Proceedings of Symposia in Pure Mathematics
Volume **00**, 1997

# Unstable Modules over the Steenrod Algebra and Cohomology of Groups

**Hans–Werner Henn**

Let $p$ be a prime. In this survey we attempt to explain how the presence of Steenrod operations in the mod - $p$ cohomology ring $H^*(G;\mathbb{F}_p)$ of a group $G$ allows to understand qualitative features of this ring, at least for a large class of groups including all finite groups but also many discrete groups like arithmetic groups, mapping class groups and automorphism groups of free groups. We will also discuss how the general theory can be applied to do actual calculations.

## 1. Quillen's F-isomorphism theorem from the point of view of unstable modules

The mod - $p$ cohomology ring of a group, even a finite group, is a very mysterious object in general even though it is understood for a fair number of interesting groups (see [AM] for a recent textbook on the subject). The first important general result on these rings is due to Evens and Venkov.

Theorem 1.1 [E1], [V]. *Let $G$ be a finite group. Then $H^*(G;\mathbb{F}_p)$ is a finitely generated (graded commutative) $\mathbb{F}_p$ - algebra.*

**1.2.** This result suggests to use the usual concepts of commutative algebra in order to study this ring. Of course, one would like to understand these concepts in terms of the group theory of $G$. Quillen's fundamental result [Q1] takes a substantial step in this direction; it describes $H^*(G;\mathbb{F}_p)$ up to nilpotent phenomena and determines the Krull dimension of $H^*(G;\mathbb{F}_p)$ and its variety (or rather its set of geometric points) in a beautiful and very satisfactory way. Quillen's approach was to understand as much as possible about $H^*(G;\mathbb{F}_p)$ by relating it via the restriction maps to the cohomology of its elementary abelian $p$ - subgroups, i.e. groups

1991 *Mathematics Subject Classification.* Primary 55S10; Secondary 20J05, 20J06.
The author was supported by Deutsche Forschungsgemeinschaft and by Max-Planck-Institut für Mathematik at Bonn.

isomorphic to $(\mathbb{Z}/p)^n$ for some natural number $n$. We recall that for an elementary abelian $p$ -group $E$ we have

$$H^*(E;\mathbb{F}_p)\cong\begin{cases} S(E^*,1) & \text{if } p=2,\\ S(E^*,2)\otimes\Lambda(E^*,1) & \text{if } p>2.\end{cases}$$

Here $E^*$ is the dual of $E$ and $S(E^*,i)$ stands for a graded symmetric algebra with $E^*$ in degree $i$ and similarly $\Lambda(E^*,i)$ for a graded exterior algebra with $E^*$ in degree $i$.

Now consider the map $H^*(G;\mathbb{F}_p)\longrightarrow\prod_E H^*(E;\mathbb{F}_p)$ with $E$ running through all elementary abelian $p$ - subgroups of $G$. It is clear that elements in the target of this map must satisfy certain compatibility conditions (arising from conjugations in $G$) to have a chance to be in its image. More precisely Quillen considered the category $\mathcal{A}_p(G)$ whose objects are the elementary abelian $p$ - subgroups of $G$ and whose morphisms are all group homorphisms which can be induced by conjugation by an element in $G$. Then $E\mapsto H^*(E;\mathbb{F}_p)$ becomes a (covariant) functor from the opposite category $\mathcal{A}_p(G)^{op}$ to graded $\mathbb{F}_p$ - algebras and the restriction homomorphisms induce a map $q_G : H^*(G;\mathbb{F}_p)\longrightarrow \lim_{\mathcal{A}_p(G)^{op}} H^*(E;\mathbb{F}_p)$. Quillen's result reads now as follows.

Theorem 1.3 [Q1]. *Let $p$ be a prime and $G$ be a finite group. Then the map*

$$q_G : H^*(G;\mathbb{F}_p)\longrightarrow \lim_{\mathcal{A}_p(G)^{op}} H^*(E;\mathbb{F}_p)$$

*has the following properties:*

- *If $x\in\operatorname{Kern} q_G$ then $x$ is nilpotent.*
- *For every $y\in\lim_{\mathcal{A}_p(G)^{op}} H^*(E;\mathbb{F}_p)$ there exists a natural number $n$ with $y^{p^n}\in\operatorname{Im} q_G$.*

A map of graded $\mathbb{F}_p$ - algebras satisfying the two conditions occuring in the theorem is called an *$F$ - isomorphism*.

**1.4.** Quillen [Q1] showed that Theorem 1.3 is true for other classes of groups as well, e.g.

a) if $G$ is a discrete group of finite virtual mod $p$ - cohomological dimension (this includes all ($S$ - ) arithmetic groups, mapping class groups and automorphism groups of free groups),

b) if $G$ is a compact Lie group and $H^*(G;\mathbb{F}_p)$ is replaced by the mod - $p$ cohomology $H^*(BG;\mathbb{F}_p)$ of its classifying space $BG$,

c) if $G$ is a profinite group and $H^*(G;\mathbb{F}_p)$ is replaced by the continuous mod - $p$ cohomology $H^*_{cts}(G;\mathbb{F}_p)$ and is assumed to be noetherian (this includes all $p$ - adic analytic groups in the sense of Lazard [Lz]).

We call a group a *Quillen - group* (at the prime $p$, to be precise) if the map $q_G$ is an $F$ - isomorphism (with the modifications that we have just indicated if $G$ is a compact Lie group or a profinite group). In the appendix we will sketch an extension of Theorem 1.3 to another large class of discrete groups. This class of groups also contains all ($S$ - ) arithmetic groups, mapping class groups and automorphism groups of free groups, and furthermore all groups in this class have a finitely generated mod $p$ - cohomology ring.

**1.5.** From Theorem 1.3 Quillen deduced that the (Krull) dimension of the algebra $H^*(G;\mathbb{F}_p)$ is equal to $r_p(G)$, the $p$ - rank of $G$, which is defined to be the maximal $n$ for which $G$ contains a subgroup isomorphic to $(\mathbb{Z}/p)^n$. Furthermore it allowed him to describe the variety of $H^*(G;\mathbb{F}_p)$, more precisely its geometric points with values in an algebraically closed field $k$, in group theoretic terms and thus to lay the basis for a very interesting subsequent development in modular representation theory, namely the theory of varieties attached to modules (see [Be1] and [E2] for textbooks on this development).

**1.6.** We will now reformulate Theorem 1.3 using the language of unstable modules over the Steenrod algebra. For this we recall that the mod - $p$ cohomology of a space is equipped with an action of the mod - $p$ Steenrod algebra $A_p$. The Steenrod algebra is a large associative (but noncommutative) graded $\mathbb{F}_p$ - algebra generated by homogeneous elements $Sq^i$, $i \geq 0$, of degree $i$ if $p = 2$, and $P^i$, $i \geq 0$, of degree $2i(p-1)$ together with the Bockstein operation $\beta$ of degree 1 if $p > 2$. The operations $Sq^0$ and $P^0$ act as identity and the other relations between these generators are known as Adem relations. We refer the reader to [SE] for more details. The cohomology of a space is a graded $A_p$ - module for which the action satisfies additional "unstability axioms" which define the categories of unstable modules resp. unstable algebras over the mod - $p$ Steenrod algebra. For a thorough discussion of these concepts the reader is referred to [Sc]; here we will just recall the basic definitions.

Definition 1.7. An *unstable* $A_p$ - module M is a graded $A_p$ - module such that

- $Sq^i x = 0$ for all $i > |x|$, if $p = 2$;
- $\beta^e P^i x = 0$ for all $2i + e > |x|$, if $p$ is odd and $e \in \{0, 1\}$.

Here and elsewhere in this paper $|x|$ denotes the degree of the element $x$.

As usual we will write $\mathcal{U}_p$ for the (abelian) category of unstable modules and $A_p$ - linear maps. Usually $p$ will be fixed and understood from the context and then we will use the abbreviations *unstable module* and $\mathcal{U}$ resp..

Definition 1.8. An *unstable* $A_p$ - algebra is an unstable $A_p$ - module $K$ together with linear maps $\varphi : K \otimes K \longrightarrow K$, $\eta : \mathbb{F}_p \longrightarrow K$ such that

- $\varphi$ and $\eta$ determine a graded $\mathbb{F}_p$ - algebra structure on $K$.
- $\varphi$ is $A_p$ - linear. (For this to make sense we recall that $A_p$ is a Hopf algebra, hence the category of graded $A_p$ - modules has an associated internal tensor product. Another way of stating this is that the product in $K$ satisfies the Cartan formula, e.g. $Sq^n(xy) = \sum_{i+j=n} Sq^i x Sq^j y$ if $p = 2$.)
- $$Sq^n x = x^2 \text{ if } p = 2 \text{ and } n = |x|,$$
  $$P^n x = x^p \text{ if } p > 2 \text{ and } 2n = |x| \ .$$

As usual we will write $\mathcal{K}_p$ for the category of unstable algebras over $A_p$; its objects are the unstable algebras and its morphisms the degree preserving maps which are both $A_p$ - linear and maps of graded algebras. As above we will often use the abbreviations *unstable algebra* and $\mathcal{K}$ if $p$ is understood from the context.

As mentioned before, the cohomology of a space, in particular the cohomology of the classifying space of a group, i.e. the cohomology of a group, is a natural example of an unstable algebra, the product $\varphi$ being given by the cup product.

The conceptual importance of the cohomology of an elementary abelian $p$ - group is underlined by the following result of Carlsson resp. Miller which in the form given here is due to Lannes and Zarati. This result marks the beginning of a flourishing period of research on unstable modules and homotopy theory of classifying spaces.

THEOREM 1.9 ([C], [M], [LZ]). *Let $V$ be an elementary abelian $p$ - group. Then $H^*(V;\mathbb{F}_p)$ is an injective object in the category $\mathcal{U}_p$.*

We remark that the only finite groups for which $H^*(G;\mathbb{F}_p)$ is injective in $\mathcal{U}_p$ are those whose $p$ - Sylow subgroup is elementary abelian [H1].

Before we can connect Theorem 1.3 with the theory of unstable modules we need the concept of a nilpotent unstable module.

DEFINITION 1.10. An unstable $A_p$ - module $N$ is called *nilpotent* iff for each $x \in N$ there exists a natural number $n$ such that

- $Sq^{2^n|x|} \cdots Sq^{|x|}x = 0$, if $p = 2$
- $P^{p^n \frac{|x|}{2}} \cdots P^{\frac{|x|}{2}} x = 0$ if $p > 2$ and $x$ has even degree.

