Max-Planck-Institut für Mathematik
Bonn

Superorbits

by

Alexander Alldridge
Joachim Hilgert
Tilmann Wurzbacher

Max-Planck-Institut für Mathematik
Preprint Series 2015 (13)

Superorbits

Alexander Alldridge
Joachim Hilgert
Tilmann Wurzbacher

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Universität zu Köln
Mathematisches Institut
Weyertal 86-90
50931 Köln
Germany

Universität Paderborn
Institut für Mathematik
33095 Paderborn
Germany

Institut É. Cartan
Université de Lorraine et C.N.R.S.
57045 Metz
France

MPIM 15-13

SUPERORBITS

ALEXANDER ALLDRIDGE, JOACHIM HILGERT, AND TILMANN WURZBACHER

Abstract. We study actions of Lie supergroups, in particular, the hitherto elusive notion of 'orbits through odd (or more general) points'. Following categorical principles, we derive a conceptual framework for their treatment and therein prove general existence theorems for the isotropy supergroups and orbits at general points. In this setting, we show that the coadjoint orbits always admit a (relative) supersymplectic structure of Kirillov–Kostant–Souriau type. Applying a generalisation of Kirillov's orbit method to the parity changed variants of the Heisenberg group, the Clifford supergroup and the odd Heisenberg supergroup, we obtain 'universal' families of representations parametrised by a supermanifold.

1. Introduction

Actions of Lie groups are legion in mathematics. In benign situations, they can be understood in terms of their 'decomposition' into orbits as their basic constituent pieces. The general philosophy of quantisation thus suggests the construction of irreducible representations from orbits on some universal G-space. This was made precise for nilpotent Lie groups by A.A. Kirillov in the form of his orbit method [33], whose phenomenal success precludes a succinct synopsis.

Lie superalgebras, or 'graded Lie algebras', as they were known at the time, were first considered in mathematics in the 1950s, in the context of deformation theory. From about ten years later, physicists used them to encode symmetries of elementary particles with different statistics, see Ref. [19] for a historical account. Supermanifolds, as the classical counterparts of supersymmetric quantum fields, were pioneered by F.A. Berezin, and with G.I. Kac, he introduced the concept of a 'graded Lie group' (or, in current terminology, a Lie supergroup) in 1970 [8]. Five years later, together with D.A. Leites, he recast the notion as that of a group object in the category of supermanifolds [9].

Almost simultaneously, B. Kostant, in his seminal paper [34], gave a detailed account of supermanifolds and Lie supergroups geared towards the orbit method. In that paper, he describes a version of prequantisation as a starting point for their representation theory. In fact, as he remarks in his note [35]: Lie supergroups are "likely to be [...] useful [objects] only insofar as one can develop a corresponding theory of harmonic analysis".

On the geometric side, the theory of homogeneous supermanifolds was quickly developed [13, 34]. However, these are clearly not general enough to reconstruct the total space of an action (consider, *e.g.* the case of an ordinary Lie group acting on a purely odd-dimensional supermanifold). However, this is all one obtains if one considers, for an action of a supergroup G on a superspace X, only orbits of

2010 *Mathematics Subject Classification.* Primary 14L30, 58A50; Secondary 14M30, 32C11, 53D50, 57S20.

Key words and phrases. Categorical quotients, Lie supergroup actions, superorbit method, supermanifold.

Research funded by Deutsche Forschungsgemeinschaft (DFG), grant nos. SFB/TR 12 (all authors), ZI 513/2-1 (A.A.), and the Institutional Strategy of the University of Cologne within the German Excellence Initiative (A.A.).

G through the ordinary points $x \in X_0$, which in the case of the coadjoint action corresponds to even elements in the dual of the Lie superalgebra.

Somewhat surprisingly, this point of view is sufficient to construct the unitary irreducible representations of a nilpotent Lie supergroup, as was shown by H. Salmasian [42]. This is, however, related to the restrictive nature of unitary representations for Lie supergroups, which are all of 'positive energy' type (see Refs. [14, 15] for a modern account of unitary representations of Lie supergroups and Ref. [41] for the interplay of unitarity and positivity).

For the purposes of harmonic analysis, this is however insufficient, even for the very basic case of the Clifford supergroup, as was shown in Ref. [5]. In this vein, it has repeatedly been observed—for the first time, by V. Kac [30, 5.5.4]—that it would be highly desirable to quantise also coadjoint orbits through odd or mixed functionals on the Lie superalgebra $\mathfrak{g}$, see also Refs. [10, 43, 44].

To understand such functionals as 'points' of $\mathfrak{g}^*$, it is crucial to broaden the notion of points. Following A. Grothendieck, a *T-valued point* of a space X is a map $T \longrightarrow X$. This idea is based on considering an ordinary point as a map $* \longrightarrow X$, allowing the (singleton) parameter space to acquire additional degrees of freedom. The G-isotropy through a T-valued point $x : T \longrightarrow X$ should then be a 'group bundle' $G_x \longrightarrow T$, and the orbit, a bundle $G \cdot x \longrightarrow T$ with a fibrewise G-action.

Based on the work of A. Grothendieck and P. Gabriel, such a framework was formulated for the study of algebraic group actions in the category of schemes by D. Mumford in his influential monograph [40]. However, to the present day, this does not seem to have been fully appreciated in the context of supergeometry. Moreover, a differentiable version of this theory is so far only partially available, although necessary for applications: Indeed, while all Lie groups are real analytic, any non-analytic (complete) vector field gives rise to an action which is not analytic (much less algebraic). Such situations are ubiquitous, particularly in the context of solvable Lie groups.

In this paper, we develop the notion of orbits through general T-valued points for Lie supergroups in the differentiable (and, simultaneously, the analytic) category. Not only does this lead to general existence statements for orbits through general T-valued points, it also allows the construction of a Kirillov–Kostant–Souriau symplectic structure (over T) on any such coadjoint orbit, and, as we show in examples, the construction of representations (on vector space bundles over T) *via* a generalisation of the Kirillov orbit method. This conceptual foundation neatly accommodates all previously known examples and unravels a vast vista of potential applications.

Let us review our main results in greater detail. Let G be a Lie supergroup acting on a supermanifold X (which might be differentiable, real or complex analytic) and $x : T \longrightarrow X$ an embedding of supermanifolds. Consider the 'rank function'

$$r_x(t) := \dim\bigl\{v \in \mathfrak{g} \bigm| \text{the germ of } x^\sharp \circ a_v \text{ vanishes at } t\bigr\}, \quad \forall t \in T_0,$$

where a_v denotes the fundamental vector field on X associated with v. Then our existence theorems on isotropies and orbits (Theorem 4.20 and Theorem 4.24) encompass the following statement (see the main text for details):

Superorbit Theorem. *The isotropy group G_x exists as a Lie supergroup over T if r_x is locally constant. Moreover, in this case, the orbit $G \cdot x \longrightarrow X$ through x exists as an equivariant local embedding of supermanifolds over T.*

In particular, this applies when the manifold T_0 underlying T is a point; in case T itself is a point, we recover the usual orbits through ordinary points [13, 16, 34].

In the situation of the coadjoint action of G on $\mathfrak{g}^*$, we write $f = x$. Our main result concerning this action is the following theorem (Theorem 5.4).

Supersymplectic Orbit Form Theorem. *Assume that r_f is locally constant. Then the coadjoint orbit $G \cdot f$ admits a canonical supersymplectic structure over T.*

Combining this methodology with the general philosophy of Kirillov's orbit method, we construct, by way of example, representations for all super versions of the three-dimensional Heisenberg group which arise by assigning arbitrary parities to the generators in the commutation relation $[x, y] = z$. In every case, we find a 'universal' parameter space T (which will be $\mathbb{A}^1$ or $\mathbb{A}^{0|1}$) and realise the representations as bundles of functors

$$\mathcal{H} \longrightarrow T$$

whose fibres are super-vector spaces. Not surprisingly, these bear a striking similarity to the Schrödinger representation.

The new feature, which we wish to emphasise, is that these representations are output by a well-defined procedure within a general conceptual framework which allows for the treatment of much more than just these examples.

We conclude the introduction by summarising the paper's contents. In Section 2, we present general categorical notions for the study of actions. In Section 3, we review categorical quotients in the setting of differentiable and analytic superspaces and suggest a weak notion of geometric quotients. In Section 4, we specialise the discussion to supermanifolds. We prepare our discussion of isotropy supergroups by generalising the notion of morphisms of constant rank to relative supermanifolds (over a possibly singular base). We prove a rank theorem in this context (Proposition 4.16); this is based on an extension of the inverse function theorem presented in the appendix (Theorem A.1). We investigate when the orbit morphism through a general point has constant rank (Theorem 4.18) and, as an application, show the representability of isotropy supergroups under general assumptions (Theorem 4.20). This gives the existence of orbits under the same assumptions (Theorem 4.24). In Section 5, we construct the relative Kirillov–Kostant–Souriau form for coadjoint orbits through general points (Theorem 5.4). Finally, in Section 6, we construct 'universal' families of representations for the super variants of the Heisenberg group.

Acknowledgements. We wish to thank the Max-Planck Institute for Mathematics in Bonn, where much of the work on this article was done, for its hospitality.

2. A categorical framework for group actions

2.1. **Categorical groups and actions.** Groups and actions can be defined quite generally for categories with finite products. In this subsection, we recall the relevant notions and give a number of examples from differents contexts, which will serve to illustrate our further elaborations.

In what follows, let $\mathbf{C}$ be a category with a terminal object $*$. For any $S, T \in \operatorname{Ob} \mathbf{C}$, let $\mathbf{C}_T^S$ be the category of objects in $\mathbf{C}$, which are under S and over T. That is, objects and morphisms are given by the commutative diagrams depicted below:

$$\begin{array}{c} S \\ \downarrow \\ X \\ \downarrow \\ T \end{array} \qquad \begin{array}{ccc} S & = & S \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & = & T \end{array}$$

Similarly, we define the categories $\mathbf{C}_T$ of objects over T and $\mathbf{C}^S$ of objects under S.

We recall the definition of group objects and actions. These concepts are well-known, see *e.g.* Ref. [37].

Definition 2.1 (Groups and actions)**.** A $\mathbf{C}$-*group* is the data of $G \in \operatorname{Ob}\mathbf{C}$, such that all non-empty finite products $G \times \cdots \times G$ exist in $\mathbf{C}$, together with morphisms

$$1 = 1_G : * \longrightarrow G, \quad i : G \longrightarrow G, \quad m : G \times G \longrightarrow G$$

satisfying for any $S \in \operatorname{Ob}\mathbf{C}$ and any $r, s, t \in_S G$ the group laws

$$1r = r1 = r, \quad rr^{-1} = 1 = r^{-1}r, \quad (rs)t = r(st),$$

where we denote $st \coloneqq m(s,t)$ and $s^{-1} \coloneqq i(s)$. In particular, $*$ is in a unique fashion a $\mathbf{C}$-group, called the *trivial* $\mathbf{C}$*-group*.

Given a $\mathbf{C}$-group G with structural morphisms 1, i, and m, we define the *opposite* $\mathbf{C}$*-group* G° to G, together with the morphisms 1 and i, and $m^\circ : G \times G \longrightarrow G$, where the latter is defined by $m^\circ(s,t) \coloneqq m(t,s)$ for all $T \in \operatorname{Ob}\mathbf{C}$ and $s, t \in_T G$.

Let $X \in \operatorname{Ob}\mathbf{C}$ and assume that the non-empty finite products $Y_1 \times \cdots \times Y_n$ exist in $\mathbf{C}$, where $Y_j = G$ or $Y_j = X$ for any j. A (left) *action* of a $\mathbf{C}$-group G in $\mathbf{C}$, interchangeably called a (left) G*-space*, consists of the data of X and a morphism

$$a : G \times X \longrightarrow X,$$

written $g \cdot x = a(g,x)$, for which we have

$$1 \cdot x = x, \quad (rs) \cdot x = r \cdot (s \cdot x)$$

for any $S \in \operatorname{Ob}\mathbf{C}$, $x \in_S X$, and $r, s \in_S G$. Slightly abusing terminology, it is sometimes the morphism a that is called an action and the space X that is called a G-space. A G°-space is called a *right* G*-space*. An action of G° is called a *right action* of G.

Remark 2.2. The data in the definition of a $\mathbf{C}$-group are not independent. Given m and 1 satisfying all above equations not involving i, there is at most one morphism i with the above conditions verified. Similarly, 1 is determined uniquely by m.

Since the Yoneda embedding preserves limits, a $\mathbf{C}$-group is the same thing as an object G of $\mathbf{C}$ whose point-functor $G(-) = \operatorname{Hom}_{\mathbf{C}}(-, G)$ is group-valued. Actions can be characterised similarly.

Example 2.3. Group objects and their actions are ubiquitous in mathematics. Since our main interest lies in supergeometry, we begin with two examples from this realm.

(i) The general linear supergroup $\mathrm{GL}(m|n)$ is a complex Lie supergroup (*i.e.* a group object in the category of complex-analytic supermanifolds). Its functor of points is given on objects T by

$$\mathrm{GL}(m|n)(T) \coloneqq \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, \begin{matrix} A \in \mathrm{GL}(m, \mathcal{O}_{\bar 0}(T)), B \in \mathcal{O}_{\bar 1}(T)^{m \times n} \\ C \in \mathcal{O}_{\bar 1}(T)^{n \times m}, D \in \mathrm{GL}(n, \mathcal{O}_{\bar 0}(T)) \end{matrix} \right\}.$$

Here, we let $\mathcal{O}_k(T) \coloneqq \Gamma(\mathcal{O}_{T,k})$, $k = \bar 0, \bar 1$. The group structure is defined by the matrix unit, matrix inversion and multiplication at the level of the point functor.

For $X = \mathbb{A}^{m|n}$, we have

$$X(T) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, a \in \mathcal{O}_{\bar 0}(T)^{m \times 1}, b \in \mathcal{O}_{\bar 1}(T)^{n \times 1} \right\}.$$

Hence, an action of $\mathrm{GL}(m|n)$ on X is given at the level of the functor of points by the multiplication of matrices with column vectors.

As another example, consider $X = \mathrm{Gr}_{p|q,m|n}$, the super-Grassmannian of $p|q$-planes in $m|n$-space (where $p \leqslant m$ and $q \leqslant n$). For affine T, the point functor takes on the form

$$X(T) = \left\{ Z \,\middle|\, Z \text{ rank } p|q \text{ direct summand of } \mathcal{O}(T)^{m|n} \right\}.$$

Again, $\mathrm{GL}(m|n)$ acts by left multiplication of matrices on column vectors. For general T (which need not be affine), the functor of points can be computed in terms of locally direct subsheaves, compare Ref. [38].

(ii) In the category $\mathbf{C}$ of $(\mathbb{K},\Bbbk)$-supermanifolds [4], where $\Bbbk\subseteq\mathbb{K}$ and both are $\mathbb{R}$ or $\mathbb{C}$, consider the affine superspace $G:=\mathbb{A}^{0|1}$ with the odd coordinate τ. Then $G(T)=\mathcal{O}_{\bar 1}(T)$, and the addition of odd superfunctions gives G the structure of a supergroup.

Let X be a manifold. The total space ΠTX of the parity reversed tangent bundle of X has the underlying manifold X and the sheaf of superfunctions $\mathcal{O}_{\Pi TX}=\Omega^\bullet_X$, the sheaf of differential forms, with the $\mathbb{Z}/2\mathbb{Z}$ grading induced by the $\mathbb{Z}$-grading.

The supermanifold ΠTX has the point functor

$$\Pi TX(T)\cong \mathrm{Hom}_{\mathbf{C}}(T\times\mathbb{A}^{0|1},X).$$

We denote elements on the left-hand side by f and the corresponding elements on the right-hand side by $\tilde f$.

We may let $x\in_T G$ act on $f\in_T \Pi TX$ by defining $x\cdot f$ *via*

$$(x\cdot f)^\sim : T\times\mathbb{A}^{0|1}\longrightarrow X:(t,y)\in_R (T\times\mathbb{A}^{0|1})\longmapsto \tilde f(t,y+x(t))\in_R X.$$

If X has local coordinates (x^a), then ΠTX has local coordinates (x^a,dx^a). If $f\in_T\Pi TX$, then in terms of the point functor above, we have

$$f^\sharp(x^a)=j^\sharp(\tilde f^\sharp(x^a)),\quad f^\sharp(dx^a)=j^\sharp\big(\tfrac{\partial}{\partial\tau}\tilde f^\sharp(x^a)\big).$$

Here, $j:T\longrightarrow T\times\mathbb{A}^{0|1}$ is the unique morphism over T defined by $j^\sharp(\tau):=0$, τ denoting the standard odd coordinate function on $\mathbb{A}^{0|1}$.

From this description, we find that the action of G on ΠTX is the morphism

$$a:G\times\Pi TX\longrightarrow\Pi TX,\quad a^\sharp(\omega)=\omega+\tau d\omega.$$

Expanding on this example a little, one may consider the action α of $(\mathbb{A}^1,+)$ on $\mathbb{A}^{0|1}$ given by dilation, *i.e.* $\alpha^\sharp(\tau)=e^t\tau$. This defines a semi-direct product group $G':=\mathbb{A}^1\ltimes\mathbb{A}^{0|1}$, and the action a considered above may be extended to G' by dilating and translating in the $\mathbb{A}^{0|1}$ argument.

In terms of local coordinates, the thus extended action is given by

$$a^\sharp(\omega)=e^{nt}(\omega+\tau d\omega),$$

for ω of degree n, compare [29, Lemma 3.4, Proposition 3.9].

(iii) Let $G:=\mathbb{A}^{0|1}$ with its standard additive structure and $X:=\mathbb{A}^{1|1}$. Then G acts on X *via* $a:G\times X\longrightarrow X$, defined by

$$a(\gamma,(y,\eta)):=(y+\gamma\eta,\eta)$$

for all R and $\gamma\in_R G$, $(y,\eta)\in_R X$. In terms of the standard coordinates γ on G and (y,η) on X, we have

$$a^\sharp(y)=y+\gamma\eta,\quad a^\sharp(\eta)=\eta.$$

Example 2.4. Complementing our examples from supergeometry, we give a list of examples for categorical groups and actions from different contexts.

(i) Let G be a $\mathbf{C}$-group. Any $X\in\mathrm{Ob}\,\mathbf{C}$ can be endowed with a natural G-action, given by taking $a:G\times X\longrightarrow X$ to be the second projection. That is, $g\cdot x:=x$ for all $T\in\mathrm{Ob}\,\mathbf{C}$, $g\in_T G$, and $x\in_T X$. This action is called *trivial.*

(ii) Any $\mathbf{C}$-group G is both a left and a right G-space, by the assignments

$$g\cdot x:=gx\quad\text{or}\quad x\cdot g:=xg,$$

respectively, for all $T\in\mathrm{Ob}\,\mathbf{C}$, $g\in_T G$, and $x\in_T X$.

(iii) Topological groups and Lie groups, and their actions on topological spaces and smooth manifolds, respectively, are examples of categorical groups and actions.

(iv) Group schemes and their actions on schemes are examples of categorical groups and actions as well. Compare, *e.g.* Refs. [21, 40].

(v) A pointed (compactly generated) topological space (W, w_0) is called an *H-group*, if it is equipped with based continuous maps $\mu : W \times W \longrightarrow W$, $e : W \longrightarrow W$ with $e(W) = w_0$, and $j : W \longrightarrow W$ such that the following holds:

$$\mu \circ (e, \mathrm{id}_W) \simeq \mu \circ (\mathrm{id}_W, e) \simeq \mathrm{id}_W,$$
$$\mu \circ (\mu \times \mathrm{id}_W) \simeq \mu \circ (\mathrm{id}_W \times \mu), \quad \mu \circ (\mathrm{id}_W, j) \simeq \mu \circ (j, \mathrm{id}_W) \simeq e,$$

where $\simeq$ denotes based homotopy equivalence, *cf.* [1, Section 2.7]. Given a pointed, compactly generated topological space (X, x_0), its based loop space ΩX is a prime example of an H-group.

In the category $\mathbf{C}$ of pointed, compactly generated topological spaces with based homotopy classes of continuous maps as morphisms, an H-group together with the homotopy classes of μ, e, and j is simply a $\mathbf{C}$-group. The basic theorem that the set $[X, W]_* = \mathrm{Hom}_{\mathbf{C}}(X, W)$ of based homotopy classes has a group structure that is natural in the variable X if and only if W is an H-group [1, Theorem 2.7.6] is an instance of Remark 2.2.

If now $(G, 1_G) = (W, w_0)$ is an H-group and (X, x_0) a pointed topological space, then a pointed continuous map $a : G \times X \longrightarrow X$ is a group action in $\mathbf{C}$ if and only if $a(1_G, \cdot)$ is pointed homotopy equivalent to id_X and the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times a} & G \times X \\ {\scriptstyle \mu \times \mathrm{id}_X} \big\downarrow & & \big\downarrow {\scriptstyle a} \\ G \times X & \xrightarrow{\quad a \quad} & X \end{array}$$

commutes up to a pointed homotopy.

(vi) In the theory of integrable systems one encounters the following situation: (M, ω) is a symplectic manifold of dimension $2n$ and $\rho : M \longrightarrow B$ is a fibration whose fibres are compact, connected Lagrangian submanifolds. Then there is a smooth fibrewise action of T^*B on M. In the above language, $T^*B \longrightarrow B$ is a group in the category of smooth manifolds over B, and it acts on $X = (M \longrightarrow B)$.

To see this latter fact, let $m \in M$, $b = \rho(m)$, and $M_b := \rho^{-1}(b)$. The dual of the differential of ρ is an injective linear map $(T_m\rho)^* : T_b^*B \longrightarrow T_m^*M$ whose image is the annihilator of $T_m(M_b)$. Since M_b is Lagrangian, the musical isomorphism $\omega_m^\flat : T_m^*M \longrightarrow T_mM$ identifies this annihilator space with $T_m(M_b)$. We thus have canonical linear isomorphisms $T_b^*B \to T_m(M_b)$ depending smoothly on m. Given $v \in T_b^*B$, we obtain a smooth vector field $\hat{v}$ on M_b.

