ON RELATIVE CANONICAL SHEAF OF A CURVE OVER DEDEKIND DOMAIN

XIAOTAO SUN

Institute of Mathematics
Academia Sinica

Beijing 100080

China

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26

D-53225 Bonn

Germany

MPI / 94-146

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INTRODUCTION

The background of this paper is the studying of base extensions of arithmetic surfaces. Let K be a number field, $\mathcal{O}_K$ the ring of integers of K, and $B = spec\ \mathcal{O}_K$. Let $f : X \to B$ be a regular arithmetic surface of genus $g \geq 2$ over B(i.e. X is a regular scheme of dimension 2, $X_K = X \times_B spec K$ is geometrically irreducible of genus $g \geq 2$). One can define a relative canonical invertible sheaf $\omega_{X/B}$ on X(see [5] for the definition)and the self-interesection number $\omega^2_{X/B}$(in the sense of Arakelov theory). A well-known result of G.Faltings says that $\omega^2_{X/B} \geq 0$ if X/B is semi-stable(i.e. every fibre of f is a semi-stable curve over the residual field). So a question is what if X/B is not semi-stable. Follow the same idea of G.Xiao in [8], we consider a base extension of X/B such that the induced arithmetic surface $\widetilde{X}/\widetilde{B}$ is semi-stable and to study the different $\lambda C = \lambda\omega^2_{X/B} - \omega^2_{\widetilde{X}/\widetilde{B}}$, where λ is the degree of base extension. However, in our case, some typical difficults of characteristic p appear, that is, the p times of an unit of our local ring may not be an unit again. Hence, to describe the conatant C, a new index (ramification index) of f has to be introduced, which can be determined by the multiplicities of irreducible components of the singular fibres in the case of characteristic zero or when the multiplicities are not multiple of the characteristic p. On the other hand, we hope to understand $\omega_{X/B}$ more much for the further studying of base extensions of arithmetic surfaces. It is well-known, in the case of function fields, $\omega_{X/B}$ can be expressed by the differential sheaves of X and B, which behave well under base extensions and easy to compute by local coordinates. Of course, it can not be so nice in our case since we have no base field. But the relative differential sheaf $\Omega^1_{X/B}$ is easy to compute by local rings. So our first result is to give a description of $\omega_{X/B}$ by $\Omega^1_{X/B}$ and the ramification divisor(which we shall define), both of which are more convenient for compution. As a by-product, it also gives a more intrinsic definition of $\omega_{X/B}$. More generally, let R be a Dedekind domain whose residual field is perfect or of characteristic zero, and X a scheme over $B = spec R$ of dimension 2. If X is flat, proper and generic smooth over B(i.e. $X \times_B spec Q(R)$ smooth over $Q(R)$)with connected fibres(i.e. $Q(R)$ is algebraically closed in $K(X)$), then we call X a curve over R. If X is regular(normal), we call X a regular(normal) curve and so on. By arithmetic surface, we always means X is considered in the sense of Arakelov theory. All the morphisms and algebras in this paper are of finit

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typ, and all the rings are noetherian domain. After a preliminary of commutative algebra in §0, the following theorem will be proved in §1.

Theorem 1.1. *Let $f : X \to B$ be a regular curve over a Dedekind domain, and let $C_1,\ldots,C_n$ be the irreducible components of singular fibres. If $\Omega^{1\vee\vee}_{X/B}$ denotes the bidual of $\Omega^1_{X/B}$, and $m_i = m(C_i)$ denotes the multiplicity of C_i in the fibre containing C_i, then there exist $r_i \geq 0$, ramification index of f at C_i, such that*

$$\omega_{X/B} \cong \Omega^{1\vee\vee}_{X/B} \otimes \mathcal{O}_X(\sum_{i=1}^{n} r_i C_i),$$

where $r_i \geq m_i - 1$, and the equality holds if and only if m_i is not a multiplicity of the characteristic of residual field of $f(C_i)$.

As the application, we use it to study base extensions of arithmetic surfaces. The following results are obtained in §2, which belongs to G.Xiao in the case of function field.

Theorem 2.1. *Let $f : X \to B = spec\, \mathcal{O}_K$ be a regular arithmetic surface, let $L \supset K$ be a finite extension of degree λ, $\mathcal{O}_L$ the ring of integers of L and $\widetilde{B} = spec\, \mathcal{O}_L$. If the induced arithmetic surface $\tilde{f} : \widetilde{X} \to \widetilde{B}$ is semi-stable, then we have*

$$\omega^2_{\widetilde{X}/\widetilde{B}} \leq \lambda \omega^2_{X/B}, \quad deg \tilde{f}_* \omega_{\widetilde{X}/\widetilde{B}} \leq \lambda deg f_* \omega_{X/B},$$

where the second inequality is valid for any metric on $f_*\omega_{X/B}$.

By the proof of the above theorem, after some computions, we have the following corollaries immediately.

Corollary 2.1. *If X/B is a regular arithmetic surface of genus g, then*

(1) $\omega^2_{X/B} \geq 0$

(2) *there is a constant $0 \leq C \leq \omega^2_{X/B}$ such that*

$$\omega_{X/B} D \geq \frac{\omega^2_{X/B} - C}{4g(g-1)} deg(D)$$

for any effective Arakelov divisor D.

Corollary 2.2. *Let $X/\mathbb{Z} = Proj\, \mathbb{Z}[X\ Y\ Z]/(X^p + Y^p - Z^p)$, then*

$$K^2_{X/\mathbf{Z}} \geq (p-1)(p-3)$$

The following corollary is a slight generalization of a theorem of S.Zhang, we removed the restriction of semi-stable.

Corollary 2.3. *If X/B is a regular arithmetic surface of genus $g \geq 2$ and not smooth over B. Then $\omega^2_{X/B} > 0$.*

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§0 Preliminary.