The justification for this terminology comes from the last property defining an unstable algebra, i.e. from the fact that in this case the condition reads $x^{p^{n+1}} = 0$. In particular, the ideal of nilpotent elements in an unstable algebra form a nilpotent module.

The following result of Lannes and Schwartz gives a characterization of nilpotent modules in terms of the injective objects $H^*(V;\mathbb{F}_p)$.

THEOREM 1.11 [LS]. *An unstable module $N$ is nilpotent if and only if $\mathrm{Hom}_{\mathcal{U}_p}(N, H^*(V;\mathbb{F}_p)) = 0$ for all elementary abelian $p$ - groups $V$.*

Theorem 1.9 and 1.11. imply.

THEOREM 1.3'. *Let $p$ be a prime. Then $G$ is a Quillen - group iff for every elementary abelian $p$ - group $V$ the map*

$$\mathrm{Hom}_{\mathcal{U}_p}(H^*(G;\mathbb{F}_p), H^*(V;\mathbb{F}_p)) \longleftarrow \mathrm{Hom}_{\mathcal{U}_p}(\lim_{\mathcal{A}_p(G)^{op}} H^*(E;\mathbb{F}_p), H^*(V;\mathbb{F}_p))$$

*induced by $q_G$ is an isomorphism.*

If we assume that the category $\mathcal{A}_p(G)$ is equivalent to a finite category (which holds for all Quillen groups whose mod - $p$ cohomology ring is finitely generated [HLS2]) then the injectivity of $H^*(V;\mathbb{F}_p)$ leads to the following isomorphism

$$\mathrm{Hom}_{\mathcal{U}_p}(\lim_{\mathcal{A}_p(G)^{op}} H^*(E;\mathbb{F}_p), H^*(V;\mathbb{F}_p)) \cong \\ \mathrm{colim}_{\mathcal{A}_p(G)} \mathrm{Hom}_{\mathcal{U}_p}(H^*(E;\mathbb{F}_p), H^*(V;\mathbb{F}_p)) \ .$$

By a theorem of Adams-Gunawardena-Miller [AGM] the natural map

$$\mathbb{F}_p[\mathrm{Hom}(V,E)] \longrightarrow \mathrm{Hom}_{\mathcal{U}_p}(H^*(E;\mathbb{F}_p), H^*(V;\mathbb{F}_p))$$

is an isomorphism where for a set $S$ the $\mathbb{F}_p$ - vector space with $S$ as a basis is denoted by $\mathbb{F}_p[S]$. Finally, it is easy to identify $\mathrm{colim}_{\mathcal{A}_p(G)} \mathbb{F}_p[\mathrm{Hom}(V,E)]$ with $\mathbb{F}_p[\mathrm{Rep}(V,G)]$ if $\mathrm{Rep}(V,G)$ denotes the set of $G$ - conjugacy classes of homomorphisms from $V$ to $G$.

In other words, we obtain the following characterization of Quillen - groups.

THEOREM 1.3". *Let $p$ be a prime and let $G$ be a group for which $\mathcal{A}_p(G)$ is equivalent to a finite category. Then $G$ is a Quillen - group iff the natural map*

$$\mathbb{F}_p[\mathrm{Rep}(V,G)] \longrightarrow \mathrm{Hom}_{\mathcal{U}_p}(H^*(G;\mathbb{F}_p), H^*(V;\mathbb{F}_p))$$

*is an isomorphism for all elementary abelian $p$ - groups $V$.*

**1.12.** Theorem 1.3" says in particular that the $F$ - isomorphism type of $H^*(G;\mathbb{F}_p)$ is completely captured by the sets $\mathrm{Rep}(V,G)$ resp. the vector spaces $\mathbb{F}_p[\mathrm{Rep}(V,G)]$ (with $V$ varying), at least if $\mathcal{A}_p(G)$ is finite up to equivalence. In particular, statements on $H^*(G;\mathbb{F}_p)$ can often be tested by the corresponding statements on these sets. For more on this point of view the reader is referred to [HLS1].

In the next section we will see how Lannes' theory of the $T$ - functor leads to a vast generalization of Theorem 1.3.

## 2. Lannes' $T$ - functor and his generalization of Quillen's theorem

Throughout this section $p$ is a fixed prime. From now on we supress the coefficients and write $H^*G$ instead of $H^*(G;\mathbb{F}_p)$, and so on.

**2.1.** Given an elementary abelian $p$ - group $V$, Lannes ([Ln1], [Ln3]) considered the functor $T_V : \mathcal{U} \longrightarrow \mathcal{U}$ characterized by the adjunction property

$$\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, N \otimes H^*V) \ .$$

One can view $T_V$ as a "cohomological analogue" of the mapping space functor $X \mapsto \mathrm{map}(BV, X)$. The existence of such a functor is quite formal, but the interesting thing here is that these functors enjoy some magical properties which make them extremely useful. Here are the two basic properties of these functors.

THEOREM 2.2 [LN1], [LN3]. *$T_V$ is an exact functor.*

THEOREM 2.3 [LN1], [LN3]. *$T_V$ commutes with tensor products. More precisely, the product on $H^*V$ induces a natural transformation $T_V(M\otimes N)\longrightarrow T_VM\otimes T_VN$ which is an isomorphism for all unstable modules $M$ and $N$.*

**2.4.** Taking $N=\mathbb{F}_p$, concentrated in dimension 0, and using that $\mathrm{Hom}_{\mathcal{U}}(L,\mathbb{F}_p)$ agrees with the linear dual $(L_0)^*$ of the homogeneous part $L^0$ of degree 0, we see that Theorem 1.3" is just the computation of $(T_V^0H^*G)^*$ with $T_V^0$ denoting the degree 0 part of $T_V$. In [Ln2] Lannes computes all of $T_VH^*G$ and this computation should be considered as a vast generalization of Quillen's theorem. To describe his result we need to introduce some more notation.

**2.5.** For a representation $\rho\in\mathrm{Rep}(V,G)$ choose a representative and by abuse of notation we will denote this representative again by $\rho$. Then consider the homomorphism

$$c_\rho : V\times C_G(\mathrm{Im}\,\rho)\longrightarrow G,\quad (v,g)\mapsto\rho(v)g\ .$$

Here $C_G(\mathrm{Im}\,\rho)$ stands for the centralizer of $\mathrm{Im}\,\rho$ in $G$. We obtain an induced map $c_\rho^*:H^*G\longrightarrow H^*V\otimes H^*C_G(\mathrm{Im}\,\rho)$ and an adjoined map $ad(c_\rho^*):T_VH^*G\longrightarrow H^*C_G(\mathrm{Im}\,\rho)$.

THEOREM 2.6 [LN2], [LN3]. *Let $G$ be a finite group and $V$ be an elementary abelian $p$ - group. Pick a representative for each element $\rho$ in $\mathrm{Rep}(V,G)$. Then the resulting map*

$$\lambda_{G,V}:T_VH^*G\longrightarrow\prod_{\rho\in\mathrm{Rep}(V,G)}H^*C_G(\mathrm{Im}\,\rho)$$

*with components $ad(c_\rho^*)$ is an isomorphism.*

**2.7.** As in the case of Theorem 1.3 this result also holds for other classes of groups, in particular it holds for all classes of groups listed in 1.4. For 1.4.a) and 1.4.b) this was shown in [Ln2] and for 1.4.c) in [H4]. Another class was considered in the appendix of [H2] (see also A.6 in this paper).

We call a group $G$ a *Lannes - group* (at the prime $p$) if the map $\lambda_{G,V}$ is an isomorphism for all elementary abelian $p$ - groups $V$ (with modifications as in 1.4 if $G$ is a compact Lie group or a profinite group). Clearly, all Lannes - groups for which $\mathcal{A}_p(G)$ is equivalent to a finite category are also Quillen - groups.

**2.8.** Theorem 2.6 has various applications to the study of group cohomology which do not involve the Steenrod algebra and which refine Quillen's results. Philosophically, Theorem 2.6 emphasizes the rôle of centralizers of elementary abelian $p$ - subgroups as a second deeper level source of information on $H^*G$, the first level being given by the category of elementary abelian $p$ - subgroups of $G$. We will elaborate on this philosophy in the next two sections.

## 3. Approximations up to higher nilpotency

We consider Theorem 1.3 as a first order approximation to $H^*G$, an approximation up to nilpotent modules. The action of the Steenrod algebra allows refinements of this approximation which are based on the notion of higher nilpotency.

DEFINITION 3.1. Let $n$ be a natural number. An unstable module $N$ is called *$n$ - nilpotent* if the following operations are locally nilpotent on $N$ (i.e. iterating them sufficiently often kills each element in $N$):

- $Sq_k : N^m \to N^{2m-k},\ x \mapsto Sq^{m-k}x$ for all $0 \leq k < n$, if $p = 2$;
- $P_k : N^{2m+e} \to N^{2mp+e-2k(p-1)},\ x \mapsto P^{m-k}x$ for all $0 \leq k < n$ if $p$ is odd and $e \in \{0,1\}$.

**3.2.** Obviously, a 1 - nilpotent module is the same as a nilpotent module. A less explicit, but more conceptual characterization is given in terms of the suspension functor. Note that for an unstable module $M$ its suspension $\Sigma M$ (defined by raising degrees by one) is again unstable, but not every unstable module is the suspension of an unstable module. An unstable module can be shown to be $n$ - nilpotent iff every finitely generated submodule of it admits a finite filtration such that the successive filtration quotients are $n$ - fold suspensions. The full subcategories of $\mathcal{U}$ with objects the $n$ - nilpotent modules are denoted by $Nil_n$.

These are localizing categories in the sense of Gabriel [Ga] and Quillen's map $q_G$ is in fact localization away from the category $Nil_1$ [HLS2], which means that the kernel and cokernel of $q_G$ are in $Nil_1$ and the target of $q_G$ is $Nil_1$ - closed, i.e. does not admit any non-trivial maps from a 1 - nilpotent module. The computation of $T_V H^*G$ in degrees less than $n$ leads to a similar desription of the localization of $H^*G$ away from the category $Nil_n$. To get at this we consider the elementary abelian $p$ - subgroups $E$ of $G$ and consider the homomorphisms $c_E : E \times C_G(E) \to G,\ (e,g) \mapsto eg$. These induce homomorphisms $(c_E)^* : H^*G \longrightarrow H^*E \otimes (H^*C_G(E))^{<n}$ where for an unstable module $M$ we denote its quotient by the submodule of elements of degree at least $n$ by $M^{<n}$. Here is a somewhat vague form of the "higher nilpotency-order" generalization of Theorem 1.3.

