On the enumeration of plane curves with two singular points

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Abstract

We study equisingular strata of curves with two singular points of prescribed types. The method of our previous work is generalized to this case. This allows to solve the enumerative problem for the class of linear singularities. In the general case this reduces the enumerative questions to the problem of collision of singular points.

The method is applied to several cases, explicit numerical results are given.

1 Introduction

1.1 The setup and the problem

We work with (complex) algebraic curves in $\mathbb{P}^2$. Identify the complete linear system $|dL| = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ (the parameter space of plane curves of degree d) with the projective space $\mathbb{P}^{N_d}_f$. Here $N_d = \binom{d+2}{2} - 1$, the subscript f corresponds to the defining equation of the curves, $f(x) = 0$.

A classical enumerative problem is: *given the singularity types $\mathbb{S}_1..\mathbb{S}_r$, "how many" curves of the system $|dL|$ possess singular points of these types?* (To make this number finite one imposes a sufficient number of generic base points. The degree d is assumed big enough to avoid various pathologies.)

In this paper we consider the case of two prescribed singular points. First recall the modern formulation of the problem. The parameter space $\mathbb{P}^{N_d}_f$ is stratified according to the singularity types of curves. The generic point of $\mathbb{P}^{N_d}_f$ corresponds to a smooth curve. The set of points corresponding to the singular curves is called *the discriminant* (Σ). It is a (projective) hypersurface in $\mathbb{P}^{N_d}_f$.

Definition 1.1 *For the given (embedded, topological, isolated) singularity types $\mathbb{S}_1..\mathbb{S}_r$, the equisingular stratum $\Sigma_{\mathbb{S}_1..\mathbb{S}_r} \subset \mathbb{P}^{N_d}_f$ is the set of points corresponding to the curves with the given singularities.*

The generic point of the discriminant lies in the stratum of nodal curves ($\Sigma = \overline{\Sigma}_{A_1}$). Other strata correspond to higher singularities. The stratum of r-nodal curves Σ_{rA_1} (whose closure $\overline{\Sigma}_{rA_1}$ contains the stratum of curves of a given genus) is the classical Severi variety. Other strata are Σ_{A_k}, Σ_{D_k}, Σ_{E_k} etc.. (For a comprehensive introduction to these equi-singular families and related notions cf. [GLSbook]).

It is well known that for sufficiently high degree d (given the singularity types $\mathbb{S}_1..\mathbb{S}_r$) the strata are non-empty, pure dimensional, of expected (co-)dimension, irreducible, smooth (quasi-projective) varieties. One sufficient condition for this is [Dimca, §I.3]: $d \geq \sum o.d.(\mathbb{S}_i) + r - 1$, here $o.d.$ are the orders of determinacy.

By construction each stratum is embedded into $\mathbb{P}^{N_d}_f$, thus a natural compactification is just the topological closure. The closures of the strata are singular in co-dimension 1. The closed stratum has the homology class $[\overline{\Sigma}_{\mathbb{S}_1..\mathbb{S}_r}] \in H_*(\mathbb{P}^{N_d}_f, \mathbb{Z})$ (in the homology of the corresponding dimension). By Poincare duality we get the cohomology class (denoted by the same letter): $[\overline{\Sigma}_{\mathbb{S}_1..\mathbb{S}_r}] \in H^*(\mathbb{P}^{N_d}_f, \mathbb{Z}) \approx \mathbb{Z}$. The degree of this class is the degree of the stratum and is *"the number" of curves in $|dL|$ possessing the prescribed singularities.*

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(The degree is obtained by the intersection with the generic plane of the complimentary dimension. This corresponds to imposing generic base points.)

In [Ker06] we proposed a method to calculate the degrees of strata $\overline{\Sigma}_{\mathbb{S}}$ of uni-singular curves. The goal of this paper is to generalize it to the enumeration of curves with two singular points (i.e. computation of the class $[\overline{\Sigma}_{\mathbb{S}_1\mathbb{S}_2}]$).

1.2 The results known and new

Since the question is completely classical, there are lots of enumerative results. We mention only a few (for a much better discussion cf. [Klei76] and [Kaz01]).

• $deg\overline{\Sigma}_{A_1}$ was known in 19'th century. $deg\overline{\Sigma}_{A_2}$ was conjectured by Enriques, proved in [Vain81] and recalculated in [Fr.Itz95] and [Al98] (there $deg\overline{\Sigma}_{A_3}$ was also obtained).

• The degrees of $\overline{\Sigma}_{rA_1}$ are (considered to be) especially important, being the generalizations of Gromov-Witten invariants: enumeration of fixed genus curves. This case was continuously attacked, with methods and results in [KlPi98, KlPi04], [Ran89, Ran02],[CapHar98-1, CapHar98-2] and [Vain03]. And of course, quantum cohomology, [KontMan94]and all that. The later approach is effective for low genus curves i.e. when the number of nodes (or higher singularities) is big. It seems to be very difficult for high genus computations (e.g. just a few singular points).

• The real breakthrough has been recently achieved in [Kaz01]-[Kaz03-2]. The proposed topological method allows (in principle) to compute the degree of *any* stratum (with lots of explicit results in [Kaz03-hab]). The drawback of his method is its generality: it solves the problem *simultaneously* for all the singularity sets of a given co-dimension. So, first one should classify the singularities (by now the classification seems to exist up to codimension 16 only). Even if this is done, one faces the problem of enumerating huge amount of cases (the number of types grows exponentially with the codimension). And of course, each computation can give a result for a specific choice of singularity types, it is not clear whether the method allows to obtain results for some *series* of singularities.

• In [Ker06] the problem was solved for curves with one singular point of arbitrary given singularity type. The proposed method gives immediate answer for some specific (series of) types (the so-called *linear*). For all other (series of) types it gives an explicit algorithm, which works well (and can be programmed if needed).

In this paper we consider the case of two singular points: $\overline{\Sigma}_{\mathbb{S}_1\mathbb{S}_2}$. We restrict mostly to the case of linear singularity types (cf. definition 2.4). The simplest examples of linear singularity types are $A_{k\leq 3}$, $D_{k\leq 6}$, $E_{k\leq 8}$, X_9, J_{10}, $Z_{k\leq 13}$ etc.

Lift the stratum to a bigger space: $\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2} \subset \mathbb{P}^{N_d}_f \times Aux$. (Here Aux is an auxiliary smooth projective variety.) Note that the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}$ is a subvariety of $\overline{\overline{\Sigma}}_{\mathbb{S}_1} \times Aux_{\mathbb{S}_2}$. Correspondingly, it is enough to calculate its class: $[\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}] \in H^*(\overline{\overline{\Sigma}}_{\mathbb{S}_1} \times Aux_{\mathbb{S}_2}, \mathbb{Z})$. The resulting class in $H^*(\mathbb{P}^{N_d}_f \times Aux, \mathbb{Z})$ is then obtained by pushforward.

Proposition 1.2 *Let $\mathbb{S}_1\mathbb{S}_2$ be linear types. Then the (appropriately lifted) stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}$ is a locally complete intersection in the space $\overline{\overline{\Sigma}}_{\mathbb{S}_1} \times Aux_{\mathbb{S}_2}$ blown up along some (smooth) loci. In particular, the cohomology class $[\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}] \in H^*(Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_1} \times Aux_{\mathbb{S}_2}))$ is the just the product (of classes) of the strict transforms of defining hypersurfaces: $\prod[\widetilde{V_i}]$.*

This method is applied to the case of two ordinary multiple points (§1.5) and to the case $\mathbb{S}, A_1$ with $\mathbb{S}$ linear (§3.2.3). Some explicit numerical results are given in Appendix.

