CONTINUED FRACTIONS AND RELATED TRANSFORMATIONS

by

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Introduction

In these lectures we describe the origin and some of the recent applications of the transfer operator method in the theory of expanding dynamical systems. Originally introduced by Sinai, Bowen and Ruelle in their work on ergodic properties of smooth dynamical systems, this method has been found to be applicable to a much wider field of problems in the theory of dynamical systems. Typical examples are the theory of zeta functions as introduced by Ruelle [R2], which we will discuss in more detail, the closely related problem of the distribution of closed orbits in hyperbolic systems, where Parry and Pollicott found an amusing analogon to the prime number theorem [PP], discussed in Pollicotts lectures [P], or finally Ruelle's recent work on resonances of Axiom A systems [R3]. In all these applications analyticity properties of different functions play an important role which are established by the transfer operator method. Another promising application of the method is to Selberg's theory of surfaces of constant negative curvature. Through his trace formula respectively his zeta function there is established a surprising connection between the spectra of the Laplacians and the length spectra of the closed geodesics of such surfaces. The main step for applying the transfer operator method in this case is Bowen's and Series' construction of a symbolic dynamics for these flows, reducing this way the dynamics to special flows over analytic expanding maps of the circle respectively the unit interval [S]. For the modular surface this map is just Gauss' continued fraction transformation which serves as the main example of this class of expanding systems. The transfer operator method for this map will be the central issue of our contribution. The above mentioned systems of 2–dimensional hyperbolic geometry, even if they are not really very exciting as physical systems, are nevertheless rather interesting from another point of view: they are highly chaotic systems where the connection between the systems quantum and classical behaviour is fully understood through Selberg's theory. Since the transfer operator method gives a rather straightforward approach to this theory one could at least hope that this method will shed new light also on the general problem of quantum chaos which treats the above relation for general dynamical systems.

In the present lectures we restrict our discussion of the transfer operator method to one dimensional expanding systems with rather smooth analytic dynamics. In this case the method gives the strongest results.

By accident the Bowen–Series maps mentioned above belong to this class of systems [S]. For a discussion of the method for systems with less smooth behaviour we refer to Ruelle's recent R. Bowen Lectures in Berkeley [R1].

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I. The Tranfer Operator Method

In this chapter we introduce the transfer operator for subshifts of finite type [K] which play an important role as mathematical models for so called lattice spin systems. It is in fact the physical theory of such systems namely classical equilibrium statistical mechanics where the origins of the transfer operator method can be found.

1. Tranfer Matrices for Lattice Spin Systems

It was one of the real deep insights of the work of Sinai, Ruelle, Bowen, Lanford et al. during the last twenty years that there is a surprising analogy in the mathematical structure of hyperbolic dynamical systems and classical spin systems on a lattice. These spin systems and their ergodic properties under lattice translations are part of general classical equilibrium statistical mechanics, one of the fundamental theories of classical physics, closely connected to names like Maxwell, Boltzmann and Gibbs, to mention only the most prominent ones. The main objects of study of this theory are systems composed of a huge number of interacting subsystems whose macroscopic behaviour one wants to understand from the underlying microscopic interactions. Since real systems like a piece of a ferromagnet are much too complicated to be described by the methods available presently to us, one has to approach the problem by discussing simple models for these systems, hoping that essential features responsible for the observed phenomena are described correctly by them. Classical lattice spin systems are among the simplest models to describe ferromagnetism: there classical spin variables are located on the sites of a lattice interacting with each other via some interaction. In the simplest case one takes a one dimensional lattice, for instance the lattice $\mathbb{Z}$ of integers. Whereas for our real world this case is not too interesting, objects in nature are in general 3 dimensional, it nevertheless plays a fundamental role in the theory of dynamical systems: this is closely related to the fact that we are interested in general in evolutions with respect to time which turns out to be 1–dimensional in our world. As soon as one wants to study the action of more general groups than $\mathbb{Z}$ or $\mathbb{R}$ on some phase space one had to consider also more complicated lattices as for instance $\mathbb{Z}^k$, $k > 1$. Spin systems on such lattices are much more complicated to describe mathematically, a fact closely related to the phenomenon of phase transitions in such systems [R2].

The best known of all classical lattice spin models are the Ising type models where the spin variable σ takes values in some finite or countable set F. We call such a system free if there is besides a possible exclusion rule for spins on neighbouring lattice sites no interaction between different spins. Depending on whether the lattice is $\mathbb{Z}$ or $\mathbb{Z}_+ = \{i \in \mathbb{Z} : i \geq 0\}$ and whether card $F < \infty$ or card $F = \infty$ the mathematical model for such a free spin system is the one – or two–sided subshift of finite respectively infinite type with alphabet F and transition matrix A describing the nearest neighbour exclusion rule. As discussed in Keane's lectures [K] A is indexed by the set $F \times F$ and its entries are either 0 or 1. We will always assume that $A^n > 0$ for some $n \in \mathbb{N}$. Since many properties of the two–sided shift can be reduced to the shift over $\mathbb{Z}_+$ [Bo], we restrict our discussion to this case. Anyhow, we are mainly interested in noninvertible dynamical systems and their symbolic dynamics leads to one–sided subshifts [K], [M], [S]. In the following we introduce some notations which are frequently used in the literature and which have their origin in classical statistical mechanics, so that to many people they appear rather strange. We hope to remedy this situation a little bit.

A sequence $\xi = (\xi_i)_{i \in \mathbb{Z}_+}$ with $\xi_i \in F$ is called a configuration of the spin system on the lattice $\mathbb{Z}_+$ and we denote the space of all such configuratious by $F^{\mathbb{Z}_+}$ which is configuration space. A configuration $\xi \in F^{\mathbb{Z}_+}$ is called allowed if

$$A_{\xi_i,\xi_{i+1}} = 1 \tag{1}$$

for all $i \in \mathbb{Z}_+$. This means spin ξ_i on lattice site i can have the spins ξ_{i-1} and ξ_{i+1} as its left resp. right neighbour for all $i \geq 1$. Let us denote by Ω_A the space of all allowed configurations. On this space acts in a natural way the lattice translation – or shift operator $\tau : \Omega_A \longrightarrow \Omega_A$ through

$$(\tau\xi)_i = \xi_{i+1} \,. \tag{2}$$

In [K] it is shown how the space Ω_A can be made a compact metric space such that τ becomes a continuous map on Ω_A . Then the pair (Ω_A,τ) defines a topological dynamical system.

Part of classical statistical mechanics of lattice spin systems is just ergodic theory of the dynamical system (Ω_A,τ) , that is the theory of measures on Ω_A invariant under the shift τ [K]. Of special interest for equilibrium statistical mechanics are the so called equilibrium states, translation invariant Gibbs states, associated to the interaction of the spins with each other; such an interaction can be described by a continuous function $A \in \mathscr{C}(\Omega_A)$, also called observable in the following, whose value at a configuration ξ just describes the interaction energy between the spin ξ_0 on lattice site $i = 0$ and the remaining spins ξ_i on sites $i \geq 1$ plus a possible selfinteraction of ξ_0 with itself. From this it is more or less obvious that the quantity

$$H_n(\xi) = \sum_{k=0}^{n-1} A(\tau^k\xi) \tag{3}$$

can be written as [R2]

$$H_n(\xi) = H(\xi_0, \dots, \xi_{n-1}) + W(\xi_0, \dots, \xi_{n-1} \,|\, \xi_n, \xi_{n+1}, \dots) \tag{4}$$

where $H(\xi_0, \dots, \xi_{n-1})$ describes the energy content of the finite configuration $(\xi_0, \dots, \xi_{n-1})$ and $W(\xi_0, \dots, \xi_{n-1} \,|\, \xi_n, \xi_{n+1}, \dots)$ is the interaction energy between the spins $(\xi_0, \dots, \xi_{n-1})$ and the spins $(\xi_n, \xi_{n+1}, \dots)$. Of fundamental importance in equilibrium statistical mechanics are the Gibbs states. They are the infinite volume limits as $n \longrightarrow \infty$ of the following finite volume probability measures μ_n

$$d\mu_n(\xi_0, \dots, \xi_{n-1}) = Z_n(A)^{-1} \exp H_n(\xi^{(n)}) \tag{5}$$

with H_n defined in (3) and $\xi^{(n)}$ any allowed configuration such that $\xi_i^{(n)} = \xi_i$ for $0 \le i \le n-1$. The choice of $\xi^{(n)}$ corresponds to the selection of certain boundary conditions for the spin system. The normalization factor $Z_n(A)$ is called the finite volume partition function and depends obviously on the boundary conditions. It has the explicit form

$$Z_n(A) = \sum_{\xi_0, \dots, \xi_{n-1} \in F} \exp H_n(\xi^{(n)}) \,. \tag{6}$$

The measures in (5) are the so called Gibb's ensembles, more precisely the canonical ensemble.

Of special interest in connexion with dynamical systems are what are called periodic boundary conditions. In this case the configuration $\xi^{(n)}$ in (5) resp. (6) is chosen as

$$\xi_{kn+i}^{(n)} = \xi_i \quad \text{for } i = 0,1, \dots, n-1 \quad \text{and all } k \in \mathbb{N} \,. \tag{7}$$

Then the partition function $Z_n(A)$ can be rewritten as

$$Z_n(A) = \sum_{\xi \in \mathrm{Fix}\, \tau^n} \exp \sum_{k=0}^{n-1} A(\tau^k \xi) \tag{8}$$

where $\mathrm{Fix}\ \tau^n = \{\xi \in \Omega_A : \tau^n \xi = \xi\}$ denotes the set of all periodic configurations with period n.

Through (5) respectively its infinite volume limit $n \longrightarrow \infty$, also called thermodynamic limit, to every observable $A \in \mathscr{C}(\Omega_A)$ there are associated one or several τ–invariant probability measures on configuration space Ω_A, which completely determine the physics or more precisely the thermodynamic behaviour of the infinitely extended spin system. The above Gibbs states are special cases of Keane's g–measures for subshifts of finite type where $g = \exp A$. What now is the relation between the above Gibbs ensembles and physical properties of the spin system, and how can the latter be extracted from them? This is exactly what the so called thermodynamic formalism is dealing with. A central role in this formalism is played by the above partition functions $Z_n(A)$ and their asymptotic behaviour in the thermodynamic limit $n \longrightarrow \infty$. More precisely, the following quantity $P(A)$ is of special interest

$$P(A) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(A) , \tag{9}$$

which is called the topological pressure of the observable A. In the physics literature this quantity, up to a sign and some factor involving the temperature, is called the free energy of the spin system. It is considered in general as a function of the temperature respectively some exterior magnetic field for a fixed given interaction. In (9) P is more generally considered a functional over the space $\mathscr{C}(\Omega_A)$, that means a function of the observable A and hence of the interaction.

The main problem now is the calculation of the pressure P for a given observable A , respectively more generally the behaviour of P as a function of A .Completely understood is the case, where the function A depends on the configuration $\xi = (\xi_i)_{i\in \mathbb{Z}_+}$ only through finitely many variables $\xi_0, \xi_1, \dots, \xi_r$. One speaks in this case also of a finite range interaction. In this case physicists found many years ago a very elegant method for solving (9): it became known in the literature as the transfer matrix method. Indeed, already Ising used this method in 1925 in his Ph.D. thesis when discussing what is nowadays called the 1–dim. Ising model: in our notation this model corresponds to a subshift of finite type with alphabet $F = \{+1, -1\}$ transition matrix $A_{\sigma,\sigma'} = 1$ for all $\sigma, \sigma' \in F$ and the following choice of the observable A : $A(\xi) = \mathscr{I}\xi_0\xi_1$ corresponding obviously to a nearest neighbour interaction. The simplest case however where the method can be applied are the free models, the subshifts of finite type with transition matrix **A** . In this case $A \equiv 0$ and $Z_n(0)$ in (8) counts just the number of allowed periodic configurations with period n :

$$Z_n(0) = \# \{\xi : \xi \in \mathrm{Fix}\, \tau^n\} . \tag{10}$$

A little thinking then shows that $Z_n(0)$ can be expressed in this case through the transition matrix **A** as

$$Z_n(0) = \mathrm{trace}\, \mathbf{A}^n . \tag{11}$$

Something similar happens in the case of the 1–dim. Ising model with $A(\xi) = \mathcal{J}\xi_0\xi_1$. If we introduce the matrix $\mathbb{L} = \mathbf{L}_{\sigma,\sigma'}$, $\sigma,\sigma' \in F = \{+1,-1\}$ with

$$\mathbb{L}_{\sigma,\sigma'} = \exp \mathcal{J}\sigma\sigma' \tag{12}$$

we find again

$$Z_n(A) = \text{trace}\, \mathbf{L}^n . \tag{13}$$

This raises the question if a representation like (13) can be found also for more general observables A . It is well known in the physics literature, that for functions A depending only on finitely many variables $\xi_0, \dots ,\xi_r$ corresponding to finite range interactions one can find indeed such a matrix $\mathbb{L} = \mathbf{L}(A)$ with

$$Z_n(A) = \text{trace}\, \mathbf{L}(A)^n . \tag{14}$$

Furthermore this matrix can be chosen to have only nonnegative entries as was the case in (11) and (13). An explicit construction for such an $\mathbb{L}$ follows from our discussion of more general transfer operators below. A positive matrix $\mathbf{L} = \mathbf{L}(A)$ fulfilling relation (14) is called a transfer matrix for the spin system with observable A . What have we achieved in this case? Quite a lot! Existence of such a transfer matrix allows a more or less complete solution of problem (9)! By the Perron–Frobenius Theorem the pressure $P(A)$ can be written simply as

$$P(A) = \log \lambda_1(\mathbf{L}) \tag{15}$$

where $\lambda_1(\mathbb{L})$ denotes the leading positive eigenvalue of $\mathbb{L}$ (assuming that at least some power of $\mathbb{L}$ is strictly positive for the strong version of the P–F Theorem to be applicable, which in fact can be shown for finite range A's) .

By (15) the problem to determine P is reduced to a purely algebraic one, namely to find the leading eigenvalue of the positive matrix $\mathbb{L}$.

For general observables $A \in \mathscr{C}(\Omega_A)$ it is not known how to construct such a transfer matrix $\mathbb{L}(A)$ respectively more general a trace class operator $\mathscr{L} = \mathscr{L}_A$ such that relation (14) holds for all n . Since the size of $\mathbb{L}(A)$ increases rapidly with the range of the finite range observables A , for infinite range observables A the transfer "matrix" $\mathbb{L}(A)$ cannot be anymore finite dimensional. In a certain sense a relation like (14) gives much more information about such a spin system than one in general wants to have: it not only describes the infinite system but also arbitrary finite approximations. The infinite system is really described by expression (15). Therefore it is very often enough to find a positive operator $\mathscr{L}_A$ whose leading eigenvalue λ_1 gives via expression (15) the pressure P(A) . That such an operator $\mathscr{L}_A$ really exists for a large class of observables A was shown by D. Ruelle. He introduced for general $A \in \mathscr{C}(\Omega_A)$ the following linear bounded operator $\mathscr{L} = \mathscr{L}_A$ on the Banach space $\mathscr{C}(\Omega_A)$ [R2]:

$$(\mathscr{L} f)(\xi) = \sum_{\sigma \in F} A_{\sigma,\xi_0} \exp A(\sigma,\xi)\, f(\sigma,\xi) \tag{16}$$

where (σ,ξ) denotes the configuration $\xi' = (\xi_i')_{i \in \mathbb{Z}_+}$ with $\xi_0' = \sigma$, $\xi_i' = \xi_{i-1}$ for $i \geq 1$.

