{ad}_{e_0} \mid \wedge^2 \mathcal{B}^* = -2 \cdot \mathrm{id}.$$

Hence $z^2(\mathcal{B})^{\mathcal{A}} = 0$. We claim that $z^2_{\mathcal{AB}} = e_0^* \wedge \mathcal{B}^*$.

Indeed, for $\omega' \in z^2_{\mathcal{AB}}, b \in \mathcal{B}$ we have

$$0 = d\omega'(e_0, e_1, b) = \omega'([e_0, e_1], b) + \omega'([e_1, b], e_0) + \omega'([b, e_0], e_1)$$

$$= 2\omega'(e_1, b) + \omega'(e_1, b) = 0.$$

Hence,

$$\omega'(e_1, \mathcal{B}) = 0,$$

moreover, the equations

$$d\omega'(e_1, a, b) = d\omega'(a, a', b) = 0$$

for $a, a' \in U$, $b \in \mathcal{B}$ are satisfied automatically. The equation

$$0 = d\omega'(e_0, a, b) = \omega'([e_0, a], b) + \omega'([a, b], e_0) + \omega'([b, e_0], a) = 2\omega'(a, b)$$

means that $\omega'(\mathcal{U}, \mathcal{B}) = 0$. Hence $\omega' \in e_0^* \wedge \mathcal{B}^*$ and $z^2_{\mathcal{AB}} = e_0^* \wedge \mathcal{B}^*$. Applying Proposition 2.10, we have $z^2(\mathcal{C}) = z^2(\mathcal{A} + \mathcal{B}) = z^2(\mathcal{A}) + e_0^* \wedge \mathcal{B}^*$. Lemma 3.3. shows that any closed form on $\mathcal{C}$ has the form

$$\lambda\omega_{can} + e_0^* \wedge v^*, \quad v^* \in \mathcal{U}^* + \mathcal{B}^*,$$

where ω_{can} is the canonical symplectic form on $\mathcal{A}$. It is degenerate if $\mathcal{B} \neq 0$. On the other hand, the bivector Λ defines a non-degenerate closed 2-form Λ^{-1} on $\mathcal{C} = \operatorname{supp}\Lambda$. Hence, $\mathcal{B} = 0$ and Lemma is proved.

The following theorem describes all closed 2-forms on Lie algebra which admits an ideal isomorphic to the elementary algebra.

Theorem 3.7. *Let $\mathcal{G}$ be a Lie algebra with semi-direct decomposition*

$$\mathcal{G} = \mathcal{F} + \mathcal{E},$$

where the ideal $\mathcal{E} = ke_0 + ke_1 + V$ is isomorphic to the elementary Lie algebra and the sublagebra $\mathcal{F}$ commutes with e_0 and has semi-direct decomposition

$$\mathcal{F} = \mathcal{A} + \mathcal{F}', \qquad \mathcal{F}' = [\mathcal{F}, \mathcal{F}], \qquad [\mathcal{A}, \mathcal{A}] = 0.$$

Then any closed 2-form ω on $\mathcal{G}$ can be written as

$$\omega = \omega_{\mathcal{F}} + \lambda\omega_{\mathrm{can}} + du^* + e_0^* \wedge a^*,$$

where $\lambda \in k$; $\omega_{\mathcal{F}}$, ω_{can} are the trivial extension to $\mathcal{G}$ of the restriction $\omega \mid \mathcal{F}$ and the canonical form of $\mathcal{E}$, $u^* \in V^*$; $a^* \in \mathcal{A}^*$. The form ω depends on $1 + \dim \mathcal{A}$ parameteres.

The form ω is non degenerate iff $\lambda \neq 0$ and the system of equations

$$\omega_{\mathcal{F}}(f, f_i) = 1/\lambda\, \sigma([f_i, u], [f, u]),$$

where f_i is a basis of $\mathcal{F}$ and $u = \sigma^{-1}u^*$ has only trivial solution.

Proof. Changing the commutative subalgebra $\mathcal{A}$ if necessary, we may assume that it commutes also with e_1.