THEOREM 3.3 [HLS3]. *Let $p$ be a prime and $G$ be a Lannes - group for which $H^*G$ is a finitely generated algebra.*

*a) The kernel of the map*

$$\widetilde{\lambda_n} : H^*G \longrightarrow \prod_{E \in \mathcal{A}_p(G)} H^*E \otimes (H^*C_G(E))^{<n}$$

*whose components are given by the maps $(c_E)^*$ is the largest $n$ - nilpotent $A_p$ - invariant submodule of $H^*G$.*

*b) The elements in the image of $\widetilde{\lambda_n}$ satisfy suitable compatibility conditions defining a subalgebra $L_n$ of $\prod_{E \in \mathcal{A}_p(G)} H^*E \otimes (H^*C_G(E))^{<n}$; the map $H^*G \longrightarrow L_n$ induced by $\widetilde{\lambda_n}$ can be identified with the localization map $\lambda_n$ away from $Nil_n$.*

*c) Furthermore, there exist natural numbers $d_0(G)$ and $d_1(G)$ such that $\lambda_n$ (and hence $\widetilde{\lambda_n}$) is a monomorphism for $n > d_0(G)$ and $\lambda_n$ is an isomorphism for $n > d_1(G)$.*

**3.4.** Note that the map $\widetilde{\lambda_1}$ is given by the restriction homomorphisms; in this case the compatibility conditions are completely determined by the category $\mathcal{A}_p(G)$ and we recover Theorem 1.3. In the general case the precise compatibility conditions are more complicated; roughly speaking they are induced by the compatibility condition coming from the variation of $E \in \mathcal{A}_p(G)$ as well as the compatibilities coming from the fact that the two homomorphisms

$$E \times E \times C_G(E) \longrightarrow E \times C_G(E)$$

given by

$$(e_1, e_2, g) \mapsto (e_1e_2, g) \text{ resp. } (e_1, e_2, g) \mapsto (e_1, e_2g)$$

are equalized by the homomorphism

$$E \times C_G(E) \to C_G(E),\ (e, g) \mapsto eg\ .$$

(We will denote the former two homomorphisms by $m_E \times id$ resp. $id \times c_E$ where we have taken the liberty to use the symbol $c_E$ again although here its target need not agree with $G$.) For more details the reader is referred to [HLS3]. Roughly speaking, Theorem 3.3 says that $L_nH^*G$ (and for $n > d_1(G)$ even $H^*G$ itself!) is determined by $\mathcal{A}_p(G)$ and the mod - $p$ cohomology of the groups $C_G(E)$ in degrees less than $n$.

In the following special case the compatibility conditions simplify drastically and they are easy to spell out.

Corollary 3.5 [HLS3]. *Let $G$ be a Lannes - group for which $H^*G$ is a finitely generated algebra. Assume that all elements of $G$ of order $p$ are central (and hence generate a central elementary abelian $p$ - subgroup $D$). Then the map $c_D$ induces a homomorphism of unstable algebras*

$$H^*G \longrightarrow Eq : H^*D \otimes (H^*G)^{<n} \rightrightarrows H^*D \otimes (H^*D \otimes H^*G)^{<n}$$

*which is localization away from $Nil_n$, and an isomorphism for $n > d_1(G)$. (Here $Eq$ stands for equalizer and the two maps in the equalizer diagram are induced by the maps $m_D \times id$ resp. $id \times c_D$.)*

As an immediate further consequence of Theorem 3.3 we get some information on the nilpotency-height of the ideal of nilpotent elements in $H^*G$ (in terms of the invariant $d_0$). For $p = 2$ this ideal agrees with the kernel $N(G)$ of the restriction map $H^*G \longrightarrow \prod_{E \in \mathcal{A}_2(G)} H^*E$, for odd primes it is closely related to $N(G)$.

Corollary 3.6 [HLS3]. *Let $G$ be a Lannes - group for which $H^*G$ is a finitely generated algebra. Then the $n$ - th power of the ideal $N(G)$ is trivial if $n > d_0(H^*G)$.*

PROOF. If $x$ is in $N(G)$ then $x$ restricts trivially to $H^*E$ for all $E \in \mathcal{A}_p(G)$ and hence, if $\widetilde{H}^*$ denotes the ideal of elements of positive degree, $c_E^*(x) \in H^*E \otimes (\widetilde{H}^*C_G(E))^{<n}$. Consequently, if $x_1, ..., x_n$ are in $N(G)$ then $c_E^*(x_1 ... x_n) = 0$ and we are done by Theorem 3.3. □

The invariants $d_0(G)$ and $d_1(G)$ are not merely theoretical constants but can be effectively bounded above. The following result gives a crude but very easy to state general estimate. In it $\dim G$ etc. refers to the the dimension of $G$ considered as a manifold. For refinements we refer the reader to sections II.2 and II.3 of [HLS3].

THEOREM 3.7 [HLS3]. *Let $G$ be a finite group or a compact Lie group (in which case we use the conventions of 1.4) and let $U$ be a unitary group such that $G$ embeds into $U$. Then we have the following inequalities*

*a)* $d_0(G) \leq \dim U - \dim G$

*b)* $d_1(G) \leq 2 \dim U - \dim G$.

**3.8.** There are also bounds for many discrete groups based on extensions of 3.7 to equivariant cohomology (see [HLS3]) but the bounds are less simple to state.

In principle these effective bounds lead to algorithms to compute $H^*G$ from knowing the category $\mathcal{A}_p(G)$ and the low dimensional cohomology of the groups $C_G(E)$. This has not been seriously tested yet but one can hope that with increasing computer power this may become a feasable method for computing the cohomology of a finite group.

The invariants $d_0$ and $d_1$ behave well with respect to certain group theoretic constructions like the wreath product construction. This can sometimes be used to keep the bounds for these invariants much smaller than those given by the Corollary. For example, in [HLS3] we show that for $p = 2$ and if $q$ is a prime power with $q \equiv 3 \bmod 4$ then $d_0(H^*GL(n, \mathbb{F}_q)) = 0$, $d_1(H^*GL(n, \mathbb{F}_q)) = 2$ and we use this to recover Quillen's computation of the mod - 2 cohomology of the groups $GL(n, \mathbb{F}_q)$ [Q2].

## 4. Commutative algebra of unstable $H^*G$ - modules; approximations of $H^*G$ up to torsion modules

Throughout this section $p$ is a fixed prime and $G$ is a Quillen - group with finitely generated mod - $p$ cohomology ring $H^*G$.

**4.1.** We will discuss how much information about $H^*G$ is carried by the cohomology of (suitable) collections $\mathcal{O}$ of centralizers of elementary abelian $p$ - subgroups of $G$. The collections will have the following two properties:

- If $E \in \mathcal{O}$ and $E'$ is conjugate to $E$ then $E' \in \mathcal{O}$
- If $E \in \mathcal{O}$ and $E \subset E'$ then $E' \in \mathcal{O}$.

We call such collections *open*. The justification for this terminology comes from the fact that the conjugacy classes of elementary abelian $p$ - subgroups $E$

are in $1-1$ correspondence with prime ideals $\wp_E \subset H^*G$ which are invariant with respect to the reduced power operations $P^i$ if $p>2$ and with respect to all $Sq^i$ if $p=2$ [Q1]; the prime ideal $\wp_E$ is given as the radical of the kernel of the restriction map $H^*G \longrightarrow H^*E$. Via this correspondence open collections correspond to intersections of open sets in the Zariski topology with the invariant part of the prime ideal spectrum of the graded ring $H^*G$.

**4.2.** If $\mathfrak{a}$ is an ideal in $H^*G$ then it is clear that the collection $\mathcal{O}(\mathfrak{a}) := \{E \in \mathcal{A}_p(G) | \mathfrak{a} \not\subset \wp_E\}$ forms an open collection. Furthermore, $\mathcal{O}(\mathfrak{a})$ obviously does not change if $\mathfrak{a}$ is replaced by the smallest radical ideal containing $\mathfrak{a}$ which in addition is invariant under the action of the reduced power operations $P^i$ if $p>2$ resp. $Sq^i$ if $p=2$; so we may and will in the sequel assume that $\mathfrak{a}$ is radical and invariant. For example, if $\mathfrak{a} = \widetilde{H}^*G$ is the ideal consisting of the elements of positive degree in $H^*G$ then $\mathcal{O}(\mathfrak{a})$ is precisely the collection of all non-trivial elementary abelian $p$ - subgroups; this can be deduced from Proposition 4.5 below.

For an open collection $\mathcal{O}$ we will denote the full subcategory of $\mathcal{A}_p(G)$ with object set $\mathcal{O}$ also by $\mathcal{O}$. The asssignment $E \mapsto H^*C_G(E)$ extends to a functor from $\mathcal{O}$ to unstable modules which have an $H^*G$ - module structure given via the restriction homomorphisms. Here is a preliminary version of the main result of this section.

THEOREM 4.3 [H2]. *Let $G$ be a Lannes - group such that $H^*G$ is a finitely generated $\mathbb{F}_p$ - algebra. Let $\mathfrak{a} \subset H^*G$ be a radical invariant ideal and let $\mathcal{O}(\mathfrak{a})$ be the associated open collection of elementary abelian $p$ - subgroups. Then the restriction maps induce a map*

$$\rho : H^*G \longrightarrow \lim_{\mathcal{O}(\mathfrak{a})} H^*C_G(E)$$

*whose kernel and cokernel are $\mathfrak{a}$ - torsion, i.e. they are annihilated by some large power of $\mathfrak{a}$.*

**4.4.** Note that from a computational point of view the two properties defining an open collection are reasonable in the following sense: If $E \subset E'$ then the centralizer $C_G(E')$ will be contained in $C_G(E)$ and will tend to be easier to understand than $C_G(E)$, both group theoretically and cohomologically. In order to apply the theorem in concrete cases one would try to compute as many $H^*C_G(E)$ as possible; the minimal $E$'s in that collection define an open collection $\mathcal{O}$ of elementary abelian $p$-subgroups (all those which contain one of the minimal ones) and chances are that $H^*C_G(E)$ can be computed for all $E \in \mathcal{O}$ and with some luck $\lim_{\mathcal{O}(\mathfrak{a})} H^*C_G(E)$ can be computed as well.