It is easy to see that these vector fields extend to a commuting family of Hamiltonian vector fields on M, and that a linearly independent set of elements of T_b^*B yields vector fields on the fibre M_b that are everywhere independent. Since M_b is compact, we obtain an action of the additive group of T_b^*B whose isotropy is a cocompact lattice Λ_b [27, Theorem 44.1].

2.2. **Isotropies at generalised points.** For many applications of group actions, the notion of isotropy groups is essential. In the categorical framework, we can consider isotropy groups through T-valued points, by following the general philosophy of base change and specialisation.

Construction 2.5 (Base change of groups and actions)**.** Let G be a $\mathbf{C}$-group, X a G-space and $T \in \mathrm{Ob}\,\mathbf{C}$. We assume that the finite products $T \times Y_1 \times \cdots \times Y_n$ exist in $\mathbf{C}$ for any choice of $Y_j = X$ or $Y_j = G$.

Consider the category $\mathbf{C}_T$. The morphism $\mathrm{id}_T : T \longrightarrow T$ is a terminal object in $\mathbf{C}_T$. Non-empty finite products in $\mathbf{C}_T$, provided they exist, are fibre products $\times_T$ over T in $\mathbf{C}$. Thus, if we denote

$$G_T := T \times G, \quad X_T := T \times X,$$

then

$$(Y_1)_T \times_T \cdots \times_T (Y_n)_T = T \times Y_1 \times \cdots \times Y_n = (Y_1 \times \cdots \times Y_n)_T$$

exist as finite products in $\mathbf{C}_T$. Thus, it makes sense to define on G_T and X_T the structure of a $\mathbf{C}_T$-group and a G_T-space, respectively. The $\mathbf{C}_T$-group structure

$$1 = 1_{G_T} : T \longrightarrow G_T, \quad i = i_{G_T} : G_T \longrightarrow G_T, \quad m = m_{G_T} : G_T \times_T G_T \longrightarrow G_T$$

on G_T is defined by the equations

$$1(t) := (t, 1), \quad (t, g)^{-1} := (t, g^{-1}), \quad (t, g)(t, h) := (t, gh)$$

for all $g, h \in_R G$ and $t \in_R T$, where we have written all morphisms in $\mathbf{C}$ and used the notational conventions from Definition 2.1.

Similarly, X_T is a G_T-space *via*

$$G_T \times_T X_T \longrightarrow X_T : (t, g) \cdot (t, x) := (t, g \cdot x)$$

for all $g \in_R G$, $x \in_R G$, and $t \in_R T$.

As we have seen, groups and actions are easily defined in the full generality of categories with terminal objects. Possibly after base change and specialisation, it will be sufficient to consider isotropy groups only through ordinary points. Their definition on the level of functors presents no difficulty.

We will define isotropy groups at ordinary points, passing to the general case of T-valued points only after base change. This definition is equivalent to the one given in Ref. [40] in the case of schemes over some base scheme.

Definition 2.6 (Isotropy group)**.** Let G be a $\mathbf{C}$-group and X a G-space. We write $X_0 := X(*)$ and call the elements of this set the *ordinary points* of X. Let $x \in X_0$. The *isotropy at* x is the functor $G_x : \mathbf{C} \longrightarrow \mathbf{Sets}$ whose object map is defined by

$$G_x(R) := \big\{ g \in_R G \;\big|\; g \cdot x = x \big\},$$

for any $R \in \operatorname{Ob} \mathbf{C}$. In other words, G_x is the fibre product defined by the following diagram in the category of set-valued functors on $\mathbf{C}$:

$$\begin{array}{ccc} G_x & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle a_x} \\ * & \xrightarrow{\;x\;} & X \end{array}$$

Here, $a_x : G \longrightarrow X$ is the *orbit morphism* defined by

$$a_x(g) := g \cdot x \tag{2.1}$$

for all $R \in \operatorname{Ob} \mathbf{C}$ and $g \in_R G$.

The functor G_x is group-valued. Indeed, let $R \in \operatorname{Ob} \mathbf{C}$. By construction, an R-valued point $g \in G_x(R)$ is just $g \in_R G$ such that $g \cdot x = x$. If $g, h \in G_x(R)$, then

$$(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x,$$

so $gh \in G_x(R)$. Taking this as the definition of the group law on G_x, we see that the canonical morphism $G_x \longrightarrow G$ preserves this operation. Since $G(R)$ is a group, so is $G_x(R)$, and this proves the assertion. In particular, if G_x is representable and the finite direct products $G_x \times \cdots \times G_x$ exist, then G_x is a $\mathbf{C}$-group.

The above definition can be combined with Construction 2.5 to give a satisfactory definition of the isotropy of an action at a T-valued point, as we now proceed to explain in detail.

Let G be a $\mathbf{C}$-group, X a G-space, and $x \in_T X$. Recall the natural bijection

$$\operatorname{Hom}_{\mathbf{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathbf{C}_T}(A, B_T) : f \longmapsto (p_A, f), \tag{2.2}$$

valid for any $(p_A : A \longrightarrow T) \in \operatorname{Ob} \mathbf{C}_T$ and any $B \in \operatorname{Ob} \mathbf{C}$.

Applying this to $A = T = *_T$, we obtain in the notation of Definition 2.6

$$(X_T)_0 = \mathrm{Hom}_{\mathbf{C}_T}(*_T, X_T) = \mathrm{Hom}_{\mathbf{C}}(T, X) = X(T).$$

Thus, we may consider x as an ordinary point of $X_T \in \mathrm{Ob}\,\mathbf{C}_T$.

By Construction 2.5, G_T is a $\mathbf{C}_T$-group and X_T is a G_T-space. In particular, we obtain an *orbit morphism* $a_x : G_T \longrightarrow X_T$ in $\mathbf{C}_T$, from Equation (2.1). It is the composite

$$T \times G \xrightarrow{(\mathrm{id}_T, x)\times \mathrm{id}_G} T \times X \times G \xrightarrow{\mathrm{id}_T \times (a\circ\sigma)} T \times X,$$

denoting the action of G on X by a, and by σ the exchange of factors, *i.e.*

$$a_x(t, g) = \big(t, g \cdot x(t)\big), \quad \forall t \in_R T, g \in_R G. \tag{2.3}$$

The objects $T = *_T$, G_T, and X_T in the category $\mathbf{C}_T$ are promoted to covariant functors on $\mathbf{C}_T$. Similarly, x and $a_x : G_T \longrightarrow X_T$ are promoted to natural transformations. We now pose the following definition.

Definition 2.7 (Isotropy functor)**.** The *isotropy functor* $G_x := (G_T)_x : \mathbf{C}_T \longrightarrow$ **Sets** is the fibre product defined by the diagram

$$\begin{array}{ccc} G_x & \longrightarrow & G_T \\ \downarrow & & \downarrow{\scriptstyle a_x} \\ T = *_T & \xrightarrow{x} & X_T \end{array}$$

in the category of set-valued functors on $\mathbf{C}_T$.

Remark 2.8. This coincides with Mumford's definition [40] in the case of $\mathbf{C} = \mathrm{Sch}_S$.

Consider now the following diagram in the category $\mathbf{C}$:

$$\begin{array}{ccc} & & T \times G \\ & & \downarrow{\scriptstyle a_x} \\ T & \xrightarrow{(\mathrm{id}_T, x)} & T \times X \end{array}$$

Its fibre product is the functor given on $R \in \mathrm{Ob}\,\mathbf{C}$ by

$$\begin{aligned} \big(T \times_{T\times X} (T \times G)\big)(R) &= \left\{ (t_1, t_2, g) \in_R (T \times T \times G) \;\middle|\; \begin{array}{c} t_1 = t_2 \\ x(t_1) = g \cdot x(t_2) \end{array} \right\} \\ &= \big\{ (t, g) \in_R (T \times G) \;\big|\; g \cdot x(t) = x(t) \big\}. \end{aligned}$$

If R comes with morphisms $R \longrightarrow T$ and $R \longrightarrow T \times G$ in $\mathbf{C}$ completing the fibre product diagram above, then we may consider $R \in \mathrm{Ob}\,\mathbf{C}_T$ *via* either of the T-projections thus obtained. The above computation then gives

$$G_x(R) = \big(T \times_{T\times X} (T \times G)\big)(R).$$

Hence, the representability of the functor $G_x = (G_T)_x$ in $\mathbf{C}_T$ is equivalent to the existence of this fibre product in $\mathbf{C}$.

Example 2.9. Recall the notation from Example 2.3 (iii). Any point $p \in X_0 = X(*)$ gives rise to $p_R \in X(R)$ and we obviously have $\gamma \cdot p_R = p_R$ for all $\gamma \in_R G$ and all $R \in \mathbf{SSp}^{\mathrm{lfg}}_{\mathbb{K}}$, see Section 3 and Ref. [4] for the terminology. Thus, we have $G_p = G$ as functors, so G_p is represented by the Lie supergroup G.

By contrast, take $T = \mathbb{A}^{0|1}$ with the odd coordinate θ and define $x \in_T X$ by

$$x^\sharp(y) := 0, \quad x^\sharp(\eta) := \theta.$$

where we might as well take any other number for $x^\sharp(y)$. That is, for any $R \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$, we have

$$x(\theta) = (0, \theta), \quad \forall \theta \in_R T.$$

In this case, the isotropy functor G_x evaluates on any $R \in \mathbf{SSp}_T^{\mathrm{lfg}}$ as

$$G_x(R) = \big\{(\theta, \gamma) \in_R (T \times G) \bigm| \gamma\theta = 0\big\}.$$

Therefore, G_x is represented by the superspace

$$\operatorname{Spec} \mathbb{K}[\theta, \gamma]/(\theta\gamma) = \big(*, \mathbb{K}[\theta, \gamma]/(\theta\gamma)\big),$$

where θ, γ are odd indeterminates. It lies over T *via* the morphism

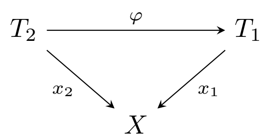
$$p : G_x \longrightarrow T, \quad p^\sharp(\theta) := \theta.$$

The group multiplication works out to be

$$m : G_x \times_T G_x \longrightarrow G_x, \quad m^\sharp(\gamma) := \gamma^1 + \gamma^2,$$

where $\gamma^i := p_i^\sharp(\gamma)$. Thus, G_x is a group object in $\mathbf{SSp}_T^{\mathrm{lfg}}$ but not given by a Lie supergroup over T.

Definition 2.10 (Specialisation of a point)**.** Let $\mathbf{C}$ be a category, T_1, T_2, X be objects in $\mathbf{C}$. Given two points $x_1 \in_{T_1} X$ and $x_2 \in_{T_2} X$, we say that x_2 is a *specialisation* of x_1 if for some morphism $\varphi : T_2 \longrightarrow T_1$ in $\mathbf{C}$, the following diagram commutes:



Proposition 2.11. *Let G be a $\mathbf{C}$-group and X a G-space. Let $x_1 \in_{T_1} X$ and $x_2 \in_{T_2} X$ such that x_2 is a specialisation of x_1. Then there is a natural isomorphism*

$$T_2 \times_{T_1} G_{x_1} = G_{x_2}$$

of **Sets***-valued functors on $\mathbf{C}_{T_2}$.*

In particular, if G_{x_1} is representable in $\mathbf{C}_{T_1}$, then G_{x_2} is representable in $\mathbf{C}_{T_2}$ if and only if the fibre product $T_2 \times_{T_1} G_{x_1}$ exists in $\mathbf{C}$.

Proof. By assumption, we have $x_2 = x_1 \circ \varphi$ for some morphism $\varphi : T_2 \longrightarrow T_1$ in $\mathbf{C}$. We compute for each $R \in \operatorname{Ob} \mathbf{C}$ and $(t, g) \in_R G_{T_2}$ that

$$g \cdot x_2(t) = g \cdot x_1(\varphi(t)),$$

so that the map $(t, g) \longmapsto (t, \varphi(t), g)$ on R-valued points defines a natural bijection

$$G_{x_2}(R) \longrightarrow \big(T_2 \times_{T_1} G_{x_1}\big)(R).$$

This proves the assertion. □

Definition 2.12 (Free G-spaces)**.** Let G be a $\mathbf{C}$-group and X a G-space. Given a T-valued point $x \in_T X$, the G-space X is called *free at x* if $(G_T)_x$ is the trivial group in the category of **Sets**-valued functors on $\mathbf{C}_T$. It is simply called *free* if it is free at any $x \in_T X$, for any $T \in \operatorname{Ob} \mathbf{C}$.

As the following corollary to Proposition 2.11 shows, it is equivalent to require that X be free at the generic point $x = \mathrm{id}_X \in_X X$.

Corollary 2.13. *Let G be a $\mathbf{C}$-group and X a G-space. Assume that X is free at the generic point $x = \mathrm{id}_X \in_X X$. Then X is free.*

2.3. **Quotients and orbits.** In this subsection, we introduce basic facts and terminology relating to quotients and orbits. For that purpose, the language of groupoids is convenient. We briefly recall it. In what follows, we let **C** be a category with all finite products.

Definition 2.14 (Groupoids)**.** Let $X \in \mathrm{Ob}\,\mathbf{C}$. A **C**-*groupoid on* X is a $\Gamma \in \mathrm{Ob}\,C$, together with morphisms $s, t : \Gamma \longrightarrow X$—called *source* and *target*—such that all finite fibre products

$$\Gamma^{(n)} := \Gamma \times_X \Gamma \times_X \cdots \times_X \Gamma = \Gamma \times_{s,X,t} \Gamma \times_{s,X,t} \cdots \times_{s,X,t} \Gamma$$

exist, and morphisms

$$1 : X \longrightarrow \Gamma, \quad i : \Gamma \longrightarrow \Gamma, \quad m : \Gamma^{(2)} \longrightarrow \Gamma$$

—where the first and third are over $X \times X$ (where we consider X as lying over $X \times X$ *via* Δ_X and Γ as lying over $X \times X$ *via* (t, s)) and the second is over the flip $\sigma : X \times X \longrightarrow X \times X$—such that the following diagrams commute:

$$\begin{array}{ccc} \Gamma^{(3)} & \xrightarrow{m\times_X \mathrm{id}} & \Gamma^{(2)} \\ {\scriptstyle \mathrm{id}\times_X m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ \Gamma^{(2)} & \xrightarrow[m]{} & \Gamma \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{(1\circ t)\times_X \mathrm{id}} & \Gamma^{(2)} \\ {\scriptstyle \mathrm{id}\times_X (1\circ s)}\big\downarrow & \searrow\!\!\!\!= & \big\downarrow{\scriptstyle m} \\ \Gamma^{(2)} & \xrightarrow[m]{} & \Gamma \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{s} & X \\ {\scriptstyle (i,\mathrm{id})}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ \Gamma^{(2)} & \xrightarrow[m]{} & \Gamma \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{(\mathrm{id},i)} & \Gamma^{(2)} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ X & \xrightarrow[1]{} & \Gamma. \end{array}$$

A morphism $\varphi : X \longrightarrow Y$ in **C** that coequalises s and t, *i.e.*

$$\varphi \circ s = \varphi \circ t : \Gamma \longrightarrow Y$$

will be called Γ-*invariant*.

A *subgroupoid* of Γ is a monomorphism $j : \Gamma' \longrightarrow \Gamma$ with the induced source and target morphisms, such that 1, $i \circ j$, and $m \circ (j \times_X j)$ factor through j.

Example 2.15. We will need the following three simple examples of groupoids.

(i) Let G be a **C**-group and X be a G-space with action morphism a. Then $\Gamma := G \times X$ is a **C**-groupoid over X, called the *action groupoid* of a. Its structural morphisms are

$$s := p_2 : \Gamma \longrightarrow X, \quad t := a : \Gamma \longrightarrow X, \quad 1 := (1_G, \mathrm{id}_X) : X \longrightarrow \Gamma,$$

as well as the inversion i and multiplication m defined by

$$i(g, x) := (g^{-1}, g \cdot x), \quad m(g_1, x, g_2) := (g_1 g_2, x), \quad \forall g_1, g_2 \in_T G, x \in_T X,$$

respectively. Here, we identify $\Gamma^{(2)} = G \times X \times G$ *via* the morphism induced by $\mathrm{id}_\Gamma \times p_1 : \Gamma \times \Gamma \longrightarrow G \times X \times G$.

(ii) Let $X \in \mathrm{Ob}\,\mathbf{C}$. Then $\Gamma := X \times X$ is a **C**-groupoid over X, called the *pair groupoid* of X. Its structural morphisms are

$$s := p_1, t := p_2 : \Gamma \longrightarrow X, \quad 1 := \Delta_X : X \longrightarrow \Gamma,$$

as well the inversion i and multiplication m defined by

$$i(x, y) := (y, x), \quad m(x, y, z) := (x, z), \quad \forall x, y, z \in_T X,$$

respectively. Here, we identify $\Gamma^{(2)} = X \times X \times X$ *via* the morphism induced by $\mathrm{id}_\Gamma \times p_2 : \Gamma \times \Gamma \longrightarrow X \times X \times X$,

(iii) Let $X \in \mathrm{Ob}\,\mathbf{C}$. By definition, an *equivalence relation* on X is a subgroupoid R of the pair groupoid of X. The definition seems to be due to Gabriel [24]. Almorox [7, Definition 2.1] was the first to adapt this definition to the case of supermanifolds.

In terms of the language of groupoids introduced above, we now recall the notion of categorical quotients [40].

Definition 2.16 (Categorical quotients)**.** Let $X \in \operatorname{Ob}\mathbf{C}$ and Γ be a $\mathbf{C}$-groupoid on X. A morphism $\pi : X \longrightarrow Q$ is called a *categorical quotient* of X by Γ if it is universal among Γ-invariant morphisms. That is, the morphism π is Γ-invariant, and for any Γ-invariant morphism $f : X \longrightarrow Y$, where $Y \in \operatorname{Ob}\mathbf{C}$, there is a unique morphism $\tilde{f} : Q \longrightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Q \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & Y \end{array}$$

By abuse of notation, we also say that Q is a categorical quotient (of X by Γ).

We say that $\pi : X \longrightarrow Q$ is a *universal categorical quotient* if for all morphisms $Q' \longrightarrow Q$, the fibre products $X' := Q' \times_Q X$ and $\Gamma' := (Q' \times Q') \times_{Q\times Q} \Gamma$ exist, and $\pi' := Q' \times_Q \pi : X' \longrightarrow Q'$ is a categorical quotient of X' by Γ'.

We use the notation X/Γ for categorical quotients. In case Γ is the action groupoid for the left (respectively, right) action of a $\mathbf{C}$-group G, we write $G\backslash X$ (respectively, X/G) for the categorical quotient (if it exists).

We now apply these notions to pointed spaces, to arrive at a definition of orbits. At this point, we have to depart from Mumford's definitions [40, Definition 0.4], since the notion of scheme-theoretic image does not apply to the setting of smooth supermanifolds that we are primarily interested in.

For any category $\mathbf{C}$ with a terminal object $*$, we define the category $\mathbf{C}^*$ of *pointed spaces* to be the category of objects and morphisms under $*$. We denote the objects $* \longrightarrow X$ in this category by (X, x).

Definition 2.17 (Categorical orbits)**.** Let G be a $\mathbf{C}$-group and X be a G-space. Let $x \in_T X$, where $T \in \operatorname{Ob}\mathbf{C}$ is arbitrary. Assume that G_x is representable in $\mathbf{C}_T$. Being a group object in that category, it is naturally pointed by the unit. Since the unit acts trivially, we have a right G_x-action on G_T in $(\mathbf{C}_T)^*$. If it exists, the categorical quotient $\pi_x : G_T \longrightarrow G_T/G_x$ in $(\mathbf{C}_T)^*$ is called the *categorical orbit* of G through x, and denoted by $\pi_x : G_T \longrightarrow G \cdot x$. If the quotient is universal categorical, then we say that the orbit is *universal categorical.*

The space X_T is pointed by

$$x_T := (\mathrm{id}_T, x) : T \longrightarrow X_T,$$

and by definition, G_x acts trivially on x_T, so if the categorical orbit exists, there is a unique pointed morphism $\tilde{a}_x : G \cdot x \longrightarrow X_T$ over T such that $\tilde{a}_x \circ \pi_x = a_x$. In order to avoid cluttering our terminology, we also refer to $\tilde{a}_x$ as the *orbit morphism* of x. Also, by definition, the categorical orbit $G \cdot x$ is pointed in $\mathbf{C}_T$, so that it comes with a section $T \longrightarrow G \cdot x$ whose composite with $\tilde{a}_x$ is x. We call this section *canonical* and will usually also denote it by x.

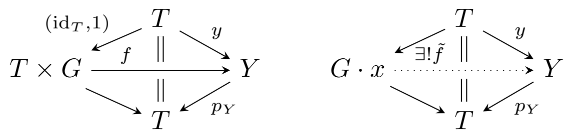
We now spell out in detail what the definition given above of an orbit through a T-valued point is. Let G be a $\mathbf{C}$-group, X a G-space in $\mathbf{C}$, $T \in \operatorname{Ob}\mathbf{C}$, and $x \in_T X$. Assume that G_x is representable in $\mathbf{C}_T$. As we have seen above, this means that the fibre product

$$G_x = T \times_{T\times X} (T \times G)$$

exists in $\mathbf{C}$. So we have in $\mathbf{C}$ a fibre product diagram

$$\begin{array}{ccc} G_x & \longrightarrow & T \times G \\ \downarrow & & \downarrow{\scriptstyle a_x} \\ T & \xrightarrow{(\mathrm{id}_T, x)} & T \times X \end{array}$$

Recall that we are working under assumption that finite products exist in $\mathbf{C}$. Then $G\cdot x$, provided it exists in $(\mathbf{C}_T)^*$, is characterised as follows: For every G_x-invariant morphism f, which fits into a commutative diagram as depicted on the left-hand side of the display below, there is a unique morphism $\tilde f$ completing the right-hand diagram commutatively:



In other words, for any such T, the set of pointed morphisms $G\cdot x\longrightarrow Y$ in $\mathbf{C}_T$ is in natural bijection to the set of morphisms $f:G_T\longrightarrow Y$, which satisfy the conditions:

$$\begin{cases} f(t,1)=y(t), & p_Y(f(t,g))=t,\\ h\cdot x(t)=x(t) \quad\Rightarrow & f(t,gh)=f(t,g)\end{cases}$$

for all $R\in\operatorname{Ob}\mathbf{C}$, $g,h\in_R G$, and $t\in_R T$. Here, we recall that the equation $h\cdot x(t)=x(t)$ characterises the R-valued points (t,h) of G_x.