By Lemmas 2.8 and 3.3, a closed form ω on $\mathcal{G}$ can be written as

$$\omega = \omega_{\mathcal{F}} + \omega_{\mathcal{E}} + \omega',$$

where $\omega_{\mathcal{F}}$, $\omega_{\mathcal{E}} = \lambda\omega_{\mathrm{can}} + e_0^* \wedge u^*$ are closed forms on $\mathcal{F}$, $\mathcal{E}$ respectively, considered as forms on $\mathcal{G}$ and $\omega' \in \mathcal{F}^* \wedge \mathcal{E}^*$ satisfies the equations (2.2), (2.3). Direct calculations show that these equations are equivalent to the following relations

$$\omega'(\mathcal{F}, e_1) = 0,\ \omega'(f, v) = u^*([f, v]),\ \omega'(\mathcal{F}', e_0) = 0$$

for all $f \in \mathcal{F}$, $v \in V$.

We can rewrite ω in the following form:

$$\omega = \omega_{\mathcal{F}} + \lambda\omega_{\mathrm{can}} + du^* + \omega'',$$

where $\omega'' \in \mathcal{E}^* \wedge \mathcal{F}^*$ satisfies the relations

$$\omega''(\mathcal{F}, ke_1 + V) = 0,\ \omega''(\mathcal{F}', e_0) = 0.$$

Hence, $\omega'' = e_0^* \wedge a^*$ for some $a^* \in \mathcal{A}^*$. It remains to study when ω is non degenerate.

We may assume that $\lambda \neq 0$, because in the opposite case e_1 belongs to the kernel of ω. Assume that a vector $z = f + \alpha e_0 + \beta e_1 + v$ belongs to the kernel of ω. Then

$$0 = \omega(z) = \omega_{\mathcal{F}}(f) + \lambda\,(\beta e_0^* - \alpha e_1^* + \sigma v) +$$

$$+ \mathrm{ad}_f^*\, u^* - \alpha u^* + \mathrm{ad}_v^*\, u^* + a^*(f)e_0^* - \alpha a^*.$$

Projecting this vector equation onto $\mathcal{F}^*$, e_0^*, e_1^*, V we obtain the following system:

$$\omega_{\mathcal{F}} f + \mathrm{ad}_v^*\, u^* - \alpha a^* = 0,$$

$$\lambda\beta + a^*(f) = 0,$$

$$\lambda\alpha = 0,$$

$$\lambda\sigma v + \mathrm{ad}_f^*\, u^* - \alpha u^* = 0.$$

Hence, $\alpha = 0$, $\beta = -1/\lambda\, a^*(f) - \lambda v = \sigma^{-1}\,\mathrm{ad}_f^*\,\sigma u = \mathrm{ad}_f u$ and the kernel is determined by solutions f of the equation

$$\omega_{\mathcal{F}} f = 1/\lambda\, \sigma u \circ \mathrm{ad}_{[f,u]}.$$

This proves Theorem.

Corollary 3.8. *For a closed 2-form ω the following conditions are equivalent :*

1) $\omega|_{\mathcal{E}} = \lambda\omega_{\mathrm{can}}$,

2) $u^* = 0$,

3) ω is the sum of eigenvectors of the operator ad_{e_0} with the eigenvalues 0 and -2.

If $[\mathcal{F}, \mathcal{E}] = V$, these conditions are equivalent to

4) $\omega(\mathcal{F}, \mathcal{E}') = \omega(\mathcal{F}, ke_1 + V) = 0$.

Corollary 3.9. *Assume that $[\mathcal{F}, \mathcal{E}] = V$. Then any closed form ω on $\mathcal{G}$ with $\omega(\mathcal{F}, \mathcal{E}) = 0$ is given by*

$$\omega = \omega_{\mathcal{F}} + \lambda\omega_{\mathrm{can}},$$

where $\omega_{\mathcal{F}}$ is a closed form on $\mathcal{F}$. It is non-degenerate iff $\lambda \neq 0$ and $\omega_{\mathcal{F}}$ is a non-degenerate closed form on $\mathcal{F}$ trivially extended to $\mathcal{G}$.

Proof. Assume that $\omega(\mathcal{F}, \mathcal{E}) = 0$. Then $a^* = 0$. Suppose that $u^* = 0$. Then there exist $f \in \mathcal{F}$ and $v \in V$ such that $u^*([f, v]) \neq 0$. Hence,

$$\omega(f, v) = du^*(f, v) = u^*([f, v]) \neq 0.$$

We come to a contradiction.

The Lie algebra $\mathcal{G}$ is called Frobenius one if it admits an exact symplectic form $\omega = d\xi$. In other words, this means that the coadjoint action of the corresponding group G has an open orbit $\mathrm{Ad}^* G\xi$.