To apply 4.3 we use that the collection $\mathcal{O}$ determines an invariant radical ideal $\mathfrak{a} := \mathfrak{a}(\mathcal{O})$ with $\mathcal{O} = \mathcal{O}(\mathfrak{a})$, namely $\mathfrak{a}(\mathcal{O}) = \bigcap_{E \notin \mathcal{O}} \wp_E$. In fact, this procedure establishes a $1-1$ corresondence between open collections $\mathcal{O}$ and radical invariant ideals [H2]. The theorem now yields an approximation of $H^*G$ up to $\mathfrak{a}$ - torsion. In particular the size (i.e. the Krull dimension, or equivalently the order of the pole of the Poincaré series at $t=1$) of the kernel of the map $\rho$ can be estimated by the size of $H^*G/\mathfrak{a}$, i.e. by the maximal rank of an $E \notin \mathcal{O}$. Furthermore, the theorem also gives a tool to construct elements in $H^*G$: for this one can take any element $z \in \lim_{\mathcal{O}(\mathfrak{a})} H^*C_G(E)$; if $x_1, \cdots x_k$ are in $\mathfrak{a}$ and $k$ is large enough then $x_1 \cdots x_k z$ is in the image of the map. Non-trivial Elements $x_i$ can often be constructed as characteristic classes and bounds for $k$ are also often available, e.g. via the methods of part II of [HLS3]. The existence of non-trivial classes $x_i \in \mathfrak{a}$ is also guaranteed by the following result.

PROPOSITION 4.5 [CAH]. *Assume $G$ is a Quillen - group and $H^*G$ is a finitely generated algebra. Let $\mathcal{O} \subset \mathcal{A}_p(G)$ be an open collection. Then there is an element $x \in H^*G$ which restricts nilpotently to $H^*E$ iff $E \notin \mathcal{O}$.*

4.6 EXAMPLES. a) In [H2] we used Theorem 4.3 to give an upper bound for the size of the kernel $K_n$ of the restriction map from $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ to $H^*(D_n;\mathbb{F}_2)$ (with $D_n$ denoting the subgroup of diagonal matrices, i.e. $D_n \cong (\mathbb{Z}[1/2]^\times)^n \cong \mathbb{Z}^n \times (\mathbb{Z}/2)^n$). This kernel is non-trivial for $n$ large, e.g. if $n = 32$ [Dw2]. If $\sigma(K_n)$ denotes the order of the pole at $t = 1$ of the Poincaré series of $K_n$, and if $n_0$ is the minimal $n$ with $K_n$ non-trivial, then 4.3 can be used to show $\sigma(K_n) \leq n - n_0 + 1$ for $n \geq n_0$. In fact, we show even more in [H2], namely $\sigma(K_n) = n - n_0 + 1$ for $n \geq n_0$.

b) In joint work with F. Cohen and Y. Xia we will investigate certain mapping class groups and will produce new classes in their cohomology by the method outlined in 4.4 above.

**4.7.** Theorem 4.3 is contained in the following more general result. To state it we need to introduce more notation.

First we need to extend the context in which we are working. We consider the category $H^*G-\mathcal{U}$ of unstable $H^*G$ - modules. Its objects are unstable modules $M$ over the Steenrod algebra equipped with the structure of a graded $H^*G$ - module such that the structure map $H^*G \otimes M \longrightarrow M$ is $A_p$ - linear, i.e. it obeys the Cartan formula. Morphisms are degree preserving homomorphisms which are both $A_p$ - linear and $H^*G$ - linear. The full subcategory whose objects are finitely generated as $H^*G$ - modules is denoted by $H^*G_{fg}-\mathcal{U}$. Natural objects of $H^*G-\mathcal{U}$ are given by $H^*G$, $H^*C_G(E)$ or more generally by the cohomology $H^*_GX := H^*(EG \times_G X)$ of the Borel construction of a $G$ - space $X$. If $X$ is a finite $G$ - $CW$ - complex and $G$ is a suitable discrete group (e.g. if $G$ is finite) then $H^*_GX$ is even in $H^*G_{fg}-\mathcal{U}$.

For an invariant ideal $\mathfrak{a}$ and an unstable $H^*G$ - module $M$ we consider

$$F_{\mathfrak{a}}M := \{x \in M | \mathfrak{a}^n x = 0 \text{ for some } n \in \mathbb{N}\} \ .$$

It is straightforward to check (using the Cartan formula) that $F_{\mathfrak{a}}$ defines a functor from $H^*G-\mathcal{U}$ resp. $H^*G_{fg}-\mathcal{U}$ to itself. This functor is left exact, and as the category $H^*G_{fg}-\mathcal{U}$ is abelian with enough injectives [H2], $F_{\mathfrak{a}}$ has right derived functors denoted by $R^iF_{\mathfrak{a}}$. If $\mathfrak{a} = \widetilde{H^*}G$ and $M$ is in $H^*G_{fg}-\mathcal{U}$ then it is clear that $F_{\mathfrak{a}}M$ is the largest finite subobject of $M$; in this case we denote the functor simply by $F$ and its derived functors by $R^iF$. For $M \in H^*G_{fg}-\mathcal{U}$ all the modules $R^iFM$ are finite.

Here is the generalization of Theorem 4.3.

THEOREM 4.8 [H2]. *Let $G$ be a Lannes - group with $H^*G$ a finitely generated algebra and $\mathfrak{a} \subset H^*G$ an invariant ideal with associated open collection $\mathcal{O}(\mathfrak{a})$. Denote the full subcategory with $\mathcal{O}(\mathfrak{a})$ as object set again by $\mathcal{O}(\mathfrak{a})$.*

*a) Then there is an exact sequence*

$$0 \longrightarrow F_{\mathfrak{a}} H^*G \longrightarrow H^*G \xrightarrow{\rho} \lim_{\mathcal{O}(\mathfrak{a})} H^*C_G(E) \longrightarrow R^1 F_{\mathfrak{a}} H^*G \longrightarrow 0$$

*in which the components of $\rho$ are induced by the restriction maps. In particular, the kernel and cokernel of $\rho$ are unstable finitely generated $\mathfrak{a}$ - torsion modules.*

*b) If* $\lim^i$ *denotes the $i$ - th right derived functor of* $\lim$ *then there are natural isomorphisms*

$$\lim^i_{\mathcal{O}(\mathfrak{a})} H^*C_G(E) \cong R^{i+1} F_{\mathfrak{a}} H^*G$$

*for all $i > 0$. In particular, $\lim^i_{\mathcal{O}(\mathfrak{a})} H^*C_G(E)$ is an unstable finitely generated $\mathfrak{a}$ - torsion module for all $i > 0$.*

The following special case of this theorem needs to be emphasized. In it $\mathcal{A}_p^*(G)$ denotes the full subcategory of $\mathcal{A}_p(G)$ consisting of all non-trivial elementary abelian $p$ - subgroups of $G$.

COROLLARY 4.9 [H2]. *Let $G$ be a Lannes - group with $H^*G$ a finitely generated algebra.*

*a) Then there is an exact sequence*

$$0 \longrightarrow FH^*G \longrightarrow H^*G \xrightarrow{\rho} \lim_{\mathcal{A}_p^*(G)} H^*C_G(E) \longrightarrow R^1 FH^*G \longrightarrow 0$$

*in which the components of $\rho$ are induced by the restriction maps. In particular, the kernel and cokernel of $\rho$ are finite.*

*b) There are natural isomorphisms*

$$\lim^i_{\mathcal{A}_p^*(G)} H^*C_G(E) \cong R^{i+1} FH^*G$$

*for all $i > 0$. In particular, $\lim^i_{\mathcal{A}_p^*(G)} H^*C_G(E)$ is finite for all $i > 0$.*

We close this section with some comments on and consequences of 4.8 and 4.9.

**4.10.** The results in [H2] are even more general; there is a version which holds for each unstable module $M \in H^*G_{fg} - \mathcal{U}$ (in fact, $H^*G$ can even be replaced by an unstable algebra $K$ as long as it is finitely generated as an $\mathbb{F}_p$ - algebra). In this generality the cohomology of the centralizers is replaced by certain "components" of $T_V M$; if $M = K = H^*G$ these components are given via Theorem 2.6 by the cohomology of the centralizers $C_G(E)$. We refer the reader to [H2] for more details. Here we just want to remark that these "components" serve in the world of unstable $H^*G$ - modules as a replacement of the sheaf associated to a module over a commutative ring in the classical context. In particular, in the world of unstable modules and unstable algebras the functor $E \mapsto H^*C_G(E)$ defined on $\mathcal{A}_p(G)$ plays the role of the structure sheaf of the ring $H^*G$, and this yields a conceptual explanation why the cohomoloy of the centralizers play such a crucial role in understanding $H^*G$. This point of view is also supported by the formal analogy of Theorem 4.8 with results of Grothendieck on local cohomology in the classical situation of modules over a noetherian commutative ring $R$ [Gr].

**4.11.** If $G$ is a finite group (or a compact Lie group) which contains an element of order $p$ then one can show [H2] that $R^i F H^*G$ is trivial for all $i \geq 0$ and therefore the map $\rho$ in Corollary 4.9 is an isomorphism and the higher limits $\lim^i_{\mathcal{A}_p^*(G)} H^*C_G(E)$ vanish for all $i > 0$. This is equivalent to a result by Jackowski and McClure [JM] resp. by Dwyer and Wilkerson [DW1] (see also section 5 below) which plays a central role in the homotopy theory of classifying spaces and $p$ - compact groups ([DW2], [JMO]). (The author apologizes that the hypothesis of the existence of an element of order $p$ in $G$ was forgotten in [H2, 3.8]; he finds this particularly embarassing because in [H3] the author pointed to this very same sloppiness in [JM].)

If $G$ is discrete or profinite then $\rho$ need no longer be an isomorphism. In [H4] Corollary 4.9 was used to compute the continuous mod - $p$ cohomology in large dimensions of $GL(p-1, \mathbb{Z}_p)$ and the $(p-1)$ - st "Morava stabilizer groups", i.e. the ($p$-Sylow subgroups of the) units in the maximal order of the division algebras over $\mathbb{Q}_p$ whose dimension is $(p-1)^2$ and whose Hasse invariant is $\frac{1}{p-1}$. For $p = 3$ additional arguments allowed to extend the calculation in large dimensions to one in all dimensions and in those cases $\rho$ turned out not to be an isomorphism. The case of $H^*(SL(3, \mathbb{Z}[1/2]); \mathbb{F}_2)$ was considered in [H3] and [H5]: Corollary 4.9 was used in [H3] to carry out the calculation in large dimensions, and building on this the calculation was completed in [H5] (see section 5 below); in this case the complete calculation showed that $\rho$ happens to be an isomorphism again.