Universal categorical orbits carry a natural action.

Proposition 2.18. *Let G be a $\mathbf{C}$-group, and (X,x) a pointed G-space in $\mathbf{C}$. If the G-orbit $G\cdot x$ exists and is universal categorical, then the morphism*

$$\pi_x\circ m:G\times G\longrightarrow G\cdot x$$

induces an action of G on $G\cdot x$. It is the unique action of G on $G\cdot x$ for which $\pi_x:G\longrightarrow G\cdot x$ is G-equivariant. Moreover, the canonical point $x:*\longrightarrow G\cdot x$ of $G\cdot x$ is invariant under the action of G_x.

Proof. By assumption, $G\cdot x$ is universal categorical, so the base change

$$\mathrm{id}\times\pi_x:G\times G\longrightarrow G\times(G\cdot x)$$

along the projection $G\times G\cdot x\longrightarrow G\cdot x$ is a categorical quotient in $\mathbf{C}$, for the groupoid

$$\Gamma' := G\times\Gamma = G\times G\times G_x$$

derived from $\Gamma=G\times G_x$. In particular, $\mathrm{id}\times\pi_x$ is an epimorphism. Applying the base change for a further copy of G, we see that so is $\mathrm{id}\times\mathrm{id}\times\pi_x$.

Consider the multiplication m of G. We have

$$\pi_x(m(g_1,g_2h))=\pi_x(g_1g_2h)=\pi_x(g_1g_2)=\pi_x(m(g_1,g_2))$$

for all $R\in\operatorname{Ob}\mathbf{C}$, $g_1,g_2\in_R G$, and $h\in_R G_x$. It follows that

$$(p_1,\pi_x\circ m):G\times G\longrightarrow G\times(G\cdot x)$$

is Γ'-invariant, and hence, there is a unique morphism

$$a_{G\cdot x}:G\times(G\cdot x)\longrightarrow G\cdot x$$

such that $a_{G\cdot x}\circ(\mathrm{id}\times\pi_x)=\pi_x\circ m$. In particular, π_x will be G-equivariant and $a_{G\cdot x}$ uniquely determined by this requirement as soon as we have established that it indeed is an action. To do so, we compute

$$\begin{aligned}a_{G\cdot x}\circ(\mathrm{id}\times a_{G\cdot x})\circ(\mathrm{id}\times\mathrm{id}\times\pi_x)&=a_{G\cdot x}\circ(\mathrm{id}\times(\pi_x\circ m))\\&=\pi_x\circ m\circ(\mathrm{id}\times m)\\&=\pi_x\circ m\circ(m\times\mathrm{id})\\&=a_{G\cdot x}\circ(m\times\pi_x)\\&=a_{G\cdot x}\circ(m\times\mathrm{id})\circ(\mathrm{id}\times\mathrm{id}\times\pi_x),\end{aligned}$$

which shows that

$$a_{G\cdot x}\circ(\mathrm{id}\times a_{G\cdot x})=a_{G\cdot x}\circ(m\times\mathrm{id}),$$

since $\mathrm{id}\times\mathrm{id}\times\pi_x$ is an epimorphism. Similarly, one has

$$a_{G\cdot x}\circ(1\times\mathrm{id})=\mathrm{id}_{G\cdot x}.$$

Hence, $a_{G\cdot x}$ is an action for which π_x is G-equivariant. We will denote it by $\cdot$, as for any action.

Finally, we verify the claim that x is G_x-fixed. By construction, π_x is pointed, so that $\pi_x(1)=x$. For $h\in_R G_x$, we compute, by use of the left G-equivariance and right G_x-invariance of π_x, that

$$h\cdot x=h\cdot\pi_x(1)=\pi_x(h\cdot 1)=\pi_x(h)=\pi_x(1)=x.$$

This completes the proof of the proposition. □

Example 2.19 (Examples of orbits). Returning to the groups and actions from Example 2.4, we explain the notion of isotropy groups and orbits in these cases. In items (i) and (ii) below, let $\mathbf{C}$ denote a category such that all finite products exist.

(i) Let G be a $\mathbf{C}$-group acting trivially on $X\in\mathrm{Ob}\,\mathbf{C}$. Then for all $x:T\longrightarrow X$ and $R\in\mathrm{Ob}\,\mathbf{C}$, we have $G_x(R)=G_T(R)$. Thus, the isotropy functor G_x is represented by $G_T=T\times G$. Here, the morphism $\pi_x=p_1:G_T\to T$ is a universal categorical orbit, as can be seen as follows: π_x is invariant with respect to the action groupoid Γ coming from the right G_T-action on G_T. Given any Γ-invariant morphism $f:G_T\to Y$ with Y over T, it uniquely factors over π_x to $\tilde f=f\circ(\mathrm{id}_T\times 1_G)$.

Furthermore, given $Q\to T$, the fibre product $Q\times_T G_T=G_Q$ exists. Moreover, $Q\times_T\pi_x=\mathrm{id}_Q\times p_1:G_Q\to Q=Q\times_T T$ is a categorical quotient by the above, since $(Q\times Q)\times_{T\times T}\Gamma$ is the action groupoid for the right G_Q-action on G_Q.

(ii) Assume given a $\mathbf{C}$-group G, viewed as a left G-space via left multiplication. For $T\in\mathrm{Ob}\,\mathbf{C}$ and $x\in_T G$, we have

$$G_x(R)=\bigl\{(t,g)\in_R G_T\bigm| g\cdot x(t)=x(t)\bigr\}=\bigl\{(t,1_G(t))\bigm| t\in_R T\bigr\}\cong T(R).$$

Thus, G_x is represented by T. Defining π_x by $\mathrm{id}_{G_T}:G_T\longrightarrow Q:=G_T$, we obtain for any Y and any G_T-invariant $f:G_T\to Y$ a unique factorisation $\tilde f:=f$. Thus, $\pi_x:G_T\longrightarrow Q$ is the categorical quotient of G_T with respect to the G_x-action. In other words, it is the categorical orbit of G through x.

Furthermore, given $Q'\in\mathrm{Ob}\,\mathbf{C}$ and $Q'\longrightarrow Q$, we have $Q'\times_Q G_T=Q'$ and $Q'\times_Q\Gamma=Q'\times_Q G_T=Q'$. The projections $\mathrm{id}_Q'\times_Q s$ and $\mathrm{id}_Q'\times_Q t$ are the identity of Q', so that $Q'\times_Q\pi_x=\mathrm{id}_{Q'}$ is the categorical quotient of Q' (the space) by Q' (the groupoid). It follows that $\pi_x:G_T\longrightarrow G_T$ is a universal categorical orbit.

(iii) Let a continuous or smooth action $a:G\times X\longrightarrow X$, respectively, of a topological group or Lie group on a topological space or a smooth manifold be given. The isotropies at $x\in X_0=X(*)=X$ are represented by the obvious set-theoretic isotropy groups, endowed with the subspace topology coming from the inclusion into G. Since these isotropies are closed, they are notably Lie subgroups in the smooth case.

Both in the topological and the smooth case, a categorical orbit through such an x is represented by the set of right cosets with respect to the isotropy group G_x, with its canonical structure of topological space or smooth manifold, respectively. For the rest of this example, let us focus on the topological case.

Then we can consider arbitrary continuous maps $x : T \longrightarrow X$, defined on some topological space T and observe that

$$G_x = \{(t, g) \in G_T \mid g \in G_{x(t)}\}$$

with the subspace topology from $T \times G$. We may define an equivalence relation $\sim$ on G_T by

$$(t, g) \sim (t', g') \;:\Longleftrightarrow\; t = t', g \cdot x(t) = g' \cdot x(t).$$

The quotient space $Q := X/\sim$ with the canonical map $\pi_x : G_T \longrightarrow Q$ satisfies the universal property of the categorical orbit of G through x.

If π_x is an open map, then it is already an universal categorical orbit. Indeed, in this case, for any $Q' \longrightarrow Q$, the projection $p_1 : Q' \times_Q G_T \longrightarrow Q'$ is open and in particular a quotient map. The map π_x is open in case $T = *$, which is the situation studied classically. In general, however, this fails to be true, as one may see in the following example: Let $G := (\mathbb{R}, +)$, $T := \mathbb{R}$, and $X := \mathbb{R}^2$. Define the action by

$$g \cdot (t, s) := (t, tg + s)$$

and set $x : T \longrightarrow X, x(t) := (t, 0)$. Then $G_x = (0 \times \mathbb{R}) \cup (\mathbb{R}^\times \times 0)$ and the projection $G_T \longrightarrow G_T/G_x$ is not open, as the saturation of an open set $U \subseteq G_T$ containing $(0, 0)$ is $(\mathbb{R} \times 0) \cup U$, which is open only if $\mathbb{R} \times 0$ is already contained in U.

The smooth case is even more subtle, since in general, the isotropy G_x might not exist as a smooth manifold over T. In Section 4, we study these questions for the category of supermanifolds. *A fortiori*, these apply to ordinary manifolds.

(iv) The existence question for isotropies and orbits in the homotopy category of pointed topological spaces leads immediately to subtle questions concerning homotopy pullbacks and homotopy orbits. We do not dwell on these matters here.

(v) From the description of the action of T^*B on M in Example 2.4 (vi), it follows immediately that for any $b \in B$, the action of T^*_bB on the fibre M_b is transitive and the orbits are n-dimensional real tori. Furthermore, the isotropy is a cocompact lattice Λ_b in T^*_bB, depending smoothly on b, *cf.* [27, Theorem 44.1]. The union of the Λ_b is total space a smooth $\mathbb{Z}^n$-principal subbundle Λ of $T^*B \longrightarrow B$.

The traditional description underlines the ensuing action-angle coordinates: Action for the base directions of B, angle for the fibre directions (compare the detailed analysis of Duistermaat [22]). In the terminology introduced above, we find that the isotropy of the generic point $x = \mathrm{id}_X : T = X \to X$ is the subgroup $G_x = M \times_B \Lambda$ of $G_T = M \times_B T^*B$.

By our results below (Theorem 4.20 and Theorem 4.24), the orbit

$$G \cdot x = G_T/G_x = (M \times_B T^*B)/(M \times_B \Lambda)$$

exists as a universal categorical quotient in the category of manifolds over M. Moreover (*loc. cit.*), it coincides with the image of the orbit morphism a_x, which is a surjective submersion. Hence, we have $G \cdot x \cong M \times_B M$ as manifolds over M.

3. Groupoid quotients of superspaces

We now apply the general setup of Section 2 to the categories of locally finitely generated superspaces and of relative supermanifolds constructed in Ref. [4]. We will start by recollecting some basic definitions, referring to the paper cited for more details.

We fix a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The category $\mathbf{SSp}_{\mathbb{K}}$ has as objects pairs $X = (X_0, \mathcal{O}_X)$ where X_0 is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-superalgebras with local stalks. Such objects are called $\mathbb{K}$-*superspaces*. Morphisms $\varphi : X \longrightarrow Y$ are again pairs $(\varphi_0, \varphi^\sharp)$ where this time, $\varphi_0 : X_0 \longrightarrow Y_0$ is a continuous map and $\varphi^\sharp : \mathcal{O}_Y \longrightarrow (\varphi_0)_*\mathcal{O}_X$ is a local morphism of $\mathbb{K}$-superalgebra sheaves.

If S is a fixed $\mathbb{K}$-superspace, the category of objects and morphisms in $\mathbf{SSp}_\mathbb{K}$ over S will be denoted by $\mathbf{SSp}_S$. Objects are denoted by X/S and morphisms by $\varphi : X/S \longrightarrow Y/S$.

Now we fix a subfield $\Bbbk$ of $\mathbb{K}$ containing $\mathbb{R}$ and a 'differentiability' class $\varpi \in \{\infty, \omega\}$. Here, ∞ means 'smooth' and ω means 'analytic' (over $\Bbbk$). We consider model spaces adapted to these data. Namely, let a finite-dimensional super-vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ over $\Bbbk$ be given, together with a compatible $\mathbb{K}$-structure on $V_{\bar 1}$. Then we may consider on the topological space $V_{\bar 0}$ the sheaf $\mathcal{C}^\varpi_{V_{\bar 0}}$ of $\mathbb{K}$-valued functions of differentiability class ϖ. We set

$$\mathbb{A}(V) \coloneqq \big(V_{\bar 0}, \mathcal{C}^\varpi_{V_{\bar 0}} \otimes_\mathbb{K} \textstyle\bigwedge(V_{\bar 1})^*\big)$$

and call this the *affine superspace* associated with V. It depends on the data of $(\mathbb{K}, \Bbbk, \varpi)$, but we will usually omit them from the notation.

By definition, a *supermanifold over* $(\mathbb{K}, \Bbbk)$ *of class* $\mathcal{C}^\varpi$ is a Hausdorff $\mathbb{K}$-superspace X which admits a cover by open subspaces which are isomorphic to open subspaces of affine superspaces. We will usually just say that X is a *supermanifold*. The full subcategory of $\mathbf{SSp}_\mathbb{K}$ comprised of these objects with be denoted by $\mathbf{SMan}_\mathbb{K}$.

In the literature, the case $\mathbb{K} = \Bbbk = \mathbb{R}$ corresponds to (smooth or real-analytic) real supermanifolds [16, 36], and the case $\mathbb{K} = \Bbbk = \mathbb{C}$ corresponds to (holomorphic) supermanifolds [16, 38]. In the case of $\mathbb{K} = \mathbb{C}$ and $\Bbbk = \mathbb{R}$, supermanifolds are also known as '*cs* manifolds' [20]. We take this opportunity to replace this unfortunate terminology with a less confusing one.

In Ref. [4], we construct a full subcategory $\mathbf{SSp}^{\mathrm{lfg}}_\mathbb{K} = \mathbf{SSp}^{\varpi,\mathrm{lfg}}_{\mathbb{K},\Bbbk}$ of $\mathbf{SSp}_\mathbb{K}$ that admits finite fibre products and contains $\mathbf{SMan}_\mathbb{K}$ as a subcategory closed under finite products. Here, 'lfg' stands for 'locally finitely generated'. For any $S \in \operatorname{Ob} \mathbf{SSp}^{\mathrm{lfg}}_\mathbb{K}$, the category of objects and morphisms over S in $\mathbf{SSp}^{\mathrm{lfg}}_\mathbb{K}$ will be denoted by $\mathbf{SSp}^{\mathrm{lfg}}_S$. Given any super-vector space V as above, we define $\mathbb{A}_S(V) \coloneqq S \times \mathbb{A}(V)$. Using these as model spaces, we arrive at a definition of *supermanifolds over* S, compare *op. cit.* We denote the corresponding full subcategory of $\mathbf{SSp}^{\mathrm{lfg}}_S$ by $\mathbf{SMan}_S$.

3.1. **Geometric versus categorical quotients.** In what follows, fix $S \in \mathbf{SSp}^{\mathrm{lfg}}_\mathbb{K}$, and let $\mathbf{C}$ be a full subcategory of $\mathbf{SSp}^{\mathrm{lfg}}_S$ admitting finite products. Particular cases are $\mathbf{C} = \mathbf{SSp}^{\mathrm{lfg}}_S$ and $\mathbf{C} = \mathbf{SMan}_S$, by [4, Corollaries 5.27, 5.42]. Furthermore, let $X \in \operatorname{Ob} \mathbf{C}$ and Γ be a groupoid over X in $\mathbf{C}$.

Proposition 3.1. *The coequaliser* $\pi : X \longrightarrow Q$ *of* $s, t : \Gamma \longrightarrow X$ *exists in* $\mathbf{SSp}_S$ *and is a regular superspace. If* $Q \in \mathbf{C}$*, then* Q *is the categorical quotient of* X *by* Γ*.*

Proof. The existence and regularity of Q is immediate from [4, Propositions 2.17, 5.5]. By definition, the morphism $\pi : X \longrightarrow Q$ is a coequaliser in $\mathbf{SSp}_S$. But since $\mathbf{C}$ is a full subcategory of $\mathbf{SSp}_S$, $\mathbf{SSp}^{\mathrm{lfg}}_S$ being full in the latter, Q is the coequaliser of s, t in $\mathbf{C}$, and thus has the quotient property as required by Definition 2.16. $\square$

Remark 3.2. We can describe the colimit Q of $s, t : \Gamma \longrightarrow X$ explicitly. Indeed, by [4, Remark 2.18], $\mathcal{O}_Q$ is the equaliser in the category $\mathbf{Sh}(Q_0)$ of sheaves on Q_0, defined by the diagram

$$\mathcal{O}_Q \xrightarrow{\ \pi^\sharp\ } \pi_{0*}\mathcal{O}_X \underset{t^\sharp}{\overset{s^\sharp}{\rightrightarrows}} (\pi_0 \circ s_0)_*\mathcal{O}_\Gamma.$$

Moreover, since the embedding of $\mathbf{SSp}_S$ in $\mathbf{SSp}$ preserves colimits, one may see easily that Q_0 is the coequaliser of $s_0, t_0 : \Gamma_0 \longrightarrow X_0$, *i.e.* the topological quotient space of X_0 by the equivalence relation generated by $s_0(\gamma) \sim t_0(\gamma)$.

Example 3.3. Recall the action from Example 2.3 (iii) and the T-valued point x from Example 2.9. Recall that the isotropy group G_x is in this case representable by the group object

$$G_x = \operatorname{Spec} \mathbb{K}[\theta, \gamma]/(\theta\gamma), \quad p^\sharp(\theta) = \theta, \quad m^\sharp(\gamma) = \gamma^1 + \gamma^2, \quad 1^\sharp(\gamma) = 0$$

in $\mathbf{SSp}_T^{\mathrm{lfg}}$, where θ, γ are odd indeterminates. In particular, it lies in $(\mathbf{SSp}_T^{\mathrm{lfg}})^*$.

Let ε be an even indeterminate and define

$$Q \coloneqq \operatorname{Spec} \mathbb{K}[\varepsilon|\theta]/(\varepsilon^2, \theta\varepsilon).$$

We then have morphisms

$$p_Q : Q \longrightarrow T, \quad p_Q^\sharp(\theta) \coloneqq \theta, \quad q : T \longrightarrow Q, \quad q^\sharp(\varepsilon) \coloneqq 0, \quad q^\sharp(\theta) \coloneqq \theta.$$

The morphism

$$\pi_x : G_T \longrightarrow Q, \quad \pi_x^\sharp(\theta) \coloneqq \theta, \quad \pi_x^\sharp(\varepsilon) \coloneqq \theta\gamma$$

is in the category $(\mathbf{SSp}_T^{\mathrm{lfg}})^*$. We claim that $\pi_x : G_T \longrightarrow Q$ is the categorical orbit of G through x.

To establish this claim, let $b : G_T \times_T G_x \longrightarrow G_T$ denote the action by right multiplication of the isotropy, *i.e.* $b^\sharp(\gamma) = \gamma^1 + \gamma^2$. We compute

$$(\pi_x \circ b)^\sharp(\varepsilon) = b^\sharp(\theta\gamma) = \theta(\gamma^1 + \gamma^2) = \theta\gamma^1 = p_1^\sharp(\theta\gamma) = (\pi_x \circ p_1)^\sharp(\varepsilon)$$

so π_x is indeed G_x-invariant. If f is a function on G_T, then

$$f = f_0 + f_\theta\theta + f_\gamma\gamma + f_{\theta\gamma}\theta\gamma$$

where $f_\alpha \in \mathbb{K}$ for $\alpha = 0, \theta, \gamma, \theta\gamma$. Then

$$b^\sharp(f) - p_1^\sharp(f) = f_\gamma\gamma^2,$$

so f is G_x-invariant if and only $f_\gamma = 0$. In this case,

$$f = \pi_x(\tilde{f}), \quad \tilde{f} = f_0 + f_\theta\theta + f_{\theta\gamma}\varepsilon,$$

and $\tilde{f}$ is unique with this property. It is easy to conclude that $\pi_x : G_T \longrightarrow Q$ is the categorical quotient of G_T by G_x, and thus the claim follows. Notice that $G \cdot x = Q$ is not a supermanifold over T.

Definition 3.4 (Weakly geometric quotients)**.** The coequaliser $\pi : X \longrightarrow Q$ of $s, t : \Gamma \longrightarrow X$ is called a *weakly geometric quotient* of X by Γ if $Q \in \operatorname{Ob} \mathbf{C}$.

Remark 3.5. The terminology is justified as follows: If G is a group scheme acting on a scheme X, then a morphism $\pi : X \longrightarrow Q$ is called a *geometric quotient* of X by G if it is the coequaliser of $p_2, a : G \times X \longrightarrow X$ in the category of locally ringed spaces, and in addition, the scheme-theoretic image of $(p_2, a) : G \times X \longrightarrow X \times X$ is $X \times_Q X$, see [40, Definition 0.6].

In terms of the above terminology, we may rephrase Proposition 3.1 as follows. The result is a generalisation of [40, Proposition 0.1].

Corollary 3.6. *Let the weakly geometric quotient Q of X by Γ exist in $\mathbf{C}$. Then Q is the categorical quotient of X by Γ in $\mathbf{C}$.*

4. Existence of superorbits

In this section, we will derive general sufficient conditions for the existence of orbits through generalised points in the category $\mathbf{SMan}_S$ of supermanifolds over S. Here and in what follows, S will denote some object of $\mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$.