The case of two ordinary multiple pointis especially important as it can be used as the starting point to solve many other cases by an indirect method: *the chain of degenerations.*

Proposition 1.3 *For all (in particular non-linear) singularity types the problem is reduced to the collision of singular points.*

Here is the idea of such a reduction. Given a pair of types $\mathbb{S}_1\mathbb{S}_2$ we want to degenerate them to the linear types $\mathbb{S}_1'\mathbb{S}_2'$ (or just to the ordinary multiple points). The corresponding procedure in the case of one singularity was described in [Ker06, §4]. Thus, naively we would get:

$$\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2} \cap \{\text{degeneration}\} = \overline{\overline{\Sigma}}_{\mathbb{S}_1'\mathbb{S}_2'}, \qquad [\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}][\text{degeneration}] = [\overline{\overline{\Sigma}}_{\mathbb{S}_1'\mathbb{S}_2'}] \in H^*(\mathbb{P}_f^{N_d} \times Aux) \tag{1}$$

As was observed in [Ker06, §2.2] the last equation in the cohomology ring has *unique* solution (i.e. the element [degeneration] $\in H^*(\mathbb{P}_f^{N_d} \times Aux)$ is "invertible"). Therefore, the needed class is restored uniquely in terms of the degeneration and the known class $[\overline{\overline{\Sigma}}_{\mathbb{S}_1'\mathbb{S}_2'}]$ of the stratum of linear types.

In reality the situation is more complicated, the equation above *always* include the residual contribution from the diagonal $\Delta = \{x = y\}$.

$$\overline{\overline{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2} \cap \{\text{degeneration}\} = \overline{\overline{\Sigma}}_{\mathbb{S}_1'\mathbb{S}_2'} \cup R_{x=y} \tag{2}$$

To advance, one should understand the geometry of this residual term. And this amounts to understanding the result of collision $\mathbb{S}_1 + \mathbb{S}_2 \to ?$. The collision problem seems to be complicated [Ker07-2]. In particular, by now we now have a method to classify the results of collisions in the case of linear singularities only.

In any case, for linear singularities this proposition gives an alternative method of computation. Some examples are given in §3.3.

1.3 Content

In §2 we fix the notations and remind some notions from singularities. In particular we define the linear singularity type (definition 2.4). In §3 we prove the proposition 1.2 and give some examples of reduction of the problem to the collision problem.

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1.5 An example: two ordinary multiple points

Let $\mathbb{S}_x = x_1^{p+1} + x_2^{p+1}$, $\mathbb{S}_y = y_1^{q+1} + y_2^{q+1}$. The ordinary multiple point is the simplest type, the first lifting (tracing the singular point) is already a *globally* complete intersection:

$$\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x) = \{(x, f)|\ f|_x^{(p)} = 0\} \subset \mathbb{P}_x^2 \times \mathbb{P}_f^{N_d} \tag{3}$$

Here $f|_x^{(p)}$ is the tensor of derivatives of order p in homogeneous coordinates, calculated at the point x. (This precisely encodes the vanishing of all the derivatives up to order p in local coordinates).

The natural candidate for the lifting of $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$ is the variety of triples (x, y, f) with f having $\mathbb{S}_y$ at y and $\mathbb{S}_x$ at x. For calculational reasons we blowup the ambient parameter space $\mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_f^{N_d}$ along the diagonal $\Delta = \{x = y\}$. Geometrically we add the line $l = \overline{xy}$ (defined by a one-form). The exceptional divisor is $E = \{(x, y, l)|\ x = y,\ l(x) = 0\}$. Thus the strict transform of the lifted stratum is defined as:

$$\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x, y, l) = \overline{\{(x, y, l, f),\quad x \neq y|\quad f|_x^{(p)} = 0 = f|_y^{(q)},\quad l(x) = 0 = l(y)\}} \subset \mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \check{\mathbb{P}}_l^2 \times \mathbb{P}_f^{N_d} \tag{4}$$

Theorem 1.4 *The strict transform $\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x,y,l)$ is a locally complete intersection. Its cohomology class in the cohomology of the ambient space $H^*(\mathbb{P}^2_x\times\mathbb{P}^2_y\times\check{\mathbb{P}}^2_l\times\mathbb{P}^{N_d}_f,\mathbb{Z})$ is:*

$$[\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x,y,l]=(L+X)(L+Y)\Big(F+(d-p)X\Big)^{\binom{p+2}{2}}\prod_{i=0}^{q}\prod_{j=0}^{q-i}\Big(F+(d-i-j)Y+iX-jL-(p+1+j-i)E\Big) \quad (5)$$

here E is the class of the exceptional divisor, for the notations of the cohomology generators cf. §2.1.1.

proof: We want to represent this as the intersection of the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)$ with hypersurfaces. At the j'th step we have a variety $M_j\stackrel{i}{\hookrightarrow}\mathbb{P}^{N_d}_f\times Aux$ and a hypersurface $V_{j+1}\subset\mathbb{P}^{N_d}_f\times Aux$. Think about the intersection $M_j\cap V_{j+1}$ as the pullback $i^*(V_{j+1})$. Then the task is to take the *strict transform* of V_{j+1}, i.e. to subtract from the total transform the part of the exceptional divisor.

The straightforward approach is just to consider the components of the tensor $f|^{(q)}_y$ and intersect with all the hypersurfaces: $\{\partial^{n_0}_0\partial^{n_1}_1\partial^{n_2}_2 f|_y=0\}_{n_0+..n_2=q}$. This will bring various complicated residual pieces. Instead, we represent this tensor condition as follows:

$$\Big\{f|^{(i)}_y(\underbrace{x..x}_{i})=0\Big\}^q_{i=0},\quad \Big\{f|^{(i+1)}_y(\underbrace{x..x}_{i}\tilde{v})=0\Big\}^{q-1}_{i=0},\quad ...\quad \Big\{f|^{(i+j)}_y(\underbrace{x..x}_{i}\underbrace{\tilde{v}..\tilde{v}}_{j})=0\Big\}^{q-j}_{i=0},\quad f|^{(q)}_y(\underbrace{\tilde{v}..\tilde{v}}_{j})=0 \quad (6)$$

Here $\tilde{v}$ is a fixed generic point, so that the points $x,y,\tilde{v}$ do not lie on one line. By direct check it is verified that for generic parameters (i.e. $y\neq x,\quad \tilde{v}\notin Span(x,y)$) these conditions are equivalent to $f|^{(q)}_y=0$. For non-generic situation each such equation will give a reducible hypersurface, correspondingly a residual term should be subtracted.

• $\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)\cap\{f|_y=0\}$. The pullback of $\{f|_y=0\}$ to $\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)$ consists of the strict transform (the closure of the part over $x\neq y$) and the exceptional divisor E (over $x=y$). To calculate the multiplicity, expand $y=x+\epsilon v$, correspondingly:

$$0=f|_y=\underbrace{f|_x+..+\epsilon^p f|^{(p)}_x(v..v)}_{vanish}+\epsilon^{p+1}f^{(p+1)}|_x(v..v)+.. \quad (7)$$

i.e. the exceptional divisor enters with the multiplicity $(p+1)$. So, the strict transform is $(f|_y=0)-(p+1)E$ and the total cohomology class: $[\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)]\Big([f|_y=0]-(p+1)E\Big)\in H^*(\mathbb{P}^{N_d}_f\times Aux)$.

