HOMOTOPY G-ALGEBRAS AND MODULI SPACE OPERAD

Murray Gerstenhaber [1]
Alexander A. Voronov [1,2]

1
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395

USA

2
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-71

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MURRAY GERSTENHABER AND ALEXANDER A. VORONOV

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Recently, shortly after a conjecture of Deligne [4], new algebraic structures on the Hochschild cochain space $V = C^\bullet(A, A)$ of an associative algebra have been discovered [8, 6]. It has also been pointed out in [8] that a similar structure takes place for the singular cochain complex $V = C^\bullet X$ of a topological space, due to Baues [2].

In this paper, we find a very general pattern which works for these two examples: in both cases, V has a natural structure of an operad. Together with a multiplication, it yields all the complicated algebraic buildup on V, see Sections 1 and 2.

The rest of the paper is dedicated to the geometry of the conjecture, which, in fact, assumed something more than mere algebraic structure:

Conjecture (Deligne). *The Hochschild cochain complex has a natural structure of an algebra over a chain operad of the little squares operad.*

In Section 4.2, we use the construction of Gromov-Witten invariants by Kontsevich to propose a way of proving the result analogous to the conjecture in the case of singular cochain complex.

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1. Homotopy G-algebra structure on an operad

1.1. The brace structure on an operad. Let $\{\mathcal{O}(n) \mid n \geq 1\}$ be an operad of vector spaces. Consider the graded vector space $\mathcal{O} = \bigoplus_n \mathcal{O}(n)$, the sum of all components of the operad. Denote by $\deg x$ the degree of an element $x \in \mathcal{O}$ and by $|x|$ the degree in the *desuspension* $\mathcal{O}[1]$ of the space $\mathcal{O}$, i.e., $\deg x = n$, $|x| = n - 1$,

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whenever $x \in \mathcal{O}(n)$. Define the following collection of multilinear operations, *braces* on $\mathcal{O}$:

$$x\{x_1, \dots, x_n\} := \sum (-1)^\varepsilon \gamma(x; \mathrm{id}, \dots, \mathrm{id}, x_1, \mathrm{id} \dots, \mathrm{id}, x_n, \mathrm{id}, \dots, \mathrm{id}) \tag{1}$$

for $x, x_1, \dots, x_n \in \mathcal{O}$, where the summation runs over all possible substitutions of $x_1, \dots, x_n$ into x in the prescribed order and $\varepsilon := \sum_{p=1}^n |x_p| i_p$, i_p being the total number of inputs in front of x_p. The braces $x\{x_1, \dots, x_n\}$ are homogeneous of degree $-n$, i.e., $\deg x\{x_1, \dots, x_n\} = \deg x + \deg x_1 + \dots + \deg x_n - n$. It will be convenient to denote $x\{y\}$ also by $x \circ y$ and understand $x\{\}$ as just x.

Remark. The sign is motivated by the example where $\mathcal{O} = \mathcal{E}nd(V)$, the *endomorphism operad* of a vector space V: $\mathcal{E}nd(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)$. Then the sign $(-1)^\varepsilon$ is picked up by rearranging the sequence of letters $x\{x_1, \dots, x_n\}(v_1, \dots, v_m)$, where $v_1, \dots, v_m \in V$, m is such that $x\{x_1, \dots, x_n\} \in \mathrm{Hom}(V^{\otimes m}, V)$, into the sequence $\gamma(x; v_1, \dots, v_{i_1}, x_1(v_{i_1+1}, \dots), \dots, v_{i_n}, x_n(v_{i_n+1}, \dots), \dots, v_m)$ in accordance with the usual sign convention.

One can immediately check the following identities:

$$\begin{aligned} x\{x_1, \dots, x_m\}\{y_1, \dots, y_n\} \\ = \sum_{0 \le i_1 \le \dots \le i_m \le n} (-1)^\varepsilon x\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots, y_{j_1}\}, y_{j_1+1}, \dots, \\ y_{i_m}, x_m\{y_{i_m+1}, \dots, y_{j_m}\}, y_{j_m+1}, \dots, y_n\}, \end{aligned} \tag{2}$$

where $\varepsilon := \sum_{p=1}^m |x_p| \sum_{q=1}^{i_p} |y_q|$, i.e., the sign is picked up by the x_i's passing through the y_j's in the shuffle.

Remark. The identity for $m = n = 1$ implies that the degree -1 bracket

$$[x, y] := x \circ y - (-1)^{|x||y|} y \circ x \tag{3}$$

defines the structure of a graded Lie algebra on $\mathcal{O}$.