Corollary 3.10. *Under the assumption of Theorem 3.7, the Lie algebra $\mathcal{G}$ is Frobenius one iff the Lie algebra $\mathcal{F}$ is Frobenius. Moreover, any exact form on $\mathcal{G}$ can be written as*

$$\omega = \omega_{\mathcal{F}} + \omega_{\mathcal{E}} = \omega_{\mathcal{F}} + d(e_1^* + v^*),$$

where $\omega_{\mathcal{F}}$ is an exact form on $\mathcal{F}$ and $v^* \in V^*$. In particular, a closed form ω is exact iff $\omega(\mathcal{F}, \mathcal{E}) = 0$ and $\omega \,|\, \mathcal{F}$ is exact.

Proof. It follows from Theorem 3.7 and Lemma 3.3.

Remark. This corollary reduces the problem of description of open coadjoint orbits of the group G with the Lie algebra $\mathcal{G}$ to the same problem for the subgroup F, corresponding to the subalgebra $\mathcal{F}$.

Denote by $z^2(\mathcal{G})$ (resp., $d\mathcal{G}^*$) the space of closed, (resp., exact) 2-forms on the Lie algebra $\mathcal{G}$ and by $H^2(\mathcal{G}) = z^2(\mathcal{G})/d\mathcal{G}^*$ the corresponding cohomology group. Remark that the space $\mathcal{A}^* \subset \mathcal{F}^*$ is the space of closed 1-forms on $\mathcal{F}$ and such forms are never exact. Using this we derive from Theorem 3.7 and Corollary 3.10 the following

Corollary 3.11. *Under the notation of Theorem 3.7, assume that the elementary algebra $\mathcal{E} = \mathcal{E}_{n+1}$ has dimension $2n + 2$. Then*

1) $\dim z^2(\mathcal{G}) = \dim z^2(\mathcal{F}) + \dim \mathcal{A} + 2n + 1$,
2) $\dim d\mathcal{G}^* = \dim d\mathcal{F}^* + 2n + 1$,
3) $\dim H^2(\mathcal{G}) = \dim H^2(\mathcal{F}) + \dim \mathcal{A} = \dim H^2(\mathcal{F}) + \dim H^1(\mathcal{F})$.
4) *If $\mathcal{F}$ admits a symplectic structure then symplectic structures on $\mathcal{G}$ depend on $\dim z^2(\mathcal{G})$ parameters.*

We say that a Poisson bivector Λ is consistent with a semi-direct decomposition $\mathcal{G} = \mathcal{F} + \mathcal{E}$ if it is a sum of two bivectors $\Lambda_{\mathcal{F}}, \Lambda_{\mathcal{E}}$ with support in $\mathcal{F}$ and $\mathcal{E}$, respectively. Then by Proposition 2.1 $\Lambda_{\mathcal{F}}$, $\Lambda_{\mathcal{E}}$ are commuting Poisson bivectors. We have

Corollary 3.12. *Under the notation of Theorem 3.7, any Poisson bivector Λ on $\mathcal{G}$ which is consistent with the decomposition $\mathcal{G} = \mathcal{F} + \mathcal{E}$ is given by*

$$\Lambda = \Lambda_{\mathcal{F}} + \Lambda_{\mathcal{E}} = \Lambda_{\mathcal{F}} + \lambda\Lambda_{\mathrm{can}} + e_1 \wedge v, \;\; \lambda \in k,$$

where $\Lambda_{\mathcal{F}}$ *is a Poisson bivector on* $\mathcal{F}$, $\Lambda_{\text{can}} = 1/2\, e_0 \wedge_1 + \Sigma p_i \wedge q_i$ *and* $v \in V$ *is a vector commuting with the subalgebra* $\operatorname{supp} \Lambda_{\mathcal{F}}$.

Proof. By Corollary 3.4, any Poisson bivector on $\mathcal{E}$ has the form

$$\Lambda_{\mathcal{E}} = \Lambda_{\mathcal{F}} + \lambda \Lambda_{\text{can}} + e_1 \wedge v$$

for some $v \in V$. Since $(\operatorname{ad} \mathcal{F})\Lambda_{\text{can}} = 0$, we have $[\Lambda_{\mathcal{F}}, \Lambda_{\text{can}}] = 0$. Hence, the bivectors $\Lambda_{\mathcal{F}}$, $\Lambda_{\mathcal{E}}$·commute iff $[\operatorname{supp} \Lambda_{\mathcal{F}}, v] = 0$. This proves Corollary.