In this section, we present some commutative algebra facts without proof, all of which can be find in [3] and [4].

Let R be a ring, M a finite R-module, and $\{m_1, ..., m_n\}$ a system of generators of M. The exact sequence

$$0 \to K \to R^n \xrightarrow{\alpha} M \to 0$$

is called the presentation of M defined by $\{m_1, ..., m_n\}$, where α maps the i-th canonical basis element e_i onto $m_i (i = 1, ..., n)$and $K = ker\alpha$. Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be a system of generators of K with $v_\lambda = (x_1^\lambda, ..., x_n^\lambda) \in R^n$ $(\lambda \in \Lambda)$. Then

$$(x_i^\lambda)_{\substack{i=1,...,n \\ \lambda\in\Lambda}}$$

is called a relation matrix of M with respect to $\{m_1, ..., m_n\}$.

Given such a matrix, let $F_i(M)$ denote the ideal of R generated by all $(n-i)$-rowed subdeterminants of the matrix $(i = 0, 1, .., n-1)$, and let $F_i(M) = R$ for $i \geq n$. You can prove that $F_i(M)$ does not depend on the special choice of the relation matrix and the choice of the generating system $\{m_1, ..., m_n\}$ of M. So we call $F_i(M)$ the i-th Fitting ideal of M.

Proposition 1. *Let M be a finite R-module, $F_i(M)$ the i-th Fitting ideal of M. Then*

(1) *For each algebra S/R we have*

$$F_i(S \otimes_R M) = S \cdot F_i(M)$$

(2) *If $N \subset R$ is a multiplicatively closed subset, then*

$$F_i(M_N) = F_i(M)_N$$

(3) *If M has rank r, then $F_i(M) = \{0\}$ for $i = 0, ..., r-1$, and $F_i(M) \neq \{o\}$ for $i \geq r$.*

Let $K := Q(R)$, and M a finite R-module such that $M_K := K \otimes_R M$ is a free K-module of some rank r. For a system of generators $\{x_1, ..., x_n\}$ of M, let

$$0 \to U \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$$

be the presentation of M corresponding to $\{x_1, ..., x_n\}$, i.e. $\beta(e_i) = x_i$ for $i = 1, ..., n$ and $U := ker\,\beta$. Clearly $\Lambda^{n-r+1}\alpha = 0$, since $U_K := K \otimes_R U$ of rank $n - r$. Then, for each $m \in \mathbb{N}$, there is a canonical R-linear map (write $F := R^n$)

$$\varphi^m : \quad \Lambda^m M \longrightarrow Hom_R(\Lambda^{n-r}U\,, \quad \Lambda^{n-r+m}F)$$

which is defined as follows: For $\omega \in \Lambda^m M$ choose a preimage $\bar{\omega} \in \Lambda^m F$ with respect to $\Lambda^m\beta$. Then

$$\varphi^m(\omega) : \quad \Lambda^{n-r}U \longrightarrow \Lambda^{n-r+m}F$$

takes any $u \in \Lambda^{n-r}U$ to $\Lambda^{n-r}\alpha(u) \wedge \bar{\omega} \in \Lambda^{n-r+m}F$. Therefore, there is a canonical commutative diagram

$$\begin{array}{ccc}
\Lambda^m M & \xrightarrow{\varphi^m} & Hom_R(\Lambda^{n-r}U, \quad \Lambda^{n-r+m}F) \\
\downarrow & & \downarrow \chi^m \\
\Lambda^m M_K & \xrightarrow{\varphi^m_K} & Hom_K(\Lambda^{n-r}U_K, \quad \Lambda^{n-r+m}F_K)
\end{array}$$

where $\chi^m(l) = id_K \otimes l$ for any $l \in Hom_R(\Lambda^{n-r}U, \quad \Lambda^{n-r+m}F)$.

You can prove the R-submodule $(\varphi^m_K)^{-1}(im\,\chi^m)$ of $\Lambda^m M_K$ is independent of the choice of the system of generators of M. If S is a R-algebra, we take $M = \Omega^1_{S/R}$, the relative differential module, then we call $\Delta^m(S/R) := (\varphi^m_K)^{-1}(im\,\chi^m)$ the m-th module of integral differential forms of S/R. For our use, we suppose $\Omega^1_{S/R}$ is of rank 1 and $S = R[x_1, ..., x_n]_N/I$. Write $P := R[x_1, ..., x_n]_N$, then there is an exact sequence

$$I/I^2 \xrightarrow{\alpha} S \otimes_P \Omega^1_{P/R} \to \Omega^1_{S/R} \to 0$$

Take $m = 1$, we have(we write $\Delta(S/R)$ for $\Delta^1(S/R)$ for simplity)

$$\Delta(S/R) \cong Hom_S(\Lambda^{n-1}I/I^2, \quad \Lambda^n(S \otimes_P \Omega^1_{P/R}))$$

Finally, we complete this section by some useful properties of the bidual of a module. Let M be a R-module and $M^\vee = Hom_R(M\,, R)$, then we say $M^{\vee\vee} := Hom_R(M^\vee\,, R)$ the bidual of M. If M is a torsion free R-module, then M and $M_\wp$ for $\wp \in spec\,R$ can be identified with their images in $M_K := M \otimes_R K$, where $K = Q(R)$ is the quotient field of R. Similarly $M^\vee$ and $M^\vee_\wp$ will be identified with their images in $M^\vee_K$, and $M^{\vee\vee}$ and $M^{\vee\vee}_\wp$ with their images in $M^{\vee\vee}_K$. We always identify M_K with $M^{\vee\vee}_K$. Then we have the following proposition.