For this operator the Ruelle–Perron–Frobenius Theorem holds [Bo]:

<u>Theorem (RPF)</u> For Hölder continuous A the operator $\mathscr{L} = \mathscr{L}_A$ has the following properties:

1) There exists $h_A \in \mathscr{C}(\Omega_A)$, $h_A > 0$ and $\lambda_1 > 0$ with $\mathscr{L}_A h_A = \lambda_1 h_A$

2) There exists a probability measure $v_A \in \mathscr{C}(\Omega_A)^*$ with $v_A \geq 0$, $v_A(h_A) = 1$ and $\mathscr{L}_A^* v_A = \lambda_1 v_A$

3) For any $f \in \mathscr{C}(\Omega_A)$

$$\lim_{n\to\infty} || \lambda_1^{-n} \mathscr{L}_A^n f - v_A(f) h_A || = 0$$

4) $P(A) = \log \lambda_1$

5) The probability measure $\mu_A = h_A \cdot v_A$ is τ–invariant and is a Gibbs state.

It is a rather simple exercise to show that the operator $\mathscr{L}_A$ for observables A with finite range, that means depending only on finitely many variables $\xi_0, \xi_1, \dots, \xi_r$, leaves invariant the subspace $\mathscr{C}_{r-1}(\Omega_A)$ of all functions depending only on the variables $\xi_0, \dots, \xi_{r-1}$. It reduces in this subspace to a matrix $\mathbb{L} = \mathbb{L}(A)$ with nonnegative entries, acting in the space $\mathbb{R}^{|F|^r}$ and indexed by

$\overbrace{F \times F \times \dots \times F}^{r} \times \overbrace{F \times F \times \dots \times F}^{r}$. Its general matrix element $\mathbb{L}_{(\xi_0, \dots, \xi_{r-1})(\sigma_0, \dots, \sigma_{r-1})}$ is given explicitly as

$$\mathbb{L}_{(\xi_0,\xi_1,\ldots,\xi_{r-1}),(\sigma_0,\ldots,\sigma_{r-1})} = \left[\prod_{i=0}^{r-2} \delta_{\xi_i,\sigma_{i+1}}\right] A_{\sigma_0,\xi_0} \exp A(\sigma_0,\sigma_1,\ldots,\sigma_{r-1},\xi_{r-1}) \tag{17}$$

which obviously is strictly positive after iterating it sufficiently many times. Indeed, this matrix is a transfer matrix for the observable A in the strong sense of relation (14) as one can verify easily. This justifies to call $\mathscr{L}_A$ the transfer operator for the spin system with observable A .

In the next section we discuss a simple example of an observable A which depends on the entire configuration $\xi \in \Omega_A$, that means is of infinite range, but nevertheless allows for a transfer operator such that relation (14) is still true with some minor modification. This example serves also as a motivation to introduce methods of analytic function theory in our discussion which play an important role in the thermodynamic formalism of such systems as developped by the physicists.

2. The Kac model and composition operators

The transfer operator $\mathscr{L}_A$ defined in (16) not only allows us to determine (at least in principle) the pressure P(A) of the observable A but several other properties of the spin system of great interest are related to this operator and its spectral properties. Let us mention only the ergodic properties of the measure μ_A under the translation τ and the closely related decay properties of correlation functions. By these one understands for arbitrary observables f , g $\in \mathscr{C}(\Omega_A)$ functions of the form

$$C_{f,g}(n) = \mu_A(g \circ \tau^n \cdot f) - \mu_A(g)\mu_A(f) . \tag{18}$$

Introducing the projector $\mathscr{P}: \mathscr{C}(\Omega_A) \longrightarrow \mathscr{C}(\Omega_A)$

$$\mathscr{P}f = v_A(f)h_A$$

with v_A resp. h_A defined as in the Ruelle–Perron–Frobenius Theorem, we find for $C_{f,g}$ the representation

$$C_{f,g}(n) = v_a\left[f \cdot \frac{\mathscr{N}^n}{\lambda_1^n} (g \cdot h_A) \right] \tag{19}$$

where $\mathscr{N}$ denotes the operator

$$\mathscr{N} = \mathscr{L}_A - \lambda_1 \mathscr{P} . \tag{20}$$

By the R–P–F Theorem we know for Hölder continuous A that

$$\lim_{n \to \infty} || \lambda_1^{-n} \mathscr{N}^n || = 0 \tag{21}$$

and hence we find in this case

$$\lim_{n \to \infty} C_{f,g}(n) = 0 . \tag{22}$$

This shows that μ_A is mixing [K] and hence certainly ergodic. How fast the decay in (22) takes place depends on the "smoothness" of the observables f and g, where smoothness is related to the way these functions depend on the variables ξ_i for large i.

In the most favourable case the spectral radius of the operator $\mathscr{N}$ in (20) then is strictly smaller than λ_1 leading to an exponential decay of such correlations. In the physics literature one talks in this case of systems with a finite correlation length.

Examples of such systems are the one dimensional lattice spin models with finite range observables A. Another nontrivial example but with an A of infinite range is provided by the Kac model [Ma3]. This model is characterized by the following data: $F = \{+1,-1\}$, $A_{\sigma,\sigma'} = 1$ for all $\sigma,\sigma' \in F$ and the observable

$$A(\xi) = \mathscr{I}\xi_0 \sum_{i=1}^{\infty} \xi_i \lambda^i . \tag{23}$$

There $\mathscr{I}$ is some real parameter and $0 < \lambda < 1$ a constant describing the asymptotic dependence of A on ξ_i for $i \longrightarrow \infty$, which obviously decays esponentially fast. The transfer operator $\mathscr{L}_A$ for this model has the simple form

$$\mathscr{L}_A f(\xi) = \sum_{\sigma=\pm 1} \exp(\mathscr{I}\sigma \sum_{i=1}^{\infty} \xi_{i-1}\lambda^i) \; f(\sigma,\xi) . \tag{24}$$

From the R–P–F Theorem we know that

$$\lim_{n\to\infty} \left\| \left[\frac{\mathscr{L}_A^n 1}{\lambda_1^n} \right] - h_A \right\| = 0 \tag{25}$$

where 1 represents the function $f(\xi) \equiv 1$.

Denoting the function $(\lambda_1^{-n} \mathscr{L}_A^n 1)(\xi)$ by $f_n(\xi)$ we see from (24) that any f_n belongs to the following subspace $A_\infty(\Omega_A)$ in $\mathscr{C}(\Omega_A)$:

$$A_\infty(\Omega_A) = \{f \in \mathscr{C}(\Omega_A) : \exists\, g \in A_\infty(D_R) : f(\xi) = g(\pi(\xi))\} \tag{26}$$

where $D_R = \{z \in \mathbb{C} : |z| < R\}$ and $\pi : \Omega_A \longrightarrow D_R$ is the map

$$\pi(\xi) = \sum_{i=1}^{\infty} \xi_{i-1}\lambda^i . \tag{27}$$

$A_\infty(D_R)$ denotes the Banach space of all functions f holomorphic in D_R and continuous on $\overline{D_R}$ with the sup norm

$$||f|| = \sup_{z \in \overline{D_R}} |f(z)| . \tag{28}$$

That the space $A_\infty(\Omega_A)$ is indeed invariant under the operator $\mathscr{L}_A$ in (24), at least for any R with $R > \frac{\lambda}{1-\lambda}$, follows from

<u>Lemma 1</u> On the space $A_\infty(\Omega_A)$ the operator $\mathscr{L}_A$ acts as

$$(\mathscr{L}_A f)(\xi) = \sum_{\sigma = \pm 1} \exp(\mathscr{F}\sigma\, \pi(\xi))\; f(\psi_\sigma(\pi(\xi)))$$

where $\psi_\sigma : D_R \longrightarrow D_R$ is the holomorphic map

$$\psi_\sigma(z) = \lambda\sigma + \lambda z . \tag{29}$$

The proof is a simple calculation together with the fact that for $R > \frac{\lambda}{1-\lambda}$ the function $\exp(\mathscr{F}\sigma z)\;\; f \circ \psi_\sigma(z)$ is in $A_\infty(D_R)$ for $f \in A_\infty(D_R)$.

This shows that $\mathscr{L}_A : A_\infty(\Omega_A) \longrightarrow A_\infty(\Omega_A)$ is a well defined bounded linear operator. What one would like to have now is that the eigenfunction h_A corresponding to the highest eigenvalue λ_1 belongs to the space $A_\infty(\Omega_A)$. Instead of working in this space one can as well study the induced action of the operator $\mathscr{L}_A$ in the space $A_\infty(D_R)$ which we denote by the same symbol:

$$\mathscr{L}_A g(z) = \sum_{\sigma = \pm 1} \exp(\mathscr{I}\sigma z)\; g(\psi_\sigma(z)) \tag{30}$$

with $\psi_\sigma : D_R \longrightarrow D_R$ defined in (29).

Lemma 2 The eigenfunction h_A belongs to the space $A_\infty(\Omega_A)$, that means there exists a function $g_A \in A_\infty(D_R)$ with

$$\mathscr{L}_A\, g_A(z) = \lambda_1 g_A(z)$$

where λ_1 is the leading eigenvalue of $\mathscr{L}_A$ in the space $\mathscr{C}(\Omega_A)$.

The proof follows from positivity and compactness properties of the operator $\mathscr{L}_A : A_\infty(D_R) \longrightarrow A_\infty(D_R)$ which we will discuss in a more general setup next.

From its definition in (30) we see that $\mathscr{L}_A$ is the sum of two operators both of which of the form

$$\mathscr{L}g(z) = \varphi(z)\; g \circ \psi(z) \tag{31}$$

acting in some Banach space of holomorphic functions over some domain $D \subset \mathbb{C}$, such that ψ maps D holomorphically inside itself and φ is a holomorphic function on D

too. To understand the functional analytic properties of $\mathscr{L}$ it is obviously enough to study the operator

$$C_\psi g(z) = g \circ \psi(z) \tag{32}$$

which is an example of a so called composition operator. These operators and their properties have been objects of intense studies up to the present day [Sh1,Sh2]. An interesting question is for instance, how the spectral behaviour of such an operator depends on the spaces of holomorphic functions on which it can be defined. There seems to take place a rather complex interplay between the way ψ maps D inside itself and the boundary behaviour of the functions on D on which C_ψ is considered to act. This can be seen already from the two extreme cases for the map ψ: $\psi(z) = z$ or $\psi(z) = z_0$. In the first case C_ψ is the identity operator whereas in the second case it is a rank one operator mapping the entire function space onto a 1–dim. subspace. In simple words the result of the work of Shapiro et al. [Sh, ShT, Sch] is essentially the following: in the different spaces of holomorphic functions over the domain D, characterized by the functions boundary behaviour, the operator C_ψ in (32) can be compact or even traceclass only if the image $\psi(\overline{D})$ of the closure $\overline{D}$ of D hits the boundary of D not too often and not too smooth. For spaces of boundary regular functions, that means those continuous up to the boundary of D, it is known [Sh] that C_ψ is compact if and only if $\psi(\overline{D})$ does not hit ∂D in any point. Let us give the argument for the Banach space $A_\infty(D_1)$. In this space a linear operator $\mathscr{L}$ is compact iff any sequence $\{f_n\}$ converging to zero pointwise contains a subsequence $\{f_j\}$ such that $\lim_{j\to\infty} ||\mathscr{L} f_j|| = 0$. Hence if C_ψ is compact the sequence $\{f_n\}$ with $f_n(z) = z^n$ which obvioulsy converges to zero pointwise must contain a subsequence $\{f_j\}$ such that $\lim_{j\to\infty} ||C_\psi f_j|| = \lim_{j\to\infty} ||\psi \cdot {}^j|| = 0$ where $\psi \cdot {}^j$ denotes the function $\psi \cdot {}^j(z) = (\psi(z))^j$. This however is possible only if $\sup_{z \in \overline{D}} |\psi(z)| < 1$. The inverse direction of our claim follows from the sequel where we will show that C_ψ has even stronger spectral properties than being compact.

<u>Example</u>: In $A_\infty(D_1)$ the operator C_ψ with $\psi(z) = \frac{z+1}{2}$ is not compact, even though $\psi(\overline{D})$ hits ∂D only in the single point $z = 1$ [Sch] .

Less restrictive for C_ψ to be compact (or even traceclass) are the conditions on ψ if one considers C_ψ as acting on spaces of functions with less regular boundary behaviour, typical examples for such spaces being the Hardy spaces $H_p(D)$. For domains D with smooth enough boundary ∂D and $1 \leq p < \infty$ the space $H_p(D)$ is defined as [D]:

$$H_p(D) = \{f: \text{ f holomorphic on D and } \int_{\partial D} |f(z)|^p dz < \infty\} . \tag{33}$$

It is known that for $\psi : D \longrightarrow D$ holomorphic the operator C_ψ is well defined and bounded on $H_p(D)$ [Sh]. Indeed one has [ShT]

<u>Theorem</u> 1 If ψ maps $\overline{D}$ inside a polygon inscribed in the boundary ∂D then C_ψ is nuclear in $H_p(D)$ in the sense of Grothendieck for all $\infty > p \geq 1$.

<u>Corollary</u> 1 Under the conditions of Theorem 1 the operator C_ψ is traceclass in the Hilbert space $H_2(D)$.

Let us briefly recall the definition of compact respectively traceclass operators in a separable Hilbert space $\mathscr{H}$.

A linear operator $\mathscr{L} : \mathscr{H} \longrightarrow \mathscr{H}$ is compact if there exist (not necessarily complete) orthonormal sets $\{f_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ and positive real numbers $\{\rho_n\}_{n=1}^N$ with $\rho_n \longrightarrow 0$ such that

$$\mathscr{L} = \sum_{n=1}^{N} \rho_n (f_n, \cdot) g_n \, , \quad 1 \leq N \leq \infty \tag{34}$$

where the sum on the right hand side converges in norm and $(\, , \,)$ denotes the scalar product in $\mathscr{H}$.

<u>Remark</u>. The numbers ρ_n are called singular values of $\mathscr{L}$.

Using this representation for compact operators in the Hilbert space $\mathscr{H}$ the trace class or nuclear operators are characterized by the property:

$$\text{L is trace class iff L is compact and } \sum_{i=1}^{N} \rho_i < \infty \, . \tag{35}$$

One then shows that any operator $\mathscr{L} : \mathscr{H} \longrightarrow \mathscr{H}$ of trace class has the property that for any orthonormal basis $\{\varphi_n\}$ of $\mathscr{H}$ the quantity $\sum_n (\varphi_n, \mathscr{L} \varphi_n)$ converges absolutely and is independent of the basis. It defines the trace–functional trace $\mathscr{L} = \sum_n (\varphi_n, \mathscr{L} \varphi_n)$ which turns out to be identical to the sum over the eigenvalues $\{\lambda_i\}$ of $\mathscr{L}$ counted according to their algebraic multiplicity. Grothendieck extended this definition of trace class to general Banach spaces: A linear operator $\mathscr{L} : B \longrightarrow B$, B an arbitrary Banach space, is called nuclear of order q, if there exist families $\{f_n\} \in B$, $\{f_n^*\} \in B^*$, $||f_n|| \leq 1$, $||f_n^*|| \leq 1$, and a sequence $\{\rho_n\}$ of complex numbers, such that

$$\mathscr{L} = \sum_n \rho_n \, f_n^*(\,.\,)\, g_n \tag{36}$$

and $q = \inf\{p \leq 1 : \sum_n |\rho_n|^p < \infty\}$. Convergence in (36) is again in the operator norm. The space B^* is the dual space of B , that is the space of continuous linear functionals on B with the usual norm. More details about Grothendieck's theory can be found in an appendix, where also the possibility for defining a trace for such nuclear operators is discussed, which is more delicate than in the Hilbert space case.