4. DECOMPOSITION OF THE BOREL SUBALGEBRA $\mathcal{B}$ INTO A SUM OF ELEMENTARY ALGEBRAS AND CLOSED 2-FORMS AND SYMPLECTIC STRUCTURES ON $\mathcal{B}$

Using the induction, we can apply the results of Section 3 to any Lie algebra $\mathcal{G}$ which is decomposed into semidirect sum

$$\mathcal{G} = \mathcal{E}^1 + \ldots + \mathcal{E}^k$$

of elementary subalgebras such that for any $i > 1$, $\mathcal{E}^1 + \ldots + \mathcal{E}^i$ is a subalgebra with the ideal $\mathcal{E}^i$ and the complementary subalgebra $\mathcal{E}^1 + \ldots + \mathcal{E}^{i-1}$.

Now we prove that the Borel subalgebra of the semisimple (complex or normal real) Lie algebra admits such semidirect decomposition (where, sometimes, also a subalgebra of the Cartan subalgebra appears).

Let $\mathcal{G}$ be a semisimple (complex) Lie algebra and R corresponding root system with respect to the Cartan subalgebra $\mathcal{H}$. Recall that a subset $Q \subset R$ is called to be closed if

$$(Q + Q) \cap R \subset Q.$$

Such subset defines a regular subalgebra $\mathcal{G}(Q)$ of $\mathcal{G}$, generated by the root vectors E_α, $\alpha \in Q$.

More generally, two closed subsets P, Q of R define the regular subalgebra $\mathcal{B} = \mathcal{G}(P) + \mathcal{G}(Q)$ with the ideal $\mathcal{G}(P)$ iff

$$(P + Q) \cap R \subset P.$$

Denote by R^+ a system of positive roots of $\mathcal{G}$ and by ρ the highest root of R^+. We set

$$R_\rho = \{\alpha \in R^+ \mid \rho - \alpha \in R^+ \cup \{0\}\} = \{\alpha \in R^+ \mid (\rho, \alpha) > 0\}$$

and

$$Q_\rho = R^+ - R_\rho = \{\alpha \in R^+ \mid (\rho, \alpha) = 0\}.$$

Proposition 4.1.

1. R_ρ, Q_ρ are closed subsets of roots and $(Q_\rho + R_\rho) \cap R \subset R_\rho$.
2. The Borel subalgebra $\mathcal{B}(\mathcal{G}) = \mathcal{H} + \mathcal{G}(R^+)$ of $\mathcal{G}$ and $\mathcal{G}(R^+)$admits semi-direct decomposition

$$\mathcal{G}(R^+) = \mathcal{E}_\rho + \mathcal{F}_\rho,$$

where $\mathcal{E}_\rho = kH_\rho + \mathcal{G}(R_\rho)$ is an ideal and $\mathcal{F}_\rho = \mathcal{H}' + \mathcal{G}(Q_\rho)$ is a subalgebra. Here H_ρ is the highest root vector and $\mathcal{H}'$ is the orthogonal complement to H_ρ into $\mathcal{H}$.

3. The ideal $\mathcal{E}_\rho$ is isomorphic to the elementary Lie algebra $\mathcal{E}_{n+1}$, where $|R_\rho| = 2n+1$ and $n = h^\vee - 2$, $h^\vee$ is the dual Coxeter number.

Proof 1. Note that the highest root ρ is always a long root and we normalize it as $(\rho, \rho) = 2$. Then the set R_ρ has the form $R_\rho = R_1 \cup \{\rho\}$ where $R_1 = \{\alpha \,|\, (\alpha, \rho) = 1\}$ since $2(\alpha, \rho)/(\rho, \rho) < 2$. Let $\gamma = \alpha + \beta \in R^+$ for $\alpha \in R^+, \beta \in R^+$. Then

If $\alpha, \beta \in R_1$, then $(\gamma, \rho) = (\alpha, \rho) + (\beta, \rho) = 2$, and $\gamma = \rho$.

If $\alpha, \beta \in Q_\rho$ then $(\gamma, \rho) = (\alpha, \rho) + (\beta, \rho) = 0 + 0 = 0$ and $\gamma \in Q_\rho$.

