# An estimate for the volume entropy of nonpositively curved graph manifolds

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**Abstract**

Let $M$ be a closed 3-dimensional graph manifold. We prove that $h(g) > 1$ for each geometrization $g$ of $M$, where $h(g)$ is the topological entropy of geodesic flow of $g$.

## 1 Introduction

Asymptotic geometry of a nonpositively curved (NPC for brevity) graph manifold is a complicated mixture of flat and hyperbolic parts which both nontrivially contribute in general picture. Recall that a NPC metric on a closed 3-dimensional graph manifold $M$ recovers the JSJ-decomposition of $M$ in the following sense. There is the unique (up to isotopy) minimal finite collection $E$ of flat geodesically embedded tori and Klein bottles which are pairwise disjoint, and the metric completion of each connected component of the complement to $E$ is a Seifert space called a block of $M$. Each block $M_v$ is fibered over a 2-orbifold $S_v$ with negative Euler characteristic, $\chi(S_v) < 0$. Furthermore, the metric locally splitts as $U \times (-\varepsilon, \varepsilon)$ along the interior of each block, where $U$ is a NPC surface, the splitting is compatible with the fibration, the fibers are closed geodesics, and the regular fibers have one and the same length $l_v > 0$ depending only on the block.

Since we are interested in asymptotic properties, which are certainly the same for any finite covering of $M$, we may assume for simplicity that $M$ is orientable, the collection $E$ consists of tori, and each block is a trivial $S^1$-bundle over a compact surface $S_v$ with boundary, $M_v = S_v \times S^1$. We also assume that the graph manifold structure of $M$ is nontrivial, i.e. $M$ itself is not a Seifert fibered space (though it may consist of one block).

The flat part of the asymptotic geometry of $M$ studied in [BS], [CK], see also [HS]. Roughly speaking, it can be described by very special geodesic rays $[0, \infty) \to M$, which terminate in no block, skip through separating tori $e \in E$ almost tangently spending most of the time near tori and moreover this time rapidly increases with each step. Though the set of such rays is

*Partially supported by RFFI Grants 99-01-00104 and 00-15-96024.

a negligible part all of the rays, it contains an important information about geometry of $M$: it was shown in [CK] how this information allows to recover (up to scaling) the marked length spectrum of closed geodesics on $S_v$ and the fiber length for each block $M_v$.

Here we study the hyperbolic part of the asymptotic geometry of $M$, assuming that the surface $U$ from the local splitting $U \times (-\varepsilon, \varepsilon)$ above has the constant curvature $K = -1$. In other words, each block fibers over a *hyperbolic* orbifold (surface) $S_v$. A NPC metric on $M$ satisfying this condition is called a *geometrization* of $M$ (because the metric of each block is modelled on $\mathrm{H}^2 \times \mathbb{R}$). Note that any geometrization of $M$ is only $C^{1,1}$-smooth being analytic along the interior of each block. It is known [L] that $M$ admits a NPC metric if and only if it admits a geometrization. Necessary and sufficient topological conditions for $M$ to carry a NPC metric were found in [BK].

The relevant metric invariant which measures hyperbolicity of a space is the volume entropy $h$. Let $\pi : X \to M$ be the universal covering, $x_0 \in X$. Recall that $h = h(X)$ is defined by

$$h = \lim_{R \to \infty} \frac{1}{R} \ln \operatorname{vol} B_R(x_0),$$

where $B_R(x)$ is the ball in $X$ of radius $R$ centered at $x$. It is well known [M] that the limit exists, $h$ is independent of the choice of $x_0$ and if $M$ is NPC then $h$ coincides with the topological entropy of geodesic flow of $M$. The volume entropy scales as $l^{-1}$, $l$ is the length. Thus the choice of a geometrization $g$ of $M$ also serves as a normalization. The sectional curvatures of $g$ satisfy $-1 \le K \le 0$. Hence $h(g) \le 2$ by comparison with $\mathrm{H}^3$. Our main result is this.