Definition 1. A *brace algebra* is a graded vector space with a collection of braces $x\{x_1, \dots, x_n\}$ of degree $-n$ satisfying the identities (2).

Thus we have made the following observation.

Proposition 1. *For every operad $\mathcal{O}$ of vector spaces, the braces* (1) *define the natural structure of a brace algebra on the underlying graded vector space $\mathcal{O}$.*

1.2. Homotopy G-algebras. A *multiplication on an operad* $\mathcal{O}$ is an element $m \in \mathcal{O}(2)$ such that $m \circ m = 0$.

Proposition 2. (1) *A multiplication on an operad is equivalent to a morphism $\mathcal{A}s \to \mathcal{O}$ of operads.*
(2) *If V is an algebra over an operad with multiplication, then V is naturally an associative algebra.*
(3) *The product*

$$x \cdot y := (-1)^{|x|+1} m\{x, y\} \tag{4}$$

of degree 0 *and the differential*

$$dx = m \circ x - (-1)^{|x|} x \circ m, \qquad d^2 = 0, \quad \deg d = 1, \tag{5}$$

define the structure of a differential graded (DG) associative algebra on $\mathcal{O}$.

Analogously, a *multiplication on a brace algebra* $V = \bigoplus_n V_n$ is an element $m \in V_2$ such that $m \circ m = 0$. It also provides V with a DG algebra structure.

Amazingly, a multiplication generates a much richer algebraic structure on an operad, as the following theorem implies. Before making the statement, we should define the algebraic structure in question.

Definition 2. A *homotopy G-algebra* is a brace algebra $V = \bigoplus_n V^n$ provided with a differential d of degree one and a dot product xy of degree 0 making V into a DG associative algebra. The dot product must satisfy the following compatibility identities:

$$(x_1 \cdot x_2)\{y_1, \dots, y_n\} = \sum_{k=0}^{n} (-1)^{\varepsilon} x_1\{y_1, \dots, y_k\} \cdot x_2\{y_{k+1}, \dots, y_n\}, \tag{6}$$

where $\varepsilon = |x_2| \sum_{p=1}^{k} |y_p|$, and

$$\begin{aligned}
& d(x\{x_1, \dots, x_{n+1}\}) - (dx)\{x_1, \dots, x_{n+1}\} \\
& - (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1| + \dots + |x_{i-1}|} x\{x_1, \dots, dx_i, \dots, x_{n+1}\} \\
& \qquad = (-1)^{|x||x_1|+1} x_1 \cdot x\{x_2, \dots, x_{n+1}\} \\
& \qquad + (-1)^{|x|} \sum_{i=1}^{n} (-1)^{|x_1| + \dots + |x_{i-1}|} x\{x_1, \dots, x_i \cdot x_{i+1}, \dots, x_{n+1}\} \\
& \qquad - x\{x_1, \dots, x_n\} \cdot x_{n+1}
\end{aligned} \tag{7}$$

Remark. 1. Note that every homotopy G-algebra is in particular a DG Lie algebra with respect to the commutator (3), which is a graded derivation of the dot product up to a null-homotopy:

$$\begin{aligned}&[x, yz] - [x, y]z - (-1)^{|x|(|y|+1)}y[x, z] \\ &\quad = (-1)^{|x|+|y|+1}(d(x\{y, z\}) - (dx)\{y, z\} - (-1)^{|x|}x\{dy, z\} - (-1)^{|x|+|y|}x\{y, dz\}).\end{aligned}$$

Moreover, the multiplication is always *homotopy* graded commutative:

$$xy - (-1)^{(|x|+1)(|y|+1)}yx = (-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|}x \circ dy). \tag{8}$$

2. Since a brace algebra V with multiplication is a DG algebra, one can define its Hochschild cochain complex $C^\bullet(V, V)$, as usually, with the differential

$$Df := d \circ f - (-1)^{|f|}f \circ d + m \circ f - (-1)^{|f|}f \circ m$$

and the cup product

$$f_1 \cup f_2 := (-1)^{|f_1|+1}m\{f_1, f_2\},$$

the degree $|f|$ meaning the desuspended total degree. (Usually, by the cup product one means the opposite product $f_2 \cup f_1$.) Then the relations (6) and (7) mean that the correspondence

$$\begin{aligned}V &\to C^\bullet(V, V), \\ x &\mapsto \sum_{n=0}^{\infty} x\{x_1, \dots, x_n\},\end{aligned}$$

the summation being in fact finite, is a morphism of DG algebras.