If $\alpha \in R_1, \beta \in Q_\rho$ then $(\gamma, \rho) = (\alpha, \rho) + (\beta, \rho) = 1$, and $\gamma \in R_1$.

This proves 1. The statement 2 follows from 1 and the remarks before Proposition 4.1.

3. From the proof of 1, it follows that $(R_1 + R_1) \cap R^+ = \{\rho\}$. Hence we can write

$$R_\rho = \{\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_n; \rho\}$$

where $\alpha_i + \beta_i = \rho, i = 1, \dots n$ are the only non trivial relations between the roots from R_ρ. This shows that $\mathcal{G}(R_\rho)$ is the Heisenberg Lie algebra. Moreover, $\mathcal{E}_\rho$ is the elementary algebra , because

$$[H_\rho, E_{\alpha_i}] = (\rho, \alpha_i)E_{\alpha_i} = E_{\alpha_i}.$$

One can check easily that $\beta_i = -S_\rho \alpha_i$ where S_ρ is the reflection in the hyperplane orthogonal to the root ρ and that $n = h^\vee - 2$ where $h^\vee$ is the dual Coxeter number. This proves 3.

Now we describe a decomposition

$$\mathcal{B}(\mathcal{G}) = \mathcal{E}_\rho + \mathcal{F}_\rho$$

of the Borel subalgebra of a semisimple Lie algebra $\mathcal{G}$ explicitly. It is sufficient to consider only simple Lie algebras. Recall that there are four series and five exceptional Lie algebras $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. The basic characteristics of these algebras are given in Table 1.

Table 1

Type of group	Rank	Coxeter number	Number of positive roots
$A_n, n \geq 1$	n	$n+1$	$n(n+1)/2$
$B_n, n \geq 2$	n	$2n$	n^2
$C_n, n \geq 3$	n	$2n$	n^2
$D_n, n \geq 4$	n	$2(n-1)$	$n(n-1)$
E_6	6	12	36
E_7	7	18	63
E_8	8	30	120
F_4	4	12	24
G_2	2	6	6

The next Table 2 enumerates the root system of each simple Lie algebra , according to the book [OV 1990], and describes the subsystem

$$R_\rho = \{\rho\} \cup R_1, \quad R_1 = \{\alpha_i, \beta_i \quad | \quad \alpha_i + \beta_i = \rho\},$$

associated with the highest root ρ.

Table 2

Type of $\mathcal{G}$	Roots	Highest root ρ	Decomposition of ρ, $\rho = \alpha_j + \beta_j$
A_n, $n \geq 1$	$e_i - e_j$	$e_1 - e_{n+1}$	$\alpha_j = e_1 - e_j$, $\beta_j = e_j - e_{n+1}$; $j = 2, \dots, n$
B_n, $n \geq 2$	$\pm e_i \pm e_j$, $\pm e_j$	$e_1 + e_2$	$\alpha_j = e_1 + e_j$, $\beta_j = e_2 - e_j$, $\tilde{\alpha}_j = e_1 - e_j$, $\tilde{\beta}_j = e_2 + e_j$, $\alpha_{2n-3} = e_1$, $\beta_{2n-3} = e_2$ $j = 3, \dots, n$
C_n, $n \geq 3$	$\pm e_i \pm e_j$, $\pm 2e_j$	$2e_1$	$\alpha_j = e_1 + e_j$, $\beta_j = e_1 - e_j$ $j = 2, \dots, n$
D_n, $n \geq 4$	$\pm e_i \pm e_j$	$e_1 + e_2$	$\alpha_j = e_1 + e_j$, $\beta_j = e_2 - e_j$ $\tilde{\alpha}_j = e_1 - e_j$, $\tilde{\beta}_j = e_2 + e_j$ $j = 3, \dots, n$
E_6	$e_i - e_j$, $\pm 2e$ $e_i + e_j + e_k \pm e$	$2e$	$\alpha_{jkl} = e + e_j + e_k + e_l$, $\beta_{jkl} = e - e_j - e_k - e_l$; $j, k, l = 1, \dots, 6$
E_7	$e_i - e_j$ $e_i + e_j + e_k + e_l$	$-e_7 + e_8$	$\alpha_j = -e_7 + e_j$, $\beta_j = e_8 - e_j$ $\alpha_{jkl} = e_8 + e_j + e_k + e_l$, $\beta_{jkl} = -e_7 - e_j - e_k - e_l$ $j, k, l = 1, \dots, 6$
E_8	$e_i - e_j$ $\pm(e_i + e_j + e_k)$	$e_1 - e_9$	$\alpha_j = e_1 - e_j$, $\beta_j = e_j - e_9$ $\alpha_{jk} = e_1 + e_j + e_k$, $\beta_{jk} = -e_9 - e_j - e_k$; $j, k = 2, \dots, 8$
F_4	$\pm e_i \pm e_j$, $\pm e_j$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$	$e_1 + e_2$	$\alpha = e_1$, $\beta = e_2$ $\alpha_j = e_1 + e_j$, $\beta_j = e_2 - e_j$, $j = 3, 4$ $\tilde{\alpha}_j = e_1 - e_j$, $\tilde{\beta}_j = e_2 + e_j$, $j = 3, 4$ $\alpha_{6,7} = \frac{1}{2}(e_1 + e_2 + e_3 \pm e_4)$ $\beta_{6,7} = \frac{1}{2}(e_1 + e_2 - e_3 \mp e_4)$
G_2	$e_i - e_j$, $\pm e_j$	$e_1 - e_3$	$\alpha_1 = e_1$, $\beta_1 = -e_3$ $\alpha_2 = e_1 - e_2$, $\beta_2 = e_2 - e_3$