Proposition 2. *If R is a normal domain and M is torsion free, then*

1. $M^\vee = \cap_{ht(\wp)=1} M^\vee_\wp$, $\quad M^{\vee\vee} = \cap_{ht(\wp)=1} M_\wp$, *where the intersection is taken for all $\wp \in spec\,R$ of heigth 1.*
2. *if M and N are torsion free R-module with $M_\wp = N_\wp$ for all $\wp \in spec\,R$ with $ht(\wp) = 1$, then*

$$M^{\vee\vee} = N^{\vee\vee}$$

§1.Relative canonical sheaf.

Let $f : X \to B = spec\, R$ be a curve over a Dedekind domain R, and L,K the function fields of X and B. We use $\Omega^1_{X/B}$ denotes the relative differential sheaf of X/B, and $\Omega^1_{L/K}$ denotes the differential module of L over K. The relative canonical sheaf $\omega_{X/B}$ and the Fitting ideal sheaf $\mathcal{F}(X/B)$ can be introduced as the following.

Definition. *The presheaves of $\omega_{X/B}$ and $\mathcal{F}(X/B)$ is defined as the following: For any affine open set $U = spec\, S$ of X, let*

$$\omega_{X/B}(U) = \Delta(S/R) \quad \mathcal{F}(X/B)(U) = F_1(\Omega^1_{S/R}).$$

By (1) and (2) of the Proposition 1, it is easy to verify that $\mathcal{F}(X/B)$ is well-defined. As for $\omega_{X/B}$, we can suppose that X is projective and

$$i : X \hookrightarrow P = \mathbb{P}^n_B$$

is the embedding. If $\mathcal{I}$ denotes the ideal sheaf of X in P, then it is clear, from the following commutative diagram, $\omega_{X/B}$ is well-defined.

$$(*)\qquad \begin{array}{ccc} \Omega^1_{X/B} & \xrightarrow{\varphi} & Hom_{\mathcal{O}_X}(\Lambda^{n-1}\mathcal{I}/\mathcal{I}^2, \quad \Lambda^n i^*\Omega^1_{P/B}) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\chi} \\ \Omega^1_{L/K} & \xrightarrow{\varphi_L} & Hom_L(\Lambda^{n-1}(\mathcal{I}/\mathcal{I}^2)_L, \quad \Lambda^n(i^*\Omega^1_{P/B})_L) \end{array}$$

where γ, χ are the canonical maps, and φ, φ_K are defined as in §0. For a coherent sheaf $\mathcal{G}$ on, we always denotes $\mathcal{G}\otimes_{\mathcal{O}_X} L$ by $\mathcal{G}_L$, and L here is considered as a constant sheaf. Since φ_L is an isomorphism, and χ is injective, we can see easily,

$$\omega_{X/B} \cong Hom_{\mathcal{O}_X}(\Lambda^{n-1}\mathcal{I}/\mathcal{I}^2, \quad \Lambda^n i^*\Omega^1_{P/B})$$

From the definition of $\omega_{X/B}$, we can see that $\omega_{X/B}$ is a subsheaf of the constant sheaf $\Omega^1_{L/K}$. The following lemma gives the relation of $\omega_{X/B}$ and the image of $\Omega^1_{X/B}$ under γ.

Lemma 1.1. *Let $f : X \to B$ be a curve over Dedekind domain, and let $T(\Omega^1_{X/B})$ denote the torsion of $\Omega^1_{X/B}$. Then*

$$\Omega^1_{X/B}/T(\Omega^1_{X/B}) \cong \omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{F}(X/B)$$

Proof. It is enough to prove the lemma locally, let $U = spec\, S$ and $S = P/I$, where $P = R[x_1, ..., x_n]_N$. We have the exact sequence

$$I/I^2 \xrightarrow{\alpha} S \otimes_P \Omega^1_{P/R} \xrightarrow{\beta} \Omega^1_{S/R} \to 0$$

Then the diagram $(*)$ becomes into

$$\begin{array}{ccc} \Omega^1_{S/R} & \xrightarrow{\varphi} & Hom_S(\Lambda^{n-1}I/I^2, \quad \Lambda^n(S \otimes_P \Omega^1_{P/R})) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\chi} \\ \Omega^1_{L/K} & \xrightarrow{\varphi_K} & Hom_L((\Lambda^{n-1}(I/I^2)_L, \quad \Lambda^n(L \otimes_P \Omega^1_{P/R})) \end{array}$$

We only need to determine $Im\,\varphi$.

Let $\{b_1, ..., b_n\}$ be a basis of $S \otimes_P \Omega^1_{P/R}$ such that $\{\omega_i = \beta(b_i)\}_{i=1,...,n}$ generate $\Omega^1_{S/R}$, and let

$$I/I^2 = S\,t_1 + \cdots + S\,t_r + S\,u_1 + \cdots + S\,u_m \quad (m \geq n-1)$$

such that $\alpha(u_1), ..., \alpha(u_m)$ is a system of generators of $\alpha(I/I^2)$ and $t_i\,(i = 1, ..., r)$ are torsion elements (i.e. $\alpha(t_i) = o$). then we have

$$\Lambda^{n-1} I/I^2 = \sum_{s+\mu=n-1} S(t_{j_1} \wedge \cdots \wedge t_{j_s} \wedge u_{i_1} \wedge \cdots \wedge u_{i_\mu})$$

$$\Lambda^{n-1}\alpha(\Lambda^{n-1} I/I^2) = \sum_{j_1,...,j_{n-1}} S\,(\alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-1}}))$$

$$\alpha(u_i) = \sum_{j=1}^{n} a_{ij} b_j \quad (i = 1, ..., m),$$

let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

be the relation matrix of $\Omega^1_{S/R}$ with respect to $\{\omega_i = \beta(b_i)\}_{i=1,...,n}$. Then, by the definition of φ,