**4.12.** Theorem 4.8 implies that the restriction map

$$\mathrm{res}_{\mathcal{O}(\mathfrak{a})} : H^*G \longrightarrow \prod_{E \in \mathcal{O}(\mathfrak{a})} H^*C_G(E)$$

is injective if and only if $F_{\mathfrak{a}} H^*G = 0$. In particular, if there exists a non-trivial element in $\mathfrak{a}$ which is not a zero-divisor then clearly $\mathrm{res}_{\mathcal{O}(\mathfrak{a})}$ is injective. This was first proved in [HLS3]. The following "converse" was proved in [CaH]: if $\mathrm{res}_{\mathcal{O}(\mathfrak{a})}$ is injective then there exists an element $x \in \mathfrak{a}$ which is not a zero-divisor; in fact, any element provided by Proposition 4.5 has this property.

**4.13.** Another extreme case of Theorem 4.8 can be considered as a generalization of a result of Duflot [Du]. She identified $(H^*G)_{\wp_E}$, the localization of $H^*G$ at a minimal invariant prime ideal $\wp_E$ (given as radical of the kernel of restriction to a maximal elementary abelian $p$ - subgroup $E$) with $((H^*C_G(E))_{\wp_E})^{N_G(E)}$, the invariants with respect to the action of the normalizer $N_G(E)$. Theorem 4.8 provides a strengthening of this result in the following sense: the maximal subgroup $E$ determines an open collection $\mathcal{O}$ consisting of all elementary abelians which are conjugate to $E$. If we abbreviate $\mathrm{Aut}_{\mathcal{A}_p(G)}(E)$ by $A$ then the inverse limit over this category can be identified with the invariants $(H^*C_G(E))^A \cong (H^*C_G(E))^{N_G(E)}$. By 4.8 the kernel and cokernel of $\rho$ are torsion with repect to the ideal $\mathfrak{a}$ consisting of all elements which restrict nilpotently to $E'$ whenever $E'$ is not conjugate to $E$. By Proposition 4.5 there exist elements in $\mathfrak{a}$ which restrict non-trivially to $E$ and therefore $\mathfrak{a}$ - torsion modules are killed by localization at $\wp_E$. Now localization at $\wp_E$ is exact and hence Duflot's result follows.

**4.14.** Corollary 4.9 is somewhat reminiscent of results on Farrell cohomology of groups of finite virtual cohomological dimension. We will come back to this in the next section.

## 5. The centralizer spectral sequence

As before $p$ is a fixed prime. In this section the coefficients need not necessarily be equal to $\mathbb{F}_p$ so we will no longer skip them from the notation.

The higher limits of Theorem 4.8 have also geometric significance; they occur as $E_2$ - term in a homotopy colimit spectral sequence. The reader uncomfortable with homotopy colimits is encouraged to consult [JMO] for an introduction to the concept. In this section homotopy colimits will be primarily discussed to the extent that they produce interesting spectral sequences; they can be defined for functors $F$ from a small category $\mathcal{C}$ to topological spaces and give a recipe to glue together the spaces $F(c)$, $c \in \mathcal{C}$, in a homotopy invariant way encoded by the category $\mathcal{C}$. Our first example is the functor $E \mapsto EG \times_G (G/C_G(E)$ from $\mathcal{A}_p(G)^{op}$ to spaces. Note that if $E$ is the trivial subgroup this functor takes value $BG$.

First we recall the following result of Jackowski and McClure which "reconstructs" $BG$ from the spaces $EG \times_G (G/C_G(E)) \simeq BC_G(E)$ with $E$ non-trivial.

Theorem 5.1 [JM]. *Let $G$ be a compact Lie group (which may be finite but is assumed to contain an element of order $p$). Then the natural map*

$$\pi : \mathrm{hocolim}_{\mathcal{A}_p^*(G)^{op}} EG \times_G G/C_G(E) \longrightarrow BG$$

*induces an isomorphism in homology with coefficients in the $p$ - local integers $\mathbb{Z}_{(p)}$.*

Starting from the result of Jackowski and McClure the following generalization was obtained in [H3]. To state it we need to introduce some notation. Let $G$ be a (topological) group and $X$ a $G$ - space. If $H$ is a subgroup of $G$ then $X^H$ denotes the subspace of $X$ fixed by all of $H$; the $p$ - singular locus $X_s$ is defined to be $X_s := \bigcup_g X^{<g>}$ where $g$ runs through all elements of $G$ of order $p$ and $< g >$ denotes the subgroup generated by $g$. The $G$ - space $X$ gives rise to a functor $E \mapsto EG \times_{C_G(E)} X^E$ from $\mathcal{A}_p(G)^{op}$ to spaces.

Theorem 5.2 [H3]. *Assume $G$ is a discrete group or a compact Lie group and $X$ is a $G$ - $CW$ - complex. If $G$ is discrete assume that the isotropy group of each $x \in X$ is finite. Then the natural map*

$$\pi : \mathrm{hocolim}_{\mathcal{A}_p^*(G)^{op}} EG \times_{C_G(E)} X^E \longrightarrow EG \times_G X_s$$

*induces an isomorphism with coefficients in $\mathbb{Z}_{(p)}$.*

Of primary interest here is the resulting homotopy colimit spectral sequence [BK].

COROLLARY 5.3 [H3]. *Assume $G$ is a discrete group or a compact Lie group and $X$ is a $G$ - $CW$ - complex. If $G$ is discrete assume that the isotropy group of each $x \in X$ is finite. Then for every $\mathbb{Z}_{(p)}$ - module $M$ there exists a first quadrant cohomological spectral sequence*

$$E_2^{s,t} = \lim\nolimits^s_{\mathcal{A}_p^*(G)} H^t_{C_G(E)}(X^E; M) \Longrightarrow H_G^{s+t}(X_s; M) \ .$$

There are strong vanishing results for this spectral sequence in case $M = \mathbb{F}_p$. In fact, in [JM] Theorem 5.1 was proved by showing via Mackey functor techniques that in this case the spectral sequence is concentrated on the vertical edge and the natural map $H^*(G; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(C_G(E); \mathbb{F}_p)$ is an isomorphism. Dwyer and Wilkerson [DW1] gave a proof of this using techniques from the theory of unstable modules and in [H2] still another such proof was given (cf. 4.11 above). In the general case the results of [H2] (cf. 4.8 above) imply:

THEOREM 5.4 [H3]. *Assume $G$ is a discrete group or a compact Lie group and $X$ is a $G$ - $CW$ - complex with finitely many equivariant cells. If $G$ is discrete assume that the isotropy group of each $x \in X$ is finite. If coefficients are in $\mathbb{F}_p$ then $E_2^{s,*}$ is a finite graded $\mathbb{F}_p$ - vector space for each $s > 0$, in particular $E_2^{s,t} = 0$ if $s > 0$ and $t$ is large.*

An additional vanishing result has been proved by Oliver. (From the point of view of 4.10 above this corresponds to the vanishing of local cohomology above the Krull dimension of the ring.)

THEOREM 5.5 [O]. *Let $G$ be any group and let $F$ be any functor from $\mathcal{A}_p^*(G)$ to $\mathbb{Z}_{(p)}$ - modules. Then $\lim^s_{\mathcal{A}_p^*(G)} F = 0$ for all $s \geq r_p(G)$, the $p$ - rank of $G$.*

As a consequence, under the assumptions of Theorem 5.4 and if $r_p(G)$ is finite then the spectral sequence is essentially concentrated on the vertical edge and it almost collapses. In particular, the edge homomorphism $H^*_G(X_s; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*_{C_G(E)}(X^E; \mathbb{F}_p)$ is an isomorphism in large dimensions.

These results have interesting consequences for computations of group cohomology. First we introduce certain classes of groups by slightly modifying a definition due to Kropholler [K].

DEFINITION 5.6. A discrete group $G$ is said to be an $\mathcal{H}_1\mathcal{F}$ - group iff there exists a $G$ - $CW$ - complex $X$ with the following properties:

- $X$ is finite dimensional and mod - $p$ acyclic,
- the isotropy group $G_x$ of each $x \in X$ is a finite group.

In this case we call $X$ an $\mathcal{H}_1\mathcal{F}$ - complex for $G$.

- If in addition $G\backslash X$ is compact, then $G$ is said to be an $\mathcal{K}_1\mathcal{F}$ - group and $X$ a $\mathcal{K}_1\mathcal{F}$ - complex for $G$.

As in [K] one can define $\mathcal{H}_\alpha\mathcal{F}$ - groups for every ordinal $\alpha$ but we will refrain from doing so because we are only interested in the case $\alpha = 1$. All groups $G$ of finite virtual cohomological dimension are $\mathcal{H}_1\mathcal{F}$ - groups and very often they are even $\mathcal{K}_1\mathcal{F}$ - groups, e.g. if $G$ is an ($S$ - ) arithmetic group [Se], a mapping class group [Ha], an outer automorphism group of a free group [CV] or a word hyperbolic group in the sense of Gromov [GH].

**5.7.** If $X$ is a mod - $p$ acyclic $G$ - $CW$ - complex then $H_G^*(X;\mathbb{F}_p)\cong H^*(G;\mathbb{F}_p)$, and if $X$ is also finite dimensional then Smith theory implies that the fixed point sets $X^E$ are also mod - $p$ acyclic and therefore $H^*_{C_G(E)}(X^E;\mathbb{F}_p)\cong H^*(C_G(E);\mathbb{F}_p)$. Furthermore $H_G^*(X;\mathbb{F}_p)\cong H_G^*(X_s;\mathbb{F}_p)$ if $*$ exceeds the dimension of $X$; more precisely, the relative group $H_G^*(X,X_s;\mathbb{F}_p)$ is naturally isomorphic to $H^*(G\backslash(X,X_s);\mathbb{F}_p)$ and therefore vanishes if $*$ exceeds the dimension of $X$. Thus we have an almost collapsing "centralizer spectral sequence" which converges not to $H^*(G;\mathbb{F}_p)$ but to the closely related object $H_G^*(X_s;\mathbb{F}_p)$. More precisely we have the following result.