Recall that subsystem of roots $S_\rho = R^+ - R_\rho$ consists of positive roots orthogonal to the root ρ, and so S_ρ is generated by simple roots orthogonal to ρ. Hence S_ρ may be easily constructed from the extended Dynkin diagram of the Lie algebra $\mathcal{G}$ which corresponds to the set of simple roots and the minimal root $(-\rho)$. The simple roots connected to the root $(-\rho)$ are not orthogonal to the root ρ. The rest roots form the extended Dynkin diagram, which generate subsystem S_ρ and the corresponding Borel subalgebra.

Note that the number of roots in R_ρ is equal to $2h^\vee - 3$, where $h^\vee$ is the dual Coxeter number.

Using these remarks, we obtain the decompositions of the Borel subalgebra indicated into Table 3. Here H'_ρ is the element of the Cartan subalgebra $\mathcal{H}$ which corresponds to $(n-1)(e_1 + e_{n+1}) - 2(e_2 + \ldots + e_n)$ under the identification $\mathcal{H} = \mathcal{H}^*$. Recall that $\dim(\mathcal{E}_n) = 2n$.

Table 3

$\mathcal{B}(A_n) =$	$\mathcal{E}_n + (\mathcal{B}(A_{n-2}) + kH'_\rho)$
$\mathcal{B}(B_n) =$	$\mathcal{E}_{2n-2} + (\mathcal{B}(B_{n-2}) + \mathcal{B}(A_1))$
$\mathcal{B}(C_n) =$	$\mathcal{E}_n + \mathcal{B}(C_{n-1})$
$\mathcal{B}(D_n) =$	$\mathcal{E}_{2n-3} + (\mathcal{B}(D_{n-2}) + \mathcal{B}(A_1))$
$\mathcal{B}(E_6) =$	$\mathcal{E}_{11} + \mathcal{B}(A_5)$
$\mathcal{B}(E_7) =$	$\mathcal{E}_{17} + \mathcal{B}(D_6)$
$\mathcal{B}(E_8) =$	$\mathcal{E}_{29} + \mathcal{B}(E_7)$
$\mathcal{B}(F_4) =$	$\mathcal{E}_8 + \mathcal{B}(C_3)$
$\mathcal{B}(G_2) =$	$\mathcal{E}_3 + \mathcal{B}(A_1)$

Using this Table, it is easy to write the explicit formulae for the decomposition of the Borel subalgebra of any semi-simple Lie algebra into the elementary subalgebras. We present the results in Table 4.