The material is organised as follows: In Subsection 4.2, we discuss at length the notion of morphisms of constant rank basic for our considerations. In particular, we characterise precisely when the orbit morphism of a generalised point is locally of constant rank. Subsequently, in Subsection 4.3, we study the isotropy of a supergroup action at a generalised point. This leads, in Subsection 4.4, to the existence of orbits through generalised points.

4.1. **Tangent sheaves of supermanifolds over S.** We briefly collect some definitions and facts concerning tangent sheaves.

Definition 4.1 (Tangent sheaf)**.** Let $p_X : X \longrightarrow S$ and $p_Y : Y \longrightarrow S$ be superspaces over S and $\varphi : X/S \longrightarrow Y/S$ a morphism over S. Let $U \subseteq X_0$ be open. An $p_{X,0}^{-1}\mathcal{O}_S$-linear sheaf map

$$v : \varphi_0^{-1}\mathcal{O}_Y|_U \longrightarrow \mathcal{O}_X|_U$$

will be called a *vector field along φ over S* (defined on U) if $v = v_{\bar 0} + v_{\bar 1}$ where

$$v_i(fg) = v_i(f)\varphi^\sharp(g) + (-1)^{i|f|}\varphi^\sharp(f)v_i(g)$$

for all $i = \bar 0, \bar 1$ and all homogeneous local sections f, g of $p_{X,0}^{-1}\mathcal{O}_Y|_U$.

The sheaf on X_0 whose local sections over U are the vector fields along φ over S defined on U will be denoted by $\mathcal{T}_{X/S\to Y/S}$ or $\mathcal{T}_{\varphi:X/S\to Y/S}$ if we wish to emphasize φ. It is an $\mathcal{O}_X$-module, and will be called the *tangent sheaf along φ over S*. In particular, we define $\mathcal{T}_{X/S} := \mathcal{T}_{\mathrm{id}_X:X/S\to X/S}$ and $\mathcal{T}_X := \mathcal{T}_{X/*}$, the *tangent sheaf of X over S* and the *tangent sheaf of X*, respectively.

Let τ be an even and θ an odd indeterminate. Whenever X is a $\mathbb{K}$-superspace, we define

$$X[\tau|\theta] := \big(X_0, \mathcal{O}_X[\tau|\theta]/(\tau^2, \tau\theta)\big).$$

There is a natural morphism $(\cdot)|_{\tau=\theta=0} : X \longrightarrow X[\tau|\theta]$ whose underlying map is the identity and whose pullback map sends τ and θ to zero.

Lemma 4.2 (Superderivations and super-dual numbers)**.** *Let X/S and Y/S be superspaces over S and $\varphi : X/S \longrightarrow Y/S$ be a morphism over S. There is a natural bijection*

$$\big\{\phi \in \mathrm{Hom}_S\big(X[\tau|\theta], Y\big) \;\big|\; \phi|_{\tau=\theta=0} = \varphi\big\} \longrightarrow \Gamma(\mathcal{T}_{X/S\longrightarrow Y/S}) : \phi \longmapsto v$$

given by the equation

$$\phi^\sharp(f) \equiv \varphi^\sharp(f) + \tau v_{\bar 0}(f) + \theta v_{\bar 1}(f) \quad (\tau^2, \tau\theta) \tag{4.1}$$

for all local sections f of $\mathcal{O}_Y$. Symbolically, we write

$$v_{\bar 0}(f) = \frac{\partial\phi^\sharp(f)}{\partial\tau} \quad \textit{and} \quad v_{\bar 1}(f) = \frac{\partial\phi^\sharp(f)}{\partial\theta}.$$

Proof. Since $X[\tau|\theta]$ is a thickening of X [4], the underlying map of ϕ is fixed by $\phi_0 = \varphi_0$. The assertion follows easily. □

Definition 4.3 (Infinitesimal flow)**.** Let $v \in \Gamma(\mathcal{T}_{X/S\longrightarrow Y/S})$. The unique morphism $\phi^v \in \mathrm{Hom}_S(X[\tau|\theta], Y)$, such that $\phi^v|_{\tau=\theta=0} = \varphi$, associated with v *via* Lemma 4.2, is called the *infinitesimal flow* of v.

The infinitesimal flow construction allows us to introduce for each fibre coordinate system a family of fibre coordinate derivations.

Construction 4.4 (Fibre coordinate derivations)**.** Let $S \in \mathbf{SSp}^{\mathrm{lfg}}_{\mathbb{K}}$ and X/S be in $\mathbf{SMan}_S$ with a global fibre coordinate system $x = (x^a)$.

By [4, Propositions 5.18, 4.19, Corollary 5.22], there are unique morphisms $\phi^a \in \mathrm{Hom}_S(X[\tau|\theta], X)$ such that

$$\phi^{a\sharp}(x^b) = \begin{cases} x^b + \tau\delta_{ab} & \text{for } |x^a| = \bar{0}, \\ x^b + \theta\delta_{ab} & \text{for } |x^a| = \bar{1}. \end{cases}$$

Evidently, we have $(\phi^a|_{\tau=\theta=0})^\sharp(x^b) = x^b$, and hence $\phi^a|_{\tau=\theta=0} = \mathrm{id}_X$.

On account of Lemma 4.2, the morphisms ϕ^a are the infinitesimal flows of unique vector fields over S, denoted by $\frac{\partial}{\partial x^a} \in \Gamma(\mathcal{T}_{X/S})$. We call these *fibre coordinate derivations* and simply *coordinate derivations* in case $S = *$.

Observe that the meaning of each individual $\frac{\partial}{\partial x^a}$ depends on the entire fibre coordinate system (x^b), and not only on the coordinate x^a.

As we shall presently see, the coordinate derivations give systems of generators for the relative tangent bundle.

Proposition 4.5 (Coordinate expression of vector fields)**.** *Let S be in $\mathbf{SSp}^{\mathrm{lfg}}_{\mathbb{K}}$, X/S be in $\mathbf{SSp}^{\mathrm{lfg}}_S$, Y/S be in $\mathbf{SMan}_S$, and $\varphi : X/S \longrightarrow Y/S$ be a morphism over S. Let (y^a) be a local fibre coordinate system on an open subset $V \subseteq Y_0$. Let $U \subseteq X_0$ be an open subset, such that $\varphi_0(U) \subseteq V$, and $v \in \mathcal{T}_{X/S\longrightarrow Y/S}(U)$. Then*

$$v = \sum\nolimits_a v(y^a)\, \varphi^\sharp \circ \frac{\partial}{\partial y^a}. \tag{4.2}$$

In particular, we have

$$\mathcal{T}_{X/S\to Y/S} = \varphi^*(\mathcal{T}_{Y/S}) \coloneqq \mathcal{O}_X \otimes_{\varphi_0^{-1}\mathcal{O}_Y} \varphi_0^{-1}\mathcal{T}_{Y/S},$$

and this $\mathcal{O}_X$-module is locally free, of rank $\mathrm{rk}_x\, \mathcal{T}_{X/S\to Y/S} = \dim_{S,\varphi_0(x)} Y$ *for $x \in X_0$.*

Proof. We may assume that (y^a) is a globally defined fibre coordinate system. Define the vector field $v' \in \varphi^*(\mathcal{T}_{Y/S})(U) \subseteq \mathcal{T}_{X/S\to Y/S}(U)$ by

$$v' \coloneqq \sum\nolimits_a v(y^a)\, \varphi^\sharp \circ \frac{\partial}{\partial y^a}.$$

Let ϕ and ϕ' be the infinitesimal flows of v and v', respectively. For any index a, we have $v'(y^a) = v(y^a)$, and hence $\phi^\sharp(y^a) = \phi'^\sharp(y^a)$. This implies that $\phi = \phi'$, by reason of [4, Propositions 5.18, 4.19, Corollary 5.22]. Hence, we have $v' = v$.

In particular, the vector fields $\varphi^\sharp \circ \frac{\partial}{\partial y^a}$ form a local basis of sections of $\mathcal{T}_{X/S\longrightarrow Y/S}$, and this readily implies the remaining assertions. ☐

Corollary 4.6 (Local freeness of $\mathcal{T}_{X/S}$)**.** *Let $S \in \mathbf{SSp}^{\mathrm{lfg}}_{\mathbb{K}}$ and $X/S \in \mathbf{SMan}_S$. Then $\mathcal{T}_{X/S}$ is locally free, with* $\mathrm{rk}_x\, \mathcal{T}_{X/S} = \dim_{S,x} X$*, for $x \in X_0$.*

A special case of the above concerns the relative tangent spaces.

Definition 4.7 (Tangent space)**.** Let $p = p_X : X \longrightarrow S$ be a superspace over S. For any point $x \in X_0$ we let $\mathfrak{m}_{X,x}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $\varkappa(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. Setting $s \coloneqq p_{X,0}(x)$, we define

$$T_x(X/S) \coloneqq \underline{\mathrm{Der}}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, \varkappa(x)),$$

the $\mathbb{Z}$-span of all homogeneous $v \in \underline{\mathrm{Hom}}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, \varkappa(x))$ such that

$$v(fg) = v(f)g(x) + (-1)^{|f||v|} f(x)v(g). \tag{4.3}$$

This is naturally a super-vector space over $\varkappa(x)$, called the *tangent space at x over S*. For $S = *$, we also write T_xX. The elements are called *tangent vectors* (over S).

As is immediate from the definitions, the tangent space coincides with the tangent sheaf over S along the morphism $(*, \varkappa(x)) \longrightarrow X$.

Corollary 4.8 (Dimension of $T_{S,x}X$)**.** *Let* $S \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$*,* X/S *be a supermanifold over* S*, and* $x \in X_0$*. Then* $\dim_{\mathbb{K}} T_{S,x}X = \dim_{S,x} X$*.*

Definition 4.9 (Tangent morphism)**.** Let $\varphi : X/S \longrightarrow Y/S$ be a morphism of superspaces over S. We define the *tangent morphism*

$$\mathcal{T}_{\varphi/S} : \mathcal{T}_{X/S} \longrightarrow \mathcal{T}_{X/S\to Y/S}$$

by setting

$$\mathcal{T}_{\varphi/S}(v) \coloneqq v \circ \varphi^\sharp$$

for any locally defined vector field v over S. In view of Proposition 4.5, if Y is in $\mathbf{SMan}_S$, then the range of $\mathcal{T}_{\varphi/S}$ is in $\varphi^*(\mathcal{T}_{Y/S})$.

Similarly, we obtain for any $x \in X_0$ a *tangent map*

$$T_x(\varphi/S) : T_x(X/S) \longrightarrow T_{\varphi_0(x)}(Y/S)$$

by setting

$$T_x(\varphi/S)(v) \coloneqq v \circ \varphi_x^\sharp$$

for any tangent vector v over S.

4.2. **Morphisms of constant rank.** In order to handle supergroup orbits through T-valued points, we will need to understand morphisms of locally constant rank in the setting of relative supermanifolds. The relevant definitions are as follows.

Definition 4.10 (Split module morphisms)**.** Let R be a superring and E, F be graded left R-modules. A morphism $f : E \longrightarrow F$ of R-modules is called *split* if $\ker f$ is a free direct summand of E and $\operatorname{coker} f$ is a free direct summand of F.

We have the following useful characterisation of splitness for a morphism.

Lemma 4.11. *Let* R *be a local supercommutative superring and* $f : R^{m|n} \longrightarrow R^{p|q}$ *an even* R*-linear map. The following are equivalent:*

(i) f *is split.*
(ii) $\ker f$ *is a direct summand of* $R^{m|n}$*.*
(iii) $\operatorname{coim} f$ *is a direct summand of* $R^{m|n}$*.*
(iv) $\operatorname{im} f$ *is a direct summand of* $R^{p|q}$*.*
(v) $\operatorname{coker} f$ *is a direct summand of* $R^{p|q}$*.*
(vi) *There are graded bases of* $R^{m|n}$ *and* $R^{p|q}$*, such that the matrix* M_f *of* f *is*

$$M_f = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. By the definitions, item (i) implies items (ii) and (v). We have the equivalence of items (ii) and (iii) and of items (iv) and (v). In case (ii) or (iii), $\ker f$ and $\operatorname{coim} f$ are projective and hence free by Kaplansky's Theorem, which says that a projective module over a local ring is free [31]. This holds similarly for $\operatorname{im} f$ and $\operatorname{coker} f$ in case of (iv) or (v). Clearly, item (vi) implies (i).

Assume that we have (ii). Then there is a homogeneous basis of $R^{m|n}$ containing a homogeneous basis of $\ker f$. With respect to this basis and any homogeneous basis of $R^{p|q}$, the matrix A_f of f has exactly $k + \ell$ non-zero columns, where $m - k|n - \ell$ is the rank of $\ker f$. Then the proof of [36, Lemma 2.3.8] shows that item (vi) holds. Working with rows, we see in a similar fashion that item (v) implies (vi). ☐

Definition 4.12 (Morphisms of constant rank)**.** Let $f : X/S \longrightarrow Y/S$ be a morphism of supermanifolds over S and $x \in X_0$. We say that f is of *locally constant rank over* S *at* x if for any x' in some open neighbourhood of x, the tangent map on stalks

$$\mathcal{T}_{f/S,x'} : \mathcal{T}_{X/S,x'} \longrightarrow (f^*\mathcal{T}_{Y/S})_{x'} = \mathcal{O}_{X,x'} \otimes_{\mathcal{O}_{Y,y'}} \mathcal{T}_{Y/S,y'},$$

where $y' = f_0(x')$, is a split morphism of graded $\mathcal{O}_{X,x'}$-modules and the rank of its kernel is independent of x'. We say f is of *locally constant rank over S* if it is of locally constant rank over S at x for any $x \in X_0$.

For our notion of constant rank morphism (which is defined on stalks) to be applicable, we need the following technical proposition, which clarifies the notion of direct summands for finite locally free sheaves.

Proposition 4.13. *Let X be a superspace and $\mathcal{F}$ a finite locally free $\mathcal{O}_X$-module. Assume that $\mathcal{E} \subseteq \mathcal{F}$ is a submodule. The following are equivalent:*

(i) *$\mathcal{E}$ is a locally direct summand of $\mathcal{F}$.*
(ii) *$\mathcal{E}$ is finite locally free and a locally direct summand of $\mathcal{F}$.*
(iii) *For any $x \in X_0$, $\mathcal{E}_x$ is a direct summand of $\mathcal{F}_x$, and the function*

$$X_0 \longrightarrow \mathbb{N}^2 : x \longmapsto d_x := \dim_{\varkappa(x)} \mathcal{E}_x/\mathfrak{m}_{X,x}\mathcal{E}_x$$

is locally constant. Here, $\varkappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is again the residue field.

Certainly, item (ii) implies (i). For the remainder of the *proof*, without loss of generality, we may assume that $\mathcal{F} = \mathcal{F}_{\bar{0}}$, so that dimensions become natural numbers (instead of pairs thereof). Moreover, shrinking X, we may assume that $\mathcal{F}$ admits a basis $(f_j)_{j \leqslant m}$ of global sections. We introduce the notation

$$E_x := \mathcal{E}_x/\mathfrak{m}_{X,x}\mathcal{E}_x, \quad F_x := \mathcal{F}_x/\mathfrak{m}_{X,x}\mathcal{F}_x,$$

and briefly postpone the rest of the proof to state and prove the following lemma.

Lemma 4.14. *Under the above assumptions, let $x \in X_0$ and $U \subseteq X_0$ be an open neighbourhood of x. Let $(e_j)_{j \leqslant k} \in \mathcal{E}(U)$, $k = d_x$. Assume there is an expression*

$$e_i = \sum_{j=1}^{k} h_{ij} f_j + r_i \tag{4.4}$$

where r_i is in the span of f_j, $j > k$, such that the image of $\det(h_{ij,x})_{1 \leqslant i,j \leqslant k}$ in $\varkappa(x)$ is non-zero. Then the following are equivalent:

(i) *For all y in some open neighbourhood $V \subseteq U$ of x, $(e_{j,y})_{j \leqslant k}$ generates $\mathcal{E}_y$.*
(ii) *For all y in some open neighbourhood $V \subseteq U$ of x, we have $d_y = k$.*

Moreover, if this is the case, and for every $y \in V$, $\mathcal{E}_y$ is a direct summand of $\mathcal{F}_y$, then $(e_j|_V)_{j \leqslant k}$ is a basis of sections for $\mathcal{E}|_V$ and $e_1|_V, \ldots, e_k|_V, f_{k+1}|_V, \ldots, f_m|_V$ is a basis of sections for $\mathcal{F}|_V$.

Proof. Let $f := \det(h_{ij})$. The assumption implies that the image of f_x in $\varkappa(x)$ is non-zero, so that the image of $(e_{j,x})_{j \leqslant k}$ in E_x is a basis. Moreover, there exists an open neighbourhood $V \subseteq U$ of x such that f is invertible on V. In particular, for any $y \in V$, the image of $(e_{j,y})_{j \leqslant k}$ in E_y is linearly independent, and $d_y = \dim E_y \geqslant k$.

Conversely, by Nakayama's Lemma, if $d_y = k$, then $(e_{j,y})_{j \leqslant k}$ is a (minimal) set of generators for $\mathcal{E}_y$. This shows the equivalence of (i) and (ii).

If now in addition, $\mathcal{E}_y$ is a direct summand, then the argument in [39, proof of Theorem 2.5] shows that $(e_{j,y})_{j \leqslant k}$ is a basis of $\mathcal{E}_y$. Hence, if this is true for every $y \in V$, then the map

$$\mathcal{O}_X^k|_V \longrightarrow \mathcal{E}|_V,$$

defined by multiplication with the row $(e_1, \ldots, e_k)$, is stalkwise and hence globally an isomorphism. In other words, $(e_j|_V)_{j \leqslant k}$ is a basis of sections for $\mathcal{E}|_V$.

Moreover, we may invert Equation (4.4) on V, so that the sections

$$e_1|_V, \ldots, e_k|_V, f_{k+1}|_V, \ldots, f_m|_V$$

generate $\mathcal{F}|_V$. Arguing as above, they form a basis of sections. □

Proof of Proposition 4.13 (continued). We retain the simplifying assumptions made above. Let us show that (i) $\Rightarrow$ (iii).

Possibly shrinking X, we may assume that $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$ for some submodule $\mathcal{E}'$. In particular, $\mathcal{E}_x$ is a direct summand of $\mathcal{F}_x$ for any $x \in X_0$. Fix $x \in X_0$, set $k := d_x$, and decompose $f_j = e_j + e'_j$ according to the splitting. We may write the e_i in terms of the f_j as in Equation (4.4).

By construction, the sections $(e_j)_{j \leqslant m}$ generate $\mathcal{E}$, so after reordering e_i and f_j, we may assume that $\det(h_{ij,x})_{1 \leqslant i,j \leqslant k}$ has non-zero image in $\varkappa(x)$. In particular, by Nakayama's Lemma, $(e_{j,x})_{j \leqslant k}$ is a set of generators for $\mathcal{E}_x$. Let $\varphi : \mathcal{O}_X^k \longrightarrow \mathcal{E}$ be given by multiplication with the row $(e_1, \dots, e_k)$.

Then φ_x is surjective, so we can choose an open neighbourhood U of x and $s_j \in \mathcal{O}_X^k(U)$, $j \leqslant m$, such that $\varphi_x(s_{j,x}) = e_{j,x}$. Shrinking U, we may assume that $\varphi(s_j) = e_j$, $j \leqslant m$. It follows that $(e_j)_{j \leqslant k}$ generates $\mathcal{E}|_U$. By Lemma 4.14, it follows that $d_y = k$ for all y in a neighbourhood $V \subseteq U$ of x, proving item (iii).

Finally, let us prove that (iii) $\Rightarrow$ (ii). Let $x \in X_0$ and set $k := d_x$. Suitably shrinking X, we may assume $d_y = k$ for all $y \in X_0$. Clearly, $m \geqslant k$. By Kaplansky's Theorem [31], $\mathcal{E}_y$ is free for any $y \in X_0$, so there exists an open neighbourhood U of x and $e_j \in \mathcal{E}(U)$, $j \leqslant k$, such that $(e_{j,x})_{j \leqslant k}$ is a basis of $\mathcal{E}_x$.

Express e_i in terms of f_j as in Equation (4.4). The image of $(e_{j,x})_{j \leqslant k}$ in F_x is linearly independent, so after reordering (f_j), we may assume that the image of $\det(h_{ij,x})_{1 \leqslant i,j \leqslant k}$ in $\varkappa(x)$ is non-zero. Then Lemma 4.14 applies, and for some open neighbourhood $V \subseteq U$ of x, $(e_j|_V)_{j \leqslant k}$ is a basis of sections for $\mathcal{E}|_V$ and $e_1|_V, \dots, e_k|_V, f_{k+1}|_V, \dots, f_m|_V$ is a basis of sections for $\mathcal{F}|_V$. But then the submodule of $\mathcal{F}|_V$ generated by the sections f_j, $k < j \leqslant m$, is complementary to $\mathcal{E}|_V$, as we may check stalkwise. This finally proves item (ii). ☐

Corollary 4.15. *Let $f : X/S \longrightarrow Y/S$ be a morphism of supermanifolds over S and $x \in X_0$. The following are equivalent:*

(i) *The morphism f has locally constant rank over S at x.*
(ii) *There is a neighbourhood $U \subseteq X_0$ of x such that for all $x' \in U$, $\ker \mathcal{T}_{f/S,x'}$ is a direct summand of $\mathcal{T}_{X/S,x'}$, and $\dim \ker T_{x'}(f/S)$ is independent of x'.*
(iii) *There exist an open neighbourhood $U \subseteq X_0$ of x and homogeneous bases of $\mathcal{T}_{X/S}|_U$ and $f^*\mathcal{T}_{Y/S}|_U$ such that the matrix M of $\mathcal{T}_{f/S}|_U$ has the form*

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. The equivalence of items (i) and (ii) follows from Lemma 4.11. Certainly, (iii) implies (i). Conversely, on applying Proposition 4.13, it follows from (i) that there is some open neighbourhood U of x such that

$$\mathcal{T}_{X/S}|_U = \ker(\mathcal{T}_{f/S})|_U \oplus \mathcal{E}, \quad f^*\mathcal{T}_{Y/S}|_U = \operatorname{im}(\mathcal{T}_{f/S})|_U \oplus \mathcal{E}'$$

and all of the above $\mathcal{O}_X|_U$-modules are free. The tangent map $\mathcal{T}_{f/S}$ induces an isomorphism $\mathcal{E} \longrightarrow \operatorname{im}(\mathcal{T}_{f/S})|_U$ of $\mathcal{O}_X|_U$-modules, so for a suitable choice of bases, this induced map is represented by the identity matrix. This gives the required block matrix expression, proving (iii). ☐

With the above definition, we generalise the rank theorem [36, Theorem 2.3.9, Proposition 3.2.9] in two respects: First, one may consider supermanifolds and morphisms over a general base superspace S. Secondly, we show the regularity not only of fibres, but also of the inverse images of subsupermanifolds of the image.