The points of this variety satisfy:

⋆ for $x\neq y$: $f|^{(p)}_x=0$ and $f|_y=0$

⋆ for $x=y$: $f|^{(p)}_x=0=f|^{(p+1)}_x(v..v)$

• In the same way do all the intersections with $\Big\{f|^{(i)}_y(\underbrace{x..x}_{i})=0\Big\}^q_{i=1}$. At each step subtract the exceptional divisor with the necessary multiplicity. The resulting cohomology class:

$$[\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)]\prod_{i=0}^{q}\Big([f|^{(i)}_y(\underbrace{x..x}_{i})=0]-[(p+1+i)E]\Big) \quad (8)$$

• Intersect now with $\Big\{f|^{(i+1)}_y(\underbrace{x..x}_{i}\tilde{v})=0\Big\}^{q-1}_{i=0}$. In addition to the exceptional divisor at each step one should subtract the contribution of the locus: $\tilde{v}\in\overline{xy}$. As the point $\tilde{v}$ is fixed, this is just a condition that the line l passes through a point $\tilde{v}\in\mathbb{P}^2$. Correspondingly the cohomology class of the resulting variety is:

$$[\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)]\prod_{i=0}^{q}\Big([f|^{(i)}_y(\underbrace{x..x}_{i})=0]-[(p+1+i)E]\Big)\prod_{i=0}^{q-1}\Big([f|^{(i+1)}_y(\underbrace{x..x}_{i}\tilde{v})=0]-[\tilde{v}\in l]-[(p+i)E]\Big) \quad (9)$$

• Do the rest of intersections, at each step subtracting (with appropriate multiplicities) the exceptional divisor and the class $[\tilde{v}\in l]$. Finally we get:

$$[\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x,y,l)] = [\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,y,l)]\prod_{i=0}^{q}\prod_{j=0}^{q-i}\Big([f|_y^{(i)}(\underbrace{x..x}_{i}\underbrace{\tilde{v}..\tilde{v}}_{j})=0] - j[\tilde{v}\in l] - [(p+1+i-j)E]\Big) \tag{10}$$

Substitute now the cohomology classes for the conditions (cf. §2). (Note that as $\tilde{v}$ is a fixed point, the condition $\tilde{v}\in l$ is just one linear condition on l, its class is L. The class of the exceptional divisor is $[E]=X+Y-L$, cf. §2.1.1.) This proves the theorem. ■

To get the solution of the enumerative problem (i.e. the degree of $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$) we should apply the Gysin homomorphism corresponding to the projection $\overline{\overline{\Sigma}}(x,y)\to\overline{\Sigma}$. In this case it means just to extract the coefficient of $X^2Y^2L^2$. In the case $\mathbb{S}_x=\mathbb{S}_y$ the resulting answer should be also divided by 2, as the singular points are indistinguishable.

Corollary 1.5 In several simplest cases the degree of $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$ is:

- $q=1.$ $deg(\overline{\Sigma}_{x_1^{p+1}+x_2^{p+1},A_1}) =$ $9\binom{p+3}{4}(d-p)^3(d+p-2)-\frac{3}{4}\binom{p+2}{3}(10p^2+39p+7)(d-p)^2+\frac{1}{2}\binom{p+2}{3}(d-p)(6+5p)$

- $q=2.$ $deg(\overline{\Sigma}_{x_1^{p+1}+x_2^{p+1},D_4}) =$ $45\binom{p+3}{4}(d-p)^3(d+p-4)+2(d-p)(8+3p+p^2)(35p^2+20p-12)$ $-\frac{5}{8}(p+1)(d-p)^2(14p^4+105p^3+147p^2+114p-80)-6(85p^2+45p-28)$

- $q=3.$ $deg(\overline{\Sigma}_{x_1^{p+1}+x_2^{p+1},X_9}) =$ $135\binom{p+3}{4}(d-p)^3(d+p-6)+2(d-p)(16+3p+p^2)(270p^2-20p-117)$ $-\frac{5}{8}(d-p)^2(54p^5+527p^4+948p^3+1853p^2-894p-1152)-14(830p^2-105p-348)$

For $p=q=1$ this gives the classical result. For $p=2,q=1$ this coincides with Kazarian's result [Kaz03-hab].

2 Some relevant notions and auxiliary results

2.1 The ambient space

2.1.1 Coordinates

We work with various projective spaces and their subvarieties. Adopt the following notation. If we denote a point in the space $\mathbb{P}^2_x$ by the letter x, then the homogeneous coordinates are (x_0,x_1,x_2). The generator of the cohomology ring of this $\mathbb{P}^2_x$ is denoted by the upper-case letter X, so that $H^*(\mathbb{P}^2_x)=\mathbb{Z}[X]/(X^3)$. Alternatively X is the first Chern class of the dual tautological bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. By the same letter we also denote the hyperplane class in homology of $\mathbb{P}^n_x$. Since it is always clear, where we speak about coordinates and where about (co)homology classes, no confusion arises. To demonstrate this, consider the hypersurface

$$V=\{(x,y,f)|\ f(x,y)=0\}\subset\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f \tag{11}$$

Here f is a bi-homogeneous polynomial of bi-degree d_x,d_y in homogeneous coordinates $(x_0,\dots,x_n)$, $(y_0,\dots,y_n)$, the coefficients of f are the homogeneous coordinates on the parameter space $\mathbb{P}^{N_d}_f$. The cohomology class of this hypersurface is

$$[V]=d_xX+d_yY+F\in H^2(\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f) \tag{12}$$

A (projective) line through the point $x\in\mathbb{P}^2_x$ is defined by a 1-form l (so that $l\in\check{\mathbb{P}}^2_l,\ l(x)=0$). Correspondingly the generator of $H^*(\check{\mathbb{P}}^2_l)$ is denoted by L.
A curve is denoted by C or the defining polynomial f, correspondingly the parameter space of curves is $\mathbb{P}^{N_d}_f$.

We often work with symmetric $p-$forms $\Omega^p \in S^p(V_3)^*$ (here $(V_3)^*$ is a 3-dimensional vector space of linear forms). Thinking of the form as of a symmetric tensor with p indices ($\Omega^{(p)}_{i_1,\dots,i_p}$), we often write $\Omega^{(p)}(\underbrace{x,\dots,x}_{k})$ as a shorthand for the tensor, multiplied k times by the point $x \in V_3$

$$\Omega^{(p)}(\underbrace{x,\dots,x}_{k}) := \sum_{0\le i_1,\dots,i_k\le 2} \Omega^{(p)}_{i_1,\dots,i_p} x_{i_1}\dots x_{i_k} \tag{13}$$

So, for example, the expression $\Omega^{(p)}(x)$ is a $(p-1)-$form. Unless stated otherwise, we assume the symmetric form $\Omega^{(p)}$ to be generic (in particular non-degenerate, i.e. the corresponding hypersurface $\{\Omega^{(p)}(\underbrace{x,\dots,x}_{p}) = 0\} \subset \mathbb{P}^n_x$ is smooth).

Symmetric forms typically occur as tensors of derivatives of order p, e.g. $f^{(p)}$. Sometimes, to emphasize the point at which the derivatives are calculated we assign it. So, e.g. $f|^{(p)}_x(\underbrace{y,\dots,y}_{k})$ means: the tensor of derivatives of order p, calculated at the point x, and contracted k times with y.