**Theorem 1.1.** *For any geometrization $g$ of a graph manifold $M$ we have $h(g) > 1$.*

*Remark 1.2.* Though the universal covering $X$ of $M$ looks much more complicated than the model space $\mathrm{H}^2 \times \mathbb{R}$, even the estimate $h(g) \ge 1 = h(\mathrm{H}^2 \times \mathbb{R})$ is not obvious and nontrivial: $X$ contains no isometrically and geodesically embedded $\mathrm{H}^2$, which would lead to $h(g) \ge 1$; on the other hand, the attempt to compare $X \to \mathrm{H}^2 \times \mathbb{R}$ via exponential maps identifying some tangent spaces fails because the Jacobian of this map is $> 1$ at some points. Finally, the estimate from [BW] for the measure theoretic entropy of geodesic flow, which is always $\le h(g)$, gives only $\pi/4 < 1$ as a lower bound for any geometrization $g$ of $M$ (if one ignores the fact that $C^{1,1}$-smoothness of $g$ is not sufficient to apply this estimate).

To prove Theorem 1.1 we use the well known fact that $h(g)$ coincides with the critical exponent of a Poincaré series

$$\mathcal{P}(t) = \sum_{\gamma \in \Gamma} e^{-t|x_0 - \gamma x_0|},$$

where the fundamental group $\Gamma = \pi_1(M)$ isometrically acts on $X$ as the deck transformation group. Actually, instead of $\mathcal{P}$ we use a modified Poincaré series $\mathcal{P}_W$, where summation is taken over a set $W$ of walls in $X$. Our proof involves three ingredients: (i) a local estimate, which is technical and used in (ii); this estimate is obtained in section 2. (ii) An accumulating procedure, which is an inductive construction of appropriately choosen broken geodesics in $X$ between the base point $x_0$ and the walls from $W$; the choice of these paths is the key point of the proof. The outcome of the accumulating procedure is a generating set for $\mathcal{P}_W$ used in (iii); the procedure is described in sect. 3. (iii) A self-similarity type argument. This part of the proof uses a standard idea from self-similarity theory to show that $\mathcal{P}_W(h)$ diverges for some $h > 1$ using the generating set obtained in (ii). This is done in sect. 4.

**Acknowledgment.** The author is grateful to W. Ballmann for useful discussions of the topic of this note.

## 2 Local estimate

Let $F$ be the universal covering of a compact hyperbolic surface $S$ with geodesic boundary. We identify $F$ with a convex subset $F \subset \mathrm{H}^2$ bounded by countably many disjoint geodesic line and fix $o \in \mathrm{H}^2 \setminus F$. Let $w_0$ be the boundary line of $F$ closest to $o$, $o_0 \in w_0$ be the point on $w_0$ closest to $o$, so that $|o - o_0| = \mathrm{dist}(o, F) =: l > 0$.

Let $A$ be the set of the boundary lines of $F$ different from $w_0$. For each $w \in A$ we denote by $o_w \in w$ the point closest to $o$. Then the geodesic segment $oo_w$ intersects $w_0$ at some point $t_w$, and for $\widetilde{l}_w = |o - o_w|$ we have

$$\widetilde{l}_w = \widetilde{l}'_w + l''_w,$$

where $\widetilde{l}'_w = |o - t_w|$, $l''_w = |t_w - o_w|$ (all distances are taken in $\mathrm{H}^2$).

Next, we identify $\mathrm{H}^2$ with $\mathrm{H}^2 \times 0 \subset \mathrm{H}^2 \times \mathbb{R}$, so that $F$ becomes a subset of $\mathrm{H}^2 \times \mathbb{R}$, and we use for it the notations introduced above. Note that the point $o_0$ is closest to $o$ from the wall $w_0 \times \mathbb{R}$. We take a geodesic line $\sigma \subset w_0 \times \mathbb{R}$ through $o_0$ which is not horizontal, i.e., $\sigma \neq w_0 \times 0$, and take $s_w \in \sigma$ with $|s_w - o_0| = |t_w - o_0|$. Now, we put $l'_w := |o - s_w|$ (the distance is taken in $\mathrm{H}^2 \times \mathbb{R}$), $\Delta_w := \widetilde{l}'_w - l'_w$.