Remark: It is common use to call a linear operator $\mathscr{L}$ in a Banach space simply nuclear, if in the representation (36) the numbers ρ_n fulfill $\sum_n |\rho_n| < \infty$. In this sense Theorem 1 has to be understood.

As for the space $A_\infty(D)$ also in the space $H_\infty(D)$ defined as

$$H_\infty(D) = \{f : f \text{ holomorphic in } D \,,\, \sup_{z \in D} |f(z)| < \infty\} \tag{37}$$

the operator C_ψ is nuclear iff ψ maps D strictly inside itself [Sch].

Let us come back now to the discussion of our transfer operator $\mathscr{L}_A$ in (30). Since the maps ψ_σ in (29) map the disc D_R for $R > \dfrac{\lambda}{1-\lambda}$ strictly inside itself it follows from the preceding discussion that the composition operators C_{ψ_σ} are nuclear, in fact of order zero (see appendix) in the space $A_\infty(D_R)$. Standard arguments about sums resp. composition of nuclear operators with bounded operators finally lead to

Lemma 3 The transfer operator $\mathscr{L}_A : A_\infty(D_R) \longrightarrow A_\infty(D_R)$ in (30) is for $R > \frac{\lambda}{1-\lambda}$ nuclear of order zero and hence of trace class.

Exercise Determine representation (36) for the operator

$$\mathscr{L} f(z) = \varphi(z) \cdot f(\rho z)$$

on $A_\infty(D_R)$, where $\rho \in \mathbb{C}$ fulfills $0 < |\rho| < 1$.

Our next aim is to determine the trace of the transfer operator $\mathscr{L}_A$. To achieve this, we need the following fixed point theorem [EH], which we formulate in a very general form:

Theorem 2 (Earle, Hamilton) If D is a bounded connected domain in some complex Banach space B and ψ is a holomorphic map of D strictly inside itself, then ψ has exactly one fixed point z^* in D and $||D\,\psi(z^*)|| < 1$.

Thereby $D\,\psi(z^*)$ denotes the derivative of ψ at the point $z = z^*$, which is a linear operator in B. The term "strictly inside itself" means that

$$\inf_{z \in D, z' \in B \setminus D} ||\psi(z) - z'|| \geq \delta > 0 .$$

Remark: For finite dimensional Banach spaces this result is rather classical [H], even if its proof is not completely trivial. The above Theorem allows a complete determination of the eigenvalues of the compostion operator C_ψ on $A_\infty(D)$ at least, if ψ maps D strictly inside itself. We restrict our discussion to the case $D \subset \mathbb{C}$, mention however, that the result can be generalized immediately to any Banach space B as long as the operator $D\psi(z^*)$ is itself nuclear [Ma1].

<u>Lemma 4</u>. If ψ maps the domain $D \subset \mathbb{C}$ strictly inside itself then the spectrum of the generalized composition operator $\mathcal{L}g = \varphi \cdot C_\psi g$ on the space $A_\infty(D)$ consists of the eigenvalues $\lambda_n = \varphi(z^*)(\psi'(z^*))^n$, $n = 0,1,\ldots$ converging for $n \longrightarrow \infty$ to the point 0, where z^* is the unique fixed point of ψ in D.

<u>Proof</u>: Since $\mathcal{L}$ is compact the spectrum of $\mathcal{L}$ is discrete with possibly 0 the only accumulation point. Assume λ to be an eigenvalue. Then we have

$$\mathcal{L}f(z) = \varphi(z)f \circ \psi(z) = \lambda f(z) .$$

At the point $z = z^*$ we find

$$\varphi(z^*)f(z^*) = \lambda f(z^*) ,$$

and hence, if $f(z^*) \neq 0$, we conclude $\lambda = \varphi(z^*)$. If on the other hand $f(z^*) = 0$ we look at the once differentiated eigenequation

$$\varphi'(z)f \circ \psi(z) + \varphi(z)\psi'(z)f'(\psi(z)) = \lambda f'(z) .$$

Taking again $z = z^*$ we get

$$\varphi(z^*)\psi'(z^*)f'(z^*) = \lambda f'(z^*) ,$$

and hence, if $f'(z^*) \neq 0$, we find $\lambda = \varphi(z^*)\psi'(z^*)$. Repeating this argument we see that any eigenvalue λ of the operator $\mathscr{L}$ must belong to the set $\{\varphi(z^*)\psi'(z^*)^n\}$. We show next that any of these numbers is a simple eigenvalue of $\mathscr{L}$. For this take any $g \in A_\infty(D_R)$ with the property $g^{(k)}(z^*) = 0$ for $0 \leq k \leq n-1$ and $g^{(n)}(z^*) \neq 0$. A straightforward calculation then shows that there is no solution in $A_\infty(D_R)$ of the equation

$$(\mathscr{L} - \varphi(z^*)\psi'(z^*)^n 1)f = g\,, \tag{38}$$

and hence $\lambda_n = \varphi(z^*)\psi'(z^*)^n$ is an eigenvalue of $\mathscr{L}$. From our previous arguments we know already that eigenfunctions φ_n belonging to this eigenvalue must fulfill the relations $\varphi_n^{(k)}(z^*) = 0 \quad 0 \leq k \leq n-1$ and $\varphi_n^{(n)}(z^*) \neq 0$. Differentiating the eigenequation

$$\mathscr{L}\varphi_n = \lambda_n \varphi_n \tag{39}$$

(n + 1)–times we find at the point $z = z^*$ that $\varphi_n^{(n+1)}(z^*)$ is uniquely determined by $\varphi_n^{(n)}(z^*)$. From this we conclude that the solution of equation (39) is, up to a constant multiplicative factor, unique. The preceding characterization of φ_n combined with equation (38) shows finally, that there cannot exist any solution to the equation

$$(\mathscr{L} - \lambda_n 1)f(z) = \varphi_n(z) \tag{40}$$

and hence besides $f = \varphi_n$ also no solution to the equation

$$(\mathscr{L} - \lambda_n 1)^k f = 0\,,\ \ k = 1,2,\ \ldots\ . \tag{41}$$

This shows that λ_n has algebraic multiplicity 1.

Knowing this way the complete eigenvalue spectrum of the operator $\mathscr{L} = \varphi \cdot C_\psi$ we can write down immediately its trace:

Corollary 2 The trace of the operator $\mathscr{L} = \varphi \cdot C_\psi$ in the space $A_\infty(D_R)$ is given by the formula

$$\text{trace } \mathscr{L} = \sum_{n=0}^{\infty} \lambda_n = \frac{\varphi(z^*)}{1 - \psi'(z^*)} . \tag{42}$$

Remark: This formula has a straightforward generalization for domains D in $\mathbb{C}^m$ and reads then

$$\text{trace } \mathscr{L} = \frac{\varphi(z^*)}{\det(1 - D\psi(z^*))}$$

where $D\psi(z^*)$ denotes the derivative of ψ at $z = z^*$.

Applying next this trace formula to the transfer operator $\mathscr{L}_A$ for the Kac model in (30) we find the finite volume partition functions $Z_n(A)$ can be written as

$$Z_n(A) = (1 - \lambda^n) \text{ trace } \mathscr{L}_A^n . \tag{43}$$

The factor $1 - \lambda^n$ can be get rid of very easily: define a second transfer operator $\tilde{\mathscr{L}}_A = \lambda \mathscr{L}_A$ which obviously is also trace class with trace $\tilde{\mathscr{L}}_A^n = \lambda^n \text{ trace } \mathscr{L}_A^n$. Hence we can represent $Z_n(A)$ finally as:

$$Z_n(A) = \text{trace } \mathscr{L}_A^n - \text{trace } \tilde{\mathscr{L}}_A^n \ . \tag{44}$$

This shows that for the Kac model the two operators $\mathscr{L}_A$ and $\tilde{\mathscr{L}}_A$ play exactly the role which for finite range functions A played the transfer matrix $\mathbf{L}(A)$.

In the next section we will discuss how far this method can be extended to more general subshifts of finite type.

3. General subshifts of finite type with nuclear transfer operators

Let us start our discussion with a slight modification of the Kac model where we allow for a nontrivial transition matrix $\mathbf{A}$, for instance $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The Ruelle operator then takes the form

$$\mathscr{L}_A f(\xi) = \sum_{\sigma = \pm 1} A_{\sigma,\xi_0} \exp(\sigma\, \mathscr{T} \sum_{i=1}^{\infty} \xi_{i-1} \lambda^i) \ f(\sigma,\xi) \ . \tag{45}$$

The right hand side depends now on the variable ξ_0 in a new way, namely through the matrix element A_{σ,ξ_0}. To cope with this situation we consider the operator $\mathscr{L}_A$ as acting in the larger space $\bigoplus_{\sigma \in F} A_\infty(\Omega_A)$ whose elements $\underline{f}$ we denote by

$$\underline{f} = (f_\sigma(\xi))_{\sigma \in F} \tag{46}$$

with $f_\sigma \in A_\infty(\Omega_A)$. The space $A_\infty(\Omega_A)$ is embedded in this space through $f_\sigma = f$ for all $\sigma \in F$, $f \in A_\infty(\Omega_A)$. On the space $\bigoplus_{\sigma \in F} A_\infty(\Omega_A)$ the operator $\mathscr{L}_A$ acts as

$$(\mathscr{L}_A f)_\sigma(\xi) = \sum_{\sigma' \in F} A_{\sigma',\sigma} \exp(\sigma' \mathscr{F} \sum_{i=1}^{\infty} \xi_{i-1}\lambda^i)\, f_{\sigma'}(\sigma',\xi)\,. \tag{47}$$

Now we can proceed as in the case $A_{\sigma,\sigma'} = 1$. The operator $\mathscr{L}_A$ in (47) induces an operator $\mathscr{L}_A$ in the Banach space $\bigoplus_{\sigma \in F} A_\infty(D_R)$

$$(\mathscr{L}_A g)_\sigma(z) = \sum_{\sigma' \in F} A_{\sigma',\sigma} \exp(\sigma' \mathscr{F} z)\, g_{\sigma'}(\psi_{\sigma'}\, z)\,, \tag{48}$$

with the maps ψ_σ defined in (29). This operator is again nuclear of order zero and its trace is given by

$$\text{trace } \mathscr{L}_A = \sum_{\sigma \in F} A_{\sigma,\sigma} \exp(\sigma\, \lambda z^*_\sigma)\, \frac{1}{1 - \psi'_\sigma(z^*_\sigma)}\,. \tag{49}$$

Introducing a second operator

$$\tilde{\mathscr{L}}_A = \lambda\, \mathscr{L}_A \tag{50}$$

we find also in this case for the partition functions $Z_n(A)$:

$$Z_n(A) = \text{trace } \mathscr{L}_A^n - \text{trace } \tilde{\mathscr{L}}_A^n\,. \tag{51}$$

It should be clear now how we can apply the preceding method to a general subshift of finite type, which fulfills the following conditions:

(T1) There exists a map $\pi : \Omega_A \longrightarrow \mathbb{R}^k$, open sets $W_\sigma \subset \mathbb{R}^k$ with $\pi(\Omega_A) \subset \bigcup_{\sigma \in F} W_\sigma$ and maps $\psi_\sigma : \bigcup_{\sigma' \in S_\sigma} W_{\sigma'} \longrightarrow W_\sigma$ with $S_\sigma = \{\sigma' \in F : A_{\sigma,\sigma'} = 1\}$ such that for any $(\sigma,\xi) \in \Omega_A$ we have

$$\pi(\sigma,\xi) = \psi_\sigma(\pi(\xi)) \in W_\sigma .$$

(T2) There exist complex neighbourhoods U_σ of W_σ in $\mathbb{C}^k$ such that ψ_σ extends to a holomorphic map of $\bigcup_{\sigma' \in S_\sigma} U_{\sigma'}$ strictly inside U_σ.

(T3) There exist holomorphic functions A_σ on U_σ such that $A(\sigma,\xi) = A_\sigma(\psi_\sigma(\pi(\xi)))$ for $\sigma \in F$.

If a subshift of finite type (Ω_A,τ) fulfills conditions (T1) – (T3) then its transfer operator $\mathscr{L}_A$ can be considered as acting on the Banach space $\bigoplus_{\sigma \in F} A_\infty(U_\sigma)$ as follows:

$$(\mathscr{L}_A g)_\sigma(z) = \sum_{\sigma' \in F} A_{\sigma',\sigma} \exp A_{\sigma'}(\psi_{\sigma'}(z))\, g_{\sigma'}(\psi_{\sigma'}(z)) . \tag{52}$$

This operator is nuclear of order zero with trace $\mathscr{L}_A$ given by the formula

$$\operatorname{trace} \mathscr{L}_A = \sum_{\sigma \in F} A_{\sigma,\sigma} \exp A_\sigma(z_\sigma^*) \frac{1}{\det(1 - D\psi_\sigma(z_\sigma^*))} . \tag{53}$$

Thereby z_σ^* denotes for $\sigma \in S_\sigma$ the fixed point of the mapping ψ_σ. If $\sigma \notin S_\sigma$ then z_σ^* is obviously not defined. This does not matter since then $A_{\sigma,\sigma} = 0$ and the corresponding term vanishes by definition. To get rid of the factor $\det(1 - D\psi_\sigma(z^*_\sigma))$ in (53) we use a well known formula from multilinear algebra which tells us that [G1]

$$\det(1 - \mathbb{L}) = \sum_{r=0}^{k} (-1)^r \operatorname{trace} \bigwedge_r \mathbb{L} \tag{54}$$

where $\bigwedge_r \mathbb{L}$ denotes the r–fold exterior product of the linear operator $\mathbb{L}$ in $\mathbb{R}^k$.

This way we are lead to a whole class of transfer operators on the Banach spaces $\bigoplus_{\sigma \in F} \bigwedge_r B(U_\sigma)$ where $\bigwedge_r B(U_\sigma)$ denotes the B–space of differential r–forms holomorphic over the domain $U_\sigma \subset \mathbb{C}^k$. An element w_r of this space has the representation

$$w_r(z) = \sum_{i_1, \dots, i_r = 1} w_{i_1 \dots i_r}(z) dz_{i_1} \wedge \dots \wedge dz_{i_r} \tag{55}$$

with $w_{i_1 \dots i_r} \in A_\infty(U_\sigma)$.