Proposition 4.16 (Rank theorem)**.** *Let X/S and Y/S be in* $\mathbf{SMan}_S$*, and $f : X/S \longrightarrow Y/S$ be a morphism of locally constant rank over S. Then the following statements hold true:*

(i) *For any* $x \in X_0$*, there is an open subset* $U \subseteq X_0$*, so that the morphism* $f|_U$ *factors as* $f|_U = j \circ p$*. Here,* $j : Y'/S \longrightarrow Y/S$ *is an injective local embedding of supermanifolds over* S *and* $p : X|_U/S \longrightarrow Y'/S$ *is a surjective submersion over* S*.*

Moreover, we may take $Y' = (Y_0', \mathcal{O}_{Y'})$*, where* $Y_0' := f_0(U)$*, endowed with the quotient topology with respect to* f_0*, and* $\mathcal{O}_{Y'} := (\mathcal{O}_Y/\mathcal{J})|_{Y_0'}$*,* $\mathcal{J} := \ker f^\sharp$*. The morphism* j *is given by taking* j_0 *equal to the embedding of* Y_0' *into* Y_0*, and* $j^\sharp$ *the quotient map with respect to the ideal* $\mathcal{J}$*.*

(ii) *If* $f' : X'/S \longrightarrow Y/S$ *is an embedding of supermanifolds over* S *with* $f_0'(X_0') \subseteq f_0(X_0)$ *and ideal* $\mathcal{J}' \supseteq \mathcal{J}$*, then the fibre product* $X' \times_Y X$ *exists as a supermanifold over* S*, and the projection* $p_2 : X' \times_Y X \longrightarrow X$ *is an embedding over* S*. We have*

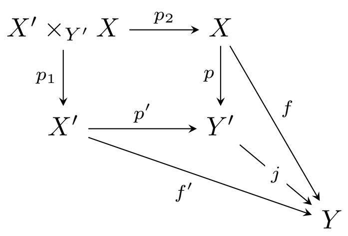
$$\dim_S(X' \times_Y X) = \dim_S X' + \dim_S X - \dim_S Y'. \tag{4.5}$$

The supermanifold Y'/S *over* S *constructed in item* (i) *is called the* image of $f|_U$*. For the assertion of item* (ii) *to hold, it is sufficient to assume that* f *has locally constant rank over* S *at any* $x \in f_0^{-1}(f_0'(x'))$*, for any* $x' \in X_0'$*.*

Proof. The statement of (i) is well-known in case $S = *$ [36, Theorem 2.3.9], in view of Corollary 4.15. By Theorem A.1, the inverse function theorem holds over a general base. Thus, in view of Corollary 4.15, the proof of the rank theorem carries over with only incremental changes to the general case.

As for (ii), the assumption clearly implies that f' factors through j to an embedding $p' : X'/S \longrightarrow Y'/S$ over S. Since p is a submersion over S, the fibre product $X' \times_{Y'} X$ exists, and has the fibre dimension stated on the right-hand side of (4.5). (See [36, Lemma 3.2.8] for the case of $S = *$, the proof of which applies in general, appealing again to Theorem A.1 and its usual corollaries.)

Since j is an injective local embedding, it is a monomorphism, and it follows that $X' \times_{Y'} X$ is actually the fibre product of f' and f. We have a commutative diagram



of morphisms over S such that the left upper square is a pullback whose lower row is an embedding. In particular, $p_{2,0}$ is injective. The image of $p_{2,0}$ is the locally closed subset $f_0^{-1}(f_0'(X_0'))$ of X_0.

To show that this map is closed, we shall show that it is proper. Let $K \subseteq X_0$ be a compact subset and $L := p_0'^{-1}(p_0(K))$, which is a compact subset of X_0'. Then $p_{2,0}^{-1}(K)$ is a closed subset of $(X' \times_Y X)_0 = X_0' \times_{Y_0'} X_0$ whose image in $X_0' \times X_0$ is contained in $L \times K$. Thus, $p_{2,0}^{-1}(K)$ is compact and $p_{2,0}$ proper, hence closed by [12, Chapter I, §10, Propositions 1 and 7]. Moreover, $p_2^\sharp$ is a surjective sheaf map. Hence, p_2 is an embedding. ☐

Remark 4.17. From the relative inverse function theorem (Theorem A.1), it is clear that the usual normal form theorems hold for submersions and immersions over S. Therefore, any morphism $f : X/S \longrightarrow Y/S$ which factors as $f = j \circ p$ where p is a submersion over S and j is an immersion over S has locally constant rank over S.

4.3. **Isotropies at generalised points.** In what follows, fix a Lie supergroup G (*i.e.* a group object in $\mathbf{SMan}_{\mathbb{K}}$) and an action $a : G \times X \longrightarrow X$ of G on a supermanifold X. Let $T \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$ and $x \in_T X$ be a T-valued point. We recall from Equation (2.3) the definition of the orbit morphism through x,

$$a_x : G_T/T = (T \times G)/T \longrightarrow X_T/T = (T \times X)/T,$$

by

$$a_x(t,g) = \big(t, a(g, x(t))\big) = \big(t, g \cdot x(t)\big), \quad \forall (t,g) \in_R G_T,$$

and for any $R \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$. When $T = *$ is the singleton space, *i.e.* $x \in X_0$ is an ordinary point, then $a_x : G \longrightarrow X$ is the usual orbit morphism, see Ref. [16].

Let $\mathfrak{g}$ be the Lie superalgebra of G, *i.e.* the set of left-invariant vector fields on G. This is a Lie superalgebra over $\mathbb{K}$. For $v \in \mathfrak{g}$, let $a_v \in \Gamma(\mathcal{T}_X)$ denote the *fundamental vector field* induced by the action. It is characterised by the equality

$$(v \otimes 1) \circ a^\sharp = (1 \otimes a_v) \circ a^\sharp. \tag{4.6}$$

Let $x \in_T X$ with $T \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$. The equation above specialises to

$$v \circ a_x^\sharp = \sigma^\sharp \circ (\mathrm{id}_G \times x)^\sharp \circ (1 \otimes a_v) \circ a^\sharp, \tag{4.7}$$

where we denote the lift of v to G_T by the same letter and the flip by σ. In the following, for any $v \in \mathfrak{g}$, we set $v(x) \coloneqq x^\sharp \circ a_v$.

We shall need to understand when the orbit morphism a_x for an arbitrary $x \in_T X$ is a morphism of locally constant rank over T. The following is a full characterisation.

Theorem 4.18. *Let $x \in_T X$ with $T \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$. Given $t \in T_0$, we define*

$$\mathfrak{g}_{x,t} \coloneqq \big\{ v \in \mathfrak{g} \bigm| v(x)_t = (x^\sharp \circ a_v)_t = 0 \big\}, \tag{4.8}$$

where $(-)_t$ denotes the germ in T_0 at t. Consider the rank function *of x, defined by*

$$r_x(t) \coloneqq \dim_{\mathbb{K}} \mathfrak{g}_{x,t}, \quad t \in T_0.$$

Then the following are equivalent:

(i) *The orbit morphism a_x has locally constant rank over T.*
(ii) *The function r_x is locally constant on T_0.*
(iii) *Any $t \in T_0$ admits a neighbourhood $U \subseteq T_0$ such that $r_x(U) \leqslant r_x(t)$.*
(iv) *The map $T_0 \longrightarrow \mathrm{Gr}(\mathfrak{g}) : t \longmapsto \mathfrak{g}_{x,t}$ to the Grassmannian variety of graded subspaces of $\mathfrak{g}$ is locally constant.*
(v) *If $v \in \mathfrak{g}$, $t \in T_0$, such that $v(x) = x^\sharp \circ a_v$ vanishes in a neighbourhood of t, then $v(x)$ vanishes on the connected component of t in T_0.*

In this case, the image of the morphism a_x has dimension over T given by

$$\dim_{T,(t,g\cdot x_0(t))} \operatorname{im} a_x = \operatorname{codim}_{\mathfrak{g}} \mathfrak{g}_{x,t} = \dim \mathfrak{g} - r_x(t), \quad t \in T_0, g \in G_0, \tag{4.9}$$

and the kernel of $\mathcal{T}_{a_x/T}$ is given by

$$\ker \mathcal{T}_{a_x/T,(t,g)} = \mathcal{O}_{G_T,(t,g)} \otimes_{\mathbb{K}} \mathfrak{g}_{x,t}, \quad \forall (t,g) \in T_0 \times G_0. \tag{4.10}$$

Proof. Fix $t \in T_0$ and choose a homogeneous basis (v_j) of $\mathfrak{g}_{x,t}$. There is an open neighbourhood $U \subseteq T_0$ of t such that $v_j(x)|_U = 0$ for all j. Hence, for any $s \in U$, we have $\mathfrak{g}_{x,t} \subseteq \mathfrak{g}_{x,s}$, so r_x is a lower semicontinuous function and items (ii) and (iii) are equivalent. Moreover, if (ii) holds, then the map $t \longmapsto \mathfrak{g}_{x,t}$ from T_0 to the Grassmannian variety of $\mathfrak{g}$ is locally constant, so (ii) implies (iv), and the converse is obvious. Moreover, (v) is just a reformulation of (iv).

Before we embark on the remainder of the proof, let us make some preliminary considerations. The Lie superalgebra $\mathfrak{g}$, being the set of left-invariant vector fields, may be canonically considered as a subset of $\mathcal{T}_{G,g}$, for any $g \in G_0$. (In general, the tangent space at a point is not a subset of the stalk of the tangent sheaf!) In fact, we have

$$\mathcal{T}_{G_T/T,(t,g)} = \mathcal{O}_{G_T,(t,g)} \otimes_{\mathbb{K}} \mathfrak{g}, \tag{4.11}$$

for all $(t,g) \in T_0 \times G_0$.

For any $v\in\mathfrak{g}$, we denote the vector field over T on X_T corresponding to the fundamental vector field a_v by the same letter. Then by Equation (4.7), we have

$$\mathcal{T}_{a_x/T}(v)=v\circ a_x^\sharp=\sigma^\sharp\circ(\mathrm{id}_G\times x)^\sharp\circ(1\otimes a_v)\circ a^\sharp.$$

Since v is left-invariant, we have

$$\begin{aligned}0&=\mathcal{T}_{a_x/T,(t,g)}(v)=(v\circ a_x^\sharp)_{(t,g)}\\ \iff\ 0&=\big((\mathrm{id}_T\times 1_G)^\sharp\circ v\circ a_x^\sharp\big)_t=\big((1_G\times x)^\sharp\circ a_v\circ a^\sharp\big)_t=(x^\sharp\circ a_v)_t\\ \iff\ 0&=(x^\sharp\circ a_v)_t=v(x)_t,\end{aligned}$$

and in particular

$$\mathfrak{g}_{x,t}\subseteq\mathcal{K}_{(t,g)},\quad \mathcal{K}:=\ker\mathcal{T}_{a_x/T}\subseteq\mathcal{T}:=\mathcal{T}_{G_T/T}.$$

We define

$$\begin{gathered}\mathcal{O}:=\mathcal{O}_{G_T},\quad \mathfrak{m}_{(t,g)}:=\mathfrak{m}_{G_T,(t,g)},\quad \mathfrak{m}_{(t,g)}^\infty:=\textstyle\bigcap_{k=1}^\infty\mathfrak{m}_{(t,g)}^k,\\ \check{\mathcal{O}}_{(t,g)}:=\mathcal{O}_{(t,g)}/\mathfrak{m}_{(t,g)}^\infty,\quad \check{\mathcal{T}}_{(t,g)}:=\mathcal{T}_{(t,g)}/\mathfrak{m}_{(t,g)}^\infty\mathcal{T}_{(t,g)},\end{gathered}$$

and let $\check{\mathcal{K}}_{(t,g)}$ be the image of $\mathcal{K}_{(t,g)}$ in $\check{\mathcal{T}}_{(t,g)}$. Then $\check{\mathcal{O}}_{(t,g)}$ is local and Noetherian [4, Lemma 3.36, Lemma 4.4], so that the submodule $\check{\mathcal{K}}_{(t,g)}$ is finitely generated.

We have $\mathfrak{g}=T_{(t,g)}(G_T/T)$, and $v\in\mathfrak{g}$ lies in the submodule $\mathcal{K}_{(t,g)}/\mathfrak{m}_{(t,g)}\mathcal{K}_{(t,g)}$ if and only if $v\in\mathfrak{g}_{x,t}$ (being left-invariant, it is determined by its value at a point). Hence, the image of $\mathfrak{g}_{x,t}$ in

$$\check{\mathcal{T}}_{(t,g)}/\mathfrak{m}_{(t,g)}\check{\mathcal{T}}_{(t,g)}=\mathcal{T}_{(t,g)}/\mathfrak{m}_{(t,g)}\mathcal{T}_{(t,g)}=T_{(t,g)}(G_T/T)$$

is precisely $\check{\mathcal{K}}_{(t,g)}/\mathfrak{m}_{(t,g)}\check{\mathcal{K}}_{(t,g)}$. Since $\check{\mathcal{K}}_{(t,g)}$ is finitely generated, Nakayama's Lemma applies, so $\check{\mathcal{K}}_{(t,g)}$ is generated by the canonical image of $\mathfrak{g}_{x,t}$ and

$$\mathcal{K}_{(t,g)}=\mathcal{O}_{(t,g)}\otimes_{\mathbb{K}}\mathfrak{g}_{x,t}+(\mathfrak{m}_{(t,g)}^\infty\mathcal{T}_{(t,g)})\cap\mathcal{K}_{(t,g)}. \tag{4.12}$$

Having concluded our preliminary discussion, let us put outselves in the situation of item (iv). We will show that the second summand in Equation (4.12) does not appear. To that end, let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{g}_{x,t}$. Let U be an open neighbourhood of (t,g), such that $\mathfrak{g}_{x,t'}=\mathfrak{g}_{x,t}$ for any $(t',g')\in U$. Consider the decomposition

$$\mathcal{T}=\mathcal{O}\otimes_{\mathbb{K}}\mathfrak{g}=\mathcal{O}\otimes_{\mathbb{K}}\mathfrak{g}_{x,t}\oplus\mathcal{O}\otimes_{\mathbb{K}}\mathfrak{h}.$$

Take $v\in\mathcal{K}_{(t,g)}$ and denote by v' the second projection of v in this decomposition.

Since the projection is linear over the ring $\mathcal{O}$, we have $v'_{(t',g')}\in\mathfrak{m}_{(t',g')}^\infty\mathcal{T}_{(t',g')}$ for all $(t',g')\in U$ where v is defined. But this implies that $v'=0$, since $\mathcal{T}$ is a locally free $\mathcal{O}$-module and $\mathcal{O}$ admits no non-zero sections which vanish to any order on an open set. Thus, the second summand in Equation (4.12) does indeed not appear. We conclude that

$$\mathcal{K}|_U=\mathcal{O}|_U\otimes_{\mathbb{K}}\mathfrak{g}_{x,t},\quad \mathcal{T}|_U=\mathcal{K}|_U\oplus\mathcal{O}|_U\otimes_{\mathbb{K}}\mathfrak{h}.$$

This proves that $\mathcal{K}=\ker\mathcal{T}_{a_x/T}$ is locally a free direct summand of $\mathcal{T}=\mathcal{T}_{G_T/T}$, and hence (i). Moreover, this proves Equation (4.10). On applying Proposition 4.16 (i), the tangent map $T_{(t,g)}(a_x/T)$ induces a surjection $\mathfrak{g}\longrightarrow T_{g\cdot x_0(t)}\operatorname{im}a_x$. Hence, we see that

$$\dim_{T,(t,g\cdot x_0(t))}\operatorname{im}a_x=\operatorname{codim}_{\mathfrak{g}}\mathfrak{g}_{x,t}=\dim\mathfrak{g}-r_x(t),$$

proving Equation (4.9).

Conversely, assume that (i) holds. Then $\mathcal{K}$ is a finite locally free direct summand of $\mathcal{T}$, by Proposition 4.13. In particular, the function

$$t\longmapsto\dim_{\mathbb{K}}\mathcal{K}_{(t,g)}/\mathfrak{m}_{(t,g)}\mathcal{K}_{(t,g)}=\dim_{\mathbb{K}}\mathfrak{g}_{x,t}=r_x(t)$$

is locally constant, which implies (ii). This concludes the proof of the theorem. □

As a special case of Theorem 4.18, we recover the well-understood case of an ordinary point [13, 16]. In fact, we can be slightly more general.

Corollary 4.19. *Let $T_0 = *$ and $x \in_T X$. Then the orbit morphism $a_x : G \longrightarrow X$ has locally constant rank.*

Proof. Since $T_0 = *$ is a singleton space, the condition (ii) in Theorem 4.18 is void and therefore automatically fulfilled. □

We now apply these general results to the problem of the representability of the isotropy group functor.

Theorem 4.20. *Let $x \in_T X$ with $T \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$. Assume that x is an embedding and the rank function r_x of x is locally constant. Then G_x is representable by a supermanifold over T of fibre dimension*

$$\dim_{T,(t,g)}(G_T)_x = r_x(t) = \dim_{\mathbb{K}} \mathfrak{g}_{x,t}. \tag{4.13}$$

The canonical morphism $G_x \longrightarrow G_T$ is a closed embedding.

Proof. By Theorem 4.18, $a_x : G_T/T \longrightarrow X_T/T$ is a morphism of locally constant rank over T, and locally, its image has fibre dimension over T given by

$$\dim_{T,t} \operatorname{im} a_x = \dim \mathfrak{g} - r_x(t) = \dim G - r_x(t).$$

In view of Proposition 4.16, it will be sufficient to prove for any superfunction f defined on an open subspace of X_T:

$$a_x^\sharp(f) = 0 \implies x_T^\sharp(f) = 0.$$

But for any supermanifold R and any $t \in_R T$, we have

$$a_x^\sharp(f)(t, 1_{G_T}) = f(t, 1_{G_T} \cdot x(t)) = f(t, x(t)) = x_T^\sharp(f)(t),$$

so this statement is manifestly verified. Hence, G_x is representable and the canonical morphism is a closed embedding. The expression for the fibre dimension of G_x follows from Equation (4.5), since $\dim_T T = 0$. □

4.4. **Orbits through generalised points.** Having discussed the representability of the isotropy group functor, we pass now to the existence of orbits. In what follows, to avoid heavy notation, we will largely eschew writing $/S$ for morphisms over S, instead mostly stating the property of being 'over S' in words.

We have the following generalisation of Godement's theorem [3, Theorem 2.6], with an essentially unchanged proof. We have added the detail that in this situation, the quotients are universal categorical.

Proposition 4.21. *Let R/S be an equivalence relation on X/S in $\mathbf{SMan}_S$, as defined in Example 2.15* (iii)*. Then the following assertions are equivalent:*

(i) *The weakly geometric quotient $\pi : X \longrightarrow X/R$ exists in $\mathbf{SMan}_S$ and, as a morphism, is a submersion over S.*

(ii) *The subsupermanifold R of $X \times_S X$ is closed, and (one of, and hence both of) $s, t : R \longrightarrow X$ are submersions over S.*

If this is the case, then $\pi : X \longrightarrow X/R$ is a universal categorical quotient. The quotient is effective*, that is, the morphism $(t, s) : R \longrightarrow X \times_{X/R} X$ is an isomorphism. Moreover, its fibre dimension is*

$$\dim_S(X/R) = 2 \dim_S X - \dim_S R. \tag{4.14}$$

Proof. Apart from the statement about universal categorical quotients, all statements are proved for $S = *$ in Refs. [3, 7]. In general, the proof carries over unchanged.

Let us prove the statement concerning universal categorical quotients. So, let the assumption of item (i) be fulfilled and set $Q \coloneqq X/R$. Then π is a submersion over S, and hence, $X' \coloneqq Q' \times_Q X$ exists in $\mathbf{SMan}_S$ for any $\psi : Q' \longrightarrow Q$, by [4, Proposition 5.41] and the normal form theorem for submersions over S (which follows from Theorem A.1). By item (ii), s is also a submersion over S. Then so is $\pi \circ s$, and $R' \coloneqq (Q' \times Q') \times_{Q \times Q} R$ exists in $\mathbf{SMan}_S$, where R lies over $Q \times Q$ *via* $(\pi \times \pi) \circ (t, s) : R \longrightarrow Q \times Q$.

First, we claim that condition (ii) holds for the equivalence relation R'/S on X'/S in $\mathbf{SMan}_S$. Note that we have a pullback diagram

$$\begin{array}{ccc} R' = Q' \times_Q R & \longrightarrow & R \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle \pi\circ s} \\ Q' & \xrightarrow{\ \psi\ } & Q \end{array}$$

Since $\pi \circ s$ is a submersion over S, so is s'. Next, consider the morphism

$$R' = (Q' \times Q') \times_{Q \times Q} R \longrightarrow X' \times_S X' = (Q' \times Q') \times_{Q \times Q} (X \times_S X).$$

It is an embedding by [4, Corollary 5.29]. Thus, the assumption (ii) is verified for R' and X', and the weakly geometric quotient $\pi' : X' \longrightarrow X'/R'$ exists in $\mathbf{SMan}_S$ and is a submersion over S. It is categorical by Corollary 3.6.