In other words, we replace the distance $\widetilde{l}'_w$ between $o$ and $t_w$ is the hyperbolic plane $\mathrm{H}^2$ by the distance $l'_w$ in $\mathrm{H}^2 \times \mathbb{R}$, which is shorter by comparison: the triangles $oo_0t_w \subset \mathrm{H}^2 \times 0$, $oo_0s_w \subset \mathrm{H}^2 \times \mathbb{R}$ both have the right angles at $o_0$, $\angle(oo_0t_w) = \frac{\pi}{2} = \angle(oo_0s_w)$, the common side $oo_0$ and equal sides $|o_0 - t_w| = |o_0 - s_w|$. Since $oo_0t_w$ lies in the hyperbolic plane $\mathrm{H}^2 \times 0$ but $oo_0s_w$ not, we have $\Delta_w > 0$ except the case $t_w = o_0 = s_w$. Now, we want to estimate accumulation of the differences $\Delta_w$ from below. The precise statement is this.

**Lemma 2.1.** *Given $l_0 > 0$, $\alpha_0 \in (0, \pi/2]$, there exists $\lambda_0 > 1$, which depends only on $l_0$, $\alpha_0$ and the compact surface $S$, so that*

$$\lambda(F, l, \alpha) := e^l \sum_{w \in A} e^{\Delta_w} e^{-\widetilde{l}_w} \geq \lambda_0,$$

*whenever $l = \operatorname{dist}(o, w_0) \geq l_0$ and the angle $\alpha$ between the lines $w_0 \times 0$ and $\sigma$ is at least $\alpha_0$, $\alpha_0 \leq \alpha \leq \pi/2$.*

*Proof.* By well known formula of hyperbolic geometry we have $e^l = (\tan \frac{\psi}{4})^{-1}$, $e^{-\widetilde{l}_w} = \tan \frac{\psi_w}{4}$, where $\psi$, $\psi_w$ are the angles under which $w_0$ respectively $w \in A$ are observed in $\mathrm{H}^2$ from $o$. The boundary at infinity $\partial_\infty F \subset \partial_\infty \mathrm{H}^2 = S^1$ coincides with the limit set of $\pi_1(S)$ represented in $\operatorname{Iso}(\mathrm{H}^2)$ as a Fuchsian group of second kind. It is well known that the Hausdorff dimension of $\partial_\infty F$ (with respect to the angle metric) is $< 1$, in particular, the Lebesgue measure of $\partial_\infty F$ is zero. Thus $\psi = \sum_{w \in A} \psi_w$. Therefore, $\tan \frac{\psi}{4} \leq \sum_{w \in A} \tan \frac{\psi_w}{4}$ and

$$e^l \sum_{w \in A} e^{-\widetilde{l}_w} = \sum_{w \in A} \tau_w \geq 1,$$

where $\tau_w = e^{l - \widetilde{l}_w}$. However, this sum, $\sum_{w \in A} \tau_w$, can be close to 1 as much as we like (taking for instance $l \to \infty$). As we noticed above, $\Delta_w > 0$ unless $s_w = t_w$. So we always have $\lambda(F, l, \alpha) > 1$. The point is that $\lambda(F, l, \alpha)$ is separated from 1 uniformly over all $l \geq l_0$, $\alpha \geq \alpha_0$.

Consider $A_0 \subset A$ consisting of all $w \in A$ with $|t_w - o_0| \geq 1$. Then $\Delta_w \geq \delta_0 > 0$ for all $w \in A_0$, where $\delta_0$ depends only on $l_0$, $\alpha_0$. We claim that

$$\sum_{w \in A_0} \tau_w \geq m_0 > 0, \tag{1}$$

where $m_0 = m_0(\widetilde{F}, l_0)$ is independent of $l$.