$$\begin{aligned} \varphi(\omega_i)(\alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-1}})) &= \alpha(u_{j_1}) \wedge \cdots \wedge \alpha(u_{j_{n-1}}) \wedge b_i \\ &= \pm |A^i_{j_1,...,j_{n-1}}|\, b_1 \wedge \cdots \wedge b_n \end{aligned}$$

where $A^i_{j_1,...,j_{n-1}}$ is obtained from A by keeping the raws with numbers $j_1, ..., j_{n-1}$ and deleting the i-th column. So, by the definition of Fitting ideal, we get

$$Im\,\varphi = F_1(\Omega^1_{S/R}) \cdot Hom_S(\Lambda^{n-1} I/I^2 \;\; \Lambda^n(S \otimes_P \Omega^1_{P/R})) \cong F_1(\Omega^1_{S/R}) \cdot \Delta(S/R)$$

But

$$\Omega^1_{S/R}/T(\Omega^1_{S/R}) \cong Im\,\gamma \cong Im\,\varphi,$$

which completes the proof.

Let A, B be two local rings with $tr.deg(Q(A)/Q(B)) = d$ and $m_A \cap B = m_B$, where m_A and m_B are maximal ideals of A and B. For any ideal I of A, we define $v_A(I)$ is the largest integer such that $I \subseteq m_A^{v_A(I)}$ and call $v_A(F_d(\Omega^1_{A/B}))$ the ramification index of A over B, denoted by $r(AB)$. We define the reduced ramification index of A over B as the following integer

$$e(AB) := \max_{(x_1,...,x_r)} \{v_A(\prod_{i=1}^{r} x_i)\,|\,(x_1, ..., x_r)\text{ are the generators of } m_B\}.$$

For any morphism $f : X \to Y$ of schemes, we call $r(\mathcal{O}_x\mathcal{O}_{f(x)})$ the ramification index of f at x, and $e(\mathcal{O}_x\mathcal{O}_{f(x)})$ the reduced ramification index of f at x. In particular, when $f : X \to B$ is a normal curve, we define the ramification divisor of f as

$$R(f) := \sum_{i=1}^{n} r(\mathcal{O}_{x_i}\mathcal{O}_{f(x_i)})\, C_i,$$

where x_i is the generic point of C_i, and the sum takes for all point of codimension 1. It is easy to see the support of $R(f)$ is contained in the fibres of f since X is generic smooth over B. If X is factorial, $R(f)$ determines an invertible sheaf on X. Especially, when X is a regular curve, for any point x of codimension 1, we have

$$(\omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{F}(X/B))_x = (\omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-R(f)))_x,$$

which means

$$(\omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{F}(X/B))^{\vee\vee} = (\omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-R(f)))^{\vee\vee},$$

i.e.,

$$\Omega^{1\vee\vee}_{X/B} \cong (\Omega^1_{x/B}/T(\Omega^1_{X/B}))^{\vee\vee} \cong \omega_{X/B} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-R(f))),$$

since $\omega_{X/B}$ is invertible. So theorem 1.1 is a corollary of the following lemma.

Lemma 1.2. *Let $f : X \to B$ be a normal curve and C an irreducible component of the fibres. If $r(Cf(C))$ denotes the ramification index of f at the generic point of C, then*

$$r(Cf(C)) \geq m(C) - 1,$$

the equality holds if and only if $m(C)$ is not a multiple of the characteristic of the residual field of $f(C)$.

Proof. Let R be the local ring of B at $f(C)$ and $m_R = (t)R$. Since X is normal, we can choose a regular point x of C such that $S = \mathcal{O}_{X,x}$ is regular and only one component of the fibre through x. If u is the local equation of C at x, then $t = a_0u^{m(C)}$ and $a_0 \notin m_S$. Since the completion and the unramification extension of R do not change $r(Cf(C))$, we can assume x is a rational point of C over $k(R)$ and S is complete. Thus, let $m_S = (u, v)S$, we have

$$S \cong \frac{R[[u,v]]}{(t - a_0u^{m(C)})R[[u,v]]}$$

and

$$\Omega^1_{S/R} = \frac{S\,du \oplus S\,dv}{((m(C)a_0u^{m(C)-1} + u^{m(C)}\frac{\partial a_0}{\partial u})du + u^{m(C)}\frac{\partial a_0}{\partial v}dv)S}$$

By the definition of Fitting ideals,

$$F_1(\Omega^1_{S/R}) = (m(C)a_0u^{m(C)-1} + u^{m(C)}\frac{\partial a_0}{\partial u}, u^{m(C)}\frac{\partial a_0}{\partial v})S$$

let $A = S_{(u)}$ be the local ring of S at $(u)S$, then $F_1(\Omega^1_{A/R}) = F_1(\Omega^1_{S/R})A$ and

$$r(Cf(C)) = r(AR) = v_A(F_1(\Omega^1_{A/R})) \geq m(C) - 1$$

the equality holds if and only if $m(C)a_0$ is an unit of A, i.e. $m(C)$ is not a multiple of the characterstic of $k(R)$, the proof is completed.

Corollary 1.1. *If $k(R)$ is prefect or of characteristic zero, then*

$$r(AR) = 0 \iff A/R \text{ is smooth}$$

Corollary 1.2. *If $f : X \to B$ is a regular curve, then*

(1) $\Omega^{1\vee\vee}_{X/B}$ *is invertible*

(2) *If all the fibres of X/B are reduced, then*

$$\omega_{x/B} \cong \Omega^{1\vee\vee}_{X/B}.$$

§2. Base extensions of arithmetic surfaces.