2.1.2 Blowup along the diagonal.

The diagonal $\Delta = \{x = y\} \subset \mathbb{P}^2_x \times \mathbb{P}^2_y$ appears constantly. (Its cohomology class is given §2.1.3.1). The blowup over the diagonal is easily described as the incidence variety.

$$\mathbb{P}^2_x \underset{\Delta}{\tilde{\times}} \mathbb{P}^2_y = \{(x,y,l)|\ l(x) = 0 = l(y)\} \overset{i}{\hookrightarrow} \mathbb{P}^2_x \times \mathbb{P}^2_y \times \check{\mathbb{P}}^2_l, \quad E_\Delta = \{x = y,\ l(x) = 0\} \subset \mathbb{P}^2_x \underset{\Delta}{\tilde{\times}} \mathbb{P}^2_y \tag{14}$$

The variety is a complete intersection, thus its cohomology class is the product $[\mathbb{P}^2_x \underset{\Delta}{\tilde{\times}} \mathbb{P}^2_y] = (L+X)(L+Y) \in H^4(\mathbb{P}^2_x \times \mathbb{P}^2_y \times \check{\mathbb{P}}^2_l)$.

We will often need the cohomology class of the exceptional locus E_Δ both as a divisor in $\mathbb{P}^2_x \underset{\Delta}{\tilde{\times}} \mathbb{P}^2_y$ and as a cycle in $\mathbb{P}^2_x \times \mathbb{P}^2_y \times \check{\mathbb{P}}^2_l$. Again, as it is a transversal intersection of the two conditions ($x = y$ and $l(x) = 0$) we get: $[E_\Delta] = (L+X)(X^2 + XY + Y^2) \in H^6(\mathbb{P}^2_x \times \mathbb{P}^2_y \times \check{\mathbb{P}}^2_l)$.

The class of the exceptional divisor in the ring $H^*(\mathbb{P}^2_x \underset{\Delta}{\tilde{\times}} \mathbb{P}^2_y)$ is $X+Y-L$. It can be obtained, for example, by noticing that the hypersurface $\begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0$ contains the exceptional divisor E_Δ and also the residual divisor $l_2 = 0$.

The two classes are related by the pushforward i_*. The identity $i_*(X+Y-L) = (L+X)(X^2+XY+Y^2) \in H^6(\mathbb{P}^2_x \times \mathbb{P}^2_y \times \check{\mathbb{P}}^2_l)$ is directly verified.

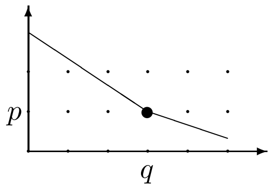
2.1.3 Cohomology classes

2.1.3.1 The cohomology class of the diagonal The diagonal $\Delta = \{x = y\} \subset \mathbb{P}^n_x \times \mathbb{P}^n_y$ appears constantly in the paper. Its class is [Fulton98]

$$[\Delta] = \sum_{i=0}^{n} X^{n-i} Y^i \in H^{2n}(\mathbb{P}^n_x \times \mathbb{P}^n_y) \tag{15}$$

For example, a condition of proportionality of two symmetric forms $f^{(p)} \sim g^{(p)}$ is just the coincidence of the corresponding points in a big projective space, thus its class is given by the above formula (with $n = \binom{p+2}{2} - 1$).

2.1.3.2 The cohomology class of the degenerating divisor We often use the cohomology classes of various divisors on the lifted strata $\overline{\overline{\Sigma}}$.
Consider the divisor of curves whose equation does not contain a particular monomial. We often need to degenerate by demanding that a monomial on the Newton diagram is absent. The condition is: a monomial $x_1^p x_2^q$ should be absent in the normal form (i.e. its coefficient must vanish). The class of this divisor was calculated in [Ker06, section A.1.2]. Considering it as a degeneration we write:



$$[\overline{\overline{\Sigma}}_1(x,l)]\Big(F+(d-p-2q)X+(q-p)L\Big)=[\overline{\overline{\Sigma}}_2(x,l)]\in H^*(\mathbb{P}^2_x\times\check{\mathbb{P}}^2_l\times\mathbb{P}^{N_d}_f) \qquad (16)$$

The class of its pull-back to the stratum $\overline{\overline{\Sigma}}_1(x,l)$ is obtained by pulling back the hyperplanes X, L, F, so it is: $i^*(F)+(d-p-2q)i^*(X)+(q-p)i^*(L)$.

2.1.3.3 On the universality and Thom polynomials According to the general philosophy of R.Thom, proved by Kazarian (cf. [Kaz03-1, Kaz03-2]), for a collection of singularities $\mathbb{S}_1..\mathbb{S}_r$ the degree of the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_1..\mathbb{S}_r}$ is expressed through the universal Thom polynomials, S_I with $I\subseteq\{\mathbb{S}_1..\mathbb{S}_r\}$, depending on the relative Chern classes of the ambient space and the linear system. For one singular point: $deg(\overline{\Sigma}_{\mathbb{S}})=S_{\mathbb{S}}$. The general expression is:

$$deg(\overline{\Sigma}_{\mathbb{S}_1..\mathbb{S}_r})=\sum_{J_1\bigsqcup..\bigsqcup J_k} S_{\mathbb{S}_{J_1}}..S_{\mathbb{S}_{J_k}} \qquad (17)$$

the sum is over all the possible different decompositions $\{\mathbb{S}_1,..,\mathbb{S}_r\}=\bigsqcup J_i$. In particular, for two singular points: $deg(\overline{\Sigma}_{\mathbb{S}_1\mathbb{S}_2})=S_{\mathbb{S}_1}S_{\mathbb{S}_2}+S_{\mathbb{S}_1,\mathbb{S}_2}$. Therefore our results give the specializations of the polynomials $S_{\mathbb{S}_i}$, to the case of a complete linear system of plane curves. (Unfortunately, the universal Thom polynomials cannot be restored from our answers.)

As the degrees $deg(\overline{\Sigma}_{\mathbb{S}_i})$ are known from [Ker06], the only unknowns are the specializations of $S_{\mathbb{S}_1,\mathbb{S}_2}$. To avoid awkward expressions we present the expressions for $S_{\mathbb{S}_x\mathbb{S}_y}$ only.

2.2 On the singularity types

Definition 2.1 *[GLSbook] Let $(C_x,x)\subset(\mathbb{C}^2_x,x)$ and $(C_y,y)\subset(\mathbb{C}^2_y,y)$ be two germs of isolated curve singularities. They are (embedded topologically) equivalent if there exist a homeomorphism $(\mathbb{C}^2_x,x)\mapsto(\mathbb{C}^2_y,y)$ mapping (C_x,x) to (C_y,y). The corresponding equivalence class is called the (embedded topological) singularity type. The variety of points (in the parameter space $\mathbb{P}^{N_d}_f$), corresponding to the curves with prescribed singularity types $\mathbb{S}_1..\mathbb{S}_r$ is called the* **equisingular stratum** *$\Sigma_{\mathbb{S}_1..\mathbb{S}_r}$.*

The topological type can be specified by a (simple polynomial) representative of the type: the **normal** form. Several simplest types are (all the notations are from [AGLV], we ignore the moduli of analytic classification):

$$\begin{array}{l} A_k: x_2^2+x_1^{k+1},\quad D_k: x_2^2x_1+x_1^{k-1},\quad E_{6k}: x_2^3+x_1^{3k+1},\quad E_{6k+1}: x_2^3+x_2x_1^{2k+1},\quad E_{6k+2}: x_2^3+x_1^{3k+2}\\ J_{k\ge1,i\ge0}: x_2^3+x_2^2x_1^k+x_1^{3k+i},\quad Z_{6k-1}: x_2^3x_1+x_1^{3k-1},\quad Z_{6k}: x_2^3x_1+x_2x_1^{2k},\quad Z_{6k+1}: x_2^3x_1+x_1^{3k}\\ X_{k\ge1,i\ge0}: x_2^4+x_2^3x_1^k+x_2^2x_1^{2k}+x_1^{4k+i},\quad W_{12k}: x_2^4+x_1^{4k+1},\quad W_{12k+1}: x_2^4+x_2x_1^{3k+1}\end{array} \qquad (18)$$

Using the normal form $f=\sum a_{\mathbf{I}}\mathbf{x}^{\mathbf{I}}$ one can draw the *Newton diagram* of the singularity. Namely, one marks the points $\mathbf{I}$ corresponding to non-vanishing monomials in f, and takes the convex hull of the sets $\mathbf{I}+\mathbb{R}^2_+$. The envelope of the convex hull (the chain of segment-faces) is the Newton diagram.