On the space $\bigoplus_{\sigma \in F} \bigwedge_r B(U_\sigma)$, $1 \le r \le k$ we define an operator $\mathcal{L}_A^{(r)}$ as

$$(\mathcal{L}_A^{(r)} w_r)_\sigma(z) = \sum_{\sigma' \in F} A_{\sigma',\sigma} \exp A_{\sigma'}(\psi_{\sigma'}(z)) \bigwedge_r D\psi_{\sigma'}(z)\, (w_r)_{\sigma'}(\psi_{\sigma'}(z)) \tag{56}$$

where $\bigwedge_r D\psi_{\sigma'}(z)$ denotes the r–fold exterior product of the linear operator $D\psi_{\sigma'}(z) : \mathbb{C}^k \longrightarrow \mathbb{C}^k$.

The trace of the operator $\mathcal{L}_A^{(r)}$ is given by the formula:

$$\operatorname{trace} \mathcal{L}_A^{(r)} = \sum_{\sigma \in F} A_{\sigma,\sigma} \exp A_\sigma(z^*_\sigma) \operatorname{trace} \bigwedge_r D\psi_\sigma(z^*_\sigma) \frac{1}{\det(1 - D\psi_\sigma(z^*_\sigma))}.$$

This together with formula (54) then shows that

$$Z_1(A) = \sum_{r=0}^{k} (-1)^r \operatorname{trace} \mathscr{L}_A^{(r)} \tag{57}$$

where $\mathscr{L}_A^{(0)}$ is identical to the transfer operator $\mathscr{L}_A$ in (52).

It certainly does not come as a big surprise that quite generally the partition functions $Z_n(A)$ of a subshift of finite type fulfilling conditions (T1) to (T3) can be expressed as

$$Z_n(A) = \sum_{r=0}^{k} (-1)^r \operatorname{trace}(\mathscr{L}_A^{(r)})^n . \tag{58}$$

Formula (51) for the Kac model now appears as a special case of formula (58): one only has to take $k = 1$ and $D\psi_\sigma(z_\sigma^*) = \lambda$.

This concludes our discussion of transfer operators for subshifts of finite type (Ω_A, τ) or in other words, one dimensional lattice spin systems. In the next chapter we turn our attention to one dimensional expanding maps of the unit interval which through symbolic dynamics are closely related to the former systems.

II. Expanding Maps and their Transfer Operators

We restrict our discussion to dimension $d = 1$ for the reasons we explained already in the introduction. Let us mention however, that most of the theory extends to expanding systems of any dimension d (See for instance [Ma4]). The transfer operator for a 1–dim. expanding map T is better known under the name "Perron–Frobenius operator" and describes how densities transform under the map. This operator corresponds exactly to the transfer operator for a subshift of finite type defined by the symbolic dynamics of T (For symbolic dynamics see contributions in [A], [K], [M], [P], [S]). One only has to apply the procedure discussed in the preceding section to define the transfer operator on some space of functions smooth in some domain D. In the present case this domain is just the phase space of T which we take as the unit interval I or some complex neighbourhood of it. The map π of condition (T1) of the preceding section, $\pi : \Omega_A \longrightarrow I$, then defines just the symbolic dynamics of T and is the first step for applying the thermodynamic formalism for spin systems as developped in the first chapter. The following discussion will be centered around the questions what kind of properties an expanding map must have so that the transfer operator method in the strong analytic form of the foregoing sections can be applied and what kind of problems can be treated by this method. In the last chapter we will finally discuss a very special but nevertheless for this meeting very interesting case of an expanding map where the transfer operator method can be pushed rather far to give rather new results.

1. The Perron–Frobenius operator

We denote by I the unit interval $[0,1]$ and consider maps $T : I \longrightarrow I$ with strong hyperbolicity properties

Definition: A map $T : I \longrightarrow I$ is called expanding, if there exists a countable partition $\mathscr{A} = \{I_i\}_{i \in \mathscr{I}}$ of I into nontrivial intervals $I_i = [t_{i-1}, t_i]$ such that

(E1) $I = \bigcup_{i \in \mathscr{I}} I_i$

(E2) $\operatorname{int} I_i \cap \operatorname{int} I_j = \phi$ if $i \neq j$

(E3) $T_i := T|_{I_i}$ is monotone and $\mathscr{C}^k$

(E4) $|(T^n)'(x)| \geq \delta > 1$ for some $n \geq 1$ and all $x \in I$.

In case the local branches T_i of T are real analytic we call T an analytic expanding map.

Ergodic properties of T with respect to an invariant Borel measure μ are closely related to spectral properties of the Perron–Frobenius (P–F) operator [LaM] of T with respect to μ. If $\mathscr{L}_1(I,d\mu)$ denotes the Banach space of μ integrable functions over I then this operator $\mathscr{L}_T : \mathscr{L}_1(I,d_\mu) \longrightarrow \mathscr{L}_1(I,d\mu)$ is defined through the equation

$$\int_I d\mu(x)\ \mathscr{L}_T f(x)g(x) = \int_I d\mu(x)\ f(x)g(Tx) \tag{1}$$

where $f \in \mathscr{L}_1(I,d\mu)$ and $g \in \mathscr{L}_\infty(I,d\mu)$ are arbitrary. This operator tells us how densities with respect to the measure μ transform under T: if $dv(x) = f(x)d\mu(x)$,

$f \in \mathscr{L}_1(I,d\mu)$, $f \geq 0$, then we find for any measurable set A :

$$T^*v(A) = v(T^{-1}A) = \int_{T^{-1}A} dv(x) = \int_I dv(x)\, \chi_{T^{-1}A}(x) =$$

$$= \int_I \chi_A(Tx) f(x) d\mu(x) = \int_I \chi_A(x) \mathscr{L}_T f(x) d\mu(x) = \int_A \mathscr{L}_T f(x) d\mu(x) \, ,$$

which shows that T^*v is absolutely continuous with respect to μ with density $\mathscr{L}_T f(x)$. From this we conclude that $v = f\mu$ is T invariant iff $\mathscr{L}_T f = f$.

Let us list some further properties of the P–F operator $\mathscr{L}_T$:

(P1) $\int_I f\, d\mu = \int_I \mathscr{L}_T f\, d\mu$ for all $f \in \mathscr{L}_1(I,d\mu)$

(P2) $\mathscr{L}_T f \geq 0$ if $f \geq 0$ (positivity) (2)

(P3) $|| \mathscr{L}_T ||_{X_1} \leq 1$.

In the following we are mainly interested in the case where μ is ordinary Lebesque measure on I , so that from now on we set $d\mu(x) = dx$. In this case the operator $\mathscr{L}_T$ has a simple explicit representation for expanding maps:

$$\mathscr{L}_T f(x) = \sum_{i \in \mathscr{I}} |\psi_i'(x)|\, f \circ \psi_i(x)\, \chi_{TI_i}(x) \tag{3}$$

where $\psi_i = T_i^{-1} : TI_i \longrightarrow I_i$ is the local inverse of T restricted to I_i and χ_{TI_i} is the characteristic function of the set TI_i. Since the functions T_i are monotone by assumption and $|(T^n)'(x)| \geq \delta > 0$ the functions ψ_i' do not change sign on TI_i and $|\psi_i'(x)|$ is therefore equal to $\varepsilon_i \psi_i'(x)$ with $\varepsilon_i = \text{sign}\, \psi_i'(x)$ on TI_i independent of x.

As soon as at least for one $i \in \mathcal{I}$ $TI_i \neq I$ $\mathcal{L}_T$ does not leave invariant the space $\mathcal{C}(I)$ of continuous functions over I since $\mathcal{L}_T f$ in this case is certainly discontinuous at least at one of the two points Tt_i or Tt_{i-1}. Hence eigenfunctions of $\mathcal{L}_T$ are at most piecewise continuous, in general even only $\mathcal{L}_1$. Here on the other hand we are interested in systems whose P–F operator has piecewise analytic eigenfunctions. This seems at first to be a rather restricted class of maps, but it turns out that such maps play a rather interesting role in hyperbolic geometry in dimension 2 and the dynamical systems there: it was shown by Bowen and Series [S] that the symbolic description of geodesic flows on surfaces of constant negative curvature involves such one dimensional maps as some kind of Poincaré map of the corresponding flows. For more details we refer to the lectures by Adler [A], Manning [M], Pollicott [P] and especially Series [S], where one can find also the references to the original literature. To introduce this class of maps, we need some more notation. If $\mathcal{A} = \{I_i\}_{i \in \mathcal{I}}$ is any countable partition of I into closed intervals $I_i = [a_{i-1}, a_i]$ we denote by

$$\partial_{\mathcal{A}} := \{a_i\}$$

the set of boundary points of the I_i, completely specifying the partition $\mathcal{A}$.

<u>Definition</u>: A partition $\mathcal{A} = \{A_i\}$ is compatible with the action of the map T iff $T\,\partial_{\mathcal{A}} \subset \partial_{\mathcal{A}}$. Denote then by $T\mathcal{A}$ the partition determined by the set

$T\,\mathscr{E}_{\mathscr{A}} = \{Ta : a \in \mathscr{E}_{\mathscr{A}}\} \cup \{0,1\}$.

We next introduce so called Markov maps:

Definition: An expanding map $T : I \longrightarrow I$ with partition $\mathscr{A} = \{I_i\}_{i\in\mathscr{I}}$ is Markov, if

(M1) $T\,\mathscr{E}_{\mathscr{A}}$ is a finite set

(M2) there exists a finite number N such that $T\,\mathscr{E}_{\mathscr{P}} \subset \mathscr{E}_{\mathscr{P}}$ if $\mathscr{P}$ denotes the partition defined by the set $\bigcup_{n=1}^{N} T^n \mathscr{E}_{\mathscr{A}} \cup \{0,1\}$.

Examples

(1) If
$$Tx = \begin{cases} 3x \mod 1 & 0 \le x \le \frac{2}{3} \\ \frac{3}{2}x - \frac{1}{2} & \frac{2}{3} \le x \le 1 , \end{cases} \tag{5}$$

then T is Markov since $\mathscr{E}_{\mathscr{A}} = \left\{0 , \frac{1}{3} , \frac{2}{3}\right\}$ and $\mathscr{E}_{\mathscr{P}} = T\,\mathscr{E}_{\mathscr{A}} = \left\{0 , \frac{1}{2} , 1\right\}$ obviously obeys: $T\,\mathscr{E}_{\mathscr{P}} \subset \mathscr{E}_{\mathscr{P}}$.

(2) If
$$Tx = \begin{cases} \frac{1}{x} \mod 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{, then} \tag{6}$$

$\mathscr{E}_{\mathscr{A}} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. If $\mathscr{E}_{\mathscr{P}} = T\,\mathscr{E}_{\mathscr{A}} = \{0,1\}$ we find : $T\,\mathscr{E}_{\mathscr{P}} \subset \mathscr{E}_{\mathscr{P}}$ and T is Markov.

Remark 1 The partition $\mathscr{P}$ constructed above for a Markov map is the minimal partition of I finer than the partition $T\mathscr{A}$ and compatible with T. We will see that this partition $\mathscr{P}$ determines the kind of smoothness an eigenfunction of the P–F operator $\mathscr{L}_T$ for T can have at most.

Remark 2 A Markov map has a second important partition namely the one determined by $\mathscr{E}_{\mathscr{P}} \cup \mathscr{E}_{\mathscr{A}}$. It is also compatible with T but finer than partition $\mathscr{P}$. This partition is in a certain sense the minimal Markov partition of the system (I,T) from which symbolic dynamics can be constructed. This is the kind of partition used in almost all of the lectures during this meeting. See especially the lectures of [P] and [S].

If $T : I \longrightarrow I$ is an expanding Markov map with partition $\mathscr{A} = \{I_i\}_{i\in\mathscr{I}}$ we denote the partition $\mathscr{P}$ constructed above by $\mathscr{P} = \{O_j\}_{j\in\mathscr{K}}$. The P–F operator $\mathscr{L}_T$ can then be written as

$$(\mathscr{L}_T f)_j(x) = \sum_{i\in\mathscr{I}} \sum_{\ell\in\mathscr{K}} \varepsilon_i \psi_i'(x)\, A^{(i)}_{j\ell}\, f_\ell(\psi_i(x)) \ \text{ if } x \in O_j , \tag{7}$$

where we introduced for $\ell \in \mathscr{K}$ and $x \in O_\ell$ the notation

$$f_\ell(x) = f|_{O_\ell}(x) \tag{8}$$

and defined for all $i \in \mathscr{I}$ and $\ell, j \in \mathscr{K}$ the transition matrices

$$A^{(i)}_{j,\ell} = \begin{cases} 1 & \text{if } \psi_i(O_j) \subset O_\ell \\ 0 & \text{otherwise} \end{cases} . \tag{9}$$

To derive expression (7) one uses the fact that for all $i \in \mathscr{I}$

$$TI_i = \bigcup_{j \in \mathscr{K}_i} O_j \tag{10}$$

$$\text{where } \mathscr{K}_i = \{j \in \mathscr{K} : \text{int } O_j \cap \text{int } TI_i \neq \phi\} . \tag{11}$$

From this one derives that $\psi_i(O_j) \subset O_k$ for $\psi_i(\text{int } O_j) \cap \text{int } O_k \neq \phi$. Simple functional analytic properties of the above P–F operator (7) can be derived for <u>analytic Markov maps</u> T . These are expanding Markov maps with a partition $\mathscr{P}$ obeying the following conditions:

(A1) $T_i = T|_{I_i}$ is real analytic

(A2) for any $O_j \in \mathscr{P}$ there exists a complex neighbourhood $U_j \subset \mathbb{C}$ with $O_j \subset U_j$ such that the mappings ψ_i extend to holomorphic maps on $\bigcup_{j \in \mathscr{K}_i} U_j$, mapping any U_j for $j \in \mathscr{K}_i$ strictly inside U_i .

See the contributions of Pollicott [P] and Series [S] for examples where such maps can arise.

For analytic Markov maps the P–F operator $\mathscr{L}_T$ in (7) obviously defines a nuclear operator of order zero on the Banach space $\bigoplus_{\ell \in \mathscr{K}} A_\infty(U_\ell)$ of piecewise analytic functions over the domains U_ℓ . The proof is the same as for subshifts of finite type fulfilling conditions (T1) – (T3). As in this latter case we can define also for expanding maps partition functions $Z_n(A)$ for abitrary observables $A \in \mathscr{C}(I)$ by

$$Z_n(A) = \sum_{x \in \mathrm{Fix}\, T^n} \exp \sum_{k=0}^{n-1} A(T^k x)\,. \tag{12}$$

A very special role in the ergodic theory of such expanding maps is then played by the function $A(x) = -\beta \log|T'(x)|$ which for piecewise analytic functions is itself piecewise analytic. In this case the partition function $Z_n(A)$ is just

$$Z_n(A) = Z_n(\beta) = \sum_{x \in \mathrm{Fix}\, T^n} \prod_{k=0}^{n-1} \frac{1}{|T'(T^k x)|^\beta}\,. \tag{13}$$

It is not too difficult to show, that for expanding Markov maps $Z_m(\beta)$ can be rewritten as

$$Z_m(\beta) = \sum_{i_1,\ldots,i_m \in \mathcal{I}} \; \sum_{\ell_1,\ldots,\ell_m \in \mathcal{K}} |(\psi_{i_1} \circ \ldots \circ \psi_{i_m})'(x^*_{i_1 \ldots i_m})|^\beta A^{(i_m)}_{\ell_1 \ell_m} \cdots A^{(i_1)}_{\ell_2 \ell_1}\,, \tag{14}$$

where $x^*_{i_1 \ldots i_m}$ is the unique fixed point of T^m with the property that

$$T^{k-1} x^*_{i_1 \ldots i_m} \in O_{\ell_k} \cap I_{i_k} \tag{15}$$

and $A^{(i_k)}_{\ell_{k+1} \ell_k} = 1$ for all $1 \le k \le m$ $(\ell_{m+1} := \ell_1)$. In the case $A^{(i_k)}_{\ell_{k+1} \ell_k} = 0$ for some k, the corresponding term has to be set equal to zero in (14).