Assuming (1), we have

$$\begin{aligned}
\lambda(F, l, \alpha) &= \sum_{w \in A} e^{\Delta_w} \tau_w \geq \sum_{w \in A_0} e^{\Delta_w} \tau_w + \sum_{w \in A \setminus A_0} \tau_w \\
&\geq e^{\delta_0} \sum_{w \in A_0} \tau_w + \sum_{w \in A \setminus A_0} \tau_w \\
&= (e^{\delta_0} - 1) \sum_{w \in A_0} \tau_w + \sum_{w \in A} \tau_w \geq (e^{\delta_0} - 1) m_0 + 1 =: \lambda_0 > 1.
\end{aligned}$$

It remains to prove (1). Let $o'_w \in w$ be the point closest to $o_0 \in w_0$. Then $\widetilde{l}_w \leq |o - o'_w| \leq l + |o_0 - o'_w|$, thus $\widetilde{l}_w - l \leq \operatorname{dist}(o_0, w)$. Hence $\tau_w \geq e^{-\operatorname{dist}(o_0, w)} \geq \frac{\widetilde{\psi}_w}{4}$, where $w$ is observed from $o_0$ under the angle $\widetilde{\psi}_w$.

The Lebesgue measure class on $\partial_\infty \mathrm{H}^2$ is independent of the choice of origin, thus $\sum_{w\in A} \widetilde{\psi}_w = \pi$.

Since $l \geq l_0$, for a sufficiently small $m_0 = m_0(S, l_0) > 0$ the sectors $S^+(m_0)$, $S^-(m_0)$, defined below, intersect no $w \in A \setminus A_0$. Here is the definition of $S^\pm(m_0)$. The common vertex $o_0$ of $S^\pm(m_0)$ divides the line $w_0$ into two opposite rays $w_0^\pm$. The sectors $S^\pm(m_0) \subset \mathrm{H}^2$ are bounded by the rays $w_0^\pm$, $s^\pm(m_0)$, where $\angle_{o_0}(s^\pm(m_0), w_0^\pm) = 2m_0$, and $s^\pm(m_0) \cap F \neq \emptyset$.

Therefore, it follows from $\sum_{w\in A} \widetilde{\psi}_w = \pi$ that $\sum_{w\in A_0} \tau_w \geq m_0$, which completes the proof. $\square$

# 3 Accumulating procedure

To describe the accumulating procedure we need some information about metric structure of the universal covering $X$ of $(M, g)$, where $g$ is a geometrization.

## 3.1 Metric structure of the universal covering

Recall (see, for example, [BS], [CK]) that $X$ can be represented as the countable union $X = \cup_v X_v$ of blocks, where each $X_v$ is a closed convex subset in $X$ isometric to the metric product $F_v \times \mathbb{R}$, and $F_v$ is the universal covering of a compact hyperbolic surface $S_v$ with geodesic boundary. Every two blocks are either disjoint or intersect over a boundary component which is a 2-flat in $X$ separating them and consequently no three blocks have a point in common. The 2-flats in $X$, which separate blocks, are called the *walls*. A wall $w$ common for blocks $X_v$, $X_{v'}$ covers a 2-torus $e \subset M$, which separates (may be locally) the blocks $M_v = \pi(X_v)$, $M_{v'} = \pi(X_{v'})$ of $M$. The metric decompositions $X_v = F_v \times \mathbb{R}$, $X_{v'} = F_{v'} \times \mathbb{R}$ do not agree on $w$, and their $\mathbb{R}$-factors induce two fibrations of $w$ by parallel geodesics. We denote by $\alpha_w$ the angle between these fibrations, $0 < \alpha_w \leq \pi/2$. Since $M$ is compact and the set $E$ of separating tori in $M$ is finite, we have $\alpha_0 := \inf_w \alpha_w > 0$, where the infimum is taken over all the walls in $X$.