Let K be a number field, $\mathcal{O}_K$ the ring of algebraic integers of K, and let $f : X \to B = spec\,\mathcal{O}_K$ be a regular arithmetic surface of genus $g \geq 2$ over B (i.e., X is a regular projective scheme of dimension 2, X_K is geometrically irreducible of genus $g \geq 2$). If $L \supset K$ is a finite extension of degree λ, then the natural morphism $\pi : \widetilde{B} = spec\,\mathcal{O}_L \to B$ is called a base extension of X/B. Now let us begin this section by the following commutative diagram:

$$\begin{array}{ccccccccc}
\widetilde{X} & \xleftarrow{\rho} & X_2 & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X \times_B \widetilde{B} & \xrightarrow{p_1} & X \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\
\widetilde{B} & = & \widetilde{B} & = & \widetilde{B} & = & \widetilde{B} & \xrightarrow{\pi} & B
\end{array}$$

where π_1 is the normalization of $X \times_B \widetilde{B}$, π_2 is the minimal desingulariztion of X_1 and ρ is the contraction of (-1)-curves in the singular fibres of f_2.

Let $\phi =: p_1 \circ \pi_1$ and $\varphi = \phi \circ \pi_2$, we call $\tilde{f} : \widetilde{X} \to \widetilde{B}$ the induced arithmetic surface of *pi*. If $\tilde{f} : \widetilde{X} \to \widetilde{B}$ is semi-stable, we say π is a semi-stablizer. For any irreducible component C_i of fibres, the multiplicity of C_i in the fibre is denoted by $m(C_i)$. We give the following easy lemma as the begining.

Lemma 2.1. *For any coherent sheaf $\mathfrak{F}$ on X, we have*

$$(\phi^*\mathfrak{F})^{\vee\vee} \cong (\phi^*(\mathfrak{F}^{\vee\vee}))^{\vee\vee}$$

Proof. Without losting the generality, we can assume $\mathfrak{F}$ is torsion free. So we only need to check

$$(\phi^*\mathfrak{F})_x \cong \phi^*(\mathfrak{F}_{\phi(x)})$$

for any codimension one point x of X_1, since $\phi(x)$ is also of codimension one at once x is a codimension one point on X_1, but this is clear. So the lemma follows, since all the isomorphisms $(\phi^*\mathfrak{F})_x \cong \phi^*(\mathfrak{F}_{\phi(x)})$ are compatible with

$$(\phi^*\mathfrak{F})_{K(X_1)} = (\phi^*\mathfrak{F}) \otimes_{\mathcal{O}_{X_1}} K(X_1) \cong \phi^*(\mathfrak{F}_{K(X)}).$$

Lemma 2.2. *Let $\bar{X} = X \times_B \widetilde{B}$, if all the fibres of $f_1 : X_1 \to \widetilde{B}$ are reduced, then*

$$(\phi^*\Omega^1_{X/B})^{\vee\vee} \cong (\Omega^1_{X_1/\widetilde{B}} \otimes_{\mathcal{O}_{X_1}} \mathcal{F}_0(X_1/\bar{X}))^{\vee\vee}.$$

Proof. Since $\phi^*\Omega^1_{X/B} = \pi_1^*(p_1^*\Omega^1_{X/B}) = \pi_1^*(\Omega^1_{\bar{X}/\widetilde{B}})$, we only need to prove

$$(\pi_1^*\Omega^1_{\bar{X}/\widetilde{B}})^{\vee\vee} \cong (\Omega^1_{X_1/\widetilde{B}} \otimes_{\mathcal{O}_{X_1}} \mathcal{F}_0(X/X_1))^{\vee\vee}$$

By the exact sequence

$$\pi_1^*\Omega^1_{\bar{X}/\widetilde{B}} \xrightarrow{\alpha} \Omega^1_{X_1/\widetilde{B}} \to \Omega^1_{X_1/\bar{X}} \to 0,$$

we know that

$$(\pi_1^*\Omega^1_{\bar{X}/\widetilde{B}})^{\vee\vee} \cong (Im\,\alpha)^{\vee\vee}$$

So it is enough to prove

$$(Im\,\alpha)_x = \mathcal{F}_0(\mathcal{O}_{X_1,x}/\mathcal{O}_{\bar{X},\pi_1(x)}) \cdot (\Omega^1_{X_1/\widetilde{B}})_x$$

for any codimension one x of X_1. Let $S = \mathcal{O}_{\bar{X},\pi_1(x)}$, $\widetilde{S} = \mathcal{O}_{X_1,x}$ and $R = \mathcal{O}_{\widetilde{B},f_1(x)}$. Then $(\Omega^1_{X_1/\widetilde{B}})_x = \Omega^1_{\widetilde{S}/R}$ is a free $\widetilde{S}$-module of rank one, since all the fibres of $f_1 : X_1 \to \widetilde{B}$ are reduced. So there exists $a \in \widetilde{S}$ such that

$$(Im\,\alpha)_x = a \cdot (\Omega^1_{X_1/\widetilde{B}})_x$$

By the definition of Fitting ideal, $(a)\widetilde{S} = \mathcal{F}_0(\widetilde{S}/S)$, the lemma is proved.

Let Γ_i be an irreducible component of the fibres of f_1 and $C_i = \phi(\Gamma_i)$, $\widetilde{S}$, S and A denote the local ring of X_1, $\bar{X}$ and X at the generic points of Γ_i, $\pi_1(\Gamma_i)$ and $\phi(\Gamma_i)$ respectively. Let $\widetilde{R}$ and R be the local rings of $\widetilde{B}$ and B at $f_1(\Gamma_i)$ and $f(C_i)$, $m_{\widetilde{R}} = (\tilde{t})\widetilde{R}$ and $m_R = (t)R$. If $m(C_i) = b_i$ and $m_A = (u)A$, then $t = a_0u^{b_i}$, a_0 is an unit of A. Since $m(\Gamma_i) = 1$, we can write $m_{\widetilde{S}} = (\tilde{t})\widetilde{S}$. Let a_i be the reduced ramification index of π at $f_1(\Gamma_i)$, i.e., $t = r_0\tilde{t}^{a_i}$, r_0 is an unit of $\widetilde{R}$.