Corollary 5.8 [H3]. *a) Assume $G$ is an $\mathcal{H}_1\mathcal{F}$ - group, $X$ an $\mathcal{H}_1\mathcal{F}$ - complex for $G$ and $M$ is a $\mathbb{Z}_{(p)}$ - module. Then there exists a first quadrant cohomological spectral sequence*

$$E_2^{s,t}=\lim{}^s_{\mathcal{A}_p^*(G)}H^t(C_G(E);M)\Longrightarrow H_G^{s+t}(X_s;M)\ .$$

*with $E_2^{s,*}=0$ for $s\geq r_p(G)$.*

*b) If $G$ is a $\mathcal{K}_1\mathcal{F}$ - group, $X$ a $\mathcal{K}_1\mathcal{F}$ - complex for $G$ and $M=\mathbb{F}_p$ then $E_2^{s,*}$ is a finite graded $\mathbb{F}_p$ - vector space for each $s>0$ and $E_2^{s,*}=0$ for $s\geq r_p(G)$. Furthermore, both the edge homomorphism $H_G^*(X_s;\mathbb{F}_p)\longrightarrow\lim_{\mathcal{A}_p^*(G)}H^*(C_G(E);\mathbb{F}_p)$ and the map $\rho: H^*(G;\mathbb{F}_p)\longrightarrow\lim_{\mathcal{A}_p^*(G)}H^*(C_G(E);\mathbb{F}_p)$ are isomorphisms in large dimensions.*

**5.9.** The corollary suggests a program for computing the mod - $p$ cohomology of a discrete $\mathcal{K}_1\mathcal{F}$ - group consisting of the following steps (each of which can be quite difficult though).

- Determine the (category of) elementary abelian $p$ - subgroups of $G$ and the centralizers of these subgroups.
- Compute the mod - $p$ cohomology of the centralizers (as a functor on $\mathcal{A}_p^*(G)$).
- Compute $H_G^*(X_s;\mathbb{F}_p)$ from the (almost collapsing) centralizer spectral sequence. (Note that the input into this spectral sequence is independant of the choice of $X$!)
- Find a suitable $\mathcal{K}_1\mathcal{F}$ - complex $X$, compute $H^*(G\backslash(X,X_s);\mathbb{F}_p)$ and evaluate the long exact mod - $p$ cohomology sequence of the pair $EG\times_G(X,X_s)$. (This part is vital for getting at the low-dimensional cohomology of $G$.)

5.10 Example. The group $G=SL(3,\mathbb{Z}[1/2])$ is known to be a $\mathcal{K}_1\mathcal{F}$ - group. For $M=\mathbb{F}_2$ this group provides a particularly favourable example for this program and was dealt with in [H3] and [H5].

The first three steps do not involve the space $X$ explicitly and are relatively easy. They were carried out in [H3]; over $\mathbb{Z}[1/2]$ all elementary abelian 2 - subgroups in $SL(n,\mathbb{Z}[1/2])$ are diagonalizable, the centralizers in case $n=3$ are small and their cohomology was known. The 2 - rank is 2, so the spectral sequence has only two columns ($s=0,1$) and degenerates into a short exact sequence

$$0 \longrightarrow \Sigma \lim^1_{\mathcal{A}_p^*(G)} H^*(C_G(E);\mathbb{F}_2) \longrightarrow H^*_G(X_s;\mathbb{F}_2) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(C_G(E);\mathbb{F}_2) \longrightarrow 0 \ .$$

Here $\Sigma$ denotes the suspension functor and appears here in order to make all maps degree preserving. The inverse limit turns out to be isomorphic to $\mathbb{F}_2[w_2,w_3]\otimes\Lambda(e_3,e_5)$, the tensor product of a polynomial algebra on the universl Stiefel Whitney classes $w_2$ and $w_3$ (coming from the embedding of $\mathbb{Z}[1/2]\subset\mathbb{R}$) and an exterior algebra on classes $e_3$ and $e_5$ of dimension 3 and 5 respectively. The $\lim^1$ - term is non-trivial; it is isomorphic to $\Sigma^3\mathbb{F}_2$.

The last step of the program was carried out in [H5]; this was quite involved and required a careful analysis of the action of $SL(3,\mathbb{Z}[1/2])$ on a suitable space $X$; such a space is given by the product $X_2\times X_\infty$ where $X_2$ denotes the Bruhat-Tits-building for $SL(3,\mathbb{Q}_2)$ and $X_\infty$ the symmetric space $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$. The relative cohomology turned out to compensate the contribution from the $\lim^1$ - term and the final result can be stated as follows.

Theorem 5.11 [H5]. *The map $\rho$ induces an isomorphism*

$$H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)\cong\mathbb{F}_2[w_2,w_3]\otimes\Lambda(e_3,e_5)\ .$$

*Furthermore the restriction map to the diagonal matrices in $SL(3,\mathbb{Z}[1/2])$ is injective.*

5.12 Example 2. Another interesting application of Theorem 5.2 (which does not strictly follow the program of 5.9) occurs in joint work in progress with J. Lannes. We study the mod - 2 cohomology of the orthogonal group (with respect to the standard bilinear form) $O(n,\mathbb{Z}[1/2])$ and show that the reduction map to $O(n,\mathbb{F}_3)$ induces an isomorphism in mod - 2 cohomology for $n\le 14$. The groups $O(n,\mathbb{Z}[1/2])$ belong to the class $\mathcal{K}_1\mathcal{F}$ and in the range $n\le 14$ we show that $X=X_s$ for a suitable "building" which serves as a $\mathcal{K}_1\mathcal{F}$ - complex for $O(n,\mathbb{Z}[1/2])$. This together with Theorem 5.2 enables us to prove by induction that the reduction map induces an isomorphism for $n\le 14$.

Similarly, if $O'(n,\mathbb{Z}[1/2])$ and $O'(n,\mathbb{F}_3)$ denote the orthogonal groups for the bilinear form $(x,y)\mapsto\Sigma_{i<n}x_iy_i+2x_ny_n$, then the reduction map induces an isomorphism for $n\le 13$ but not for $n=14$. This can be used to show that the restriction map in mod - 2 cohomology from $GL(n,\mathbb{Z}[1/2])$ to its subgroup $D_n$ of diagonal matrices is not injective for $n\ge 14$ thus improving on Dwyer's result [Dw2] (see 4.6 above) which required $n\ge 32$.

**5.13.** The "centralizer spectral sequence" has certain similarities with a "normalizer spectral sequence" in Farrell cohomology for groups of finite virtual cohomological dimension. Recall that above the virtual cohomological dimension Farrell cohomology (with any coefficients) agrees for such groups with ordinary cohomology and that there is a spectral sequence whose $E_1$ - term is given by the Farrell cohomology of intersections of normalizers of elementary abelian $p$ - subgroups of $G$ and which converges to the Farrell cohomology of $G$ [Br]. The relation between the two constructions leading to these spectral sequences has been investigated in work of [Sl] and more recently [Dw1]. However, the input in both spectral sequences is different. The input in the centralizer spectral sequence has a homological nature (namely derived functors of lim) and we believe that it is therefore easier to handle. The example of $SL(3, \mathbb{Z}[1/2])$ and $p = 2$ gives some evidence to this claim.

Given that the two spectral sequences abut to the same object (at least in large dimensions) one starts wondering whether they might agree from some stage on. There is a bit of experimental evidence in this direction. E.g. in the case of $SL(3, \mathbb{Z}[1/2])$ the two spectral sequences agree in large total degree from $E_2$ on. Results by Chun Nip Lee [Le] on the edge homomorphism of the spectral sequence give some more evidence.

We summarize by expressing our belief that the "centralizer spectral sequence" is more effective for cohomology with trivial coefficients $\mathbb{F}_p$. Its $E_2$ - term is more conceptual and its abutment is more geometrical (the cohomology of the Borel construction $EG \times_G X_s$ vs. the Farrell cohomology whose geometrical interpretation in low dimension is less clear). In fact, this geometric information was quite useful in computing the low dimensional cohomology of $SL(3, \mathbb{Z}[1/2])$ [H5].

## 6. The Landweber-Stong Conjecture

Another very nice application of the theory of unstable modules to a problem related to group cohomology concerns the Landweber-Stong-Conjecture [LSt]. The conjecture was apparently motivated by an attempt to better understand the depth of cohomology rings of finite groups (although it doesn't directly apply to such rings except in cases where the $p$ - Sylow subgroup is elementary abelian in which case the depth is well understood); the conjecture gives a characterization of the depth of rings of invariants as follows.

Let $G$ be a finite group, $k$ a finite field of order $q$, $V$ an $n$ - dimensional $k[G]$ - module and let $R = S[V]^G$ be the graded ring of invariants with respect to the action of $G$ on the symmetric algebra of $V$ (with $V$ in degree 2 if $q$ is odd and in degree 1 otherwise). Then $R$ is a module over the Dickson algebra $D(V) := S[V]^{\mathrm{Aut}\, V}$ which is well-known to be a polynomial algebra on the Dickson generators $c_1, \cdots, c_n$ of degree $\epsilon(q^n - q^{n-i})$ with $\epsilon = 2$ if $q$ is odd and $\epsilon = 1$ otherwise. The conjecture claims that the depth of $R$ is the maximal $k$ such that the sequence $c_1, \cdots, c_k$ is a regular sequence for $R$. (We recall that the depth is defined to be the maximum of the lengths of all possible regular sequences of elements of positive degree.)

This conjecture has recently been proved by Bourguiba and Zarati [BZ] in the case of prime fields by first generalizing it to a statement on the depth of unstable modules over the Dickson algebra and then using deep results on unstable modules, most notably Lannes' $T$ - functor and the related functor Fix. This is quite remarkable in so far as the conjecture does not involve Steenrod operations at all. A variant of their proof which we believe to be a bit simpler is presented in [H6]. Both proofs can be generalized to cover the case of any finite field. Details can be found in [Bo].

## Appendix: $F$ - isomorphisms and finiteness results for certain infinite discrete groups

In this appendix we generalize Quillen's $F$ - isomorphism theorem to a certain class of discrete groups. In fact, the proof of this generalization merely consists of a presentation of Quillen's theorem and its proof which was suggested by Lannes' (unpublished) proof [Ln2] of Theorem 2.6. We also give a very brief outline how Lannes proves Theorem 2.6 in the hope that he himself will still give a more detailed account in the near future. Finally we show how certain geometric and homological conditions on a group $G$ imply that its mod - $p$ cohomology ring is noetherian and its category $\mathcal{A}_p(G)$ is equivalent to a finite one, or in other words that it contains only a finite number of conjugacy classes of elementary abelian $p$ - subgroups. In this section all coefficients will be in $\mathbb{F}_p$ again and will be skipped from the notation.