Theorem 3. *A multiplication on an operad $\mathcal{O}(n)$ defines the structure of a homotopy G-algebra on $\mathcal{O} = \oplus\mathcal{O}(n)$. A multiplication on a brace algebra is equivalent to the structure of a homotopy G-algebra on it.*

Proof. The differential, the dot product and the braces have been already defined. What remains is to check the compatibility identities. In view of (4) amd (5), they both are particular cases of (2). □

2. Applications: Hochschild cochains and singular cochains

2.1. Hochschild complex. Applying this theorem to the Hochschild cochain complex $C^\bullet(A, A)$, which is at the same time the endomorphism operad $\mathcal{E}nd(A)(n) = C^n(A, A)$ and whose multiplication cocycle $m(a, b) := ab$, $m \in C^2(A, A)$, is a multiplication on this operad, we obtain the following result conjectured by Deligne [4] and proved in [6, 8].

Corollary 4. *The Hochschild complex $C^\bullet(A, A)$ of an associative algebra A has a natural structure of homotopy G-algebra.*

It is clear that the product obtained this way is the usual cup product

$$(x \cup y)(a_1, \dots, a_{k+l}) = x(a_1, \dots, a_k)y(a_{k+1}, \dots, a_{k+l}) \tag{9}$$

altered by the sign $(-1)^{(|x|+1)(|y|+1)}$. The bracket $[z, z]$ plays the role of a primary obstruction in deformation theory. It was introduced by Gerstenhaber in [5]. The higher braces for the Hochschild complex have been introduced in Getzler's work [7], where they are used to define the Hochschild cohomology of a homotopy associative algebra.

2.2. G-algebras. The structure inherited by the Hochschild cohomology was introduced in [5] and has been discovered in a number of places in mathematics and physics since then. A *G-algebra* is a graded vector space H with a dot product xy defining the structure of a graded commutative algebra and with a bracket $[x, y]$ of degree -1 defining the structure of a graded Lie, such that the bracket with an element is a derivation of the dot product:

$$[x, yz] = [x, y]z + (-1)^{|x|(|y|+1)}y[x, z].$$

In other words, a G-algebra is a specific graded version of a Poisson algebra.

Corollary 5. *The dot product and the bracket $[x, y] := x \circ y - (-1)^{|x||y|} y \circ x$ define the structure of a G-algebra on the Hochschild cohomology $H^\bullet(A, A)$ of an associative algebra A.*

Proof. A simple computation shows that the identity (1) yields the Jacobi identity for the bracket. Equation (8) implies that the differential is a derivation of the bracket:

$$d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = 0. \tag{10}$$

Therefore, even before passing to cohomology, the Hochschild complex forms a DG Lie algebra with respect to the bracket and a DG associative algebra with respect to the dot product.

Thus, we will be through if we see that

(1) the two operations take cocycles into cocycles and are independent of the choice of representatives of cohomology classes,
(2) the dot product is graded commutative and
(3) the bracket is a derivation of the dot product.

It easy to observe Fact (1) from Proposition 2(3) and (10), Fact (2) from the homotopy commutativity (8) at the cochain level. Fact (3) has already been mentioned in Remark after (7). □

2.3. Singular cochain complex. Let $C^\bullet X$ be the singular cochain complex of a topological space (or a simplicial set) X. For an n-simplex $\sigma : \Delta(n) \to X$, $\Delta(n)$ being the standard n-simplex, let $\sigma(n_0, \dots, n_k)$ denote its face spanned by the vertices $n_0, \dots, n_k$, where $i \mapsto n_i$ is an injective monotone function $\Delta(k) \to \Delta(n)$. The singular cochain complex $C^\bullet X$ has a natural operad structure $S(n) = C^n X$ defined by the compositions:

$$\begin{aligned}\gamma : S(k) \otimes S(n_1) \otimes \dots \otimes S(n_k) &\to S(n_1 + \dots + n_k),\\ \gamma(\varphi; \varphi_1, \dots \varphi_k)(\sigma) &:= \varphi(\sigma(0, n_1, n_1 + n_2, \dots, n_1 + \dots + n_k))\\ &\qquad \varphi_1(\sigma(0, 1, \dots, n_1))\varphi_2(\sigma(n_1, \dots, n_1 + n_2)) \dots\\ &\qquad \varphi_k(\sigma(n_1 + \dots + n_{k-1}, \dots, n_1 + \dots + n_k)).\end{aligned}$$

This automatically yields the structure of a brace algebra on $C^\bullet X$, according to Proposition 1. Define the multiplication $m \in C^2 X$ as

$$m(\sigma) := 1 \qquad \text{for any 2-simplex } \sigma.$$

Then Theorem 3 immediately recovers the following result of Baues [2].