Definition 2.2 *[GLSbook]*
• *The singular germ is called Newton-non-degenerate with respect to its diagram if the truncation of its polynomial to every face of the diagram is non-degenerate (i.e. the truncated polynomial has no singular*

points in the torus $(C^*)^2$).
• *The germ is called generalized Newton-non-degenerate if it can be brought to a Newton-non-degenerate form by a locally analytic transformation.*
• *The singular type is called Newton-non-degenerate if it has a (generalized) Newton-non-degenerate representative.*

For Newton-non-degenerate types the normal form is always chosen to be Newton-non-degenerate . So, the Newton-non-degenerate type $\mathbb{S}$ can be specified by giving the Newton diagram of its normal form $\mathbb{D}_{\mathbb{S}}$.

Newton-non-degeneracy implies strong restrictions on the tangent cone:

Proposition 2.3 *Let $T_C = \{(l_1, p_1)...(l_k, p_k)\}$ be the tangent cone of the germ $C = \cup C_j$ (here all the tangents l_i are different, p_i are the multiplicities, so that $\sum_i p_i = mult(C)$). If the germ is generalized Newton-non-degenerate then $p_i > 1$ for at most two tangents l_i.*

So, for a generalized Newton-non-degenerate germ there are at most two distinguished tangents. We always orient the coordinate axes along these tangents.

In the tangent cone of the singularity $T_C = (l_1^{p_1}...l_k^{p_k})$, consider the lines appearing with the multiplicity 1. They correspond to smooth branches, not tangent to any other branch of the singularity. We call such branches *free*. Call the tangents to the non-free branches the *non-free tangents*.

As we consider the topological types, one could expect that to bring a germ to the Newton diagram of the normal form, one needs local homeomorphisms. However for curves the locally analytic transformation always suffice. In this paper we restrict consideration further to the types for which only linear transformations suffice.

Definition 2.4 *[Ker06] A (generalized Newton-non-degenerate) singular germ is called linear if it can be brought to the Newton diagram of its type by projective transformations only (or linear transformations in the local coordinate system centered at the singular point). A linear stratum is the equisingular stratum, whose open dense part consists of linear germs. The topological type is called linear if the corresponding stratum is linear.*

The linear types happen to be abundant due to the following observation

Proposition 2.5 *[Ker06, section 3.1] The Newton-non-degenerate topological type is linear iff every segment of the Newton diagram has the bounded slope: $\frac{1}{2} \leq tg(\alpha) \leq 2$.*

Example 2.6 The simplest class of examples of linear singularities is defined by the series: $f = x^p + y^q, \ p \leq q \leq 2p$. In general, for a given series only for a few types of singularities the strata can be linear. In the low modality cases the linear types are:
• Simple singularities (no moduli): $A_{1\leq k\leq 3}, \ \ D_{4\leq k\leq 6}, \ \ E_{6\leq k\leq 8}$
• Unimodal singularities: $X_9 (= X_{1,0}), \ \ J_{10} (= J_{2,0}), \ \ Z_{11\leq k\leq 13}, \ \ W_{12\leq k\leq 13}$
• Bimodal: $Z_{1,0}, \ \ W_{1,0}, \ \ W_{1,1}, \ \ W_{17}, \ \ W_{18}$

Most singularity types are nonlinear. For example if a curve has an A_4 point, the best we can do by projective transformations is to bring it to the Newton diagram of A_3 $a_{0,2}x_2^2 + a_{2,1}x_2x_1^2 + a_{4,0}x_1^4$.

This quasi-homogeneous form is degenerated ($a_{2,1}^2 = 4a_{0,2}a_{4,0}$) and by quadratic (nonlinear!) change of coordinates the normal form of A_4 is achieved.

By the finite determinacy theorem the topological type of the germ is fixed by a finite jet of the defining series. Namely, for every type $\mathbb{S}$, there exists k such that for all bigger $n \geq k$: $jet_n(f_1)$ has type $\mathbb{S}$ causes f_1 has type $\mathbb{S}$. The minimal such k is called: *the order of determinacy*. E.g. $o.d.(A_k) = k+1$, $o.d.(D_k) = k-1$. The classical theorem says: if $m^{k+1} \subset m^2 Jac(f)$ then $o.d.(f) \leq k$.

3 The method

Our approach is most naive and classical. In a sense it is a brute-force calculation. Correspondingly it is often long and cumbersome. The advantages of the method are:
• The method gives a recursive algorithm, consisting of routine parts.
• The method seems to be more effective than other approaches (to the best of our knowledge). In particular, in Appendix we present the results for some *series* of types (as compared to single, isolated results previously known).
• As the final result we obtain the multi-degree of the (partial resolution of the) stratum $\overline{\Sigma}_{\mathbb{S}A_1}$. (The actual degree is just a particular coefficient in a big polynomial.) This multi-degree contains many important numerical invariants, e.g. enumeration of $(\mathbb{S}_1, \mathbb{S}_2)$ with one or two singular points restricted to some curves, or with some conditions on the tangents to the branches. More generally: when the parameters of the singularity (the points, the tangents, the conics osculating to the branches etc.) are restricted to a subvariety of their original ambient space. So, this solves a whole class of enumerative problems.

3.1 The case of one singular point

The uni-singular case was solved in [Ker06] by partial resolutions of the strata $\overline{\widetilde{\Sigma}}_{\mathbb{S}}\to\overline{\Sigma}_{\mathbb{S}}$. Namely, the strata $\Sigma_{\mathbb{S}} \subset \mathbb{P}^{N_d}_f$ were lifted to a bigger ambient space $\widetilde{\Sigma}_{\mathbb{S}} \subset Aux \times \mathbb{P}^{N_d}_f$ (*Aux* for auxiliary). This was done by taking into account parameters of the singularity (singular point, tangent cone, osculating conics etc.) For *linear* singularities (def. 2.4) the lifted strata are smooth locally complete intersections defined by explicit equations. This enabled to calculate their cohomology classes $[\overline{\widetilde{\Sigma}}_{\mathbb{S}}] \in H^*(Aux \times \mathbb{P}^{N_d}_f)$. For *non-linear* singularities the problem was reduced to the linear case, by a chain of degenerations.

Once the class of the lifted stratum $[\overline{\widetilde{\Sigma}}_{\mathbb{S}}]$ is known, the class of the original stratum $[\overline{\Sigma}_{\mathbb{S}}]$ is obtained by projection $\overline{\widetilde{\Sigma}}\to\overline{\Sigma}$ (Gysin homomorphism). It amounts to extracting a particular coefficient from the big polynomial. So, the class $[\overline{\Sigma}_{\mathbb{S}}]$ is completely fixed by $[\overline{\widetilde{\Sigma}}_{\mathbb{S}}]$. Similarly, in the following we will be interested in the cohomology classes of the *lifted* strata $[\overline{\widetilde{\Sigma}}_{\mathbb{S}_1\mathbb{S}_2}]$.