To apply now the theory of transfer operators we define generalized P–F operators

$$\mathscr{L}_\beta^{(s)} : \bigoplus_{\ell\in\mathscr{K}} A_\infty(U_\ell) \longrightarrow \bigoplus_{\ell\in\mathscr{K}} A_\infty(U_\ell) \quad \text{for} \quad s = 0,1$$

as follows

$$(\mathscr{L}_\beta^{(s)} f)_j(z) = \sum_{i\in\mathscr{I}} \sum_{\ell\in\mathscr{K}} (\varepsilon_i \psi_i'(z))^\beta \, \psi_i'(z)^s \, A_{j,\ell}^{(i)} \, f_\ell(\psi_i(z)) \, . \tag{16}$$

As long as $|\mathscr{I}|$ is finite (remember $\mathscr{K}$ is finite by definition!) and $\psi_i'(z) \neq 0$ on $\bigcup_{j\in\mathscr{K}_i} U_j$ the operators $\mathscr{L}_\beta^{(s)}$, $s = 0,1$ are nuclear for all $\beta \in \mathbb{C}$. If on the other hand $|\mathscr{I}| = \infty$ the range of $\beta' s$ such that $\mathscr{L}_\beta^{(s)}$ is nuclear has to be investigated in more detail. This will be done later for the continued fraction transformation, where $\mathscr{I} = \mathbb{N}$.

Applying next the trace formula (53) in ch. I. we find

$$\operatorname{trace} \mathscr{L}_\beta^{(s)} = \sum_{i\in\mathscr{I}} \sum_{\ell\in\mathscr{K}} A_{\ell,\ell}^{(i)} \, |\psi_i'(z_i^*)|^\beta \, \psi_i'(z_i^*)^s \, \frac{1}{1-\psi_i'(z_i^*)} \tag{17}$$

where z_i^* denotes the unique fixed point of $T_i = T\big|_{I_i}$ respectively ψ_i in the interval O_ℓ with $\psi_i(O_\ell) \subset O_\ell$. From this we conclude

$$Z_1(\beta) = \operatorname{trace} \mathscr{L}_\beta^{(0)} - \operatorname{trace} \mathscr{L}_\beta^{(1)} \, . \tag{18}$$

It is a little bit cumbersome but straightforward to show for general $m \in \mathbb{N}$:

$$Z_m(\beta) = \operatorname{trace} \mathscr{L}_\beta^{(0)m} - \operatorname{trace} \mathscr{L}_\beta^{(1)m} \, . \tag{19}$$

This shows piecewise analytic Markov maps possess nuclear transfer operators (generalized Perron–Frobenius operators) which give a simple description of their partition functions $Z_m(\beta)$. In the next section we apply these results to zeta functions assciated with such systems.

2. Ruelle's zeta functions for expanding maps

We restrict our discussion to analytic expanding Markov maps introduced in the last section. It should be mentioned that the theory of zeta functions has been developped for quite general dynamical systems [R], but the results are most complete in the case considered here. These functions play a fundamental role in Parry's and Pollicott's work on the distribution of closed orbits in hyperbolic systems [P]. If $T : I \longrightarrow I$ is such a Markov map and $A : I \longrightarrow \mathbb{C}$ some function then we defined already the partition functions

$$Z_n(A) = \sum_{x \in \mathrm{Fix}\, T^n} \exp \sum_{k=0}^{n-1} A(T^k x) .$$

These numbers can be put together in a very elegant way in some kind of generating function, the so called zeta function

$$\zeta(z,A) := \exp \sum_{k=1}^{\infty} \frac{1}{k} z^k Z_k(A) , \tag{20}$$

which in this generality was introduced by D. Ruelle [R] for general hyperbolic dynamical systems. In the special case $A \equiv 0$, the above formal power series in z is just the Artin–Mazur function [ArM] for T, since in this case $Z_n(0) = \# \{x \in I : T^n x = x\}$ again simply counts periodic points of T of period n. Quite a lot is known about analyticity properties of the function $\zeta(z,A)$ both as a function of $z \in \mathbb{C}$ and of A in some Banach space of functions [R]. In general there is some disc D around $z = 0$ such that $\zeta(z,A)$ for fixed A is meromorphic in D. For our class of maps this result can be improved quite a lot:

Theorem 3 If $T : I \longrightarrow I$ is an analytic Markov map with partition $\mathscr{A} = \{I_i\}_{i \in \mathscr{I}}$ and local inverses ψ_i, $i \in \mathscr{I}$, and if the functions $\exp A \circ \psi_i$ extend to holomorphic functions on $\bigcup_{j \in \mathscr{K}_i} U_j$, then the Ruelle function $\zeta(z,A)$ has a meromorphic extension to the entire complex z–plane. This extension is given by the formula

$$\zeta(z,A) = \frac{\det(1 - z\, \mathscr{L}_A^{(1)})}{\det(1 - z\, \mathscr{L}_A^{(0)})}$$

Proof: Consider the generalized P–F operators $\mathscr{L}_A^{(s)}$, $s = 0,1$ defined on $\bigoplus_{\ell \in \mathscr{K}} A_\infty(U_\ell)$ by

$$(\mathscr{L}_A^{(s)} g)_j(z) = \sum_{i \in \mathscr{I}} \sum_{\ell \in \mathscr{K}} \exp A(\psi_i(z))\, (\psi_i'(z))^s\, A_{j\ell}^{(i)}\, g_\ell(\psi_i(z)) . \tag{21}$$

One then shows again

$$Z_m(A) = \operatorname{trace} \mathscr{L}_A^{(0)m} - \operatorname{trace} \mathscr{L}_A^{(1)m} .$$

Using next the fomula

$$\det(1 - z\,\mathscr{L}) = \exp \operatorname{trace} \log(1 - z\,\mathscr{L}) \tag{22}$$

valid for the Fredholm determinant of a nuclear operator of order zero [G1], we find

$$\zeta(z,A) = \frac{\det(1 - z\,\mathscr{L}_A^{(1)})}{\det(1 - z\,\mathscr{L}_A^{(0)})}.$$

By Grothendieck's theory for nuclear operators the right hand side obviously is meromorphic in the entire z–plane as the quotient of two entire functions.

Instead of studying the function $\zeta(z,A)$ in the variable $z \in \mathbb{C}$ for fixed A, we can also consider the function $\zeta(1,\beta A)$ for fixed A and β varying in the complex plane. Of special interest again is the case $A = -\log|T'(x)|$, which we mentioned already in the last section: there is a close connection between this function and the Selberg zeta function for geodesic flows on surfaces of constant negative curvature, if one takes for T the appropriate expanding maps. These matters are discussed in the lectures by Pollicott [P].

By applying exactly the same arguments as above and also the fact, that $\det(1 - \mathscr{L}_\beta)$ is holomorphic in β as long as the nuclear operator $\mathscr{L}_\beta$ depends itself analytically on β in some domain, we arrive at

<u>Theorem 4</u> If $T : I \longrightarrow I$ is an analytic Markov map with finite partition $\mathscr{A} = \{I_i\}_{i \in \mathscr{I}}$, $|\mathscr{I}| < \infty$, then the function

$$\zeta(1,\beta) = \exp \sum_{m=1}^{\infty} \frac{1}{m} Z_m(\beta) \tag{23}$$

with $Z_m(\beta) = \sum_{x \in \mathrm{Fix}\, T^m} \exp -\beta \sum_{k=0}^{m-1} \log |T'(T^k x)|$ extends to a meromorphic function in the whole complex β–plane . This extension can be written as

$$\zeta(1,\beta) = \frac{\det(1 - \mathscr{L}_\beta^{(1)})}{\det(1 - \mathscr{L}_\beta^{(0)})} \tag{24}$$

<u>Proof</u>: We have to take simply the generalized P–F operators $\mathscr{L}_\beta^{(s)}$ defined in (16) of ch. II.

<u>Remark</u>: Obviously, the poles of this function have to be found among those values of β for which the operator $\mathscr{L}_\beta^{(s)}$ has $\lambda = 1$ as an eigenvalue. Their multiplicity determines the order of the pole.

In the next chapter we are going to apply our general results to the continued fraction map $Tx = \frac{1}{x} \bmod 1$. This map plays an important role in number theory in connection with continued fraction expansions. But this map is also closely related to the symbolic description of the geodesic flow on the so called modular surface as found by Artin [Ar] whose work was continued quite recently by Series, Bowen, Adler et al. For more details see their contributions [A], [S].

III. The Continued Fraction Transformation (Gauss–Map)

1. Perron–Frobenius operators

The transformation $T : I \longrightarrow I$

$$Tx = \begin{cases} \frac{1}{x} \bmod 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{1}$$

plays a crucial role in number theory through its relation to the continued fraction expansion of any real in the unit interval I :

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}} \quad , \ n_i \in \mathbb{N} \ . \tag{2}$$

For short, we will write this as $x = [n_1, n_2, \dots]$. It is known that this expansion is finite iff x is rational. In this case the expansion is not unique since $n + 1 = n + \frac{1}{1}$. For irrational x however the expansion does not determinate and it is also unique. From its definition we find

$$T^k [n_1, n_2, \dots] = [n_{k+1}, \dots] \ , \ k = 0,1,2, \dots \tag{3}$$

and $n_k = [(T^{k-1}x)^{-1}]$ where $[x]$ denotes the largest integer $\leq x$. From this we can see immediately that the distribution of the entries n_k in the expansion (2) of x is closely related to the ergodic proporties of the dynamical system T in (1). Obviously, the Gauss map T is an analytic expanding Markov map defined in sect. 1 of ch. II: For the partition $\mathscr{A} = \{I_n\}_{n\in\mathbb{N}}$ with

$$I_n = \left[\frac{1}{n+1}, \frac{1}{n}\right] \tag{4}$$

we find: $T\big|_{I_n}(x) = T_n(x) = \frac{1}{x} - n$ is analytic in $x \neq 0$, and $|(T^2)'(x)| \geq 4 > 1$ for all $x \in I$.

Furthermore we get for all $n \in \mathbb{N}$: $TI_n = I$, so that $\chi_{TI_n} \equiv 1$ for all $n \in \mathbb{N}$. The inverse maps $\psi_i = T_i^{-1} : I \longrightarrow I_i$ have the explicit form

$$\psi_i(x) = \frac{1}{x+i}\ , \tag{5}$$

and hence are meromorphic in the entire z–plane with a simple pole at $z = -i$.

Since $T\,\partial\mathscr{A} = \{0,1\}$ and hence $T\,\partial\mathscr{P} = \partial\mathscr{P}$ if $\partial\mathscr{P} = \{0,1\}$, the partition $\mathscr{P}$ is the trivial partition $\mathscr{P} = \{I\}$. Therefore the generalized P–F operators $\mathscr{L}_\beta^{(s)}$ as defined in (16) of ch. II. have the form

$$\mathscr{L}_\beta^{(s)} f(z) = \sum_{i=1}^{\infty} (-1)^s \left[\frac{1}{z+i}\right]^{2\beta+2s} f\left[\frac{1}{z+i}\right] \tag{6}$$

acting on the space $A_\infty(U_1)$ where U_1 is the disc

$$U_1 = \left\{z \in \mathbb{C} : |z-1| < \tfrac{3}{2}\right\}\ . \tag{7}$$

It is easy to verify that $\psi_i(\overline{U_1}) \subset U_1$ for all $i \in \mathbb{N}$. With this choice of U_1 the operators $\mathscr{L}_\beta^{(s)}$ define nuclear operators of order zero on the space $A_\infty(U_1)$ for all β

with $\operatorname{Re}\beta > \frac{1}{2}$ for $s = 0$ respectively $\operatorname{Re}\beta > -\frac{1}{2}$ for $s = 1$.

Before we are going to discuss the different zeta functions for this system, let us investigate a little bit in more detail the above operators $\mathscr{L}_\beta^{(s)}$. For $s = 1$ the operator $\mathscr{L}_\beta^{(0)}$ is the ordinary Perron–Frobenius operator for T with respect to dx, which perhaps was known already to Gauss. In fact, he must have known at least the eigenfunction belonging to the leading eigenvalue λ_1 of $\mathscr{L}_{\beta=1}^{(0)}$, which by property (P1) of the P–F operator in (2) of ch. II must be equal 1. This eigenfunction h is the invariant density of the map T and turns out to be given by

$$h(x) = \frac{1}{\log 2}\,\frac{1}{x+1}\,. \tag{8}$$

and defines what is called Gauss measure for T.

Obviously, the function $h(z)$ belongs to the space $A_\infty(U_1)$. In a letter to Laplace Gauss stated the result, that the asymptotic probability for the event $T^n x < a$ in the limit $n \longrightarrow \infty$ is given by the formula: $P(T^n x < a) = \frac{1}{\log 2}\log(1+a)$. In modern terminology this simply says

$$\lim_{n\to\infty} \mu(T^{-n}[0,a]) = \lim_{n\to\infty} \int_{T^{-n}[0,a]} d\mu(x) = \frac{1}{\log 2}\int_0^a \frac{1}{x+1}\,dx\,, \tag{9}$$

where μ denotes any normalized measure on I absolutely continuous with respect to Lebesque. By relation (1) in ch. II we can write this also as

$$\lim_{n\to\infty} \int_I \chi_{[0,a]}(x)\left[\mathscr{L}_1^{(0)n} g\right](x)dx = \int_I \chi_{[0,a]}(x)h(x)dx\,, \tag{10}$$

and the result of Gauss is a special case of the asymptotic behaviour of the operator $\mathscr{L}_1^{(0)}$ under iterations. Unfortunately, it never became known how Gauss derived his result in (9). In his letter to Laplace he also posed the problem of determining the rate of convergence to the asymptotic law. A first proof of relation (9) was given much later by R. Kuzmin [Ku] in (1928), who also showed the error for finite n to be bounded by $q^{\sqrt{n}}$ for some $0 < q < 1$. This result was improved by P. Levi to q^n with $0 < q < 0.68$ [Le]. In the meantime the number q has been determined numerically even up to 20 decimal places [W] as

$$q \approx 0.30366300289873265860 \ldots\ldots \quad (11)$$

In the space $A_\infty(U_1)$ Kuzmin's (or better Levy's) Theorem follows from spectral properties of the operator $\mathscr{L}_\beta^{(0)}$ valid for real $\beta > \frac{1}{2}$:

Theorem 5 The operator $\mathscr{L}_\beta^{(0)} : A_\infty(U_1) \longrightarrow A_\infty(U_1)$ has a positive leading eigenvalue $\lambda_1(\beta)$ which is simple and strictly larger than all other eigenvalues in absolute value. The corresponding eigenfunction $h_\beta \in A_\infty(U_1)$ is strictly positive on $\overline{U}_1 \cap \mathbb{R}$. The adjoint operator $\mathscr{L}_\beta^{(0)*} : A_\infty^*(U_1) \longrightarrow A_\infty^*(U_1)$ has a positive eigenfunctional ℓ_β^* with eigenvalue $\lambda_1(\beta)$ with $\ell_\beta^*(f) > 0$ if $f > 0$ on $\overline{U}_1 \cap \mathbb{R}$. If $\mathscr{P}_\beta$ denotes the projector

$$\mathscr{P}_\beta = \ell_\beta^* \otimes h_\beta \,, \quad (\ell_\beta^*(h_\beta) = 1\,!) \quad (12)$$

then $\mathscr{L}_\beta^{(0)}$ has the representation

$$\mathscr{L}_\beta^{(0)} = \lambda_1(\beta)\,\mathscr{P}_\beta + \mathscr{N}_\beta \quad (13)$$

with $\mathcal{P}_\beta \cdot \mathcal{N}_\beta = \mathcal{N}_\beta \cdot \mathcal{P}_\beta = 0$. The spectral radius of $\mathcal{N}_\beta$ is strictly smaller than $\lambda_1(\beta)$.