Corollary 6. *The singular cochain complex $C^\bullet X$ of a topological space X has a natural structure of homotopy G-algebra.*

The dot product determined by the multiplication m as in (4) is nothing but the familiar cup product, up to the sign $(-1)^{(|\varphi|+1)(|\psi|+1)}$,

$$\varphi \cup \psi = \sum_{k=0}^{n} \varphi(\sigma(0, \dots, k))\psi(\sigma(k, \dots, n))$$

and the differential determined by (5) is merely the familiar coboundary operator, up to the sign $(-1)^{|\varphi|+1}$,

$$(d\varphi)(\sigma) = \sum_{k=0}^{n+1} (-1)^k \varphi(\sigma(0, 1, \dots, \hat{k}, \dots, n+1)).$$

Moreover, the lower brace $\varphi \circ \psi$ is the Steenrod operation $\varphi \cup_1 \psi$ and higher braces are multilinear generalizations of it.

3. Generalities

3.1. A_∞ version. Here we would like to consider A_∞ generalizations of our results in the spirit of [8]. An *A_∞-multiplication on an operad $\mathcal{O}$ (a brace algebra V)* is a formal sum $m = m_1 + m_2 + m_3 + \dots$, each $m_n \in \mathcal{O}(n)$ (or V_n), such that $m \circ m = 0$. In this case (5) is also a differential, but not homogeneous, $d = d_0 + d_1 + \dots$. As above, an A_∞-multiplication on an operad $\mathcal{O}$ defines

(1) a morphism $\mathcal{A}_\infty \to \mathcal{O}$ of operads, where $\mathcal{A}_\infty$ is the A_∞ (homotopy associative) operad, see [9],

(2) a natural structure of A_∞-algebra on each algebra over the operad $\mathcal{O}$,
(3) the structure of an A_∞-algebra on $\mathcal{O}$ itself with higher products M_n defined by the formula:

$$M_n(x_1, \dots, x_n) := m\{x_1, \dots, x_n\}, \qquad \text{for } n > 1,$$
$$M_1(x) := dx := m \circ x - (-1)^{|x|} x \circ m.$$

For any A_∞-algebra V the same formulas define the structure of an A_∞-algebra on the Hochschild complex $C^\bullet(V, V)$, see Getzler [7].

Theorem 7. *An A_∞-multiplication on an operad $\mathcal{O}(n)$ defines the following A_∞ version of a homotopy G-algebra on $V = \mathcal{O} = \oplus \mathcal{O}(n)$. It is a brace algebra V and an A_∞-algebra at the same time, such that the correspondence*

$$V \to C^\bullet(V, V),$$
$$x \mapsto \sum_{n=0}^{\infty} x\{x_1, \dots, x_n\},$$

is a morphism of A_∞-algebras.

3.2. Bar interpretation. In this section we want to make a translation of the algebraic notions introduced above into the dual language of bar constructions, following ideas of Getzler-Jones' work [8]. For a graded vector space $V = \bigoplus_n V_n$, let $V[-1]$, $V[-1]_n := V_{n-1}$, be its *suspension* and

$$\mathrm{B}\, V = \bigoplus_{n=0}^{\infty} (V[-1])^{\otimes n},$$

the *bar coalgebra* with the usual coproduct

$$\Delta[x_1|\dots|x_n] = \sum_{i=0}^{n} [x_1|\dots|x_i] \otimes [x_{i+1}|\dots|x_n],$$

$[x_1|\dots|x_k]$ denoting an element of $V[-1]^{\otimes k} \subset \mathrm{B}\, V$.

We call a product $\mathrm{B}\, V \otimes \mathrm{B}\, V \xrightarrow{\cup} \mathrm{B}\, V$ *left nonincreasing* if $\deg(x \cup y) \geq \deg x$, where by definition $\deg[x_1|\dots|x_n] = n$.