## 3.2 Modified Poincaré series

We fix a wall $w^* \subset X$, take a block $X_{v^*} \subset X$, for which $w^*$ is a boundary wall, and take a base point $x_0 \in w^*$. We denote by $W_0$ the set of the boundary walls of $X_{v^*}$ different from $w^*$; by $W_n$, $n \geq 1$ the set of walls in $X$ at the combinatorial distance $n+1$ from $w^*$, that is, $w \in W_n$ if and only if any geodesic segment in $X$ between $x_0$ and $w$ intersects $n$ walls over its interior, including a wall from $W_0$. Note that $W = \cup_{n\geq 0} W_n$ consists of all walls in $X$, which lie on one and the same side of $w^*$ as $X_{v^*}$. Now, we define a modified Poincaré series as follows

$$\mathcal{P}_W(t) = \sum_{w \in W} e^{-t\,\mathrm{dist}(x_0,w)}.$$

Comparing $P_W$ with $\mathcal{P}(t) = \sum_{\gamma\in\Gamma} e^{-t|x_0-\gamma x_0|}$, one easily obtains from triangle inequality that $\mathcal{P}(t) \geq e^{-D}\mathcal{P}_W(t)$, where $D > 0$ is the maximal diameter of the tori $e \in E$. Recall that the critical exponent of $\mathcal{P}$ is defined as the infimum of $t \in \mathbb{R}$ for which $\mathcal{P}(t) < \infty$. Therefore, the critical exponent $\overline{h}$ of $\mathcal{P}_W$ satisfies $\overline{h} \leq h(g)$, and to prove Theorem 1.1 it suffices to show that $\overline{h} > 1$.

### 3.3 Special broken-geodesic paths

**Proposition 3.1.** *For each $n \geq 0$ we have*

$$\mathcal{P}_n(1) := \sum_{w \in W_n} e^{-\,\mathrm{dist}(x_0,w)} \geq \frac{\pi}{4}\lambda_0^n,$$

*where $\lambda_0 > 1$ is the constant from Lemma 2.1.*

*Proof.* Induction over $n$. For each $n \geq 0$ and each wall $w \in W_n$ we produce a broken geodesic $\xi_w$ in $X$ between $x_0$ and $w$ as follows. For $w \in W_0$ we put $\xi_w = x_0x_w$, where $x_w \in w$ is the point closest to $x_0$. Note that $x_0x_w$ lies in a horizontal slice $F_{v^*} \times \{r_0\}$ of the block $X_{v^*} = F_{v^*} \times \mathbb{R}$.

Assume that for all $k$, $0 \leq k \leq n-1$ and all $w \in W_k$ the broken geodesic $\xi_w$ is already defined, $\xi_w$ is a juxtaposition $\eta_{w_0}\eta_{w_1}\dots\eta'_{w_k}$ of geodesic segments with the sequence $w_i \in W_i$ of walls leading to $w = w_k$. Moreover, we assume that each segment of $\xi_w$ lies in a block and connects its different boundary components, the last segment $\eta'_{w_k}$ lies in a horizontal slice of the block, which contains it, and $\eta'_{w_k}$ is orthogonal to the wall $w$.

Take $w \in W_n$. Then there is a unique $\overline{w} \in W_{n-1}$ which precedes $w$. By the assumption, the last edge $\eta'_{\overline{w}} = s_{\overline{w}}x_{\overline{w}}$ of $\xi_{\overline{w}}$ lies in a horizontal slice $F_{v'} \times \{r_{v'}\}$ of its block $X_{v'} = F_{v'} \times \mathbb{R}$ and it is orthogonal to the wall $\overline{w}$ (at the end point $x_{\overline{w}}$). The important feature of our construction is that $\xi_w$ contains all segments of $\xi_{\overline{w}}$ but the last one $\eta'_{\overline{w}}$.