Exercise: Determine the eigenfunctional ℓ_1^* .

From this Kuzmin's Theorem then follows as a simple corollary

Corollary 3: If $\mathcal{L}_\beta^{(0)}$ is the generalized P–F operator for the Gauss map in the space $A_\infty(U_1)$ then

$$\| \lambda_1(\beta)^{-n} \mathcal{L}_\beta^{(0)n} - \mathcal{P}_\beta \| \le q_\beta^n$$

where $q_\beta = \left| \frac{\lambda_2(\beta)}{\lambda_1(\beta)} \right| < 1$ and $\lambda_2(\beta)$ is the second highest eigenvalue of $\mathcal{L}_\beta^{(0)}$ in absolute value. If $\beta = 1$ then $\lambda_1(1) = 1$ and hence $q_1 = |\lambda_2(1)|$ [MaR1], [MaR2].

The proof of Theorem 5 is a Perron–Frobenius type of argument based on positivity properties of the operator $\mathcal{L}_\beta^{(0)}$. What positivity really means in the setup of infinite dimensional Banach spaces we are going to explain next.

Definition A set K in the real Banach space B is called a proper cone, if $\rho f \in K$ for all $f \in K$ and all $\rho \ge 0$ and if $K \cap -K = \{0\}$. A proper cone is called reproducing if $B = K - K$, that is every $g \in B$ has a representation $g = f_1 - f_2$, $f_i \in K$, $i = 1,2$. Given such a proper, reproducing cone K in B we can define positive operators with respect to K :

Definition: A linear operator $\mathcal{L}\colon B \longrightarrow B$ is positive with respect to K if $\mathcal{L}K \subset K$.

In the following we assume the cone K to have nonemty interior $\operatorname{int} K = \overset{\circ}{K}$.

Definition: A positive operator $\mathcal{L}: B \longrightarrow B$ is called u_0–positive with $u_0 \in \overset{\circ}{K}$, if there exist for every $0 \not\equiv f \in K$ a number $p \in \mathbb{N}$ and reals $\alpha,\beta > 0$, such that

$$\beta\, u_0 \le \mathcal{L}^p f \le \alpha\, u_0 ,$$

where the order $\le$ is defined by $K : f \le g \Leftrightarrow g - f \in K$. For u_0–positive compact operators one has a Perron–Frobenius Theorem [Kr]:

Theorem 6 (Krasnoselskii) If $\mathcal{L}: B \longrightarrow B$ is a compact u_0–positive operator with respect to the cone K such that $\beta\, u_0 \le \mathcal{L}^p u_0 \le \alpha\, u_0$, then there exists exactly one eigenvector $h_1 \in \overset{\circ}{K}$ and a $\lambda_1 > 0$ such that $\mathcal{L} h_1 = \lambda_1 h_1$. The eigenvalue λ_1 is simple, in absolute value strictly larger than all other eigenvalues of $\mathcal{L}$ and fulfills the bounds $\beta^{1/p} \le \lambda_1 \le \alpha^{1/p}$. For any $f \in B$ one has $\lim_{n\to\infty} \lambda_1^{-n} \mathcal{L}^n f = c(f) h_1$ where $\mathcal{L}^* c = \lambda_1 c$.

It turns out that operator $\mathcal{L}_\beta^{(0)}$ in (6) of this chapter is u_0–positive with respect to the following cone K:

$$K = \{f \in A_\infty(U_1) : f_{|U_1 \cap \mathbb{R}} \ge 0\} \tag{14}$$

which is obviously proper, reproducing and has non empty interior.

Theorem 7 The generalized P–F operator $\mathscr{L}_\beta^{(0)}$ is for real $\beta > \frac{1}{2}$ u_0–positive with respect to the cone K. Its leading eigenvalue $\lambda_1(\beta)$ fulfills a minimax principle

$$\min_{x \in \overline{U}_1 \cap \mathbb{R}} \max_{f \in \overset{\circ}{K}} \frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)} = \lambda_1(\beta) = \max_{x \in \overline{U}_1 \cap \mathbb{R}} \min_{f \in \overset{\circ}{K}} \frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)} . \tag{15}$$

Proof (idea): Take $u_0(x) = 1$, which is certainly in $\overset{\circ}{K}$. The bound $\mathscr{L}_\beta^{(0)} f \le \alpha(f)$ is trivial. To establish a lower bound $\beta(f) \le \mathscr{L}_\beta^{(0)p} f$ for some $p \ge 1$ for $f \in K \backslash \{0\}$, one assumes that for every $p \in \mathbb{N}$ there exists a point $x \in \overline{U}_1 \cap \mathbb{R}$ such that $\mathscr{L}_\beta^{(0)p} f(x) = 0$. Using the explicit form of $\mathscr{L}_\beta^{(0)}$ one then shows that this is possible only if $f \equiv 0$. To get the minimax principle one argues as follows: if $f \in \overset{\circ}{K}$ then also $\mathscr{L}_\beta^{(0)} f \in \overset{\circ}{K}$, hence the function $\frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)}$ takes its minimum and its maximum in $\overline{U}_1 \cap \mathbb{R}$. Then obviously

$$\min_{x \in \overline{U}_1 \cap \mathbb{R}} \left[\frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)} \right] f \le \mathscr{L}_\beta^{(0)} f \le \max_{x \in \overline{U}_1 \cap \mathbb{R}} \left[\frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)} \right] f .$$

This being true for all $f \in \overset{\circ}{K}$, we conclude

$$\sup_{f \in \overset{\circ}{K}} \min_x \left[\frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)} \right] \le \lambda_1(\beta) \le \inf_{f \in \overset{\circ}{K}} \max_x \frac{\mathscr{L}_\beta^{(0)} f(x)}{f(x)}$$

by referring to the bound for λ_1 in Theorem 6. Since by the same Theorem h_1 belongs to $\overset{\circ}{K}$ we get the minimax principle for $\lambda_1(\beta)$.

This minimax principle leads to simple rigorous bounds for the eigenvalue $\lambda_1(\beta)$:

$$\min_{x \in \overline{U}_1 \cap \mathbb{R}} \frac{\mathcal{L}_\beta^{(0)} f(x)}{f(x)} \leq \lambda_1(\beta) \leq \max_{x \in \overline{U}_1 \cap \mathbb{R}} \frac{\mathcal{L}_\beta^{(0)} f(x)}{f(x)}, \tag{16}$$

where f is arbitrary in $\overset{\circ}{K}$. For $f = 1$ this gives for instance

$$\zeta\left[\, 2\beta, \tfrac{7}{2} \,\right] \leq \lambda_1(\beta) \leq \zeta\left[\, 2\beta; \tfrac{1}{2} \,\right] \tag{17}$$

where $\zeta(z;q) = \sum\limits_{i=0}^{\infty} \frac{1}{(q+i)}z$ is the Hurwitz zeta function.

In the special case $\beta = 1$ we succeeded in [MaR2] applying the same technique also to the second highest eigenvalue $\lambda_2(\beta)$. To achieve this, one introduces the B–space $A_{1,\infty}(U_1) \subset A_\infty(U_1)$ of all f's in $A_\infty(U_1)$ which together with their first derivative $\frac{d}{dz} f(z)$ are continuous on $\overline{U}_1$. Since any eigenfunction of $\mathcal{L}_1^{(0)}$ different from h_1 must lie in the kernel of the projector $\mathcal{P}_{\beta=1}$ onto h_1 we can restrict our discussion to the following space

$$A_{1,\infty}^{\perp}(U_1) = \{f \in A_{1,\infty}(U_1) : \mathcal{P}_1 f = 0\}\,. \tag{18}$$

Since $\mathcal{L}_1^{(0)} \mathcal{P}_1 = \mathcal{P}_1 \mathcal{L}_1^{(0)}$ this space is invariant under $\mathcal{L}_1^{(0)}$, and obviously, $\mathcal{L}_1^{(0)}$ restricted to this space is indentical to the operator $\mathcal{N}_1$ of Theorem 5. To define then a cone left invariant by $\mathcal{N}_1$ we proceed as follows: any $f \in A_{1,\infty}^{\perp}$ can be written as

$f = h_1 \cdot \hat{f}$, where h_1 is the eigenfunction of $\mathscr{L}^{(0)}_{\beta=1}$ with leading eigenvalue $\lambda_1(1) = 1$, which obviously is nonvanishing on $\overline{U}_1$. Then define the cone C as follows

$$C = \{f \in A^{\perp}_{1,\infty}(U_1) : \hat{f}'(x) \geq 0 \text{ on } \overline{U}_1 \cap \mathbb{R}\} . \tag{19}$$

In [MaR2] we proved

<u>Theorem 8</u> The operator$-\mathscr{N}_1$ is u_0–positive in the Banach space $A^{\perp}_{1,\infty}(U_1)$ with respect to the cone C , where $u_0(z) = 1 - h_1(z) \in \overset{\circ}{C}$.

From this it follows that at least for $\beta = 1$ the eigenvalue $\lambda_2(\beta)$ is again simple and real, in fact negative. It can be determined from the minimax principle:

$$\max_{f \in \overset{\circ}{C}} \min_{x \in \overline{U}_1 \cap \mathbb{R}} \frac{(-V\hat{f})'(x)}{\hat{f}'(x)} = \lambda_2(1) = \min_{f \in \overset{\circ}{C}} \max_{x \in \overline{U}_1 \cap \mathbb{R}} \frac{(-V\hat{f})'(x)}{\hat{f}'(x)} \tag{20}$$

$$\text{where } V\hat{f}(z) = (z+1) \sum_{n=1}^{\infty} \frac{1}{z+n} \frac{1}{z+n+1} \hat{f}\left(\frac{1}{z+n}\right) . \tag{21}$$

<u>Open problems</u>:

1) Prove that all eigenvalues of the operator $\mathscr{L}^{(0)}_1$ are simple.

2) Do there exist invariant cones analogous to the cone C for $\mathscr{N}_\beta$ if $\beta \neq 1$?

2. Generalized transfer operators in Hardy spaces

In the foregoing sections we discussed spectral properties of generalized P–F operators in Banach spaces of boundary regular holomorphic functions. In this section we try to restrict the space further to functions holomorphic in entire half planes which however need not be so regular at the boundaries of such domains. For certain Hilbert spaces of such functions it turns out that the operators $\mathscr{L}_\beta^{(0)}$ are isomorphic to very simple integral operators $\mathscr{K}_\beta$ with kernel just the Bessel functions. To derive this we proceed as follows: From the explicit form of the operator $\mathscr{L}_\beta = \mathscr{L}_\beta^{(0)}$ in (6) we see that any eigenfunction of $\mathscr{L}_\beta$ in the space $A_\infty(U_1)$ must be holomorphic and bounded in every of the half planes

$$H_{-1+\delta} = \{z \in \mathbb{C} : \mathrm{Re}\, z > -1 + \delta\} \tag{22}$$

for $\delta > 0$. It is therefore quite natural to introduce an $\mathscr{L}_\beta$ invariant space $\mathscr{H}$ of such functions. This can be done via a generalized Laplace transform:

$$f(z) = \int_0^\infty dm(s) e^{-sz} \varphi(s) \tag{23}$$

where $dm(s)$ is some measure on $\mathbb{R}_+$ which will be determined shortly. The function φ should belong to some space of square integrable functions over $\mathbb{R}_+$ with respect to the measure μ. Since the space $\mathscr{H}$ we are looking for should be $\mathscr{L}_\beta$ invariant we apply $\mathscr{L}_\beta$ to f in (23) and find

$$\mathscr{L}_\beta f(z) = \sum_{n=1}^\infty \left[\frac{1}{z+n}\right]^{2\beta} \int_0^\infty dm(s) e^{-s \frac{1}{z+n}} \varphi(s) . \tag{24}$$

Obviously for $\operatorname{Re}\beta > \frac{1}{2}$ the sum $\sum_{n=1}^{\infty} \left[\frac{1}{z+n}\right]^{2\beta} e^{-s\frac{1}{z+n}}$ is uniformly convergent in $s \in \mathbb{R}_+$ and summation and integration can be interchanged. This sum however can be rewritten also as

$$\sum_{n=1}^{\infty} \left[\frac{1}{z+n}\right]^{2\beta} e^{-s\frac{1}{z+n}} = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \zeta(k+2\beta; z+1) \tag{25}$$

where $\zeta(z; q) = \sum_{n=0}^{\infty} \left[\frac{1}{q+n}\right]^{z}$ is the Hurwitz zeta function. For $\operatorname{Re} z > 1$ this function can be represented also as [Gr]

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1} e^{-qt}}{1-e^{-t}}\, dt \tag{26}$$

and hence relation (25) can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{z+n}\right]^{2\beta} e^{-\frac{s}{z+n}} = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \frac{1}{\Gamma(k+2\beta)} \int_0^{\infty} \frac{t^{k+2\beta-1} e^{-zt}}{e^t - 1}\, dt\,. \tag{27}$$

Inserting this into expression (24) we find

$$\mathscr{L}_\beta f(z) = \int_0^{\infty} dm(s)\varphi(s) \int_0^{\infty} dt\, \frac{t^{2\beta-1}}{e^t - 1}\, e^{-zt} \sum_{k=0}^{\infty} \frac{(-st)^k}{k!\,\Gamma(k+2\beta)}\,.$$

The sum in this expression can be performed explicitly to give [GR]:

$$\sum_{k=0}^{\infty} \frac{(-st)^k}{k!\,\Gamma(k+2\beta)} = \frac{\mathcal{J}_{2\beta-1}(2\sqrt{ts})}{\sqrt{ts}^{\,2\beta-1}} . \tag{28}$$

Inserting this finally gives

$$\mathcal{L}_\beta f(z) = \int_0^\infty dt\, \frac{t^{2\beta-1}}{e^t - 1}\, e^{-zt} \int_0^\infty dm(s)\, \frac{\mathcal{J}_{2\beta-1}(2\sqrt{t\cdot s})}{(ts)^{\beta-\frac{1}{2}}}\, \varphi(s) . \tag{29}$$

With $\varphi(s) = s^{\beta-\frac{1}{2}}\tilde{\varphi}(s)$ we find for

$$f(z) = \int_0^\infty dm(s) e^{-sz}\, s^{\beta-\frac{1}{2}}\tilde{\varphi}(s) .$$

$$\mathcal{L}_\beta f(z) = \int_0^\infty dt\, \frac{t^{\beta-\frac{1}{2}}}{e^t-1}\, e^{-zt} \int_0^\infty dm(s)\; \mathcal{J}_{2\beta-1}(2\sqrt{ts})\tilde{\varphi}(s) .$$

Chosing therefore the measure

$$dm(s) = \frac{ds}{e^s - 1} \tag{30}$$

we get

$$\mathcal{L}_\beta f(z) = \int_0^\infty dm(t) t^{\beta-\frac{1}{2}} (\mathcal{K}_\beta \tilde{\varphi})(t) \tag{31}$$

with

$$\mathcal{K}_\beta \tilde{\varphi}(t) = \int_0^\infty dm(s)\, \mathcal{J}_{2\beta-1}(2\sqrt{ts})\tilde{\varphi}(s) .$$

From this we conclude

Lemma 4: If $\mathcal{H}_\beta$ denotes the space of all functions f holomorphic in the half plane $H_{-\frac{1}{2}}$ and bounded in every half plane $H_{-\frac{1}{2}+\delta}$ for $\delta > 0$, which have a representation

$$f(z) = \int_0^\infty dm(s) s^{\beta-\frac{1}{2}} e^{-sz}\tilde{\varphi}(s) \tag{32}$$

with $\tilde{\varphi} \in \mathcal{L}_2(\mathbb{R}_+, dm)$, then $\mathcal{L}_\beta$ leaves this space invariant.