Lemma 8. *The structure of a brace algebra on a graded vector space V is equivalent to the structure of a bialgebra on the bar coalgebra $\mathrm{B}\, V$ defined by a left nonincreasing product.*

Proof. A product $\cup$ on $\mathrm{B}\, V$ compatible with the coproduct Δ and satisfying $\deg(x \cup y) \geq \deg x$ determines the braces uniquely by the formula

$$[x_1|\dots|x_m] \cup [y_1|\dots|y_n]$$
$$= \sum (-1)^\epsilon [y_1|\dots|y_{i_1}|x_1\{y_{i_1+1}, \dots\}|\dots|y_{i_m}|x_m\{y_{i_m+1}, \dots\}|\dots|y_n],$$

where the sign is the same as in (2). The associativity of the product is then equivalent to the relations (2). $\square$

Let V be a brace algebra, $\mathrm{B}\,V$ the corresponding bar bialgebra. A DG-*bialgebra* is a bialgebra with a degree -1 differential which is simultaneously a derivation and a coderivation.

Lemma 9. *An A_∞-multiplication on a brace algebra V is equivalent to the structure of a DG-bialgebra on the bar bialgebra $\mathrm{B}\,V$.*

Proof. An A_∞-multiplication m on V is equivalent to a codifferential δ on $\mathrm{B}\,V$, as has been well-known since Stasheff [14]. That δ is a derivation is equivalent to the compatibility condition of Theorem 7. $\square$

4. Topological and mirror reflections

4.1. Moduli spaces and little squares. As predicted by Deligne [4], the structure of a homotopy G-algebra on the Hochschild complex arises from an action of a chain complex of the little squares operad. The following combinatorial version of this statement was proved by Getzler and Jones. Consider Fox-Neuwirth's cellular partition of the configuration spaces $F(2,n)$ of n points in $\mathbb{R}^2$: cells are labelled by ordered partitions of the set $\{1,\dots,n\}$ into subsets with orderings within each subset. This reflects grouping points lying on common vertical lines on the plane and ordering the points lexicographically. Take the quotient cell complex $K_\bullet\mathcal{M}(n)$ by the action of translations $\mathbb{R}^2$ and dilations $\mathbb{R}_+^*$ and assemble these quotient spaces into a cellular operad $K_\bullet\underline{\mathcal{M}}(n)$. The resulting space $\underline{\mathcal{M}}(n)$ is a circle bundle over the real compactification $\underline{\mathcal{M}}_{0,n}$ of the moduli space $\mathcal{M}_{0,n}$ of n-punctured curves of genus zero, see [1, 8, 10, 11]. The space $\underline{\mathcal{M}}(n)$ can be also interpreted as a "decorated" moduli space, see next section. Cells in this cellular operad $K_\bullet\underline{\mathcal{M}}(n)$ are enumerated by pairs (T,p), where T is a tree with n initial vertices and one terminal vertex, labelling a component of the boundary of $\underline{\mathcal{M}}(n)$, and p is a partition, as above, of the set $\mathrm{in}(v)$ of incoming vertices for each vertex v of the tree T.

In [8], it is shown that a complex V is an A_∞ homotopy G-algebra, iff it is an algebra over the operad $K_\bullet\underline{\mathcal{M}}(n)$ satisfying the following condition. The structure mappings

$$K_\bullet\underline{\mathcal{M}}(n) \to \mathrm{Hom}(V^{\otimes n}, V),$$

of the algebra V over the operad $K_\bullet\underline{\mathcal{M}}(n)$ send all cells in $K_\bullet\mathcal{M}(n)$ to zero, except cells of two kinds:

(1) $(\delta_n, ((i_1),(i_2,\dots,i_n)))$, where δ_n is the corolla, the tree with one root and n edges, connecting it to the remaining n vertices, corresponding to the configuration where the points $i_2,\dots,i_n$ sit on a vertical line, the i_kth point being below the i_{k+1}st, and the i_1st point is in the half-plane to the left of the line;

(2) $(\delta_n, ((i_1, i_2, \ldots, i_n))$, corresponding to the configuration where all the points sit on a single vertical line, the i_kth point being below the i_{k+1}st.

Cells of the first kind map to the braces $x_{i_1}\{x_{i_2}, \ldots, x_{i_n}\}$, $n \geq 1$, and cells of the second kind map to the A_∞ products $M_n(x_{i_1}, \ldots, x_{i_n})$, $n > 1$.

Thus, the conditions of Theorem 7 and, in particular, when $M_n = 0$ for $n > 2$ the relations (6) and (7), follow from the combinatorial structure of the cell complex.