Table 4

$\mathcal{B}(A_n) =$	$\mathcal{E}_n + \mathcal{E}_{n-2} + \ldots + \mathcal{E}_2(\text{or}\mathcal{E}_1) + \mathcal{H}_m; \quad m = [\frac{n}{2}]$
$\mathcal{B}(B_n) =$	$\mathcal{E}_{2n-2} + \mathcal{E}_{2n-6} + \ldots + \mathcal{E}_4(\text{or}\mathcal{E}_2) + m\mathcal{E}_1, \quad m = [\frac{n+1}{2}]$
$\mathcal{B}(C_n) =$	$\mathcal{E}_n + \mathcal{E}_{n-1} + \ldots + \mathcal{E}_2 + \mathcal{E}_1$
$\mathcal{B}(D_n) =$	$\mathcal{E}_{2n-3} + \mathcal{E}_{2n-7} + \ldots + \mathcal{E}_5 + (m+1)\mathcal{E}_1, \quad n = 2m$
$\mathcal{B}(D_n) =$	$\mathcal{E}_{2n-3} + \mathcal{E}_{2n-7} + \ldots + \mathcal{E}_3 + m\mathcal{E}_1 + \mathcal{H}_1, \quad n = 2m+1$
$\mathcal{B}(E_6) =$	$\mathcal{E}_{11} + \mathcal{E}_5 + \mathcal{E}_3 + \mathcal{E}_1 + \mathcal{H}_2$
$\mathcal{B}(E_7) =$	$\mathcal{E}_{17} + \mathcal{E}_9 + \mathcal{E}_5 + 4\mathcal{E}_1$
$\mathcal{B}(E_8) =$	$\mathcal{E}_{29} + \mathcal{E}_{17} + \mathcal{E}_9 + \mathcal{E}_5 + 4\mathcal{E}_1$
$\mathcal{B}(F_4) =$	$\mathcal{E}_8 + \mathcal{E}_3 + \mathcal{E}_2 + \mathcal{E}_1$
$\mathcal{B}(G_2) =$	$\mathcal{E}_3 + \mathcal{E}_1$

Here $\mathcal{H}_m$ is the subalgebra of the dimension m of the Cartan subalgebra.

Using the results of section 3, we derive now some corollaries from these results.

By Corollary 3.10, any subalgebra $\mathcal{B}$ which admits a decomposition into a semi-direct sum of the elementary subalgebra is a Frobenius Lie algebra. This means that the coadjoint action of the corresponding Lie group has an open orbit or, in other words, $\mathcal{B}$ has an exact symplectic form. Checking Table 4 and using Corollary 3.11, we get

Proposition 4.2.

1) *The Borel subalgebra of a simple Lie algebra $\mathcal{G}$ is Frobenius iff $\mathcal{G}$ is different from A_n, D_{2m+1} and E_6.*
2) *The minimal dimension of the kernel of an exact 2-form (which is equal to the codimension of a regular coadjoint orbit) is equal to $m = [n/2]$ for $\mathcal{B}(A_n)$, 1 for $\mathcal{B}(D_{2m+1})$ and 2 for $\mathcal{B}(E_6)$.*
3) *The Borel subalgebra admits a symplectic form iff it has even dimension. In the opposite case it admits a closed 2-form with one-dimensional kernel.*

Recall that for the elementary Lie algebra $\mathcal{E}_n$ the dimension of the space of closed 2-forms is equal to $2n-1$ and $H^2(\mathcal{E}_n) = 0$, since any closed 2-form is exact (Lemma 3.3). Now we calculate the cohomology $H^2(\mathcal{B}(\mathcal{G}))$ for each simple Lie algebra $\mathcal{G}$.

Proposition 4.3.

Let $\mathcal{G}$ be a simple Lie algebra of rank n. Then

$$\dim H^2(\mathcal{B}(\mathcal{G})) = n(n-1)/2.$$

Proof. Let $\mathcal{B} = \mathcal{E}^1 + \ldots + \mathcal{E}^p + \mathcal{H}_q$ be a decomposition of the Lie algebra $\mathcal{B}(\mathcal{G})$ of rank n into semi-direct sum of elementary Lie algebras and the commutative q-dimensional Lie algebra $\mathcal{H}_q$.

Then Corollary 3.11 implies the following formula for the dimension of the second cohomology group:

$$\dim H^2(\mathcal{B}) = (n-1) + \ldots + (n-p) + q(q-1)/2 = (2n-p-1)p/2 + q(q-1)/2.$$

For the Frobenius Borel algebra, $q = 0, p = n$ and we get the Proposition. Using this formula we check Proposition also for the cases $\mathcal{G} = A_{2m+1}$, A_{2m}, D_{2m+1} and E_6.