Proof: The operator $\mathcal{K}_\beta : \mathcal{L}_2(\mathbb{R}_+, dm) \longrightarrow \mathcal{L}_2(\mathbb{R}_+, dm)$ is bounded. Since its kernel $\mathcal{J}_{2\beta-1}(2\sqrt{st})$ is integrable with respect to dm the operator K_β is even trace class (as a Hilbert space operator). The space $\mathcal{H}_\beta$ can obviously be made a Hilbert space by introducing the scalar product

$$(f_1, f_2) = \int_0^\infty dm(s) \tilde{\varphi}_1^*(s) \tilde{\varphi}_2(s) \tag{33}$$

if $f_i(z) = \int_0^\infty dm(s) s^{\beta - \frac{1}{2}} e^{-sz} \tilde{\varphi}_i(s)$, $\tilde{\varphi}_i \in \mathscr{L}_2(\mathbb{R}_+, dm)$.

We can give a more direct description of this Hilbert space without using explicitly representation (32):

Theorem 9 For $\operatorname{Re}\beta \geq 1$ the space $\mathscr{H}_\beta$ is identical to the generalized Hardy space $\mathscr{H}^{(2)}_{\operatorname{Re}\beta}$ of functions f holomorphic in $H_{-\frac{1}{2}}$ belonging to the Hardy space $\mathscr{H}^{(2)}\left[H_{-\frac{1}{2}+\delta}\right]$ for any $\delta > 0$ such that

$$\int_0^\infty x^{2\operatorname{Re}\beta-2}\, dx \int_{-\infty}^{+\infty} dy\left(|f(x - \tfrac{1}{2} + iy)|^2 - |f(x + iy)|^2 \right) < \infty . \tag{34}$$

Ordinary Hardy space $\mathscr{H}^{(2)}(H_\alpha)$ over the half plane $\operatorname{Re} z > \alpha$ is defined as

$$\mathscr{H}^{(2)}(H_\alpha) = \{f : f \text{ holom. in } H_\alpha \text{, bounded in } H_{\alpha+\varepsilon} \text{ for all } \varepsilon > 0 \\ \text{and } \int_{-\infty}^{+\infty} dy\ |f(\alpha + iy)|^2 < \infty\} . \tag{35}$$

Proof: A simple calculation using essentially Plancherel's Theorem shows that for $f \in \mathscr{H}_\beta$ as in (32)

$$\frac{1}{\pi}\int_0^\infty x^{2\mathrm{Re}\,\beta-2}\,dx \int_{-\infty}^{+\infty} dy(|f(x-\tfrac{1}{2}+iy)|^2-|f(x+iy)|^2)=$$

$$=\frac{\Gamma(2\mathrm{Re}\,\beta-1)}{2^{2\mathrm{Re}\,\beta-2}}\cdot\int_0^\infty dm(s)\,|\tilde{\varphi}(s)|^2<\infty$$

holds. To show that any $f\in\mathscr{H}^{(2)}_{\mathrm{Re}\,\beta}$ has a representation as in (32) with $\tilde{\varphi}\in\mathscr{L}_2(\mathbb{R}_+,dm)$ is a straightforward generalization of the classical Paley–Wiener Theorem [D] for functions in $\mathscr{H}^{(2)}(H_\alpha)$. Details are given in [Ma2].

For $\beta\in\mathbb{C}$ with $\mathrm{Re}\,\beta>\frac{1}{2}$ one defines spaces $\mathscr{H}^{(2)}_{\mathrm{Re}\,\beta}$ as those holomorphic functions in $H_{-\frac{1}{2}}$ which vanish for $\mathrm{Re}\,z\longrightarrow\infty$ and have the property that $f'(z)$ belongs to $\mathscr{H}^{(2)}_{\mathrm{Re}\,\beta+1}$ as in Theorem 9. Obviously, the space $\mathscr{H}_\beta$ is again identical to the space $\mathscr{H}^{(2)}_{\mathrm{Re}\,\beta}$. The case $\beta=1$ has been discussed in [MaR1]. Using next arguments very similar to the ones used for $\beta=1$ in [MaR2] one proves

Theorem 10 The spectrum $\sigma(\mathscr{L}_\beta)$ of $\mathscr{L}_\beta: A_\infty(U_1)\longrightarrow A_\infty(U_1)$ and the spectrum $\tilde{\sigma}(\mathscr{L}_\beta)$ of $\mathscr{L}_\beta:\mathscr{H}_\beta\longrightarrow\mathscr{H}_\beta$ are identical and equal to $\sigma(\mathscr{K}_\beta)$ of the integral operator $\mathscr{K}_\beta$ with kernel $\mathscr{J}_{2\beta-1}(2\sqrt{st})$ in $\mathscr{L}_2(\mathbb{R}_+,dm)$.

An immediate consequence of this is

Corollary 4 For real β with $\beta>\frac{1}{2}$, all eigenvalues of $\mathscr{L}_\beta: A_\infty(U_1)\longrightarrow A_\infty(U_1)$ are real.

Conjecture: For general real β, $\beta>\frac{1}{2}$, the eigenvalues are simple.

3. Zetafunctions for the Gauss map

Since $Tx = \frac{1}{x} \bmod 1$ is an analytic expanding Markov map as introduced and discussed in ch. II, we can immediately apply the general results about zeta functions for such maps as described in sect. 2 of this ch. II. We only have to take care of the fact that for this map the set $\mathrm{Fix}\, T^n$ of all periodic points of period n has infinite elements for all $n \in \mathbb{N}$: Obviously, $x \in \mathrm{Fix}\, T^n$ iff the continued fraction expansion of x is periodic of period n, that means if $x = [k_1, k_2, \dots]$ then $x \in \mathrm{Fix}\, T^n$ iff $k_{i+n} = k_i$ for all $i \in \mathbb{N}$. Such x's we write as

$$x = [\,\overline{k_1, \dots, k_n}\,]\,. \tag{36}$$

Consequently, the partition functions

$$Z_n(A) = \sum_{x \in \mathrm{Fix}\, T^n} \exp \sum_{k=0}^{n-1} A(T^k x)$$

are not well defined for general A. Inserting the set $\mathrm{Fix}\, T^n = \{x = [\,\overline{k_1, \dots, k_n}\,], k_i \in \mathbb{N}\}$ we find

$$Z_n(A) = \sum_{i_1, \dots, i_n \in \mathbb{N}} \prod_{k=0}^{n-1} \exp A([\,\overline{i_{1+k}, \dots, i_n, i_1, \dots, i_{1+k-1}}\,])\,,$$

or if we introduce the function $\varphi(x) = \exp A(x)$

$$Z_n(\varphi) = \sum_{i_1,\ldots,i_n \in \mathbb{N}} \prod_{k=0}^{n-1} \varphi([\overline{i_{1+k},\ldots,i_n,i_1,\ldots,i_{1+k-1}}]) \tag{37}$$

To ensure convergence of these sums it is certainly sufficient the function $|\varphi(x)|$ to behave like $|x|^{\delta}$ for $x \longrightarrow 0$ for some $\delta > 1$. To be able to apply again the techniques of analytic function theory, obviously, the function φ has to have some mild analyticity properties: the functions $\varphi \circ \psi_i(z) = \varphi\left[\frac{1}{z+n}\right]$ must be holomorphic in the disc U_1. Obviously, the function $\varphi(x) = \exp - \beta \log|T'(x)| = x^{2\beta}$ has this property, since the functions $\left[\frac{1}{z+i}\right]^{2\beta}$ are in $A_\infty(U_1)$ for all $i \in \mathbb{N}$ and furthermore $|\varphi(x)| \sim |x|^{2\beta}$ for $x \longrightarrow 0$, so that convergence in (37) is guaranteed for $\mathrm{Re}\,\beta > \frac{1}{2}$. Summarizing this discussion we hence get for the Gauss map T:

<u>Theorem 11</u> [Ma5] If $\varphi : I \longrightarrow \mathbb{C}$ is such that $\varphi \circ \psi_i \in A_\infty(U_1)$ for all $i \in \mathbb{N}$ and $|\varphi(x)| \sim |x|^{\delta}$ as $x \longrightarrow 0$ for some $\delta > 1$, then the zeta function

$$\zeta(z,\varphi) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\varphi)$$

extends to a meromorphic function in the entire z–plane. This extension is given by

$$\zeta(z,\varphi) = \frac{\det\left[1 - z\mathscr{L}_\varphi^{(1)}\right]}{\det\left[1 - z\mathscr{L}_\varphi^{(0)}\right]},$$

where the nuclear operators $\mathscr{L}_\varphi^{(s)}$, $s = 0,1$ are defined as

$$\mathscr{L}_{\varphi}^{(s)}f(z) = \sum_{i=1}^{\infty} \varphi\left[\frac{1}{z+i}\right](-1)^{s} f\left[\frac{1}{z+i}\right]. \tag{38}$$

As mentioned already several times, of special interest is the case $\varphi(x) = \exp - \beta \log|T'(x)| = x^{2\beta}$. Then the above operators (38) are just the generalized P–F operators of section 1 of the present chapter:

$$\mathscr{L}_{\beta}^{(s)}f(z) = \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta+2s}(-1)^{s} f\left[\frac{1}{z+i}\right]. \tag{39}$$

For $\operatorname{Re}\beta > \frac{1}{2}$ these are nuclear operators of order zero and depend analytically on β . Hence we can apply Theorem 4 in section II. 2 to find

<u>Theorem 12</u> The function $\zeta(1,\beta) = \exp \sum_{m=1}^{\infty} \frac{1}{m} Z_m(\beta)$ with $Z_n(\beta) = \sum_{x\in \operatorname{Fix} T^n} \prod_{k=0}^{n-1} (T^k x)^{2\beta}$ extends to a meromorphic function in the complex β–half plane $\operatorname{Re}\beta > \frac{1}{2}$ through the formula

$$\zeta(1,\beta) = \frac{\det\left[1 - \mathscr{L}_{\beta}^{(1)}\right]}{\det\left[1 - \mathscr{L}_{\beta}^{(0)}\right]},$$

whose poles are among the β–values , for which $\mathscr{L}_{\beta}^{(0)}$ has eigenvalue $\lambda = 1$.

We will show next that the function $\zeta(1,\beta)$ is meromorphic even in the entire β–plane . To achieve this we have to find meromorphic continuations of the functions

$\det(1-\mathscr{L}_\beta^{(s)})$. Since the arguments for $s = 0$ and $s = 1$ are identical, we restrict the discussion to the case $s = 0$. For simplicity we write again $\mathscr{L}_\beta$ for $\mathscr{L}_\beta^{(0)}$, which is the operator

$$\mathscr{L}_\beta f(z) = \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} f\left[\frac{1}{z+i}\right]. \tag{40}$$

The idea is to extend this operator to the whole β–plane. This can be done step by step as follows: we write $\mathscr{L}_\beta$ in a slightly different way as

$$\mathscr{L}_\beta f(z) = \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} \left[f(0) + \left[f\left[\frac{1}{z+i}\right] - f(0)\right]\right]. \tag{41}$$

For $\operatorname{Re}\beta > \frac{1}{2}$ this gives

$$\mathscr{L}_\beta f(z) = f(0)\zeta(2\beta;z+1) + \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} \left[f\left[\frac{1}{z+i}\right] - f(0)\right]. \tag{42}$$

That means, $\mathscr{L}_\beta$ is the sum of the finite rank operator

$$\hat{\mathscr{L}}_\beta f(z) = f(0)\zeta(2\beta;z+1) \tag{43}$$

which is trivially nuclear of order zero in $A_\infty(U_1)$, and another nuclear operator

$$\tilde{\mathscr{L}}_\beta f(z) = \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} \left[f\left[\frac{1}{z+i}\right] - f(0)\right]. \tag{44}$$

The zeta function $\zeta(2\beta\,;\,z+1)$ is for any $z \in U_1$ a meromorphic function in the entire β–plane with the only pole at $\beta = \frac{1}{2}$, which is simple and has residue $\frac{1}{2}$. From this we conclude that the operator $\hat{\mathscr{L}}_\beta$ in (43) is a nuclear operator meromorphic in the entire β–plane with the property:

$$\lim_{\beta \to \frac{1}{2}} \left[\hat{\mathscr{L}}_\beta f(z) - \frac{1}{2\beta - 1} f(0) \right] = -\psi(z+1) f(0)\,, \tag{45}$$

where ψ denotes the function $\psi(x) = \frac{d}{dx} \log \Gamma(x)$. The operator $\tilde{\mathscr{L}}_\beta$ in (44) on the other hand is nuclear of order zero in the domain $\operatorname{Re}\beta > 0$. This comes from the fact that

$$\left| f\left[\frac{1}{z+i} \right] - f(0) \right| \le c \frac{1}{|z+i|} \quad \text{for all } i \ge M \text{ and } M \text{ large enough.}$$

The foregoing discussion shows that the operator $\mathscr{L}_\beta$ in (42) defines an analytic continuation of the operator $\mathscr{L}_\beta$ in (40) which is nuclear of order zero in the domain $\operatorname{Re}\beta > 0$ with a simple pole at the point $\beta = \frac{1}{2}$, determined by equation (45). Quite generally we can continue the operator $\mathscr{L}_\beta$ in (40) meromorphically into the whole β–plane as follows: for any $N \in \mathbb{N}$ we decompose $\mathscr{L}_\beta$ into two pieces

$$\begin{aligned} \mathscr{L}_\beta f(z) = \sum_{k=0}^{N} \sum_{i=1}^{\infty} \left[\frac{1}{z+i} \right]^{2\beta} \frac{f^{(k)}(0)}{k!} \left[\frac{1}{z+i} \right]^{k} + \\ + \sum_{i=1}^{\infty} \left[\frac{1}{z+i} \right]^{2\beta} f_N \left[\frac{1}{z+i} \right], \end{aligned} \tag{46}$$

where $f_N(z)$ denotes the rest term in Taylor's expansion of f around the point $z = 0$:

$$f_N(z) = f(z) - \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} z^k . \tag{47}$$

Since the Taylor expansion for $f \in A_\infty(U_1)$ is uniformly convergent for instance for all z with $|z| \le \frac{1}{4}$ we find that for $|z| \le \frac{1}{4}$

$$|f_N(z)| \le C|z|^{N+1} . \tag{48}$$

Expression (46) can be simplified to give

$$\mathscr{L}_\beta f(z) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} \zeta(2\beta + k; z + 1) + \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} f_N \left[\frac{1}{z+i}\right] . \tag{49}$$

The first term in (40) defines a finite rank operator $\hat{\mathscr{L}}_{\beta,N}$

$$\hat{\mathscr{L}}_{\beta,N} f(z) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} \zeta(2\beta + k; z + 1) , \tag{50}$$

meromorphic in the entire β–plane with simple poles at the points $2\beta + k = 1$. The behaviour of the operator $\hat{\mathscr{L}}_{\beta,N}$ for $\beta \longrightarrow \frac{1-k}{2}$ $0 \le k \le n$ is the following:

$$\lim_{\beta \to \frac{1-k}{2}} \left[\hat{\mathscr{L}}_{\beta,N} f(z) - \frac{1}{2\beta + k - 1} \frac{f^{(k)}(0)}{k!} \right] = - \psi(z+1) \frac{f^{(k)}(0)}{k!} + \tag{51}$$

$$+ \sum_{\ell=0,\, \ell \ne k}^{N} \frac{f^{(\ell)}(0)}{\ell!} \zeta(1 + \ell - k; z + 1) .$$

This shows that the residue of $\hat{\mathscr{L}}_{\beta,N}$ at the point $\beta \longrightarrow \frac{1-k}{2}$ is the following operator of rank 1 in the space $A_\infty(U_1)$

$$\mathscr{N}_k\, f(z) = \frac{1}{2}\, \frac{f^{(k)}(0)}{k!} .$$

Obviously this operator is nilpotent for all $k \geq 1$ that means $\mathscr{N}_k^2 \equiv 0$ for $k \geq 1$.