Let $X_v \subset X$ be the other block adjacent to $\overline{w}$, in particular, the walls $w$, $\overline{w}$ are its boundary components. Recall that the metric splitting $X_v = F_v \times \mathbb{R}$ does not agree with that of $X_{v'}$ along of $\overline{w}$. Let $F_v \times \{r_v\}$ be the horizontal slice of $X_v$ which contains $x_{\overline{w}}$ on the corresponding boundary component. The boundary lines of the slices $F_{v'} \times \{r_{v'}\}$, $F_v \times \{r_v\}$, which lie in $\overline{w}$, contain $x_{\overline{w}}$ and form an angle $\alpha_{\overline{w}} \in (\alpha_0, \pi/2]$.

To construct $\xi_w$ we do the following. We take an isometric copy $F_v \subset \mathrm{H}^2 \times\{r_{v'}\}$ of $F_v \times \{r_v\}$ (by rotating the last by the angle $\alpha_{\overline{w}}$), where $F_{v'} \times \{r_{v'}\} \subset \mathrm{H}^2 \times\{r_{v'}\}$, so that $F_{v'} \times \{r_{v'}\}$ and $F_v$ are sitting in the hyperbolic plane $\mathrm{H}^2 \times\{r_{v'}\}$ and they are adjacent along the common boundary component, for which we use the same notation $\overline{w}$. Now, we connect the initial point $s_{\overline{w}}$ of the last edge $\eta'_{\overline{w}} \subset \xi_{\overline{w}}$ with the boundary component of $F_v$ corresponding to $w$ by the shortest geodesic segment $s_{\overline{w}}x'_w \subset \mathrm{H}^2 \times \{r_{v'}\}$ and take $t_w = s_{\overline{w}}x'_w \cap \overline{w}$. The segment $t_w x'_w$ turned back to $F_v \times \{r_v\}$ gives the last segment $\eta'_w = s_w x_w$ of $\xi_w$. So $s_w \in \overline{w} \cap F_v \times \{r_v\}$, $|s_w - x_{\overline{w}}| = |t_w - x_{\overline{w}}|$ and $\eta'_w \subset F_v \times \{r_v\}$ is orthogonal to $w$ at $x_w$. To complete the construction of $\xi_w$, we delete the last segment $\eta'_{\overline{w}} \subset \xi_{\overline{w}}$ replacing it by $\eta_{\overline{w}}\eta'_w$, where $\eta_{\overline{w}} = s_{\overline{w}}s_w \subset X_{v'}$. Clearly, so constructed $\xi_w$ has all properties advertised above.

Let $l = |s_{\overline{w}} - x_{\overline{w}}| = L(\eta'_{\overline{w}})$ be the length of the last segment of $\xi_{\overline{w}}$, $l'_w = L(\eta_{\overline{w}})$, $l''_w = L(\eta'_w)$. Then we have

$$L(\xi_w) = L(\xi_{\overline{w}}) - l + l'_w + l''_w = L(\xi_{\overline{w}}) - l + \widetilde{l}_w - \Delta_w,$$

where $\widetilde{l}_w = L(s_{\overline{w}}x'_w)$, $\widetilde{l}'_w = L(s_{\overline{w}}t_w)$ so that $\widetilde{l}_w = \widetilde{l}'_w + l''_w$, and $\Delta_w = \widetilde{l}'_w - l'_w$. We obtained the same configuration which was studied in sect. 2, and we are going to apply Lemma 2.1 to estimate $\mathcal{P}_n(1)$ from below. Note that $l$ being the length of a segment in $X$ connecting different boundary components of a block is separated from 0 by some positive constant $l_0$, which depends only on $M$, $l \geq l_0 > 0$.

For $n = 0$ we have

$$\mathcal{P}_0(1) = \sum_{w \in W_0} e^{-|x_0 - x_w|} = \sum_{w \in W_0} \tan \frac{\psi_w}{4} \geq \frac{\pi}{4},$$

where $\psi_w$ is the angle under which the boundary component $w$ of $X_{v^*}$ is observed from $x_0$ (in the horizontal direction).