Now the calculation of the dimension of the space of closed 2-forms reduces to the calculation of the dimension of the space of exact 2-form. For the Lie algebra with a semidirect decomposition $\mathcal{B} = \mathcal{E}_n + \mathcal{F}$ we have

$$\dim d\,\mathcal{B}^* = 2n - 1 + \dim d\,\mathcal{F}^*$$

by Theorem 3.7 and Corollary 3.11. More generally, for the Lie algebra with a semidirect decomposition

$$\mathcal{B} = \mathcal{E}_{n_1} + \ldots + \mathcal{E}_{n_p} + \mathcal{H}_q$$

we get formula

$$\dim d\,\mathcal{B}^* = \sum_{i=1}^{p} (2n_i - 1),$$

i.e $\dim d\,\mathcal{B}^*$ is equal to

the number of positive roots of algebra $\mathcal{G}$. Using this formula we calculate the dimension of the space of exact 2-forms $d\mathcal{B}(\mathcal{G})^*$ and the space $z^2\ (\mathcal{B}(\mathcal{G}))$ of closed 2-forms for all simple Lie algebras $\mathcal{G}$. The results are presented in Table 5.

Table 5

Type of group $\mathcal{G}$	$\dim z^2(\mathcal{B}(\mathcal{G}))$	$\dim H^2(\mathcal{B}(\mathcal{G}))$	$\dim d\mathcal{B}(\mathcal{G})^*$
A_n, $n \geq 1$	n^2	$n(n-1)/2$	$n(n+1)/2$
B_n, $n \geq 2$	$n(3n-1)/2$	$n(n-1)/2$	n^2
C_n, $n \geq 3$	$n(3n-1)/2$	$n(n-1)/2$	n^2
D_n, $n \geq 4$	$3n(n-1)/2$	$n(n-1)/2$	$n(n-1)$
E_6	51	15	36
E_7	84	21	63
E_8	148	28	120
F_4	30	6	24
G_2	7	1	6

Let

$$\mathcal{B}(\mathcal{G}) = \mathcal{E}_{n_1} + \mathcal{E}_{n_2} + \ldots + \mathcal{E}_{n_k} + \mathcal{H}_m$$

be the decomposition of the Borel subalgebra of the simple Lie algebra $\mathcal{G}$ into the sum of elementary Lie algebras and, may be, the commutative Lie algebra, described in Table 3. Denote by Λ_i the canonical Poisson bivector on elementary subalgebra $\mathcal{E}_i$ and by Λ_0 any bivector on the commutative subalgebra $\mathcal{H}_m$. Then Corollary 3.12 implies the following result.

Proposition 4.4. *The Poisson bivectors Λ_i, $i \geq 0$ mutually commute and define the Poisson bivector*

$$\Lambda = \Lambda_1 + \ldots + \Lambda_k + \Lambda_0$$

on the Borel subalgebra $\mathcal{B}(\mathcal{G})$.

References

[BD 1982] Belavin A.A. and Drinfel'd V.G., *Solutions of the classical Yang–Baxter equation*, Funct. Anal. Appl. **16**, No 3, 1–29

[BD 1983] Belavin A.A. and Drinfel'd V.G., *On the classical Yang–Baxter equation for simple Lie algebras* , Funct. Anal. Appl. **17**, No 3, 69–70

[CGR 1994] Cahen M., Gutt S. and Rawnsley J., *Some remarks on the classifications of Poisson-Lie groups*, in Contemp. Math. **179**, Symplectic Geometry and Quantization, pp. 1-15

[Dr 1983] Drinfel'd V.G., *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the Yang–Baxter equations*, DAN SSSR **268**, No 2, 285–287

[GD 1980] Gel'fand I.M. and Dorfman I.Ya., *Schouten brackets and the classical Yang–Baxter equation*, Funct. Anal. Appl. **14**, No 3, 71–74

[GS 1984] Guillemin V. and Sternberg S., *Symplectic Techniques in Physics*, Cambridge Univ. Press

[OV 1990] Onishchik A.L. and Vinberg E.B., *Lie Groups and Algebraic Groups*, Springer-Verlag; Berlin, Heidelberg, New York

[P-S 1962] Pyatetsky-Shapiro I.I., *The structure of j-algebras*, Izv. Akad. Nauk SSSR **26**, 107–124

[P-S 1966] Pyatetsky-Shapiro I.I., *The Geometry of Bounded Domains*

[S-T-S 1983] Semenov-Tyan-Shanskii M. A., *What is a classical R-matrix?*, Funct. Anal. Appl. **17**, 259–272

[St 1991] Stolin A., *On rational solutions of the classical Yang-Baxter equation*, Stockholm University Preprint