Lemma 2.3. *Let $R(\pi_1)$ be the ramification divisor of $X_1 \xrightarrow{\pi_1} \bar{X}$ and p denote the characteristic of $k(R)$. Then*

(1) $r(\widetilde{S}S) \leq \frac{a_i}{b_i}$, *and* $r(\widetilde{S}S) = 0$ *if* $p \nmid b_i$.
(2) *If $R(f)$ denotes the ramification divisor of f and*

$$\phi^*(R(f)) - R(\pi_1) = \sum_{i=1}^{m} k_i\Gamma_i$$

then $k_i \geq 0$.

Proof. From the commutative digram

$$\begin{array}{ccc} \widetilde{S} & \longleftarrow & A \\ \uparrow & & \uparrow \\ \widetilde{R} & \longleftarrow & R \end{array}$$

we know $(t)\widetilde{S} = (u^{b_i})\widetilde{S}$, the reduced ramification index $e(\widetilde{S}A) = \frac{a_i}{b_i}$. So, if (1) is proved, plus lemma 1.2, (2) is clear.

m_S is generated by $\tilde{t}$ and u with $r_0\tilde{t}^{a_i} = a_0u^{b_i}$, r_0 and a_0 are also units of S. Let $s_0 = r_0^{-1}a_0$, then

$$(\frac{\tilde{t}^{\frac{a_i}{b_i}}}{u})^{b_i} = s_0\,, \quad \text{in } Q(S)$$

So $y = \frac{\tilde{t}^{\frac{a_i}{b_i}}}{u} \in \widetilde{S}$ since $\widetilde{S}$ is the integral closure of S in $Q(S)$. If S_1 denotes the localization of $S[y]$ at $m_{\widetilde{S}} \cap S[y]$, where $S[y]$ denotes the algebra generated by y and S in $\widetilde{S}$, then $m_{S_1} = (\tilde{t})S_1$ and S_1 is already a discrete valuation ring in $Q(S)$. So $\widetilde{S} = S_1$, which implies

$$\Omega^1_{\widetilde{S}/S} = \widetilde{S} \cdot d_{\widetilde{S}/S}(y)$$

By $uy = \tilde{t}^{e_i}$, we have $ud_{\widetilde{S}/S}(y) = 0$, so

$$r(\widetilde{S}S) \leq v_{\widetilde{S}}(u) = e_i = \frac{a_i}{b_i}$$

From $y^{b_i} = s_0$, we know $b_iy^{b_i-1}d_{\widetilde{S}/S}(y) = 0$. If $p \nmid b_i$, then

$$b_iy^{b_i-1} \notin m_{\widetilde{S}}\,, \text{ and } \Omega^1_{\widetilde{S}/S} = 0$$

So $r(\widetilde{S}S) = 0$, we have proved the lemma.

Theorem 2.1. *If π is a semi-stablizer of degree λ, then*

$$\omega^2_{\tilde{X}/\tilde{B}} \leq \lambda\omega^2_{X/B} \quad deg\, \tilde{f}_*\omega_{\tilde{X}/\tilde{B}} \leq \lambda deg\, f_*\omega_{X/B}$$

Proof. Since $\phi^*(\Omega^{1\vee\vee}_{X/B})$ is invertible, by lemma 2.1, we have

$$\phi^*(\Omega^{1\vee\vee}_{X/B}) \cong (\phi^*\Omega^1_{X/B})^{\vee\vee}$$

From lemma 2.2,

$$(\phi^*\Omega^1_{X/B})^{\vee\vee} \cong \Omega^{1\vee\vee}_{X_1/\tilde{B}} \otimes_{\mathcal{O}_{X_1}} \mathcal{F}(X_1/\bar{X})^{\vee\vee}$$

Let $U = X_1 - \{\text{singularities of } X_1\}$, then, on U, $\Omega^{1\vee\vee}_{X_1/\tilde{B}}$ and $\mathcal{F}(X_1/\bar{X})^{\vee\vee}$ are invertible such that

$$\phi^*\omega_{X/B} \cong \omega_{X_1/\tilde{B}} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(\sum_{i=1}^{m} k_i\Gamma_i) \quad \text{on } U$$

So there exists a divisor D_1 on X_2 and $\pi_2(D_1)$ = closed points on X_1 such that

$$\varphi^*\omega_{X/B} \cong \omega_{X_2/\tilde{B}} \otimes \mathcal{O}_{X_2}(D_1 + \sum_{i=1}^{m} k_i\Gamma_i)$$

where $\varphi = \phi \cdot \pi_2$ and we still use Γ_i to denote the strict image of Γ_i on X_2. Let $K_{X/B}$ and $K_{X_2/\tilde{B}}$ denote the weil divisors of $\omega_{X/B}$ and $\omega_{X_2/\tilde{B}}$, we forget the metric this moment, thus

$$\varphi^*K_{X/B} = K_{X_2/\tilde{B}} + \sum_{i=1}^{m} k_i\Gamma_i + D_1$$

we claim $D_1 \geq 0$, in fact, let $D_1 = D_1' - D_1''$, where $D_1' \geq 0$ and $D_1'' \geq 0$ have no common components. If $D_1'' > 0$, then there is an irreducible component Γ of D_1'' such that $\Gamma D_1'' < 0$ since $(D_1'')^2 < 0$, which implies