**A.1.** To state our generalization of Quillen's theorem we need to introduce certain functors from $\mathcal{A}_p(G)^{op}$ to the category $\mathcal{E}ns$ of sets. The category of such functors will also be denoted by $\mathcal{A}_p(G) - \mathcal{E}ns$; its objects will be called $\mathcal{A}_p(G)$ - sets.

The functor $E \mapsto H^*E$ will simply be denoted by $H^*$.

If $X$ is a $G$ - space then the centralizer $C_G(E)$ acts on the $E$ - fixed points $X^E$ and, if $\pi_0$ denotes the set of path components, we can form the set $\pi_0(X^E)/C_G(E)$. Then it easy to check that the assignment $E \mapsto \pi_0(X^E)/C_G(E)$ extends canonically to a functor $F_X : \mathcal{A}_p(G)^{op} \to \mathcal{E}ns$. If we want to emphasize the group as well we will denote this functor by $F_X^G$.

**A.2.** We recall that a homomorphism $\varphi : A \longrightarrow B$ of $\mathbb{F}_p$ - algebras is called a *uniform $F$ - isomorphism* if there exists a natural number $n$ such that

- If $x \in \operatorname{Kern}\varphi$ then $x^{p^n} = 0$ and
- if $y \in B$ then $y^{p^n}$ is in the image of $\varphi$.

Uniformity is automatically guaranteed if $A$ and $B$ are finitely generated $\mathbb{F}_p$ - algebras and $\varphi$ is an $F$ - isomorphism but not otherwise.

**A.3.** For a $G$ - space $X$ we obtain a map $q_{G,X}$ from $H_G^*X$ to the natural transformations $\mathrm{Nat}_{\mathcal{A}_p(G)-\mathcal{E}ns}(F_X, H^*)$ of $\mathcal{A}_p(G)$ - sets as follows: For $c \in \pi_0(X^E)$ we choose a point $x_c$ in the component of $c$. The inclusions $\{x_c\} \subset X^E \subset X$ on the space level and $E \subset G$ on the group level induce a map $c^* : H_G^*X \longrightarrow H^*E$ which one can easily check to be independent of the choice of $x_c$. If $u \in H_G^*X$ then the $E$ - th component of $q_{G,X}(u)$ maps $c$ to $c^*(u)$; it is straightforward to check that $q_{G,X}(u)$ is a natural transformation from $F_X$ to $H^*$. Furthermore the target of $q_{G,X}$ is a graded $\mathbb{F}_p$ - algebra (because $H^*$ takes values in graded $\mathbb{F}_p$ - algebras) and the map $q_{G,X}$ is a homomorphism of graded $\mathbb{F}_p$ - algebras.

Here is our generalization of Quillen's theorem.

THEOREM A.4. *Assume $G$ is a discrete group and $X$ is a $G$ - CW - complex of finite dimension such that all isotropy groups of points in $X$ are finite of bounded order $N$. Then the natural map*

$$q_{G,X} : H_G^*X \longrightarrow \mathrm{Nat}_{\mathcal{A}_p(G)-\mathcal{E}ns}(F_X, H^*)$$

*is a uniform $F$ - isomorphism.*

If $X$ is a point then the functor $F_X$ is the constant functor with value a point; in this case the target is nothing but $\lim_{\mathcal{A}_p(G)^{op}} H^*$. If $X$ is mod - $p$ acyclic then so is $X^E$ by Smith theory, and again the functor $F_X$ is constant with value a point and the target identifies with $\lim_{\mathcal{A}_p(G)^{op}} H^*$. This proves the following Corollary.

Corollary A.5. *Assume $G$ is an $\mathcal{H}_1\mathcal{F}$ - group and there is a bound $N$ for the order of finite subgroups in $G$. Then $G$ is a Quillen - group, and in fact, the map $H^*G \longrightarrow \lim_{\mathcal{A}_p(G)^{op}} H^*E$ is a uniform $F$ - isomorphism.*

Sketch of proof of Theorem A.4. Abbreviate the source of $q_{G,X}$ by $L_1(X)$ and its target by $L_2(X)$. Then both $L_1$ and $L_2$ are functors from $G$ - spaces to graded $\mathbb{F}_p$ - algebras which are invariant with respect to $G$ - homotopy equivalences and turn arbitrary coproducts into products. Furthermore $L_2$ turns a homotopy pushout into a pullback of graded $\mathbb{F}_p$ - algebras; $L_1$ associates to a homotopy pushout a Mayer-Vietoris sequence and this means (cf. Proposition 3.2 in [Q1]) that up to (uniform) $F$ - isomorphism it also turns a homotopy pushout into a pullback.

Consequently, we may concentrate on the case of an orbit $X = G/K$ where $K$ is a finite group of order at most $N$. It suffices to show that for such $X$ the map $q_{G,X}$ is a uniform $F$ - isomorphism with an $n$ which is independent of $K$. (There is a subtle point to observe here because we have to deal with arbitrary products of $F$ - isomorphisms and such a product need not be an $F$ - isomorphism in general. However, if there are only finitely many isomorphism classes of factors then there is no problem, and this holds in our case because there are only finitely many isomorphism classes of groups $K$ whose order is bounded by $N$.)

For $X = G/K$ we have the following "change of groups isomorphisms": the functors $L_1(X) = H^*_G(G/K)$ resp. $L_2(X) = \mathrm{Nat}_{\mathcal{A}_p(G)-\mathcal{E}ns}(F^G_{G/K}, H^*)$ can be identified with $H^*K$ resp. $\mathrm{Nat}_{\mathcal{A}_p(K)-\mathcal{E}ns}(F^K_{pt}, H^*)$ and the map $q_{G,G/K}$ with $q_{K,pt}$. This map is known to be a uniform $F$ - isomorphism by Quillen's Theorem; because the order of $K$ is bounded by $N$ there are only finitely many maps $q_{K,pt}$ to consider, so we may find an $n$ which works for all orbits $G/K$. (Note that there might be infinitely many conjugacy classes of $K$'s but this does not matter!). □

**A.6.** a) We have relied in the sketch of the proof on Quillen's theorem for finite groups. In fact, the proof for finite groups is carried out in the same spirit. Using Quillen's trick (cf. section 6 in [Q1]) of embedding $G$ into a unitary group $U(n)$ and working with the projection $X \times F \longrightarrow X$ (with $F = U(n)/S$, the quotient of $U(n)$ by the subgroup $S$ generated by diagonal matrices of order $p$), one reduces the case of a general $G$ - space to the case of an orbit $X = G/E$ with $E$ elementary abelian. Then one identifies $q_{G,G/E}$ with $q_{E,pt}$ as above; finally one checks that the map $q_{E,pt}$ is a (genuine!) isomorphism.

b) If $G$ satisfies the assumptions of A.5 then Lannes' Theorem 2.6 holds as well, and its proof follows the same strategy, in particular it involves again a generalization to the case of finite dimensional $G$ - $CW$ - complexes for which all isotropy groups are finite of bounded order (cf. the appendix of [H2] if $G$ is a $\mathcal{K}_1\mathcal{F}$ - group).

Now the functor $L_1(X)$ is given as $T_V H_G^* X$ and $L_2(X)$ as $\prod_{\mathrm{Rep}(V,G)} H^*_{C_G(E)} X^E$. Then both functors are defined on $G$ - spaces and take values in graded vector spaces (or unstable modules, unstable algebras etc. ...). They are invariant with respect to $G$ - homotopy equivalences and associate to a homotopy pushout Mayer-Vietoris sequences (because $T_V$ is exact). $L_2$ turns arbitrary coproducts into products and $L_1$ does so if there are only finitely many different abstract isomorphism classes of isotropy subgroups (the subtle point here is that $T_V$ does not commute with arbitrary products unless there are only finitely many different isomorphism classes of factors in the product). Therefore it is enough to consider the case of an orbit, and by Quillen's trick again, we may even assume that the orbit is of the form $X = G/E$ with $E$ elementary abelian. Finally one checks that there are again "change of groups isomorphisms" as above and it remains to consider the case of an elementary abelian group $E$ acting on a point which can be checked directly.

**A.7.** We finish by giving two applications of Theorem A.4 to $\mathcal{H}_1\mathcal{F}$ - groups $G$ which are also of type $FP_\infty$. Recall that a discrete $G$ is said to be of type $FP_\infty$ if the trivial module $\mathbb{Z}$ admits a projective resolution by finitely generated projective $G$ - modules. For example, it is well known that a $\mathcal{K}_1\mathcal{F}$ - group has type $FP_\infty$.

For groups in $\mathcal{K}_1\mathcal{F}$ it is also known, say by Smith theory, that there are only finitely many conjugacy classes of finite $p$ - subgroups and that the mod - $p$ cohomology ring is finitely generated as an algebra [H2]. Here we offer the following results.

Theorem A.8. *Assume that $G$ is an $\mathcal{H}_1\mathcal{F}$ - group of type $FP_\infty$. Then there are only finitely many conjugacy classes of elementary abelian $p$ - subgroups of $G$.*

Theorem A.9. *Assume that $G$ is an $\mathcal{H}_1\mathcal{F}$ - group of type $FP_\infty$. Then the mod - $p$ cohomology ring of $G$ is finitely generated as an algebra.*

Theorem A.8 was known to me in the spring of 1995 and has since been proved independently and with very different methods by Benson [Be2]. Results of [Be2] also enter the proof of A.9 in a crucial way.

Proof of Theorem A.8. Under the stated assumptions on $G$ Kropholler [K] proved that the order of finite subgroups of $G$ is bounded by some number $N$. So we can apply Theorem A.4 to the finite dimensional mod - $p$ acyclic complex $X$ on which $G$ acts with finite isotropy groups.