By inductive assumption we have

$$\mathcal{P}_{n-1}(1) \geq \sum_{w \in W_{n-1}} e^{-L(\xi_w)} \geq \frac{\pi}{4}\lambda_0^{n-1}.$$

Represent $W_n = \cup_{\overline{w}} W_{n,\overline{w}}$, where the union is taken over all $\overline{w} \in W_{n-1}$ and each $w \in W_{n,\overline{w}}$ follows $\overline{w}$. Then

$$\mathcal{P}_n(1) \geq \sum_{w \in W_n} e^{-L(\xi_w)} = \sum_{\overline{w} \in W_{n-1}} e^{-L(\xi_{\overline{w}})} e^{l} \sum_{w \in W_{n,\overline{w}}} e^{\Delta_w} e^{-\widetilde{l}_w}.$$

Applying Lemma 2.1 with $A = W_{n,\overline{w}}$, we obtain

$$\mathcal{P}_n(1) \geq \lambda_0 \sum_{\overline{w} \in W_{n-1}} e^{-L(\xi_{\overline{w}})} \geq \frac{\pi}{4}\lambda_0^n,$$

which completes the proof. $\square$

## 4 Self-similarity argument

The constant $\lambda_0 > 1$ from Proposition 3.1 depends only on some metric data of $M$. Thus there is $n \in \mathbb{N}$, $n = n(M)$, so that $\frac{\pi}{4}\lambda_0^n > 1$. It follows from Proposition 3.1 that

$$\mathcal{P}_n(\overline{h}) = \sum_{w \in W_n} e^{-\overline{h}\,\mathrm{dist}(x_0,w)} \geq 1$$

for some $\overline{h} > 1$. Furthermore, taking $n$ sufficiently large, we can find $\overline{h} > 1$ with $\mathcal{P}_n(\overline{h}) \geq 1$ for any choice of the initial block $X_{v^*}$, its wall $w^*$ and the base point $x_0 \in w^*$, since the set of choices up to isometries of $X$ is compact.

We fix $n \in \mathbb{N}$ with this property and select a subset $W^* \subset W$, $W^* = \cup_{k \geq 1} W_k^*$, where $W_k^* = W_{kn}$. The set $W_1^*$ serves as the generating set for $W^*$. Connecting $x_0$ with each wall $w \in W_1^*$ by the shortest geodesic segment $x_0 x_w$, we obtain new base points $x_w \in w$ (these $x_w$ may be different from $x_w$ constructed in the proof of Proposition 3.1). By induction, we find a base point $x_w \in w$ for each $w \in W_k^*$, $k \geq 1$ with the property $\mathrm{dist}(x_{\overline{w}}, w) = |x_{\overline{w}} - x_w|$, where $\overline{w} \in W_{k-1}^*$ precedes $w$. Furthermore, by the choice of $n$ and $\overline{h}$ we have

$$\sum_w e^{-\overline{h}|x_{\overline{w}} - x_w|} \geq 1,$$

for each $\overline{w} \in W_{k-1}^*$, where the summation is taken over all $w \in W_k^*$ which follow $\overline{w}$. Since $\mathrm{dist}(x_0, w) \leq \mathrm{dist}(x_0, x_w) \leq \mathrm{dist}(x_0, x_{\overline{w}}) + |x_{\overline{w}} - x_w|$, we obtain

$$\begin{aligned}
\mathcal{P}_{W_k^*}(\overline{h}) &= \sum_{w \in W_k^*} e^{-\overline{h}\,\mathrm{dist}(x_0,w)} \\
&\geq \sum_{w \in W_{k-1}^*} e^{-\overline{h}\,\mathrm{dist}(x_0,x_w)} \geq \sum_{w \in W_n} e^{-\overline{h}\,\mathrm{dist}(x_0,x_w)} \geq 1
\end{aligned}$$

for each $k \geq 1$. Therefore, the modified Poincaré series

$$\mathcal{P}_W(\overline{h}) \geq \mathcal{P}_{W^*}(\overline{h}) = \sum_{k \geq 1} \mathcal{P}_{W_k^*}(\overline{h})$$

diverges at $\overline{h}$ and hence $h(g) \geq \overline{h} > 1$. This completes the proof of Theorem 1.1.

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