$$0 = \Gamma\varphi^*K_{X/B} = \Gamma K_{X_2/\tilde{B}} + \Gamma\sum_{i=1}^{m} k_i\Gamma_i + \Gamma D_1' - \Gamma D_1''$$

i.e., $\Gamma K_{X_2/\tilde{B}} < 0$, it is impossible since π_2 is the minimal desingularization of X_1. On the other hand,

$$K_{X_2/\tilde{B}} = \rho^*K_{\tilde{X}/\tilde{B}} + D_2$$

$D_2 \geq 0$ is composed by (-1)-curves. Let

$$V = D_1 + D_2 + \sum_{i=1}^{m} k_i\Gamma_i$$

then V is an effective vertical divisor on X_2 and

$$\varphi^*\omega_{X/B} \otimes_{\mathcal{O}_{X_2}} \rho^*\omega^{-1}_{\tilde{X}/\tilde{B}} \cong \mathcal{O}_{X_2}(V) \tag{2.1}$$

Now we suppose $\omega_{X/B}$, $\omega_{X_2/\tilde{B}}$ and $\omega_{\tilde{X}/\tilde{B}}$ have the metrics such that the residue maps (we only write out for X/B the other is same)

$$(\Omega^1_{X_v/K_v} \otimes \mathcal{O}_{X_v}(P))|_P \cong \mathbb{C}$$

are isometric for all infinit places v and points P (see [2] and [5] for the detail). Since

$$\varphi^*\omega_{X/B}|_w = \Omega^1_{X_{2\,w}/L_w} = \rho^*\omega_{\tilde{X}/\tilde{B}}|_w$$

for all infinit places w of L, we know that the isomorphism (2.1) is an isometric if we give the trivial metric on $\mathcal{O}_{X_2}(V)$. So we have (in the sense of Arakelov intersection)

$$\lambda K^2_{X/B} - K^2_{\tilde{X}/\tilde{B}} = V\varphi^*K_{X/B} + V\rho^*K_{\tilde{X}/\tilde{B}}$$

where $K_{X/B}$ and $K_{\tilde{X}/\tilde{B}}$ denote the Arakelov divisors of $\omega_{X/B}$ and $\omega_{\tilde{X}/\tilde{B}}$. Let

$$C = \frac{1}{\lambda}(V\varphi^*K_{X/B} + V\rho^*K_{\tilde{X}/\tilde{B}})$$

then $C \geq 0$, since V is an effective vertical divisor. So the first inequality is proved. By $\rho^*\omega_{\widetilde{X}/\widetilde{B}} \hookrightarrow \varphi^*\omega_{X/B}$, we have

$$\tilde{f}_*\omega_{\widetilde{X}/\widetilde{B}} = f_{2*}\rho^*\omega_{\widetilde{X}/\widetilde{B}} \hookrightarrow f_{2*}\varphi^*\omega_{X/B} = \pi^* f_*\omega_{X/B}$$

which proves the second inequality. Since both sheaves are the same at the infinit places, the inequality has nothing to do with metric. We complete the proof of theorem 2.1.

Remark 2.1. *If $C = 0$, i.e., $\lambda K^2_{X/B} = K^2_{\widetilde{X}/\widetilde{B}}$, then $D_2 = 0$ (X_2 is minimal), X_1 has only rational double points and $k_i = 0$, i.e., $(r_i - 1)\frac{a_i}{b_i} \leq 0$, which implies $r_i \leq 1$. So $b_i = m(C_i)$ can not be a multiple of the characteristic of $k(f(C_i))$ (otherwise $r_i \geq b_i$ by the lemma 1.2), but this implies $r(\widetilde{S}S) = 0$ by the lemma 2.3, so $R(f) = 0$, i.e., all the fibres of X/B are reduced. Clearly, the converse is also true. Thus $C = 0$ if and only if all the fibres of X/B are reduced, X_1 has only rational double points and X_2 is minimal.*

Corollary 2.1. *If X/B is a regular arithmetic surface of genus g, then*

(1) $\omega^2_{X/B} \geq 0$

(2) *there exists a constant $0 \leq C \leq \omega^2_{X/B}$ such that*

$$\omega_{X/B}D \geq \frac{\omega^2_{X/B} - C}{4g(g-1)} deg(D)$$

for any effective Arakelov divisor.

Proof. Fix a semi-stablizer π of degree λ, let C be the constant such that

$$\lambda\omega^2_{X/B} - \omega^2_{\widetilde{X}/\widetilde{B}} = \lambda C,$$

then, by the result of [2] and our theorem 2.1, $\omega^2_{X/B} \geq 0$ and

$$\omega^2_{X/B} = 0 \iff \omega^2_{\widetilde{X}/\widetilde{B}} = 0, \ C = 0$$

For any effective divisor D (we can suppose D is an irreducible horizontal divisor), we have

$$\varphi^*\omega_{X/B}\varphi^*D = \rho^*\omega_{\widetilde{X}/\widetilde{B}}\varphi^*D + V\varphi^*D \geq \omega_{\widetilde{X}/\widetilde{B}}\rho_*\varphi^*D$$

Since $\widetilde{X}/\widetilde{B}$ is semi-stable, by the result of [2], we get

$$\lambda\omega_{X/B}D \geq \frac{\omega^2_{\widetilde{X}/\widetilde{B}}}{4g(g-1)} deg\varphi^*D$$