3.2 Two singular points of linear types

A natural approach to the case of two singularities is as follows:
• For the types $\mathbb{S}_x, \mathbb{S}_y$ (here $x, y \in \mathbb{P}^2$ are the points) consider the liftings $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}, \overline{\widetilde{\Sigma}}_{\mathbb{S}_y}$ (as above) and define:

$$\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y} := \overline{\left\{(C, \begin{matrix}\{x, l^x_i\} \\ \{y, l^y_j\}\end{matrix} \quad x \neq y)|\, C \in \overline{\widetilde{\Sigma}}_{\mathbb{S}_x} \cap \overline{\widetilde{\Sigma}}_{\mathbb{S}_y}\right\}} \subset Aux_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y} \times \mathbb{P}^{N_d}_f \tag{19}$$

here l^x_i, l^y_j are the non-free tangents (cf. §2.2). So, the stratum is defined outside the diagonal $x = y$ and the closure consists of all the possible results of collision of two singularities.
• Try to relate the defining ideals $I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y})$, $I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x})$, $I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_y})$. Try to relate the cohomology classes $[\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}]$, $[\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}]$, $[\overline{\widetilde{\Sigma}}_{\mathbb{S}_y}]$. The difficulty here is that naive intersection contains a residual variety over the diagonal $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x} \cap \overline{\widetilde{\Sigma}}_{\mathbb{S}_y} = \overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y} \cup R_{x=y}$. (Alternatively: $I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}) \supsetneq < I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}), I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_y}) >$.) Its contribution should be subtracted from the product of classes $[\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}] \times [\overline{\widetilde{\Sigma}}_{\mathbb{S}_y}]$. But the dimension of this residual piece is *always bigger* than the dimension of the needed stratum: $dim(R_{x=y}) > dim(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y})$.

3.2.1 Intersections with hypersurfaces

A way to repair this situation is to split the intersection into a step-by-step procedure of intersection with hypersurfaces (cf. [StuVog82], [vGas89, vGas91]).

$\star$ Suppose the lifted stratum is a (globally) complete intersection of hypersurfaces: $\overline{\overline{\Sigma}}_{\mathbb{S}_y} = \cap_{i=1}^k V_i$. At each step of the intersection we have (set theoretically)

$$\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j} V_i} \cap V_{j+1} = \overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j+1} V_i} \cup R_{j+1}, \qquad \overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y} \subsetneq \overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{k-1} V_i} \subsetneq \overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{k-2} V_i} \subsetneq .. \subsetneq \overline{\overline{\Sigma}}_{\mathbb{S}_x V_1} \subsetneq \overline{\overline{\Sigma}}_{\mathbb{S}_x} \tag{20}$$

Here $\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j+1} V_i}$ is an irreducible variety and its dimension drops (precisely) by one with each intersection. R_j is the residual piece produced at the j'th step (it contains all the non-enumerative contributions). The key point is: *as the initial variety is irreducible and the intersection is by a hypersurface, the resulting variety is pure dimensional,* $dim(\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j+1} V_i}) = dim(R_{j+1})$. Therefore, the contribution of the residual piece can be subtracted: $[\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j+1} V_i}] = [\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{j} V_i}][V_{j+1}] - [R_{j+1}]$ (up to multiplicities). By repeating this procedure we calculate the needed class $[\overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y}]$.

$\star$ Suppose the variety is a locally complete intersection $\overline{\overline{\Sigma}}_{\mathbb{S}_y} = \cap_{i=1}^k V_i$ and its defining equations are known (the case of linear singularities). Represent it locally by the minimal collection of hypersurfaces $\cap_{i=1}^k V_i \subsetneq \cap_{i=1}^{k'} V_i$. Perform the above procedure to get the class $[\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{k'} V_i}]$. And now subtract the residual contributions (due to non-globally complete intersection): $[\overline{\overline{\Sigma}}_{\mathbb{S}_x \cap_{i=1}^{k'} V_i}] = [\overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y}] + [\overline{\overline{\Sigma}}_{\mathbb{S}_x R}]$.

3.2.2 The classes of residual pieces

The classes of residual pieces can be computed by the thorough check of the intersection geometry. The process is greatly simplified by the following trick.

Consider the stratum $\overline{\Sigma}_{\mathbb{S}_x \mathbb{S}_y}$ as a projective subvariety of $\overline{\Sigma}_{\mathbb{S}_x}$. Correspondingly, we start from a lifting $\overline{\overline{\Sigma}}_{\mathbb{S}_x}$, add some parameters of the singularity: $\overline{\overline{\Sigma}}_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y}$ and define the lifting $\overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y} \subset \overline{\overline{\Sigma}}_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y} \subset \mathbb{P}_f^{N_d} \times Aux$. So, we calculate the class $[\overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y}] \in H^*(\overline{\overline{\Sigma}}_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y})$ and then take its pushforward.

Blowup the space $\overline{\overline{\Sigma}}_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y}$ along some smooth loci and the consider the strict transform of the defining hypersurfaces in $Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x} \times Aux_{\mathbb{S}_y})$.

In more details: let $\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x, \{l_i^x\})$ be the lifting for the linear singularity $\mathbb{S}_x$. Assign the second point y, the line $l = \overline{xy}$ and the non-free tangent lines $\{l_j^y\}$. So, consider:

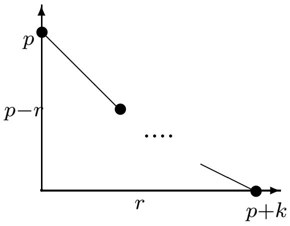
$$\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x, \{l_i^x\}, y, \{l_j^y\}, l) = \Big\{(x, \{l_i^x\}, f) \in \overline{\overline{\Sigma}}_{\mathbb{S}_x}(x, \{l_i^x\}), \quad x \in l \ni y, \quad \{y \in l_j^y\}\Big\} \tag{21}$$

The projection $\overline{\overline{\Sigma}}_{\mathbb{S}_x}(x, \{l_i^x\}, y, \{l_j^y\}, l) \to \overline{\overline{\Sigma}}_{\mathbb{S}_x}(x, \{l_i^x\}, y, \{l_j^y\})$ is the blowup over the diagonal $x = y$. We blow up further along the cycles: $\{x = y,\ l_i^x = l\}$, $\{x = y,\ l_i^x = l_j^y\}$. Next blow up along the triple coincidence loci: $\{x = y,\ l_i^x = l = l_j^y\}$, $\{x = y,\ l_i^x = l_j^x\}$, $\{x = y,\ l_i^y = l_j^y\}$. and so on. The process stops very quickly as $\mathbb{S}_x \mathbb{S}_y$ are linear (so there are at most two different non-free tangent lines for each type). Denote the corresponding blown up variety by $Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$.

The defining condition of $\mathbb{S}_y$ are of the form: $f|_y^{(q)}(v_1..v_q) = 0$ (here v_i are either vectors along the non-free tangents or some generic vectors). Pull back the first of these hypersurfaces V_1 to $Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$, i.e. consider its total transform. By expanding around the diagonal $y = x + \epsilon v$ and all the other coincidence loci one sees that the residual piece is precisely the exceptional divisor (with appropriate multiplicities). Remove them to get the strict transform: $\widetilde{V_1} \cap Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$. In the same way one continues with all the V_i's to get: $\overline{\overline{\Sigma}}_{\mathbb{S}_x \mathbb{S}_y} = Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x}) \cap \widetilde{V_1}$ ...