The second term in the representation (46) on the other hand defines a nuclear operator $\tilde{\mathscr{L}}_{\beta,N} : A_\infty(U_1) \longrightarrow A_\infty(U_1)$ with

$$\tilde{\mathscr{L}}_{\beta,N}\, f(z) = \sum_{i=1}^{\infty} \left[\frac{1}{z+i}\right]^{2\beta} f_N\left[\frac{1}{z+i}\right] \tag{52}$$

with f_N defined in (47).

Because of (48) the operator $\tilde{\mathscr{L}}_{\beta,N}$ is nuclear of order zero and holomorphic in β in the half plane $\mathrm{Re}\beta > -\frac{N}{2}$. Since the above arguments hold for any $N \in \mathbb{N}$, we have shown

<u>Theorem 13</u> The operators $\mathscr{L}_\beta^{(s)}$ in (39) have meromorphic continuations as nuclear operators of order zero into the entire complex β–plane with simple poles at the points $\beta = \frac{1-k}{2}$, $k = 2s, 2s+1, \ldots$, and residue the rank 1 operator $\mathscr{N}_k\, f = \frac{1}{2}\frac{1}{k!} f^{(k)}(0)$. The Fredholm determinants $\det(1 - \mathscr{L}_\beta^{(s)})$ are meromorphic in the entire β–plane with simple poles at the points $\beta = \frac{1-k}{2}$, $k = 2s, 2s+1, 2s+2, \ldots$.

Let us apply this result to the following function ζ_T which is a little modification of Riemann's zeta function:

$$\zeta_T(\beta) := \sum_{x \in \mathrm{Fix}\, T} x^{2\beta} = \sum_{i \in \mathbb{N}} [\,\bar{i}\,]^{2\beta}, \tag{53}$$

with $[\,\bar{i}\,] = [i,i,i, \ldots]$ the irrational number $x > 0$, which obeys the equation $x^2 + ix = 1$. Since

$$\zeta_T(\beta) = \operatorname{trace} \mathscr{L}_\beta^{(0)} - \operatorname{trace} \mathscr{L}_\beta^{(1)} \tag{54}$$

the analyticity properties of $\zeta_T(\beta)$ are determined by the analyticity properties of the two traces in (54).

<u>Lemma 5</u> The trace of the operator $\mathscr{L}_\beta^{(s)}$ is meromorphic in the entire β plane with a simple pole at the point $\beta = \frac{1}{2} - s$.

<u>Proof</u>: Since $\mathscr{L}_\beta^{(s)} = (-1)^s(\hat{\mathscr{L}}_{\beta+s,N} + \tilde{\mathscr{L}}_{\beta+s,N})$, where the operators on the right have been defined in (50) resp. (52), we find

$$\operatorname{trace} \mathscr{L}_\beta^{(s)} = (-1)^s(\operatorname{trace} \hat{\mathscr{L}}_{\beta+s,N} + \operatorname{trace} \tilde{\mathscr{L}}_{\beta+s,N}). \tag{55}$$

The trace of the operator $\hat{\mathscr{L}}_{\beta+s,N}$ can be determined explicitly, since this operator is a finite sum of rank 1 operators:

$$\operatorname{trace} \hat{\mathscr{L}}_{\beta+s,N} = \sum_{k=0}^{N} \frac{1}{k!} \frac{d^k}{dz^k} \zeta(2\beta + 2s; z + 1)\Big|_{z=0}. \tag{56}$$

Using the formula [GR]:

$$\frac{d}{dz}\ \zeta(q,z) = -q\zeta(q+1;z)\ , \tag{57}$$

we find:

$$\operatorname{trace} \hat{\mathscr{L}}_{\beta+s,N} = \sum_{k=0}^{N} (-1)^k \frac{1}{k!} (2\beta + 2s) \dots (2\beta + 2s + k - 1)\zeta(2\beta + 2s + k)\ , \tag{58}$$

with $\zeta(x)$ ordinary Riemann's zeta function. Since this function has only one pole, namely at the point $x = 1$, which is simple, we conclude from (58): trace $\hat{\mathscr{L}}_{\beta+s,N}$ is meromorphic in the entire β–plane with simple pole at the point $2\beta + 2s = 1$ with residue $\frac{1}{2}$. On the other hand we know, that the operator $\tilde{\mathscr{L}}_{\beta+s,N}$ is holomorphic in β in the half plane $\operatorname{Re}\beta > -\frac{N}{2} - s$ and hence trace $\tilde{\mathscr{L}}_{\beta+s,N}$ is holomorphic in this region too. Since representation (55) holds for any $N \in \mathbb{N}$ Lemma 5 is proved.

From this we then get

<u>Theorem 14</u> The function $\zeta_T(\beta) = \sum_{i \in \mathbb{N}} [\,\bar{i}\,]^{\beta}$ has a meromorphic continuation into the entire β–plane with simple poles at the points $\beta = \pm 1$ with residues 1.

<u>Problem</u> Does there hold something like Riemann's conjecture about the position of the zero's of the function ζ_T ? Does the function ζ_T fulfill some functional equation? Since the function $i \longrightarrow [\,\bar{i}\,]$ is not multiplicative it is not obvious why this should be true.

Theorem 13 allows also an improvement of Theorem 12:

Theorem 15 The function $\zeta(1,\beta)$ in Theorem 12 has a meromorphic extension to the entire β–plane with trivial zero's at the points $\beta = 0$ resp. $\beta = \frac{1}{2}$. The nontrivial poles respectively zero's of $\zeta(1,\beta)$ are among the points β such that $\mathscr{L}_\beta^{(0)}$ respectively $\mathscr{L}_\beta^{(1)}$ have $\lambda = 1$ among their eigenvalues.

Remark: We know already that $\mathscr{L}_\beta^{(0)}$ has $\lambda = 1$ as an eigenvalue for $\beta = 1$: the eigenfunction is just the density of the Gauss measure. Our discussion above also shows that the operator $\mathscr{L}_1^{(1)}$ has a spectral radius strictly smaller than 1. Hence $\beta = 1$ is a simple pole of the function $\zeta(1,\beta)$. A similar argument shows that $\zeta(1,\beta)$ has no other pole on the real axis for $\beta > 1$ nor any zero for $\beta \geq 0$. Of special interest are the poles of $\zeta(1,\beta)$ on the line $\operatorname{Re}\beta = \frac{1}{2}$: one expects a close relation between these numbers and the eigenvalues of the hyperbolic Laplacian Δ_Γ on the modular surface M_Γ , $\Gamma = PSL(2,\mathbb{Z})$. This is related to recent work of Pollicott on Selberg's theory for compact hyperbolic surfaces via transfer operators. The function $\zeta(1,\beta)$, where the Gauss map is replaced by the Bowen–Series boundary map [S] for the corresponding compact surface, is then closely related to Selberg's zeta function for this surface. Its poles on the line $\operatorname{Re}\beta = \frac{1}{2}$ hence determine the spectrum of Δ_Γ completely. This way it is, at least in principle, possible to determine these eigenvalues through the spectrum of the corresponding transfer operators. Since the Bowen–Series maps belong to the class of analytic expanding Markov maps the methods developped above should be of some help. Since for the modular surface M_Γ the Gauss map is more or less the Series–Bowen map one should expect $\zeta(1,\beta)$ and its poles on the line $\beta = \frac{1}{2} + i\,s$ to be closely related to the spectrum of $-\Delta_\Gamma$. Since M_Γ for $\Gamma = PSL(2,\mathbb{Z})$ is not compact its Laplacian has continuous spectrum, into which there are embedded infinitely many eigenvalues. Not much seems to be known about these numbers. It would be nice if more could be learned about them through our transfer operators $\mathscr{L}_\beta^{(s)}$ by using the theory developped above.

Appendix

Grothendieck's Theory of Nuclear Operators

This theory generalizes nuclear resp. trace class operators to arbitrary Banach spaces [G2].

If B is an arbitrary B–space and B^* its dual, that means the space of bounded functionals on B, the projective topological tensor product $B^* \hat{\otimes}_\pi B$ is the completion of the ordinary tensor product $B^* \otimes B$ under the norm

$$||X||_\pi = \inf \sum_{\{i\}} ||e_i^*|| \; ||e_i|| \tag{A1}$$

where the infimum is taken over all finite representations $X = \sum_{\{i\}} e_i^* \otimes e_i \in B^* \otimes B$.

The elements $X \in B^* \hat{\otimes}_\pi B$ are called Fredholm–kernels and any such X has a representation

$$X = \sum_{\{i\}} \lambda_i e_i^* \otimes e_i \tag{A2}$$

with $e_i \in B$, $e_i^* \in B^*$ such that $||e_i|| = ||e_i^*|| = 1$ and $\{\lambda_i\} \in \ell_1$, that means $\sum_{\{i\}} |\lambda_i| < \infty$.

Every such X defines in a canonical way a linear operator $\mathscr{L}_X : B \longrightarrow B$ through

$$\mathscr{L}_X f = \sum_{\{i\}} \lambda_i e_i^*(f) e_i . \tag{A3}$$

On the space $B^* \hat{\otimes}_\pi B$ of Fredholm kernels there exists a canonical linear functional, the trace:

$$\text{trace } X = \sum_{\{i\}} \lambda_i e_i^*(e_i) . \tag{A4}$$

One can define nuclear operators in an arbitrary Banach space B as follows:

Definition 1 A linear bounded operator $\mathscr{L} : B \longrightarrow B$ is nuclear, if there exists a Fredholm kernel $X \in B^* \hat{\otimes}_\pi B$ with $\mathscr{L} = \mathscr{L}_X$.

An interesting class of nuclear operators are the p–summable ones. To define these we need

Definition 2 A Fredholm kernel $X \in B^* \hat{\otimes}_\pi B$ is called p–summable $(0 < p \leq 1)$ if X has a representation $X = \sum_{\{i\}} \lambda_i e_i^* \otimes e_i$ with $\{\lambda_i\} \in \ell_p$, that means $\sum_{\{i\}} |\lambda_i|^p < \infty$.

A nuclear operator $\mathscr{L}$ is p–summable, if there exists a p–summable $X \in B^* \hat{\otimes}_\pi B$ with

$$\mathscr{L} = \mathscr{L}_X .$$

The order of a Fredholm kernel X is the infimum q of all $0 < p \leq 1$ such that X is p–summable . Since a nuclear operator $\mathscr{L}$ can have more than one Fredholm kernel with $\mathscr{L} = \mathscr{L}_X$ the trace of $\mathscr{L}$ cannot be defined in general. For nuclear operators of order $\leq \frac{2}{3}$ Grothendieck proved however

Theorem A1 If $\mathscr{L}$ is nuclear of order $\leq \frac{2}{3}$ then $\mathscr{L}$ has a trace with trace $\mathscr{L} = \sum_{\{i\}} \rho_i$, where ρ_i are the eigenvalues of $\mathscr{L}$ counted according to their algebraic multiplicities. The Fredholm determinant $\det(1 - z\mathscr{L})$ is an entire function of z given by the formula $\det(1 - z\mathscr{L}) = \prod_i (1 - \rho_i z)$. For this Fredholm determinant the formula

$$\det(1 - z\mathscr{L}) = \exp \operatorname{trace} \log(1 - z\mathscr{L})$$

is true. If $\mathscr{L} = \mathscr{L}(\beta)$ and the dependence on β is holomorphic for β in some domain D then $\det(1 - \mathscr{L}(\beta))$ is holomorphic in D .

For special Banach spaces, for instance those of holomorphic functions over domains in $\mathbb{C}^n$, every nuclear operator $\mathscr{L}$ is of order zero and hence of trace class [G2].

The notion of nuclear operator can be generalized to Frechet spaces, complete metric topological spaces. Among them there is a class of spaces, so called nuclear spaces, which have the nice property that every bounded map of such a space $\mathscr{F}$ into an arbitrary Banach space B is nuclear.

A typical example of such a nuclear space is the space $\mathscr{H}(D)$ of all holomorphic functions over some domain D in $\mathbb{C}^n$ whose topology is defined by the seminorms $|| \; ||_K$, K compact in D :

$$||f||_K = \sup_{z \in K} |f(z)| .$$

By using nuclearity of the space $\mathscr{H}(D)$ one proves in an elegant way nuclearity of the composition operator $C_\psi f = f \circ \psi$ if ψ maps D strictly inside itself: consider namely the operator $C_\psi : \mathscr{H}(D) \longrightarrow A_\infty(D)$. One shows that under the above condition C_ψ is bounded and hence nuclear: we have only to find a neighbourhood of zero in $\mathscr{H}(D)$ which is mapped into a bounded set in $A_\infty(D)$. For this define

$$U_M(0) = \{f \in \mathscr{H}(D) : \sup_{z \in K} |f(z)| < M\}$$

where a compactum K is chosen such that

$$\psi(\overline{D}) \subseteq K \subset D .$$

But then we find for all $f \in U_M(0)$

$$||C_\psi f|| = \sup_{z \in \overline{D}} |f \circ \psi(z)| \le \sup_{z \in K} |f(z)| < M$$

and hence $C_\psi U_M(0)$ is bounded in $A_\infty(D)$. Composing C_ψ with the bounded injection

$$i : A_\infty(D) \longrightarrow \mathscr{H}(D) \qquad i(f) = f$$

we find $C_\psi \circ i : A_\infty(D) \longrightarrow A_\infty(D)$ is nuclear. More details about nuclear spaces and nuclear operators on Frechet spaces one finds in [G2].

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