The morphism $p_1 = \mathrm{id}_{Q'} \times_Q \pi : X' \longrightarrow Q'$ is manifestly R'-invariant, so that there is a unique morphism

$$\varphi : X'/R' \longrightarrow Q', \quad \varphi \circ \pi' = \mathrm{id}_{Q'} \times_Q \pi.$$

Since so is p_1, φ is a surjective submersion.

To see that it is a local isomorphism, we compute the dimensions of the supermanifolds over S in question. On one hand, we have

$$\dim_S Q = 2 \dim_S X - \dim_S R,$$

and on the other, we have

$$\begin{aligned} \dim_S X'/R' &= 2 \dim_S X' - \dim_S R' \\ &= 2(\dim_{Q'} X' + \dim_S Q') - (\dim_{Q' \times Q'} R' + 2 \dim_S Q') \\ &= 2 \dim_Q' X' - \dim_{Q' \times Q'} R' = 2 \dim_Q X - \dim_{Q \times Q} R \\ &= 2(\dim_Q X + \dim_S Q) - (\dim_{Q \times Q} R + 2 \dim_S Q) \\ &= 2 \dim_S X - \dim_S R \end{aligned}$$

Upon invoking the inverse function theorem (Theorem A.1), this proves that φ is a local isomorphism over S. Finally, we need to show that φ_0 is injective.

To that end, let $q_j' \in Q_0'$, $x_j \in X_0$, such that $\psi_0(q_j') = \pi_0(x_j')$. Assume that $\varphi_0(\pi_0'(q_1', x_1)) = \varphi_0(\pi_0'(q_2', x_2))$, so that $q_1' = q_2'$, because

$$\varphi_0 \circ \pi_0' = p_{1,0} : X_0' = Q_0' \times_{Q_0} X_0 \longrightarrow Q_0'.$$

It follows that $\pi_0(x_1) = \psi_0(q_1') = \psi_0(q_2') = \pi_0(x_2)$, so that $(x_1, x_2) \in R_0$, since π is an effective quotient. Then $(q_1', q_2', x_1, x_2) \in R_0'$, so that $\pi_0'(q_1', x_1) = \pi_0'(q_2', x_2)$, proving the injectivity. □

We now wish to apply this proposition to supergroup actions. Thus, fix a Lie supergroup G and a G-supermanifold X. Let $x \in_T X$, where T is some supermanifold. We assume that G_x is representable in $\mathbf{SMan}_T$ and that the canonical morphism $G_x \longrightarrow G_T$ is an embedding over T (automatically closed).

We define an equivalence relation R_x on G_T by

$$R_x \coloneqq G_T \times_T G_x, \quad i : R_x \longrightarrow G_T \times_T G_T,$$

where i is given by

$$i(g,g') \coloneqq (g,gg'), \quad \forall (g,g') \in_{T'/T} \big(G_T \times_T G_x\big)/T,$$

and for any supermanifold T'/T over T. It is straightforward to check that i is an embedding and indeed, that R_x is an equivalence relation.

Proposition 4.22. *Let G_x be representable in $\mathbf{SMan}_T$ and the canonical morphism $G_x \longrightarrow G_T$ be an embedding. Then the weakly geometric quotient $\pi_x : G_T \longrightarrow G \cdot x$ of G_T by G_x exists in $\mathbf{SMan}_T$. It is universal categorical, effective, and a submersion over T. Its fibre dimension is*

$$\dim_T G \cdot x = \dim G - \dim_T G_x. \tag{4.15}$$

Proof. The underlying map of $G_x \longrightarrow G_T$ is injective and a homeomorphism onto its closed image, so it is proper. Therefore, the map underlying the morphism $i : R_x \longrightarrow G_T \times_T G_T$ is closed. The first projection s of R_x is obviously a submersion over T. Then Proposition 4.21 applies, and we reach our conclusion. Equation (4.15) follows from Equation (4.14), since $\dim_T R_x = \dim G + \dim_T G_x$. □

Definition 4.23. In the situation of Proposition 4.22, we say that the *orbit $G \cdot x$ of G through x exists.* By abuse of language, the morphism $\tilde a_x : G \cdot x \longrightarrow X_T$ over T induced by a_x will also be called the *orbit morphism.*

Combining this fact with our previous results, we get the following theorem.

Theorem 4.24. *Let $x : T \longrightarrow X$ be an embedding such that the rank function r_x is locally constant. Then the orbit $G \cdot x$ exists, and its fibre dimension over T is*

$$\dim_{T,(t,g\cdot x_0(t))} G \cdot x = \dim G - r_x(t), \quad \forall (t,g) \in (G_T)_0 = T_0 \times G_0. \tag{4.16}$$

Moreover, the morphism $\tilde a_x$ has locally constant rank over T and if $U \subseteq X_0$ is open such that $a_x|_U$ admits an image in the sense of Proposition 4.16, then so does $\tilde a_x|_{\pi_{x,0}(U)}$ and these images coincide.

Proof. Combining Theorem 4.20 with Proposition 4.22, we see that $G \cdot x$ exists. Equation (4.16) follows from Equation (4.15) and Equation (4.13).

The orbit map a_x is locally of constant rank, so by Proposition 4.16, locally in G_T, it factors as $j \circ p$, where p is a surjective submersion and j is an injective local embedding. By the universal property, p factors through the canonical projection $\pi_x : G_T \longrightarrow G \cdot x$ to a submersion, so that $\tilde a_x$ admits a similar factorisation. It follows by Remark 4.17 that $\tilde a_x$ has locally constant rank.

Moreover, since $\pi_{x,0}$ is surjective and $\pi_x^\sharp$ is an injective sheaf map, it follows that $\operatorname{im} a_x|_U = \operatorname{im} \tilde a_x|_{\pi_{x,0}(U)}$ whenever one of the two is defined. □

5. Coadjoint superorbits and their super-symplectic forms

In this section, we construct the Kirillov–Kostant–Souriau form in the setting coadjoint superorbits through general points. We will follow the notation and conventions of Sections 3–4 and Ref. [6], only briefly recalling the basic ingredients of the latter reference.

Let G be a Lie supergroup—*i.e.* a group object in $\mathbf{SMan}_{\mathbb{K}} = \mathbf{SMan}^{\omega}_{\mathbb{K},\Bbbk}$—with Lie superalgebra $\mathfrak{g}$. We set $\mathfrak{g}_{\Bbbk} \coloneqq \mathfrak{g}_{\Bbbk,\bar 0} \oplus \mathfrak{g}_{\bar 1}$, where $\mathfrak{g}_{\Bbbk,\bar 0}$ is the Lie algebra of G_0. (Note that the latter is a $\Bbbk$-form of $\mathfrak{g}_{\bar 0}$.) The dual $\mathbb{K}$-super vector space of $\mathfrak{g}$ will be denoted by $\mathfrak{g}^*$. Let $\mathfrak{g}^*_{\Bbbk}$ be the set of $\mathbb{K}$-linear functionals $f = f_{\bar 0} \oplus f_{\bar 1} \in \mathfrak{g}^*$ such that $f_{\bar 0}(\mathfrak{g}_{\Bbbk,\bar 0}) \subseteq \Bbbk$. We denote the adjoint action of G on $\mathbb{A}(\mathfrak{g}_{\Bbbk})$ by Ad.

The *coadjoint action* is defined by

$$\big\langle \mathrm{Ad}^*(g)(f), x\big\rangle \coloneqq \big\langle f, \mathrm{Ad}(g^{-1})(x)\big\rangle, \quad \forall g \in_T G, x \in_T \mathbb{A}(\mathfrak{g}_{\Bbbk}), f \in_T \mathbb{A}(\mathfrak{g}^*_{\Bbbk}),$$

where $\langle \cdot,\cdot\rangle$ denotes the canonical pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$.

5.1. **The super-symplectic Kirillov–Kostant–Souriau form.** Let $T \in \mathbf{SMan}_{\mathbb{K}}$ and $f \in_T \mathbb{A}(\mathfrak{g}_{\Bbbk}^*)$. Assume that f is an embedding and the rank function r_f is locally constant, so that the orbit $G \cdot f$ exists in $\mathbf{SMan}_T$, by Theorem 4.24. Then by Equation (4.10), there is a short exact sequence

$$0 \longrightarrow p^*\mathfrak{g}_f \longrightarrow \mathcal{O}_{G_T} \otimes_{\mathbb{K}} \mathfrak{g} = \mathcal{T}_{G_T/T} \xrightarrow{\mathcal{T}_{\pi_f/T}} \mathcal{T}_{G_T/T \to G\cdot f/T} \longrightarrow 0$$

of $\mathcal{O}_{G_T}$-modules. Here, $p : G_T \longrightarrow T$ is the structural projection and $\mathfrak{g}_f$ is the $\mathcal{O}_T$-module defined by

$$\mathfrak{g}_f(U) := \{ v \in \mathfrak{g}_T(U) \mid v(f)|_U = (f^\sharp \circ a_v)|_U = 0 \} \tag{5.1}$$

for any open $U \subseteq T_0$. Here, we let $\mathfrak{g}_T := \mathcal{O}_T \otimes_{\mathbb{K}} \mathfrak{g}$.

We define an even super-skew symmetric tensor $\tilde\omega_f$,

$$\tilde\omega_f : \mathcal{T}_{G_T/T\to G\cdot f/T} \otimes_{\mathcal{O}_{G_T}} \mathcal{T}_{G_T/T\to G\cdot f/T} \longrightarrow \mathcal{O}_{G_T},$$

by the formula

$$\tilde\omega_f\big(\mathcal{T}_{\pi_f/T}(v), \mathcal{T}_{\pi_f/T}(w)\big) := \big\langle f, [v,w]\big\rangle, \quad \forall v, w \in (\mathcal{O}_{G_T} \otimes \mathfrak{g})(U),$$

where $U \subseteq T_0 \times G_0$ is open, and we identify f with a section of $\mathcal{O}_T \otimes \mathfrak{g}$ *via* the natural bijection

$$\mathrm{Hom}\big(T, \mathbb{A}(\mathfrak{g}_{\Bbbk}^*)\big) \longrightarrow \Gamma\big((\mathcal{O}_T \otimes \mathfrak{g}^*)_{\Bbbk,\bar 0}\big),$$

compare [4, Corollary 4.26, Proposition 5.18]. The identification is *via*

$$f^\sharp(x) = \langle f, x\rangle, \quad \forall x \in \mathfrak{g} \subseteq \Gamma(\mathcal{O}_{\mathbb{A}(\mathfrak{g}_{\Bbbk}^*)}).$$

Lemma 5.1. *The 2-form $\tilde\omega_f$ is well-defined and non-degenerate.*

Proof. Let $v \in \mathfrak{g}$ be homogeneous and $x \in \mathfrak{g} \subseteq \Gamma(\mathcal{O}_{\mathbb{A}(\mathfrak{g}_{\Bbbk}^*)})$. Then we compute for all supermanifolds R and all $\mu \in_R \mathbb{A}(\mathfrak{g}_{\Bbbk}^*)$ that

$$\begin{aligned} a_v(x)(\mu) &= \frac{d}{d\tau}\Big|_{\tau=0} \big\langle \mathrm{Ad}^*(\phi^v)(\mu), x\big\rangle \\ &= \frac{d}{d\tau}\Big|_{\tau=0} \big\langle \mu, \mathrm{Ad}(\phi^{-v})(x)\big\rangle = -\big\langle \mu, [v,x]\big\rangle = -\mu(\mathrm{ad}(v)(x)). \end{aligned}$$

Here, we let $|\tau| = |v|$ and follow the conventions of Definition 4.3.

Note that vector fields are uniquely determined by their action on systems of local fibre coordinates, by Proposition 4.5. Moreover, any homogeneous basis of $\mathfrak{g}$ contained in $\mathfrak{g}_{\Bbbk}$ defines a global fibre coordinate system on $\mathbb{A}_T(\mathfrak{g}_{\Bbbk}^*)$.

Therefore, a vector field $v \in \mathfrak{g}$ defines a section of $p^*\mathfrak{g}_f$ if and only if we have

$$0 = v(f)(x) = f^\sharp(a_v(x)) = -\big\langle f, \mathrm{ad}(v)(x)\big\rangle = (-1)^{|f||v|}\big\langle \mathrm{ad}^*(v)(f), x\big\rangle$$

for all $x \in \mathfrak{g}$, *i.e.* when $\mathrm{ad}^*(v)(f) = 0$. (Here, we interpret f as the section $p^\sharp(f)$ of $\mathcal{O}_{G_T} \otimes \mathfrak{g}^*$.) Since

$$v(f) = 0 \iff \mathcal{T}_{\pi_f/T}(v) = 0,$$

we have

$$\tilde\omega_f\big(\mathcal{T}_{\pi_f/T}(v), \mathcal{T}_{\pi_f/T}(w)\big) = \big\langle f, [v,w]\big\rangle = -(-1)^{|v||f|}\big\langle \mathrm{ad}^*(v)(f), w\big\rangle$$

for all homogeneous local sections v, w of $\mathcal{O}_{G_T} \otimes \mathfrak{g}$. Because the tensor $\tilde\omega_f$ is super-skew symmetric, it follows that $\tilde\omega_f$ is well-defined. But the same computation shows that it is non-degenerate, so we have proved the assertion. □

Since $G \cdot f \in \mathbf{SMan}_T$, we have

$$\mathcal{T}_{G_T/T\to G\cdot f/T} = \pi_f^*(\mathcal{T}_{G\cdot f/T}),$$

by Proposition 4.5, so we may ask whether $\tilde\omega_f$ is induced by some tensor ω_f on $G \cdot f$. Indeed, this is the case, as we presently show.

The module inverse image and direct image functors $((\pi_f)^*, (\pi_f)_{0*})$ form an adjoint pair, so there is a natural bijection

$$\mathrm{Hom}_{\mathcal{O}_{G\cdot f}}\big(\textstyle\bigwedge^2 \mathcal{T}_{G\cdot f/T}, (\pi_f)_{0*}\mathcal{O}_{G_T}\big) \xrightarrow{\pi_f^*} \mathrm{Hom}_{\mathcal{O}_{G_T}}\big(\textstyle\bigwedge^2 \mathcal{T}_{G_T/T\to G\cdot f/T}, \mathcal{O}_{G_T}\big).$$

Proposition 5.2. *There is a unique even super-skew symmetric tensor*

$$\omega_f : \mathcal{T}_{G\cdot f/T} \otimes_{\mathcal{O}_{G\cdot f}} \mathcal{T}_{G\cdot f/T} \longrightarrow \mathcal{O}_{G\cdot f}$$

such that $\pi_f^*(\omega_f) = \tilde{\omega}_f$.

Proof. By the above, there is a unique even super-skew symmetric tensor

$$\omega_f : \mathcal{T}_{G\cdot f/T} \otimes_{\mathcal{O}_{G\cdot f}} \mathcal{T}_{G\cdot f/T} \longrightarrow (\pi_f)_{0*}\mathcal{O}_{G_T},$$

such that $\pi_f^*(\omega_f) = \tilde{\omega}_f$. We need to show that it takes values in the subsheaf $\mathcal{O}_{G\cdot f}$.

But $G\cdot f = G_T/G_f$ is a weakly geometric quotient by Proposition 4.22 and Theorem 4.24, so that by Remark 3.2, we have

$$\mathcal{O}_{G\cdot f} = ((\pi_f)_{0*}\mathcal{O}_{G_T})^{G_f}.$$

It thus remains to prove that ω_f takes values in the sheaf of invariants.

To that end, fix a homogeneous basis (x_a) of $\mathfrak{g}$ contained in $\mathfrak{g}_{\Bbbk}$. Take any $v, w \in \mathcal{T}_{G\cdot f/T}(U)$, where $U \subseteq (G\cdot f)_0$ is open and define $V := (\pi_f)_0^{-1}(U) \subseteq T_0 \times G_0$. We may write $\pi_f^\sharp \circ v = \sum_a v_a(x_a \circ \pi_f^\sharp)$ for some $v_a \in \mathcal{O}_{G_T}(V)$, $|v_a| = |x_a| + |v|$, and similarly for w.

Denote by (t, g, h) the generic point of $G_T|_V \times_T G_f|_V$. We compute for any superfunction k on $G\cdot f$, defined on an open subset of U, that

$$(\pi_f^\sharp \circ v)(k)(t, gh) = v(k)\big((t, gh)\cdot f(t)\big) = v(k)\big((t, g)\cdot f(t)\big) = (\pi_f^\sharp \circ v)(k)(t, g).$$

Here, we are using the fact that $G\cdot f$ is a universal categorical quotient (Theorem 4.24), so that, by Proposition 2.18, it admits a G-action for which π_f is equivariant and f, considered as a T-valued point of $G\cdot f$, is fixed by G_f.

On the other hand, using results from Ref. [6], we have

$$\begin{aligned}\sum_a \big(v_a(x_a \circ \pi_f^\sharp)(k)\big)(t, gh) &= \sum_a v_a(t, gh)\frac{d}{d\tau}\Big|_{\tau=0} k\big((t, gh\exp(\tau x_a))\cdot f(t)\big)\\ &= \sum_a v_a(t, gh)\frac{d}{d\tau}\Big|_{\tau=0} k\big((t, g\exp(\tau\,\mathrm{Ad}(h)(x_a))h)\cdot f(t)\big)\\ &= \sum_a v_a(t, gh)\frac{d}{d\tau}\Big|_{\tau=0} k\big((t, g\exp(\tau\,\mathrm{Ad}(h)(x_a)))\cdot f(t)\big)\\ &= \sum_a v_a(t, gh)(\mathrm{Ad}(h)(x_a)\circ \pi_f^\sharp)(k)(t, g).\end{aligned}$$

Combining both computations, we arrive at the equality

$$\sum_a v_a(t, gh)(\mathrm{Ad}(h)(x_a)\circ \pi_f^\sharp) = \sum_a v_a(t, g)(x_a \circ \pi_f^\sharp) \tag{5.2}$$

of vector fields over T along the morphism

$$\pi_f \circ m_{G_T} = \pi_f \circ p_1 : G_T \times_T G_f \longrightarrow G\cdot f.$$

Using Equation (5.2), we may compute

$$\begin{aligned}\omega_f(v,w)(t,gh) &= \tilde{\omega}_f(\pi_f^\sharp\circ v,\pi_f^\sharp\circ w)(t,gh)\\ &= \sum_{ab}(-1)^{|x_a||x_b|}(v_aw_b)(t,gh)\big\langle f(t),[x_a,x_b]\big\rangle\\ &= \sum_{ab}(-1)^{|x_a||x_b|}(v_aw_b)(t,gh)\big\langle f(t),[\mathrm{Ad}(h)(x_a),\mathrm{Ad}(h)(x_b)]\big\rangle\\ &= \sum_{ab}(-1)^{|x_a||x_b|}(v_aw_b)(t,g)\big\langle f(t),[x_a,x_b]\big\rangle\\ &= \tilde{\omega}_f(\pi_f^\sharp\circ v,\pi_f^\sharp\circ w)(t,g)=\omega_f(v,w)(t,g),\end{aligned}$$

which shows that indeed, $\omega_f(v,w)$ is right G_f-invariant, and hence, that ω_f takes values in the sheaf $\mathcal{O}_{G\cdot f}$, as desired. □

We may consider ω_f as a global section of $\Omega^2_{G\cdot f/T}=\bigwedge^2\Omega^1_{G\cdot f/T}$, *i.e.* a 2-form over T. We show that it is closed.

Proposition 5.3. *The 2-form ω_f over T is relatively closed.*

Proof. The element of $\Gamma(\mathcal{O}_{G_T}\otimes\mathfrak{g}^*)$ corresponding to f is a left G-invariant 1-form (which is, moreover, even and real-valued). We show that it gives a potential for the pullback of ω_f. To that end, we follow ideas of Chevalley–Eilenberg [17].

Let $v,w\in\mathfrak{g}$. Denote by $d=d_{G_T/T}$ the relative differential. Then

$$\iota_w d+d\iota_w=\mathcal{L}_w,$$

where ι_v, $|\iota_v|=|v|$, denotes relative contraction, and $\mathcal{L}_v$, $|\mathcal{L}_v|=|v|$, denotes the relative Lie derivative. We have

$$\begin{aligned}df(v,w)&=(-1)^{|v||w|}\iota_w\iota_v df=(-1)^{|v||w|}\iota_w(\mathcal{L}_vf)\\ &=-[\mathcal{L}_v,\iota_w]f=-\iota_{[v,w]}f=-\big\langle f,[v,w]\big\rangle=-\tilde{\omega}_f(\mathcal{T}_{\pi_f/T}(v),\mathcal{T}_{\pi_f/T}(w)),\end{aligned}$$

since $\iota_wf=\langle f,w\rangle\in\Gamma(\mathcal{O}_T)$, so that $d\iota_wf=0=\mathcal{L}_v\iota_wf$. Since both sides of the equation are $\mathcal{O}_{G_T}$-bilinear, the equation

$$\tilde{\omega}_f\big(\mathcal{T}_{\pi_f/T}(v),\mathcal{T}_{\pi_f/T}(w)\big)=-df(v,w)$$

holds for any vector fields v,w on G_T over T, defined on some open subset. But since $\tilde{\omega}_f=\pi_f^*(\omega_f)$ by Proposition 5.2, we have

$$\pi_f^\sharp(\omega_f)(v,w)=\tilde{\omega}_f\big(\mathcal{T}_{\pi_f/T}(v),\mathcal{T}_{\pi_f/T}(w)\big)=-df(v,w)$$

for any vector fields v,w on G_T over T. Thus,

$$\pi_f^\sharp(d\omega_f)=d\pi_f^\sharp(\omega_f)=-d^2f=0.$$

Since $\pi_f^\sharp$ is an injective sheaf map, we conclude that $d\omega_f=0$. □

We summarise the above results in the following theorem.

Theorem 5.4. *Let G be a Lie supergroup with Lie superalgebra $\mathfrak{g}$. Let T be a supermanifold and $f:T\longrightarrow\mathbb{A}(\mathfrak{g}^*_{\mathbb{k}})$ an embedding such that the rank function r_f is locally constant.*

Then the coadjoint orbit $G\cdot f$ exists, is universal categorical, and with the Kirillov–Kostant–Souriau form ω_f, $G\cdot f$ is a supersymplectic supermanifold over T.