By the definition of C, we have

$$\omega_{X/B}D \geq \frac{\omega^2_{X/B} - C}{4g(g-1)} deg(D)$$

and

$$\omega^2_{X/B} - C = 0 \iff \omega^2_{\widetilde{X}/\widetilde{B}} = 0$$

Remark 2.2. *Some other results of arithmetic surface in the semi-stable case can also be generalized to nonsemi-stable case, but we omit here. For example, the theorem 3 of C. Soule in [6] and some results of Bost, J.-B. in [1]. It would be interesting to give a detail description of C, which need to study the singularities produced by base extension. A more general question is to study the finit ramification coverings of a two dimensional regular scheme and to understand the relations between the singlarities of the finit covering and the singularities of the branch locus (the image of ramification divisor). We hope to come to these problems again in the other occasion.*

Corollary 2.2. *Let $X/\mathbb{Z} = Proj\,\mathbb{Z}[X\,Y\,Z]/(X^p + Y^p - Z^p)$ and $\pi : B \to \mathbb{Z}$ a semi-stablizer of degree λ. Then*

$$K_{X/\mathbf{Z}}^2 \geq \frac{1}{\lambda}\omega_{\tilde{X}/B}^2 + (p-1)(p-3)$$

In particular,

$$K_{X/\mathbf{Z}}^2 \geq (p-1)(p-3)$$

Proof. $X/\mathbb{Z}$ has only one singular fibre F at p, $F = pC$. From the proof of theorem 2.1, we have

$$\lambda K_{X/\mathbf{Z}}^2 - K_{\tilde{X}/B}^2 \geq \phi^* K_{X/\mathbf{Z}}(\phi^* R(f) - R(\pi_1))$$

Now the local equation of C is $u = x + y - 1$ (we consider the affine case) and

$$\Omega_{X/\mathbf{Z}}^1 = \frac{\mathbb{Z}[X\,Y]dX + \mathbb{Z}[X\,Y]dY}{(pX^{p-1}dX + pY^{p-1}dY)\mathbb{Z}[X\,Y]}$$

Thus

$$F_1(\Omega_{X/\mathbf{Z}}^1) = (px^{p-1},\, py^{p-1})\mathbb{Z}[x\,y]$$

which implies

$$r(Cf(C)) = p + min\,(v_{(u)}(x^{p-1}),\, v_{(u)}(y^{p-1}))$$

where $v_{(u)}$ denotes the valuation determined by local ring $\mathbb{Z}[x\,y]_{(u)}$. But $x + y = 1 + u$, so x and y can not belong to (u) at the same time, i.e., $r(Cf(C)) = p$. Hence

$$\phi^* R(f) - R(\pi_1) \geq \phi^*((p-1)C)$$

and

$$\lambda K_{X/\mathbf{Z}}^2 - K_{\tilde{X}/B}^2 \geq \lambda(p-1)K_{X/\mathbf{Z}}C$$

But $K_{X/\mathbf{Z}}C = \frac{1}{p}K_{X/\mathbf{Z}}F = \frac{2g-2}{p} = p - 3$, we have

$$K_{X/\mathbf{Z}}^2 \geq \frac{1}{\lambda}K_{\tilde{X}/B}^2 + (p-1)(p-3)$$

which complete the proof of corollary.

Remark 2.3. *It is easy to see our lemma 2.3. (2) means* $\pi_1^* K_{\tilde{X}/\tilde{B}} - K_{X_1/\tilde{B}}$ *is an effective divisor on the regular part of* X_1. *In fact, our theorem 2.1 holds for any base extensions once we know this fact. Professor E. Kunz told me in a letter that it can be proved by the theory of regular differential forms and* $\pi_1^* K_{\tilde{X}/\tilde{B}} - K_{X_1/\tilde{B}}$ *is the conductor of* π_1. *Thus the lemma 2.3 tell us that if* $m(C_i)$ *is not a multiple of the characteristic of* $k(f(C_i))$ *(for example, the case of characteristic zero), the conductor of* π_1 *is* $\phi^* R(f)$. *In fact, in this case, we can prove lemma 2.3 for any base extension (*X_1 *may have multiple fibres). A question is to determine the conductor of* π_1 *by* $R(f)$ *and* $R(f_1)$ *in general case and to prove* $r(\widetilde{S}S) = 0$ *when* $p|b_i$ *in lemma 2.3, or give an example in which* $r(\widetilde{S}S) \neq 0$.

With this remark in mind, we complete the paper by a generalization of S.Zhang's result. The theorem of S.Zhang said: If X/B is semi-stable and not smooth of genus $g \geq 2$, then $\omega^2_{X/B} > 0$ (see [7] or [9]). We remove the restriction of semi-stable as the following

Corollary 2.3. *If* $f : X \to B$ *is a regular arithmetic surface of genus* $g \geq 2$ *and not smooth, then* $\omega^2_{X/B} > 0$.

Proof. Let π be the base extension of X/B such that $\widetilde{X}/\widetilde{B}$ is semstable. By our Remark 2.1, $\omega^2_{X/B} = 0$ if and only if $C = 0$ and $\omega^2_{\widetilde{X}/\widetilde{B}} = 0$, which implies $\widetilde{X} = X_2$ is smooth over $\widetilde{B}$ by Zhang's theorem. So $X_1 = X_2$ (otherwise, $X_2/\widetilde{B}$ will have a singular fibre with two irreducible components at least). Since X_1 is regular, the conductor ideal sheaf of π_1 is locally principle, i.e. an invertible sheaf, whose divisor is $\phi^* R(f)$ by our lemma 2.3, which is zero since all of the fibres of X/B are reducible. Thus $X_1 = \bar{X} = X \times_B \widetilde{B}$ is smooth, that is, the support of $\mathcal{F}_1(\Omega^1_{\bar{X}/\widetilde{B}}) = p_1^* \mathcal{F}_1(\Omega^1_{X/B})$ is empty. It is impossible since X/B is not smooth and p_1 is flat.

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, BEIJING 100080, CHINA

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN-STRAβE 26, 53225 BONN, GERMANY

E-mail address: xsun@mpim-bonn.mpg.de