3.2.3 Example: the case $\mathbb{S}_x, A_1$

In this case we can give quite general answer. Let $T_C = (l_1^{p_1}..l_r^{p_r})$ be the tangent cone of $\mathbb{S}_x$. We consider first the case of *one non-free tangent* (i.e. $p_i > 1$ for only one l_i).



Let p be the multiplicity, r be the number of free branches, so the expansion of the defining equation starts as $f = x_1^p + x_1^{p-r}x_2^r + terms\ in\ m_{p+1}$. The type $\mathbb{S}_x$ is assumed to be linear, let $p+k$ be its order of determinacy. So the corresponding Newton diagram has the shape shown on the picture. In particular, the monomials x_2^q for $q < p+k$ are absent.

Let l_x denote the non-free tangent line of $\mathbb{S}_x$. The initial lifted variety is:

$$\overline{\overline{\Sigma}}_{\mathbb{S}_x A_1}(x,y,l,l_x) = \overline{\left\{ \begin{array}{l} (x,y,l,l_x,f),\quad x \neq y \\ (x,l_x,f) \in \overline{\overline{\Sigma}}_{\mathbb{S}_x}(x,l_x) \end{array} \middle| \ f|_y^{(1)} = 0, \quad l(x) = 0 = l(y) \right\}} \subset Aux \times \mathbb{P}_f^{N_d} \tag{22}$$

Here $Aux = \mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \check{\mathbb{P}}_l^2 \times \check{\mathbb{P}}_{l_x}^2$ is the auxiliary space. According to the general method we blow Aux along the cycle $x = y,\ l = l_x$. Let E_{Δ_l} denote the exceptional divisor, let $\pi^*(E_\Delta)$ denote the total transform of the first exceptional divisor (over the diagonal $x = y$). Let $Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$ denote the strict transform of the stratum.

As in the example of ordinary multiple point(§1.5) we represent the defining conditions of the node as: $f|_y = 0 = f^{(1)}|_y(x) = f^{(1)}|_y(v_x)$ where $x \neq v_x \in l_x$. As in that case the computation consists of intersecting with the hypersurfaces and subtracting the exceptional divisor.

• The total transform of the hypersurface $\{f|_y = 0\} \subset Bl(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$ is reducible, with exceptional divisor over the diagonal $x = y$. To check this expand: $y = x + \epsilon v$ and expand the equation.

$$0 = f|_y = \underbrace{f|_x + .. + \epsilon^{p-1} f^{(p-1)}|_x(v..v)}_{vanish} + \epsilon^p f|_x^{(p)}(v..v) + .. \tag{23}$$

For generic v, v_x (i.e. $l \neq l_x$) the term $f|_x^{(p)}(v..v)$ does not vanish, so the residual piece is $\pi^*(pE_\Delta)$. Over the degeneracy locus $\{x = y,\ l = l_x\}$ the first non-vanishing term in the expansion is $\epsilon^{p+k} f^{(p+k)}|_x(v..v)$. Correspondingly, the strict transform of $\{f|_y = 0\}$ is: $\pi^*(\{f|_y = 0\}) - \pi^*(pE_\Delta) - kE_{\Delta_l}$. The points of this hypersurface correspond to:

⋆ $\{f|_y = 0\}$ for $x \neq y$
⋆ $f|_x^{(p)}(v..v) = 0$ for $x = y,\ l \neq l_x$
⋆ $f^{(p+k)}|_x(v..v) = 0$ for $x = y,\ l = l_x$

• To find the strict transform of $f^{(1)}|_y(x) = 0$ inside $\overline{\overline{\overline{\Sigma}}_{\mathbb{S}_x} \underset{x \neq y}{\cap} \{f|_y = 0\}}$ expand

$$0 = f^{(1)}|_y(x) = \underbrace{f|_x + .. + \epsilon^p f|_x^{(p)}(v..v)}_{vanish} + \epsilon^p f|_x^{(p+1)}(v..v) + .. \tag{24}$$

As above we get that the strict transform of $\{f^{(1)}|_y = 0\}$ is: $\pi^*(\{f|_y^{(1)}(x) = 0\}) - \pi^*((p+1)E_\Delta) - kE_{\Delta_l}$.

• The last hypersurface is treated similarly. Here there are two cases:

⋆ $r = 0$, i.e. there are no free branches. Then the strict transform is: $\pi^*(\{f|_y^{(1)}(v_x) = 0\}) - (l = l_x) - \pi^*((p+1)E_\Delta) - (k+1)E_{\Delta_l}$

• $r > 0$: $\pi^*(\{f|_y^{(1)}(v_x) = 0\}) - (l = l_x) - \pi^*(pE_\Delta) - (k+2)E_{\Delta_l}$

Note that here $(l = l_x)$ is a divisor and the strict transform is pure-dimensional (as it should be).

In the general case of several non-free tangents, one should blow up the space over all the simple coincidence loci: $x = y, l = l_i$, then over the double coincidences: $x = y, l_j = l = l_i$, etc.. Then one proceeds similarly to the case of one non-free tangent.

3.3 Approach by degenerations

In general the equations are not known or the variety is not a locally complete intersection (the case of non-linear singularities). Apply the degeneration procedure: $\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y} \to \overline{\overline{\Sigma}}_{\mathbb{S}'_x\mathbb{S}'_y}$ (as described in [Ker06]) to arrive at simpler singularity types. (The trivial choice is ordinary multiple points of sufficiently high multiplicities). As in the above cases the degeneration is done by a chain of hypersurface sections:

$$\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}\cap V_1=\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y\cap V_1}+R_1^{x=y}\ldots\qquad \overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y\cap_{i=1}^{k-1}V_i}\cap V_k=\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}'_y}+R_k^{x=y} \tag{25}$$

so, at each step we get a pure dimensional variety and the contribution of the residual pieces can be subtracted. As was explained in §1.2 this reduces the enumerative problem to the geometry of of the pieces $R_i^{x=y}$.

When this geometry is easily understood (e.g. the case when both initial types are linear), the problem is solved. So, for linear types we get an alternative solution of the problem. We show below some examples. The methodod degeneration has been described in details in [Ker06]. Recall that the cohomology class of the degenerating divisor that forces to vanish the coefficient of $x_1^p x_2^q$ is $(d-p-2q)X+F+(q-p)L$ (cf. §2.1.3.2).

We consider some examples, following the chain of degenerations:

$$\begin{gathered} x_1^{p-1}x_2+x_2^{p+1}\to x_1^p+x_2^{p+1}\to x_1^{p+1}+x_2^{p+1}\\ x_1^{p-1}x_2+x_1x_2^p+x_2^{p+2}\overset{p\geq 3}{\to} x_1^p+x_1x_2^p+x_2^{p+2}\to x_1^{p+1}+x_1x_2^p+x_2^{p+2}\end{gathered} \tag{26}$$

• $\mathbb{S}_x=x_1^p+x_2^{p+1}$, $\mathbb{S}_y=y_1^{q+1}+y_2^{q+1}$. $\mathbb{S}_x\to\mathbb{S}_x{}'=x_1^{p+1}+x_2^{p+1}$. (For $p=2$: A_2,A_1, for $p=3$: E_6,A_1). Degenerate by demanding that $f|_x^{(p)}=0$. The cohomology class of the corresponding divisor is $(d-p)X+F-pL$. As the result get the stratum $\widetilde{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$ and a residual piece over the diagonal. The piece occurs because the restriction to the diagonal has a component of curves with a point of multiplicity $p+1$. The corresponding type $\mathbb{S}_x{}''=(x_1^{p-q}+x_2^{p-q})(x_1^{q+1}+x_2^{2q+1})$ is obtained by collision [Ker07-2]. So, this piece should be subtracted. Its multiplicity is $q+1$ as is obtained below. Therefore we get the equation for cohomology classes:

$$[\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x,y,l,l^x)]((d-p)X+F-pL)=[\overline{\overline{\Sigma}}_{\mathbb{S}_x{}'\mathbb{S}_y}(x,y,l,l^x)]+(q+1)[x=y,\ x\in l\ \overline{\overline{\Sigma}}_{\mathbb{S}_x{}''}(x,l^x)] \tag{27}$$

The numerical results for some simple cases are given in Appendix.