6. Application: The orbit method for supergroups of Heisenberg type

This section offers an application of our general theory of coadjoint orbits to the geometric construction of representations. By way of example, we show how the formalism can be applied to give a certain 'universal' T-families of representations of two graded variants of the 3-dimensional Heisenberg group. At this point, we will not address the issue to which extent unitary structures exist on these families, nor in which precise sense they are universal. We intend to treat these issues in forthcoming work, together with an extension to more general Lie supergroups.

Let us consider the Lie superalgebra $\mathfrak{g}$ over $\mathbb{K}$ spanned by homogeneous vectors x, y, z satisfying the unique non-zero relation

$$[x, y] = z.$$

When x, y, z are even, $\mathfrak{g}$ is the classical Heisenberg algebra of dimension $3|0$. When x, y are odd, z must be even. The central element z spans a copy of $\mathbb{K}$, so $\mathfrak{g}$ is a unital Lie algebra in the sense of Ref. [2], and its unital enveloping algebra $\mathfrak{U}(\mathfrak{g})/(1-z)$ is the Clifford algebra $\mathrm{Cliff}(2, \mathbb{K})$. (NB: We will use a different normalisation below.) For this reason, $\mathfrak{g}$ is called the Clifford–Lie superalgebra, and its representation theory was studied *e.g.* in Refs. [5, 42]. However, the construction of the representations used there is *ad hoc*. Below, we show how they arise in a natural fashion.

A third possibility, which does not seem to have been considered before, is that x, y are of distinct parity (but see Ref. [23]). In this case, z is odd. As we show below, besides characters, there exists a family of representations (which happen to be finite-dimensional) parametrised by $T = \mathbb{A}^{0|1}$, which bear a striking resemblance to the Schrödinger representation of the Heisenberg group.

6.1. **Parity-independent computations.**

A number of computations concerning the Lie superalgebra $\mathfrak{g}$ of Heisenberg type introduced above can be performed independently of the parity of its elements. We begin with the coadjoint representation of $\mathfrak{g}$. Let x^*, y^*, z^* be the dual basis of x, y, z. In terms of this basis, we have

$$\mathrm{ad}^*(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -(-1)^{|x||z|} \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}^*(y) = \begin{pmatrix} 0 & 0 & (-1)^{|y|} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}^*(z) = 0.$$

Recall the definitions given at the beginning of Section 4. We will consider the field $\Bbbk = \mathbb{R}$, since we are mainly interested in super versions of real Lie groups. A Lie supergroup G (*i.e.* a group object in the category of supermanifolds over $(\mathbb{K}, \mathbb{R})$ of class $\mathcal{C}^\varpi$) with Lie superalgebra $\mathfrak{g}$ is uniquely determined by the choice of a real Lie group G_0 whose Lie algebra is a real form $\mathfrak{g}_{\mathbb{R},\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$, compare Ref. [6].

We fix $\mathfrak{g}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R},\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ by setting $\mathfrak{g}_{\mathbb{R},\bar{0}} := \mathfrak{g}_{\bar{0}} \cap \langle x, y, z \rangle_{\mathbb{R}}$. Let G be the connected and simply connected Lie supergroup whose Lie superalgebra is $\mathfrak{g}$ and whose Lie group has Lie algebra $\mathfrak{g}_{\mathbb{R},\bar{0}}$. Unless $\mathfrak{g}$ is purely even, G_0 is the additive group of $\mathbb{R}$.

Let T be any supermanifold and $f = \alpha x^* + \beta y^* + \gamma z^* \in_T \mathbb{A}(\mathfrak{g}_{\mathbb{R}}^*)$. Then the orbit exists if the orbit map a_f attached to f has locally constant rank, by Theorem 4.24. Observe that

$$\mathrm{ad}^*(ax + by + cz)(f) = -a\gamma y^* + (-1)^{|y|(1+|z|)} b\gamma x^*$$

for $v = ax + by + cz \in \mathfrak{g}$, where $a, b, c \in \mathbb{K}$. Hence, we have $v \in \mathfrak{g}_{f,t}$ for some $t \in T_0$ if and only if

$$a\gamma_t = b\gamma_t = 0,$$

where γ_t denotes the germ of γ at the point t. (Compare the definition of $\mathfrak{g}_{f,t}$ from Equation (4.8).) Thus, $a = b = 0$ whenever $\gamma_t \neq 0$, and a, b are arbitary otherwise.

That is, we have

$$\mathfrak{g}_{f,t} = \begin{cases} \mathbb{K}z & \gamma_t \neq 0, \\ \mathfrak{g} & \gamma_t = 0. \end{cases} \tag{6.1}$$

Hence, assuming T_0 to be connected, Theorem 4.18 shows that the orbit map a_f is of locally constant rank if and only if $\operatorname{supp}\gamma = \{t|\gamma_t \neq 0\} \in \{\varnothing, T_0\}$. Moreover, the orbit will be up to isomorphism over T independent of α and β. In what follows, we will therefore assume that $\alpha = \beta = 0$.

We will be mostly interested in the case where $\operatorname{supp}\gamma = T_0$, since the orbits in the case $\gamma = 0$ will just be $G \cdot f = T$, as G is connected. Hence, let $\operatorname{supp}\gamma = T_0$. The orbit will again be independent, up to an isomorphism over T, of the choice of γ. However, there is in general no canonical choice for γ, so we will keep this information. As we shall see below, this is sensible since γ will play the role of a scaling parameter in the representation associated with the orbit.

To compute the coadjoint action, we realise G in matrix form and $\mathfrak{g}$ as left-invariant vector fields on G. For any $R \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$, consider 3×3 matrices with entries in $\mathcal{O}_R$. We fix the parity on the matrices by decreeing that the rows and columns of nos. $1, 2, 3$ have parities depending on those of x, y, z according to Table 1.

Table 1. Parity distribution for the supergroups of Heisenberg type

$\lvert x\rvert$	$\lvert y\rvert$	$\lvert z\rvert$	1	2	3
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$

Then matrices of the form

$$\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}$$

are even if and only if $|a'| = |x|$, $|b'| = |y|$, and $|c'| = |z|$. Let $G'(R)$ be the set of these matrices where in addition $\{a', b', c'\} \subseteq \Gamma(\mathcal{O}_{R,\mathbb{R}})$. Clearly, by defining the group multiplication by the multiplication of matrices, G' is the point functor of a Lie supergroup. As we shall show presently, it is isomorphic to G. Since $G'_0 = G_0$ is the additive group of $\mathbb{R}$, unless G is purely even, it will be sufficient to show that the Lie superalgebra of left-invariant vector fields on G' is precisely $\mathfrak{g}$.

Let (a, b, c) be the coordinate system on G defined on points by

$$h \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} := \begin{cases} (-1)^{|x|} a' & h = a, \\ (-1)^{|y|} b' & h = b, \\ (-1)^{|z|} c' & h = c. \end{cases}$$

Note that this sign convention is natural in the following sense: Consider the supermanifold G' as the affine superspace of strictly upper triangular matrices. Then a, b, c are the linear superfunctions which constitute the dual basis to the standard basis (E_{12}, E_{13}, E_{23}) of elementary matrices.

Let $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}$ be the coordinate derivations given by the coordinate system (a, b, c). Let R_x, R_y, R_z be the left-invariant vector fields on G' determined by

$$R_x(1_{G'}) = \frac{\partial}{\partial a}(1_{G'}), \quad R_y(1_{G'}) = \frac{\partial}{\partial b}(1_{G'}), \quad R_z := [R_x, R_y],$$

where write $R_x(1_{G'})$ for $1_{G'}^{\sharp} \circ R_x$, *etc.*

We now proceed to compute these explicitly. Let $\phi^x : *[\tau_x] \longrightarrow G'$ be the infinitesimal flow of $R_x(1_{G'})$, where $|\tau_x| = |x|$. (Compare Definition 4.3.) For any function h on G', we have

$$\frac{\partial}{\partial\tau_x}\Big|_{\tau_x=0} h\begin{pmatrix}1 & (-1)^{|x|}\tau_x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \Big(\frac{\partial}{\partial a}h\Big)(1_{G'}),$$

as one sees by inserting the coordinates $h = a, b, c$. Thus, we have

$$(\phi^x)^\sharp(h) = h\begin{pmatrix}1 & (-1)^{|x|}\tau_x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Similarly, we obtain

$$(\phi^y)^\sharp(h) = h\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & (-1)^{|y|}\tau_y\\ 0 & 0 & 1\end{pmatrix}$$

for the infinitesimal flow ϕ^y of $R_y(1_{G'})$.

We compute

$$(R_xh)\begin{pmatrix}1 & a' & c'\\ 0 & 1 & b'\\ 0 & 0 & 1\end{pmatrix} = \frac{\partial}{\partial\tau_x}\Big|_{\tau_x=0} h\begin{pmatrix}1 & a' + (-1)^{|x|}\tau_x & c'\\ 0 & 1 & b'\\ 0 & 0 & 1\end{pmatrix},$$

$$(R_yh)\begin{pmatrix}1 & a' & c'\\ 0 & 1 & b'\\ 0 & 0 & 1\end{pmatrix} = \frac{\partial}{\partial\tau_y}\Big|_{\tau_y=0} h\begin{pmatrix}1 & a' & (-1)^{|y|}a'\tau_y + c'\\ 0 & 1 & (-1)^{|y|}\tau_y + b'\\ 0 & 0 & 1\end{pmatrix},$$

by again inserting the coordinates for h. We obtain

$$R_x = \frac{\partial}{\partial a}, \quad R_y = \frac{\partial}{\partial b} + (-1)^{|x||y|}a\frac{\partial}{\partial c}. \tag{6.2}$$

Here, we have used the parity identity $|x| + |y| + |z| = \bar{0}$. From these expressions, we see immediately that

$$R_z = [R_x, R_y] = (-1)^{|x||y|}\left[\frac{\partial}{\partial a}, a\frac{\partial}{\partial c}\right] = (-1)^{|x||y|}\frac{\partial}{\partial c}, \tag{6.3}$$

and that this is the only non-zero bracket between the vector fields R_x, R_y, R_z. The sign $(-1)^{|x||y|}$ that appears in the case of $|x| = |y| = \bar{1}$ is an artefact of the parity distribution which is non-standard in that case.

Since R_x, R_y, R_z are linearly independent, they span the Lie superalgebra of G', and it follows that $G \cong G'$. In what follows, we will identify these two supergroups. Moreover, we will identify x, y, z with R_x, R_y, R_z, respectively.

For further use below, we note that the right-invariant vector fields L_x, L_y, L_z, defined by

$$L_v := -i_G^\sharp \circ R_v \circ i_G^\sharp, \quad v = x, y, z,$$

take on the form

$$L_x = \frac{\partial}{\partial a} + b\frac{\partial}{\partial c}, \quad L_y = \frac{\partial}{\partial b}, \quad L_z = (-1)^{|x||y|}\frac{\partial}{\partial c}. \tag{6.4}$$

One immediately checks the bracket relation $[L_x, L_y] = -L_z$.

We now calculate the adjoint action of G in terms of the matrix presentation. Let $R \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$ and $(g, v) \in_R G \times \mathbb{A}^{\mathbb{K}}(\mathfrak{g})$ (*cf.* Ref. [6] for the notation), where we write

$$g = \begin{pmatrix}1 & a' & c'\\ 0 & 1 & b'\\ 0 & 0 & 1\end{pmatrix}, \quad v = \xi x(1_G) + \eta y(1_G) + \zeta z(1_G) \in \Gamma((1_G(g)^*\mathcal{T}_G)_{\bar{0}}).$$

According to the definition of a, b, and c, the generic point $\mathrm{id}_G \in_G G$ is

$$\mathrm{id}_G = \begin{pmatrix} 1 & (-1)^{|x|}a & (-1)^{|z|}c \\ 0 & 1 & (-1)^{|y|}b \\ 0 & 0 & 1 \end{pmatrix}.$$

Denoting the diagonal morphism of R by Δ_R, we compute, for any function h on G, that

$$\begin{aligned}\mathrm{Ad}(g)(v)(h) &= \Delta_R^\sharp(1\otimes v\otimes 1)h(g\,\mathrm{id}_G\,g^{-1})\\ &= \Delta_R^\sharp(1\otimes v\otimes 1)\,h\begin{pmatrix} 1 & (-1)^{|x|}a & (-1)^{|z|}c+(-1)^{|y|}a'b-(-1)^{|x|}ab' \\ 0 & 1 & (-1)^{|y|}b \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

To evaluate this further, we insert a, b, c for h. For $h = a, b$, Equation (6.2) tells us that we get ξ and η, respectively. For $h = c$, we get, upon applying Equation (6.3):

$$(-1)^{|x||y|}\zeta + (-1)^{|x|(|y|+\bar 1)}\eta a' - (-1)^{|y|}\xi b'.$$

Thus, identifying x with $x(1_G)$, *etc.*, and writing v in columns, we find

$$\mathrm{Ad}\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\xi x\\ \eta y\\ \zeta z\end{pmatrix} = \begin{pmatrix}\xi x\\ \eta y\\ (\zeta+(-1)^{|x|}\eta a'-(-1)^{(|x|+\bar 1)|y|}\xi b')z\end{pmatrix}.$$

One may verify the correctness of this result by rederiving the bracket relation

$$\begin{aligned}[x,y] &= \frac{\partial}{\partial\tau_y}\Big|_{\tau_y=0}(-1)^{|x||y|}[x,\tau_y y]\\ &= \frac{\partial^2}{\partial\tau_y\partial\tau_x}\Big|_{\tau_x=\tau_y=0}(-1)^{|x||y|}\,\mathrm{Ad}\begin{pmatrix} 1 & (-1)^{|x|}\tau_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(\tau_y y)\\ &= \frac{\partial^2}{\partial\tau_y\partial\tau_x}\Big|_{\tau_x=\tau_y=0}(-1)^{|x||y|}(-1)^{|x||y|+|x|}\tau_y((-1)^{|x|}\tau_x) = z.\end{aligned}$$

It is now straightforward if somewhat tedious to derive

$$\mathrm{Ad}^*\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\xi^* x^*\\ \eta^* y^*\\ \zeta^* z^*\end{pmatrix} = \begin{pmatrix}(\xi^*+(-1)^{|y|(|x|+\bar 1)}b'\zeta^*)x^*\\ (\eta^*-(-1)^{|x|}a'\zeta^*)y^*\\ \zeta^* z^*\end{pmatrix} \tag{6.5}$$

for any

$$(g,v^*)\in_R G\times\mathbb{A}^{\mathbb{K}}(\mathfrak{g}^*),\quad g=\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix},\quad v^*=\begin{pmatrix}\xi^* x^*\\ \eta^* y^*\\ \zeta^* z^*\end{pmatrix}.$$

As for the adjoint action, we make a sanity check:

$$\begin{aligned}\mathrm{ad}^*(x)(z^*) &= \frac{\partial^2}{\partial\tau_z\partial\tau_x}\Big|_{\tau_x=\tau_z=0}(-1)^{|x||z|}\,\mathrm{Ad}^*\begin{pmatrix} 1 & (-1)^{|x|}\tau_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(\tau_z z^*)\\ &= \frac{\partial^2}{\partial\tau_z\partial\tau_x}\Big|_{\tau_x=\tau_z=0}(-1)^{|x||z|+|x|}(-(-1)^{|x|}\tau_x)\tau_z z^* = -(-1)^{|x||z|}z^*,\end{aligned}$$

which is in agreement with our previous computations.

Returning to our point $f = \gamma z^* \in_T \mathbb{A}(\mathfrak{g}_{\mathbb{R}}^*)$, we have

$$(t,g)\in_R G_f \iff a't^\sharp(\gamma) = b't^\sharp(\gamma) = 0,\quad g=\begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.6}$$

Moreover, the orbit map $a_f : G_T \longrightarrow \mathbb{A}(\mathfrak{g}_{\mathbb{R}}^*)$ takes the form

$$(a_f)^\sharp(x) = \gamma b, \quad (a_f)^\sharp(y) = -(-1)^{|x||z|}\gamma a, \quad (a_f)^\sharp(z) = \gamma,$$

in terms of coordinates a, b, c on G and the (linear) coordinates x, y, z on $\mathbb{A}(\mathfrak{g}_{\mathbb{R}}^*)$, given by

$$h\begin{pmatrix}\xi^* x^* \\ \eta^* y^* \\ \zeta^* z^*\end{pmatrix} = \begin{cases}(-1)^{|x|}\xi^* x(x^*) = \xi^* & h = x,\\ (-1)^{|y|}\eta^* y(y^*) = \eta^* & h = y,\\ (-1)^{|z|}\zeta^* z(z^*) = \zeta^* & h = z.\end{cases}$$

We will now analyse this further, separately in the two cases in which G is not a Lie group (*i.e.* when at least one of x, y, z is odd).

6.2. **The Clifford supergroup of dimension** $1|2$**.** Assume that $|x| = |y| = \bar{1}$. In this case, G is called the *Clifford supergroup*. This case has been given a definitive treatment by Neeb and Salmasian [41, 42]. Our emphasis here will be to put in the general context the orbit method. Moreover, we shall obtain the full family of Clifford modules for any non-trivial central character in one sweep.

We will take $T \coloneqq \mathbb{A}^1$ and $\gamma \coloneqq u$, the standard coordinate function on $\mathbb{A}^1$, so $f = \gamma z^* : T \longrightarrow \mathbb{A}(\mathfrak{g}_{\Bbbk}^*)$. The Lie supergroup G_f over $T = T_0$ is completely determined by its underlying Lie group $(G_f)_0$ over T_0 and its Lie superalgebra $\mathfrak{g}_f$, defined in Equation (5.1). In view of Equation (6.1), we have $\mathfrak{g}_f = \mathcal{O}_T z$. For $R = *$, the condition in Equation (6.6) is void. We conclude that the point functor of G_f is given by

$$G_f(R) = \left\{\left(t, \left(\begin{smallmatrix}1 & 0 & c' \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right)\right) \,\middle|\, t, c' \in \Gamma(\mathcal{O}_{R,\mathbb{R},\bar{0}})\right\},$$

so that $G_f \cong \mathbb{A}^1_T$ with the standard addition of $\mathbb{A}^1$ as multiplication over T.

The orbit $G \cdot f = G_T/G_f$ is $T \times \mathbb{A}^{0|2}$ with fibre coordinates a, b. The local embedding $\tilde{a}_f : G \cdot f \longrightarrow \mathbb{A}(\mathfrak{g}_{\Bbbk}^*)_T$ over T is given by

$$(\tilde{a}_f)^\sharp(x) = \gamma b, \quad (\tilde{a}_f)^\sharp(y) = -\gamma a, \quad (\tilde{a}_f)^\sharp(z) = \gamma.$$

Proceeding for $G \cdot f$ analogous to the general philosophy of geometric quantisation or Kirillov's orbit method, *cf.* Ref. [18], we now construct a T-family of representations. This approach demands us to choose a *polarising subalgebra.* To avoid reality problems, we consider the case of $\mathbb{K} = \mathbb{C}$. In the real case, we would have to complexify function sheaves.

A polarising subalgebra corresponds here to the preimage $\mathfrak{h}$ in $\mathfrak{g}_T = \mathcal{O}_T \otimes \mathfrak{g}$ of a locally direct submodule of $\mathfrak{g}_T/\mathfrak{g}_f$ which is maximally totally isotropic with respect to the supersymplectic form induced by ω_f. We will consider the case of

$$\mathfrak{h} \coloneqq \langle x, z\rangle_{\mathcal{O}_T}.$$

The image in $\mathfrak{g}_T/\mathfrak{g}_f$ is indeed maximally totally isotropic.

Extending the general philosophy of geometric quantisation to our situation, the space of $\mathfrak{h}$-polarised sections of the canonical line bundle on $G \cdot f$ is a T-family of G-representations that should be considered a quantisation of $(G \cdot f, \omega_f, \mathfrak{h})$. In the case at hand, it corresponds to the functor $\mathcal{H}$, defined on objects $R \in \mathbf{SSp}_{\mathbb{C}}^{\mathrm{lfg}}$ by

$$\mathcal{H}(R) \coloneqq \left\{(t, \psi) \,\middle|\, t \in_R T, \psi \in \Gamma((t^*\mathcal{O}_{G_T})_{\bar{0}}), R_x\psi = 0, R_z\psi = -it^\sharp(\gamma)\psi\right\},$$

where we write R_x, *etc.*, in order to emphasise that we are acting by left-invariant vector fields. By Equations (6.2) and (6.3), the conditions on ψ amount to

$$\psi = \varphi e^{it^\sharp(\gamma)c}$$

where $\varphi \in \Gamma(\mathcal{O}_{R\times\mathbb{A}^{0|1},\bar{0}})$, and we consider b as fibre coordinate on $(R \times \mathbb{A}^{0|1})/R$. Thus, ψ admits an expansion in the powers b^0, b^1 of b, with coefficients in functions on R. Thus, $\mathcal{H}$ is the point functor of the trivial $\mathbb{C}$-vector bundle over T of rank $1|1 = \dim \Gamma(\mathcal{O}_{\mathbb{A}^{0|1}})$. We denote the corresponding $\mathcal{O}_T$-module by the same letter.

The action of $\mathfrak{g}$ on $\mathcal{H}$ is given by right-invariant vector fields. Indeed, for $v \in \mathfrak{g}$, $\psi \in \mathcal{H}(U)$, we set $\pi(v) := -L_v\psi$. By Equation (6.4) and the conditions on ψ, we readily obtain

$$\pi(x) = -ib\gamma, \quad \pi(y) = -\frac{\partial}{\partial b}, \quad \pi(z) = i\gamma.$$

Since the supercommutator of $\pi(x)$ and $\pi(y)$ is an anticommutator, we recognise this as the 'fermionic Fock space' or 'spinor module' of the $\mathcal{O}_T$-Clifford algebra $\mathrm{Cliff}(2, \mathcal{O}_T) := (\mathcal{O}_T \otimes \mathfrak{U}(\mathfrak{g}))/(z - i\gamma \cdot 1)$. That is, we have a trivial bundle of 'spinor' modules $\mathbb{C}^{1|1}$ over the base space $\mathbb{R}$, where the central character on the fibre at $t \in \mathbb{R}$ is $i\gamma(t) = it$. (The fibres at $t \neq 0$ are unital algebra representations of $\mathrm{Cliff}(2, \mathbb{C})$, whereas the fibre at 0 is a non-unital representation.)