The target of the map $q_{G,X}$ identifies with $\lim_{\mathcal{A}_p(G)^{op}} H^*$. Now assume that there are infinitely many conjugacy classes of elementary abelian $p$ - subgroups of $G$. Because the order of finite subgroups is bounded this means that there must be infinitely many maximal elementary abelian $p$ - subgroups of $G$ of the same rank $k$ (although $k$ itself need not necessarily be maximal). For each such conjugacy class $(D)$ the top Dickson invariant $c_k$ determines a class $y_{(D)}$ in the inverse limit in dimension $\epsilon(p^k - 1)$ with $\epsilon = 1$ if $p = 2$ and $\epsilon = 2$ otherwise; namely one defines a family $(x_E)$ by putting $x_E = 0$ if $E$ is not in the given conjugacy class and $x_E = c_k$ if $E$ is in the given conjugacy class. The different classes $y_{(D)}$ are all linearly independent, so the inverse limit is infinite dimensional as $\mathbb{F}_p$ - vector space in degree $\epsilon(p^k - 1)$. Because $q_{G,X}$ is a uniform $F$ - isomorphism we derive that $H^*G$ is infinite dimensional as $\mathbb{F}_p$ - vector space in degree $n\epsilon(p^k-1)$ for some sufficiently large $n$. This, however, is in contradiction with the assumption that $G$ is of type $FP_\infty$. $\square$

Sketch of proof of Theorem A.9. The crucial input here is the following result of Benson [Be2].

If $G$ is of type $FP_\infty$ and is an $\mathcal{H}_1\mathcal{F}$ - group then the trivial module $\mathbb{F}_p$ belongs to the smallest class $\mathcal{C}$ of $\mathbb{F}_p[G]$ - modules with the following properties:

• If $E$ is an elementary abelian $p$ - subgroup and $N$ is a finitely generated $\mathbb{F}_p[E]$ - module then the induced module $\mathrm{Ind}_E^G(N)$ belongs to $\mathcal{C}$.

• $M_1 \oplus M_2$ belongs to $\mathcal{C}$ if and only if both $M_1$ and $M_2$ belong to $\mathcal{C}$.

• If $M_1$ and $M_2$ belong to $\mathcal{C}$ and $f : M_1 \longrightarrow M_2$ is a surjective $\mathbb{F}_p[G]$ - module homomorphism then $\mathrm{Kern}\, f$ is in $\mathcal{C}$.

• If $M_1$ and $M_2$ belong to $\mathcal{C}$ and $f : M_1 \longrightarrow M_2$ is an injective $\mathbb{F}_p[G]$ - module homomorphism then $\mathrm{Coker}\, f$ is in $\mathcal{C}$.

With this result we proceed more or less as in the appendix of [H2]. The category $\mathcal{A}_p(G)$ is equivalent to a finite category by Theorem A.4 and using this it is not difficult to show that the inverse limit $\Lambda := \lim_{\mathcal{A}_p(G)^{op}} H^*$ is a finitely generated algebra. Because of the uniformicity of the $F$ - isomorphism in Theorem A.4 a suitably often iterated $p$ - th power of $\Lambda$ is a finitely generated subalgebra $\Lambda'$ of $H^*G$.

Next one observes that the cohomology $H^*E$ of every elementary abelian $p$ - subgroup $E$ is finitely generated as a module over $\Lambda$ and therefore also over $\Lambda'$. Then the same holds for $H^*(E,N) \cong H^*(G, \mathrm{Ind}_E^G(N))$ for every finitely generated $\mathbb{F}_p[E]$ - module $N$ and therefore $H^*(G,M)$ is finitely generated over $\Lambda'$ for every module $M$ in the class $\mathcal{C}$. In particular, $H^*(G,\mathbb{F}_p)$ is finitely generated over the finitely generated algebra $\Lambda'$ and is therefore also a finitely generated algebra. $\square$

## References

[AGM] J. F. Adams, J. H. C. Gunawardena and H. R. Miller, *The Segal conjecture for elementary abelian p–groups*, Topology **24** (1985), 435–460.

[AM] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der mathematischen Wissenschaften 309, Springer Verlag, 1994.

[Be1] D. Benson, *Representations and cohomology II: Cohomology of groups and modules*, Cambridge studies in advanced mathematics 31, Cambridge University Press, 1991.

[Be2] D. Benson, *Complexity and varieties for infinite groups*, Preprint 1996.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304 Springer, 1972.

[Bo] D. Bourguiba, *Profondeur et algèbre de Steenrod*, Thèse, Université de Tunis 1997.

[BZ] D. Bourguiba and S. Zarati, *On the depth of an unstable $H^*V$ - $A$ - module which is of finite type as an $H^*V$ - module*, to appear in Invent. Math..

[Br] K. Brown, *Groups of virtually finite dimension*, London Math. Soc. LNS 36, Homological Group Theory (edited by C.T.C. Wall), Cambridge University Press, 1979, pp. 27-70.

[CaH] J. F. Carlson and H.-W. Henn, *Cohomological detection and regular elements in group cohomology*, Proc. Amer. Math. Soc. **124** (1996), 665–670.

[C] G. Carlsson, *G. B. Segal's Burnside ring conjecture for* $(\mathbb{Z}/2)^k$, Topology **22** (1985), 83-103.

[CV] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. math. **84** (1986), 91-119.

[Du] J. Duflot, *Localization of equivariant cohomology rings*, Trans. Amer. Math. Soc. **284** (1984), 91-105.

[Dw1] W.G. Dwyer, *Sharp homology decompositions for classifying spaces of finite groups*, Preprint 1996.

[Dw2] W.G. Dwyer, *Exotic cohomology for* $GL_n(\mathbb{Z}[1/2])$, Preprint 1996.

[DW1] W. Dwyer and C. Wilkerson, *A cohomology decomposition theorem*, Topology **31** (1992), 433-443.

[DW2] W. G. Dwyer and C. W. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Annals of Mathematics **139** (1994), 395-442.

[E1] L. Evens, *The cohomology ring of a finite group*, Trans. AMS **101** (1961), 224-239.

[E2] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs, Clarendon Press, 1991.

[Ga] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. Fr. **90** (1962), 323-448.

[Gr] A. Grothendieck (notes by R. Hartshorne), *Local cohomology*, Springer Lecture Notes in Math. 41, 1967.

[GH] E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhail Gromov*, Progress in Mathematics 83, Birkhäuser, 1990.

[Ha] J. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. math. **84** (1986), 157-176.

[H1] H.-W. Henn, *Classifying spaces with injective mod - p cohomology*, Comm. Math. Helv. **64** (1989), 200–206.

[H2] H.-W. Henn, *Commutative algebra of unstable $K$ - modules, Lannes' $T$ - functor and equivariant mod - p - cohomology*, Journal für die reine und angewandte Mathematik **478** (1996), 189-215.

[H3] H.-W. Henn, *Centralizers of elementary abelian p - subgroups, the Borel construction of the singular locus and applications to the cohomology of discrete groups*, Topology **36** (1997), 271-286.

[H4] H.-W. Henn, *Centralizers of elementary abelian p - subgroups and mod - p cohomology of profinite groups*, Heft Nr. **102** Forschergruppe Topologie und nichtkommutative Geometrie Univ. Heidelberg, Oktober 1994, to appear in Duke Mathematical Journal.

[H5] H.-W. Henn, *The cohomology of* $SL(3,\mathbb{Z}[1/2])$, Heft Nr. **136** Forschergruppe Topologie und nichtkommutative Geometrie Univ. Heidelberg, Mai 1996.

[H6] H.-W. Henn, *A variant of the proof of the Landweber-Stong-Conjecture*, these Proceedings.

[HLS1] H.-W. Henn, J. Lannes and L. Schwartz, *Analytic functors, unstable algebras and cohomology of classifying spaces*, Contemp. Mathematics **96** (1989), 197–220.

[HLS2] H.-W. Henn, J. Lannes and L. Schwartz, *The categories of unstable modules and unstable algebras modulo nilpotent objects*, American Journal of Math. **115** (1993), 1053-1106.

[HLS3] H.-W. Henn, J. Lannes and L. Schwartz, *Localization of unstable $A$ - modules and equivariant mod - p cohomology*, Math. Ann. **301** (1995), 23-68.

[JM] S. Jackowski and J. McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), 113-132.

[JMO] S. Jackowski, J. McClure and R. Oliver, *Homotopy theory of classifying spaces of compact Lie groups*, Algebraic topology and its applications, Springer-Verlag, 1994, pp. 81–123.

[K] P. Kropholler, *On groups of type* $(FP)_\infty$, Journal of pure and applied algebra **90** (1993), 55-67.

[LSt] P. S. Landweber and R. E. Stong, *The depth of rings of invariants over finite fields*, Springer Lect. Notes in Math. 1240, 1987, pp. 259-274.

[Ln1] J. Lannes, *Sur la cohomologie modulo p des p-groupes abéliens élémentaires*, Proceedings of the Durham Symposium on Homotopy theory 1985, Cambridge University Press 1987.

[Ln2] J. Lannes, *Cohomology of groups and function spaces*, Preprint 1986.

[Ln3] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Publ. Math. IHES **75** (1992), 135-244.

[LS] J. Lannes and L. Schwartz, *Sur la structure des $A$ - modules instables injectifs*, Topology **28** (1989), 153-169.

[LZ] J. Lannes and S. Zarati, *Sur les $\mathcal{U}$-injectifs*, Ann. Scient. Ec. Norm. Sup. **19** (1986), 303-333.

[Lz] M. Lazard, *Groupes analytiques $p$ - adiques*, Publ. Math. IHES **26** (1965).

[Le] C. N. Lee, *Farrell cohomology and centralizers of elementary abelian $p$ - subgroups*, Preprint.

[M] H. Miller, *The Sullivan Conjecture on maps from classifying spaces*, Ann. of Math. **120** (1984), 39-87.

[O] R. Oliver, *Higher limits via Steinberg representations*, Comm. in Algebra **22** (1994), 1381-1393.

[Q1] D. Quillen, *The spectrum of an equivariant cohomology ring I, II*, Ann. of Math. **94** (1971), 549-572, 573-602.

[Q2] D. Quillen, *On the cohomology and $K$ - theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552-586.

[Sc] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, University of Chicago Press, 1994.

[Se] J. P. Serre, *Arithmetic groups*, London Math. Soc. LNS 36, Homological Group Theory (edited by C.T.C. Wall), Cambridge University Press, 1979, pp. 105-136.

[Sl] J. Slomińska, *Homotopy colimits on E-I-categories*, Algebraic Topology, Poznań 1989 Proceedings, Springer Lecture Notes in Math. 1474, 1991.

[SE] N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies 50, Princeton University Press, 1962.

[V] B. B. Venkov, *Cohomology algebras for some classifying space*, Dokl. Akad. Nauk SSSR **127** (1959), 943–944.



Max-Planck-Institut für Mathematik, Gottfried Claren Strasse 26, D–53225 Bonn, Fed. Rep. of Germany

*E-mail address*: `henn@mpim-bonn.mpg.de`