• $x_1^p+x_1x_2^p+x_2^{p+2}\to x_1^{p+1}+x_1x_2^p+x_2^{p+2}$. (For $p=2$: A_3,A_1, for $p=3$: E_7,A_1) First we should calculate the class of the (auxiliary) stratum $\widetilde{\Sigma}_{x_1^{p+1}+x_1x_2^p+x_2^{p+2},A_1}$. This is done by degeneration of the ordinary multiple point. At this step no residual piece over the diagonal is produced.

Next, the type $x_1^p+x_1x_2^p+x_2^{p+2}$ is degenerated (by demanding that the coefficient of x_1^p vanish). A residual piece occurs over the diagonal (collision of $x_1^p+x_1x_2^p+x_2^{p+2}$ and a node), its type is: $x_1^p x_2+x_1^2x_2^{p-1}+x_2^{p+2}$. Subtracting its cohomology class (with multiplicity 2) gives the final result.

• $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+2}$. (For $p=1$: A_3,A_1, for $p=2$: D_5,A_1.) This stratum is treated by degeneration of $\widetilde{\Sigma}_{x_1^{p+1}+x_2^{p+1},A_1}$. The first degeneration $x_1^{p+1}+x_2^{p+1}\to x_1^{p+1}+x_1x_2^p+x_2^{p+2}$ brings no residual piece. The second $x_1^{p+1}+x_1x_2^p+x_2^{p+2}\to x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+2}$ brings the residual piece over the diagonal when the tangent to the singular point at x passes through the point y also. This piece is: $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+3}$, as always it should be subtracted with multiplicity 2.

• $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+3}$. (For $p=2$: D_6,A_1.) The cases $p=1,2$ are exceptional here due to specific coincidences on Newton diagram. This stratum is treated by degeneration of $\widetilde{\Sigma}_{x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+2},A_1}$. There residual piece over the diagonal occurs when the tangent line at x passes through y. The type is: $x_1^{p+1}+x_1^2x_2^{p-1}+x_2^{p+4}$ therefore it is irrelevant (by codimension).

3.3.1 The multiplicity of the piece over diagonal

It is computed in the standard way: as the multiplicity of the intersection of two varieties. Say the degenerating hypersurface is defined by $f|_x^{(p)}(\underbrace{v..v}_{p})=0$ (with residual pieces subtracted). We should obtain the local equations of the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}$ near the diagonal $x=y$. This is done by expansion $y=x+\epsilon v$ and the corresponding expansion of the equations depending on y. In our case $\mathbb{S}_y = y_1^{q+1}+y_2^{q+1}$, so the equations (outside $x=y$) are $f^{(q)}|_y=0$. When approaching the diagonal the flat limit should be taken (cf. [Ker07-2, §3.1]). In particular in this case the resulting system of series is:

$$f|_x^{(p-1)}=0,\quad f|_x^{(p)}(\underbrace{v..v}_{p-q})+..=0,\quad f|_x^{(p+1)}(\underbrace{v..v}_{p+2-q})+..=0,\quad f|_x^{(p+2)}(\underbrace{v..v}_{p+4-q})+..=0\quad ..,f|_x^{(p+q)}(\underbrace{v..v}_{p+q})+..=0 \tag{28}$$

Form here one get by contraction with v:

$$\begin{gathered} f|_x^{(p)}(\underbrace{v..v}_{p})+\epsilon^{q+1}f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1})..=0,\quad f|_x^{(p+1)}(\underbrace{v..v}_{p+1})+\epsilon^{p+q+1}f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1})..=0,\\ f|_x^{(p+2)}(\underbrace{v..v}_{p+2})+\epsilon^{p+q+1}f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1})..=0\quad ..,f|_x^{(p+q)}(\underbrace{v..v}_{p+q})+..=0 \end{gathered} \tag{29}$$

Thus, intersection with $f|_x^{(p)}(\underbrace{v..v}_{p})=0$ gives the piece of multiplicity $q+1$.

A Some numerical results

Below are listed some specifications of Thom polynomials for linear types. As was emphasized in §2.1.3.3 we give the polynomials $S_{\mathbb{S}_x\mathbb{S}_y}$ only. The degree of the stratum $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$ is then $S_{\mathbb{S}_x}S_{\mathbb{S}_y}-S_{\mathbb{S}_x\mathbb{S}_y}$. The singular types are specified by their normal forms or by the standard notation.

- $S_{x_1^{p+1}+x_2^{p+1},A_1} = -\frac{3}{4}\binom{p+2}{3}(d-p)^2(3p+4)(p^2+3p+4)+3\binom{p+2}{3}(d-p)(5p+6)$

- $S_{x_1^{p+1}+x_2^{p+1},D_4} = -\frac{5}{8}(d-p)^2(p+1)(3p-1)(p^2+3p+8)(p^2+3p+10)+2(d-p)(p^2+3p+8)(35p^2+20p-12)-6(85p^2+45p-28)$

- $S_{x_1^{p+1}+x_2^{p+1},X_9} = -\frac{5}{8}(d-p)^2(3p+2)(3p-2)(p^2+3p+16)(p^2+3p+18)+2(d-p)(p^2+3p+16)(270p^2-20p-117)-14(830p^2-105p-348)$

- $S_{x_1^{p}+x_2^{p+1},A_1} = -\frac{3}{8}p^3\Big(p(3+p)(d-p)^2(p^2+3p-2)+4(p-1)(d-p)(p^2+3p-2)-8p\Big)$

- $S_{(x_1^{p-1}+x_2^{p})(x_1+x_2^2),A_1} = -9\binom{p+3}{4}p(d-p)^2(4+p+2p^2)-\frac{3}{2}p^2(3+p)(d-p)(p^3-3p^2-p-8)+3p(p^4+3p^3+3p^2+4p-4)$

- $S_{(x_1^{p-1}+x_2^{p-1})(x_1^2+x_2^3),A_1} = \begin{array}{l} -\frac{(d-p)^2}{4}(3p^2+p+2)(p^2+3p+6)(p^2+3p+4)(p+1) \\ +\frac{d-p}{2}(p^2+3p+4)(3p^4+23p^3+30p^2+28p+12)-12-30p-28p^2-21p^3-5p^4 \end{array}$

- $S_{(x_1^{p-1}+x_2^{p-1})(x_1^2+x_2^4),A_1} = \begin{array}{l} -\frac{(d-p)^2}{4}(p^2+3p+6)(p^2+3p+8)(9p^3+18p^2+16p+8) \\ +3\frac{d-p}{2}(p+1)(p^2+3p+6)(3p^3+20p^2+18p+16)-3(32+72p+78p^2+51p^3+10p^4) \end{array}$

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