6.3. **The odd Heisenberg supergroup of dimension $1|2$.** Assume now that $|x| = \bar{0}$, $|y| = |z| = \bar{1}$. In this case, we call G the *odd Heisenberg supergroup*, since it is an central extension of the Abelian Lie supergroup $\mathbb{A}^{1|1}$ with respect to a 2-cocycle corresponding to an odd supersymplectic form.

We will take $T := \mathbb{A}^{0|1}$ and $\gamma := \theta$, the standard coordinate function on $\mathbb{A}^{0|1}$, so $f = \theta z^* : T \longrightarrow \mathbb{A}(\mathfrak{g}^*_{\Bbbk})$. Arguing as in the last subsection, we conclude that

$$G_f(R) = \left\{ \left(t, \left(\begin{smallmatrix} 1 & 0 & c' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) \;\middle|\; t, c' \in \Gamma(\mathcal{O}_{R,\bar{1}}) \right\},$$

so that $G_f \cong T \times \mathbb{A}^{0|1}$ where $\mathbb{A}^{0|1}$ carries its usual additive structure. The orbit $G \cdot f = G_T/G_f$ is $T \times \mathbb{A}^{1|1}$ with fibre coordinates a, b. The local embedding $\tilde{a}_f : G \cdot f \longrightarrow \mathbb{A}(\mathfrak{g}^*_{\mathbb{R}})_T$ over T is given by

$$(\tilde{a}_f)^\sharp(x) = \gamma b, \quad (\tilde{a}_f)^\sharp(y) = -\gamma a, \quad (\tilde{a}_f)^\sharp(z) = \gamma.$$

As a polarising subalgebra, we again choose

$$\mathfrak{h} := \langle x, z\rangle_{\mathcal{O}_T}.$$

Once again, we define $\mathcal{H}$ on objects $R \in \mathbf{SSp}^{\mathrm{lfg}}_{\mathbb{C}}$ by

$$\mathcal{H}(R) := \left\{(t, \psi) \;\middle|\; t \in_R T, \psi \in \Gamma((t^*\mathcal{O}_{G_T})_{\bar{0}}), R_x\psi = 0, R_z\psi = -it^\sharp(\gamma)\psi\right\}$$

we see that the condition on ψ amounts to

$$\psi = \varphi e^{it^\sharp(\gamma)c} = \varphi(1 + it^\sharp(\gamma)c),$$

where $\varphi \in \Gamma(\mathcal{O}_{R\times\mathbb{A}^{0|1},\bar{0}})$ admits a finite expansion in b with coefficients in functions on R. The corresponding sheaf is $\mathcal{H} = \mathcal{O}_T \otimes \mathbb{C}^{1|1}$, representing the trivial rank $1|1$ $\mathbb{C}$-vector bundle over $T = \mathbb{A}^{0|1}$.

Then one readily computes the representation $\pi(v)\psi = -L_v\psi$ of $\mathfrak{g}$ by right-invariant vector fields:

$$\pi(x) = i\gamma b, \quad \pi(y) = -\frac{\partial}{\partial b}, \quad \pi(z) = i\gamma.$$

Since the supercommutator of $\pi(x)$ and $\pi(y)$ is an ordinary commutator, this is a parity reversed Schrödinger representation. This becomes even more apparent when we write out the integrated action $\pi(t, g)(t, \psi) = (t, \pi(g)_t\psi)$ of G on R-valued points of $\mathcal{H}$:

$$\pi\left(\begin{smallmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{smallmatrix}\right)_t \psi = \varphi' e^{it^\sharp(\gamma)c}, \quad \varphi'(b) = e^{it^\sharp(\gamma)(a'b + c' + a'b')}\varphi(b - b'),$$

where ψ and φ are related by $\psi = \varphi e^{it^\sharp(\gamma)c}$.

Appendix A. The relative inverse function theorem

In this appendix, we prove a relative version of the inverse function theorem, valid for an arbitrary base $S \in \mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$. This was used heavily in Section 4.

Theorem A.1 (Inverse function theorem)**.** *Let $X/S, Y/S \in \mathbf{SMan}_S$ and $\varphi : X/S \longrightarrow Y/S$ be a morphism over S. For any $x \in X_0$, the following are equivalent:*

(i) *There is an open neighbourhood $U_0 \subseteq X_0$ of x so that $V_0 := \varphi_0(U_0) \subseteq Y_0$ is open, and $\varphi : X|_U \longrightarrow Y|_V$ is an isomorphism.*
(ii) *The germ $(\mathcal{T}_{\varphi/S})_x : \mathcal{T}_{X/S,x} \longrightarrow \mathcal{T}_{Y/S,\varphi_0(x)}$ is invertible.*
(iii) *The map $T_{S,x}\varphi : T_{S,x}X \longrightarrow T_{S,\varphi_0(x)}Y$ is invertible.*

Basic for the *proof* is the following proposition, allowing us to locally extend morphisms of full rank to supermanifold neighbourhoods.

Proposition A.2. *Let X and Y be open subspaces of $\mathbb{A}_S^{p|q}/S$. Consider a morphism*

$$\varphi : X/S \longrightarrow Y/S$$

over S such that $\operatorname{rk} T_{S,x}\varphi = p|q$ for all $x \in X_0$.

Then for any $x \in X_0$, $y = \varphi_0(x)$, there exist open subspaces $U \subseteq X$, $V \subseteq Y$, $x \in U_0$, $\varphi_0(U_0) = V_0$, a supermanifold S', supermanifolds $X', Y' \in \mathbf{SMan}_{S'}$ over S' of dimension $\dim_{S'} X' = \dim_{S'} Y' = p|q$, an embedding $i : S \longrightarrow S'$, and embeddings $j_X : U \longrightarrow X'$, $j_Y : V \longrightarrow Y'$ over i which fit into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & Y' \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ U & \xrightarrow{\varphi} & V \end{array}$$

of morphisms over S' such that $\operatorname{rk} T_{S',x'}\varphi' = p|q$ for all $x' \in X'_0$.

In the *proof* of the proposition, we will use the following lemma.

Lemma A.3. *Let $\varphi : X/S \to Y/S$ be a morphism of supermanifolds over S. For any pair $m|n$ of non-negative integers, the set*

$$\{x \in X_0 \mid \operatorname{rk} T_{S,x}\varphi \geqslant m|n\}$$

is open. Here, we write $p|q \geqslant m|n$ if and only if $p \geqslant m$ and $q \geqslant n$.

Proof. In local fibre coordinates, $T_{S,x}\varphi$ is represented by the Jacobian matrix $\mathrm{Jac}_S(\varphi)(x)$, which is a continuous function of x. Since the rank of the upper or lower diagonal block of a block matrix is a lower semicontinuous function and the finite intersection of open sets remains open, the assertion follows. □

Proof of Proposition A.2. By assumption, S is in $\mathbf{SSp}_{\mathbb{K}}^{\mathrm{lfg}}$, so by definition [4], it locally admits embeddings into some $\mathbb{A}^{r|s}$. Passing to open subspaces, we may assume that there is an embedding $i : S \longrightarrow S' := \mathbb{A}^{r|s}$.

We define $\tilde{X} := \mathbb{A}_{S'}^{p|q}$ and $\tilde{Y} := \mathbb{A}_{\mathbb{K},S'}^{p|q}$, *cf.* Ref. [4]. That is, $\tilde{X}$ is the affine superspace over S' of fibre dimension $p|q$ and $\tilde{Y}$ is the functor given on objects $T/S' \in \mathbf{SSp}_{S'}^{\mathrm{lfg}}$ by

$$\tilde{Y}(T/S') = \Gamma\big(\mathcal{O}_{T,\bar{0}}^p \times \mathcal{O}_{T,\bar{1}}^q\big).$$

We may define the morphism $j_X : X \longrightarrow \tilde{X}$ over i as the restriction of $i \times \mathrm{id}_{\mathbb{A}^{p|q}}$ to X. By the Yoneda Lemma, $\mathrm{Hom}_{S'}(Y, \tilde{Y}) = \tilde{Y}(Y/S')$. Thus, $j' : Y/S' \longrightarrow \tilde{Y}/S'$ is determined by the tuple (x^a) of standard fibre coordinate functions on $\mathbb{A}_S^{p|q}$, restricted to Y.

Since the morphism j_X is an embedding, $j_X^\sharp$ is an surjective sheaf map, and there exist an open subspace $X' \subseteq \tilde{X}$ and functions $(\varphi'^a) \in \Gamma(\mathcal{O}_{X'})$, $|\varphi'^a| = |x^a|$, such that $j_{X,0}(x) \in X'_0$ and $j_X^\sharp(\varphi'^a) = \varphi^\sharp(j'^\sharp(x^a))$. By the definition of $\tilde{Y}$, the tuple (φ'^a) defines a morphism $\tilde{\varphi} : X' \longrightarrow \tilde{Y}$ over S'.

In view of Lemma A.3, we may assume that the matrix $\big(\frac{\partial \varphi'^b}{\partial x^a}(x)\big)$ over $\mathbb{K}$ has rank $p|q$ for every $x \in X'_0$. In particular, the upper left principal $p \times p$ block is invertible. We may now apply the inverse function theorem to the map $\varphi'_0 : X'_0 \longrightarrow \mathbb{K}^p$ whose components are the functions underlying the even entries of (φ'^a). Possibly after shrinking X'_0, it follows that $Y'_0 \coloneqq \varphi'_0(X_0)$ is a locally closed $\Bbbk$-submanifold of $\mathbb{K}^p$ of dimension p and differentiability class ϖ, such that the restriction of the even coordinates in (x^a) to Y'_0 defines a coordinate system, and that φ'_0 is an isomorphism of $\Bbbk$-manifolds of class ϖ.

Define $Y' \coloneqq Y'_0 \times \mathbb{A}_{S'}^{0|q}$ and a morphism $\varphi' : X' \longrightarrow Y'$ by $\varphi'^\sharp(x^a) \coloneqq \varphi'^a$ for all a. There is a natural map $Y' \longrightarrow \tilde{Y}$ over S', and j' manifestly factors through it to a morphism $j_Y : V \longrightarrow Y'$ where $V \subseteq Y$ is a maximal open subspace. These morphisms fit into a commutative diagram as in the assertion, since

$$j_X^\sharp(\varphi'^\sharp(x^a)) = \varphi^\sharp(j_Y^\sharp(x^a)),$$

and in particular, $V_0 = \varphi_0(U_0)$, where $U_0 \coloneqq j_{X,0}^{-1}(X'_0)$.

Moreover, the Jacobian matrix of φ' has full rank by construction. To see that j_Y is an embedding, notice that $j_{Y,0}$ is injective and that moreover, there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\;j_Y\;} & Y' \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \mathbb{A}_S^{p|q} & \xrightarrow{\;i\times \mathrm{id}_{\mathbb{A}^{p|q}}\;} & \mathbb{A}_{S'}^{p|q} \end{array}$$

where the left column is the canonical embedding, the right column x is the coordinate chart defined by (x^a), and the lower row is an embedding by the construction of $\mathbb{A}_S^{p|q}$ in Ref. [4]. This proves the claim. □

Remark A.4. The only slightly non-trivial point in the proof is that one may not assume in general that the functions φ'^a be $\Bbbk$-valued. This is circumvented by the use of the classical inverse function theorem.

We finally come to the proof of the relative inverse function theorem.

Proof of Theorem A.1. Certainly, the condition (i) implies (ii) and (ii) implies (iii).

To prove the remaining implication, we may by Proposition A.2 assume that $X, Y \subseteq \mathbb{A}^{p+r|q+s}$ and $S \subseteq \mathbb{A}^{r|s}$ are open subspaces, that the structural morphisms of X/S and Y/S are the respective restrictions of the projection $\mathbb{A}^{p+r|q+s} \longrightarrow \mathbb{A}^{r|s}$, and that the Jacobian of φ over S is invertible at every point.

Since the absolute Jacobian of φ (*i.e.* the Jacobian over $*$) has triangular block form with the Jacobian of the S-part being the identity, we may drop the relative setting and assume that $S = *$. The Jacobians of the canonical embeddings j_{X_0} and j_{Y_0} of the underlying spaces are injective, so by assumption, φ_0 has invertible Jacobian at some point. Hence, φ_0 is invertible after possibly shrinking the domains.

Because we have $X = X_0 \times \mathbb{A}^{0|q}$ and $Y = Y_0 \times \mathbb{A}^{0|q}$, we may upon postcomposing with $\varphi_0^{-1} \times \mathrm{id}_{\mathbb{A}^{0|q}}$ assume that $\varphi_0 = \mathrm{id}_{X_0}$. Rephrasing this setting, we may assume that $S \subseteq \mathbb{A}^p$ is an open subspace and $X = Y = S \times \mathbb{A}^{0|q}$ where the structural morphism is the projection $\pi : \mathbb{A}^{p|q} \longrightarrow \mathbb{A}^p$. This has put us back in a relative setting, but the only non-linearity now comes from the odd directions.

We now expand

$$\varphi^\sharp(\theta^a) \equiv \sum_b f_{ab}\theta^b \pmod{I^2}$$

where $f_{ab} = \pi^\sharp(h_{ab})$, $h_{ab} \in \Gamma(\mathcal{O}_S)$, and I is the ideal of $\Gamma(\mathcal{O}_X)$ generated by the standard odd coordinate functions (θ^a). Then for any $z \in S_0 = X_0$,

$$(f_{ab}(z)) = \mathrm{Jac}_S(\varphi)(z) = T_{S,z}\varphi$$

is invertible, so that the inverse matrix (f^{ab}) exists.

Define $\psi : Y/S \longrightarrow X/S$ by

$$\psi^\sharp(\theta^a) := \sum_b f^{ab}\theta^a.$$

Since φ is over S, we have

$$\varphi^\sharp(\psi^\sharp(\theta^a)) = \sum_b f^{ab}\varphi^\sharp(\theta^b) \equiv \theta^a \pmod{I^2}.$$

But then $(\psi \circ \varphi)^\sharp = \mathrm{id} + \delta$ where δ is $\mathcal{O}_{X_0}$-linear and $\delta(I) \subseteq I^2$, so that $\delta^{q+1} = 0$. It follows that $(\psi \circ \varphi)^\sharp$ is invertible, and in particular, φ possesses a left inverse. Applying the above argument to the latter, we obtain a right inverse of φ. It follows that φ is invertible in a neighbourhood of x. □

References

[1] M. Aguilar, S. Gitler, and C. Prieto, *Algebraic Topology from a Homotopical Viewpoint*, Universitext, Springer-Verlag, New York, 2002.

[2] M. Andler and S. Sahi, *Equivariant cohomology and tensor categories* (2008), available at `arXiv:0802.1038`.

[3] A. Alldridge and J. Hilgert, *Invariant Berezin integration on homogeneous supermanifolds*, J. Lie Theory **20** (2010), no. 1, 65–91.

[4] A. Alldridge, J. Hilgert, and T. Wurzbacher, *Singular superspaces*, Math. Z. **278** (2014), 441–492.

[5] A. Alldridge, J. Hilgert, and M. Laubinger, *Harmonic analysis on Heisenberg-Clifford Lie supergroups*, J. Lond. Math. Soc. (2) **87** (2013), no. 2, 561–585, DOI 10.1112/jlms/jds058.

[6] A. Alldridge and W. Palzer, *Asymptotics of spherical superfunctions on rank one Riemannian symmetric superspaces*, Documenta Math. **19** (2014), 1317–1366.

[7] A.L. Almorox, *Supergauge theories in graded manifolds*, Differential geometric methods in mathematical physics (Salamanca, 1985), Lecture Notes in Math., vol. 1251, Springer, Berlin, 1987, pp. 114–136, DOI 10.1007/BFb0077318.

[8] F.A. Berezin and G.I. Kac, *Lie groups with commuting and anticommuting parameters*, Mat. Sb. (N.S.) **82 (124)** (1970), no. 3, 343–359 (Russian); English transl., Math. USSR Sbornik **11**, no. 3, 311–325.

[9] F.A. Berezin and D.A. Leĭtes, *Supermanifolds*, Dokl. Akad. Nauk SSSR **224** (1975), no. 3, 505–508 (Russian); English transl., Soviet Math. Dokl. **16** (1976), no. 5, 1218–1222.

[10] P. Bieliavsky, A. de Goursac, and G. Tuynman, *Deformation quantization for Heisenberg supergroup*, J. Funct. Anal. **263** (2012), no. 3, 549–603, DOI 10.1016/j.jfa.2012.05.002.

[11] A. Borel, *Semisimple Groups and Riemannian Symmetric Spaces*, Texts and Readings in Mathematics, vol. 16, Hindustan Book Agency, New Delhi, 1998.

[12] N. Bourbaki, *Elements of mathematics. General topology. Part 1*, Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[13] C.P. Boyer and O.A. Sánchez-Valenzuela, *Lie supergroup actions on supermanifolds*, Trans. Amer. Math. Soc. **323** (1991), no. 1, 151–175, DOI 10.2307/2001621.

[14] C. Carmeli, G. Cassinelli, A. Toigo, and V.S. Varadarajan, *Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles*, Comm. Math. Phys. **263** (2006), no. 1, 217–258, DOI 10.1007/s00220-005-1452-0.

[15] C. Carmeli, G. Cassinelli, A. Toigo, and V. S. Varadarajan, *Erratum to: Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles*, Comm. Math. Phys. **307** (2011), no. 2, 565–566, DOI 10.1007/s00220-011-1332-8.

[16] C. Carmeli, L. Caston, and R. Fioresi, *Mathematical Foundations of Supersymmetry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011.

[17] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.

[18] J.-L. Clerc, P. Eymard, J. Faraut, M. Raïs, and R. Takahashi, *Analyse harmonique*, Les cours du C.I.M.P.A., CIMPA/ICPAM, Nice, 1982.

[19] L. Corwin, Y. Ne'eman, and S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)*, Rev. Modern Phys. **47** (1975), 573–603.

[20] P. Deligne and J.W. Morgan, *Notes on supersymmetry*, Quantum Fields and Strings: A Course for Mathematicians, 1999, pp. 41–98.

[21] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970.

[22] J.J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. **33** (1980), no. 6, 687–706, DOI 10.1002/cpa.3160330602.

[23] A.M. Frydryszak, *Q-representations and unitary representations of the super-Heisenberg group and harmonic superanalysis*, Journal of Physics: Conference Series **563** (2014), no. 1, 012010.

[24] P. Gabriel, *Construction de préschémas quotient*, Schémas en Groupes (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1963/64), Inst. Hautes Études Sci., Paris, 1963, pp. 251–286 (French).

[25] S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[26] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I.*, Grundlehren der mathematischen Wissenschaften, vol. 166, Springer-Verlag, Berlin, 1971.

[27] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[28] S. Helgason, *Groups and Geometric Analysis*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000. Corrected reprint of the 1984 original.

[29] H. Hohnhold, M. Kreck, S. Stolz, and P. Teichner, *Differential forms and 0-dimensional supersymmetric field theories*, Quantum Topol. **2** (2011), no. 1, 1–41, DOI 10.4171/QT/12.

[30] V.G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96.

[31] I. Kaplansky, *Projective modules*, Ann. of Math (2) **68** (1958), 372–377.

[32] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren, vol. 292, Springer-Verlag, Berlin, 1994.

[33] A.A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110 (Russian); English transl., Russian Math. Surv. **17**, 53–104, DOI doi:10.1070/RM1962v017n04ABEH004118.

[34] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), Lecture Notes in Math., vol. 570, Springer, Berlin, 1977, pp. 177–306.

[35] ———, *Harmonic analysis on graded (or super) Lie groups*, Group theoretical methods in physics (Sixth Internat. Colloq., Tübingen, 1977), Lecture Notes in Phys., vol. 79, Springer, Berlin-New York, 1978, pp. 47–50.

[36] D.A. Leites, *Introduction to the theory of supermanifolds*, Uspekhi Mat. Nauk **1** (1980), 3–57 (Russian); English transl., Russian Math. Surveys **35** (1980), no. 1, 1–64.

[37] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[38] Y.I. Manin, *Gauge Field Theory and Complex Geometry*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 289, Springer-Verlag, Berlin, 1997.

[39] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.

[40] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Ergebnisse der Mathematik, vol. 34, Springer-Verlag, Berlin, 1994.

[41] K.-H. Neeb and H. Salmasian, *Lie supergroups, unitary representations, and invariant cones*, Supersymmetry in mathematics and physics, Lecture Notes in Math., vol. 2027, Springer, Heidelberg, 2011, pp. 195–239, DOI 10.1007/978-3-642-21744-9_10.

[42] H. Salmasian, *Unitary representations of nilpotent super Lie groups*, Comm. Math. Phys. **297** (2010), no. 1, 189–227, DOI 10.1007/s00220-010-1035-6.

[43] G.M. Tuynman, *Super symplectic geometry and prequantization*, J. Geom. Phys. **60** (2010), no. 12, 1919–1939, DOI 10.1016/j.geomphys.2010.06.009.

[44] ———, *Super Heisenberg orbits: a case study*, XXIX Workshop on Geometric Methods in Physics, AIP Conf. Proc., vol. 1307, Amer. Inst. Phys., Melville, NY, 2010, pp. 181–184.

Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany
E-mail address: alldridg@math.uni-koeln.de

Universität Paderborn, Institut für Mathematik, 33095 Paderborn, Germany
E-mail address: hilgert@math.upb.de

Institut É. Cartan, Université de Lorraine et C.N.R.S., 57045 Metz, France
Current address: Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany
E-mail address: tilmann.wurzbacher@univ-lorraine.fr