This construction of Getzler and Jones is essentially combinatorial and the question of a natural topological construction, perhaps, similar to those which come from quantum field theory (cf. [10]), where algebraic operations are obtained by integration over cycles, remains a mystery. At least in known examples, we anticipate that other cells give rise to nonzero multilinear operations.

Wish 10. *In all the examples, where a homotopy G-algebra structure occurs, e.g., the Hochschild complex or the singular cochain complex, the structure extends nontrivially to a natural structure of an algebra over the operad $K_\bullet \underline{\mathcal{M}}(n)$.*

This operad is closer to the homotopy G-operad in the sense of Ginzburg-Kapranov [9]. The latter operad is the operad cobar construction for the Koszul dual to the Gerstenhaber operad, which is the homology operad $H_\bullet \underline{\mathcal{M}}(n)$, see [3, 8].

4.2. The construction. Here we are going to sketch how to make Wish 10 come true in case of the homotopy G-algebra $C^\bullet X$. Our construction is a real version of Kontsevich's contsruction of Gromov-Witten invariants in [12, 13]. The difference is that we replace the moduli spaces $\overline{\mathcal{M}}_{0,n+1}$, which are compact complex manifolds, with the spaces $\underline{\mathcal{M}}(n)$, which are circle bundles over the real compactifications $\underline{\mathcal{M}}_{0,n+1}$, which are compact real manifolds with corners.

Let (X, ω) be a compact manifold with a sufficiently positive Kähler form ω. We will replace singular cochains $C^\bullet X$ with smooth forms

$$V = \Omega^\bullet X.$$

We want to define the natural structure of an algebra over the chain operad $C_\bullet \underline{\mathcal{M}}(n)$ on V. In particular, this will yield the structure of an algebra over the cellular chain operad $K_\bullet \underline{\mathcal{M}}(n)$ on V, giving a solution of Wish 10.

Let $\underline{\mathcal{M}}(n)(X, \beta)$ be the moduli space of *stable holomorphic maps* $(C; p_1, \ldots, p_{n+1}; \tau_1, \ldots, \tau_m, \tau_\infty; \phi)$ from a (degenerated) curve C of genus 0 to X:

$$\phi : C \to X$$

mapping the fundamental class of C to a given homology class $\beta \in H_2(X, \mathbb{Z})$. Here the curve C has $n+1$ punctures $p_1, \ldots, p_{n+1}$ and all the singularities of C must be m double points. For each i, $1 \leq i \leq m$, τ_i is the choice of a tangent direction at the ith double point to the irreducible component that is farther away from the "root", i.e., from the component of C containing the puncture $\infty := p_{n+1}$. τ_∞ is a tangent direction at ∞. The stability of a map is understood in the sense of Kontsevich

[12, 13]: each irreducible component of C contracted to a point by ϕ must be stable, i.e., admit no infinitesimal automorphisms. Because of finite group of automorphisms, the moduli space $\underline{\mathcal{M}}(n)(X,\beta)$ is only a compact stack. In some cases, e.g., when X is a homogeneous space, it is a smooth stack with corners. Let us assume it is one.

Notice that the configuration space $\underline{\mathcal{M}}(n)$ considered in Section 4.1 is the same as the moduli space of data $(C; p_1, \dots, p_{n+1}; \tau_1, \dots, \tau_m, \tau_\infty)$ as above, except that all components of C must be stable, cf. [8, 10]. The operad composition is given by attaching the ∞ punctures on curves to the other punctures on another curve, remembering the tangent direction at each new double point. Let

$$\pi : \underline{\mathcal{M}}(n)(X,\beta) \to \underline{\mathcal{M}}(n)$$

denote the forgetful map of the space $\underline{\mathcal{M}}(n)(X,\beta)$ of maps to the space $\underline{\mathcal{M}}(n)$ of curves.

There is a universal (evaluation) map

$$\Phi : \mathcal{C} \to X$$

from the universal curve over $\underline{\mathcal{M}}(n)(X,\beta)$ to the manifold X. The natural projection $\mathcal{C} \to \underline{\mathcal{M}}(n)(X,\beta)$ admits $n+1$ canonical sections $s_1, \dots, s_{n+1}$, sending a point of the moduli to the ith puncture on the universal curve.

Now we are ready to define the structure of an algebra over the operad $C_\bullet \underline{\mathcal{M}}(n)$ on V. Let $(\varphi, \psi) := \int_X \varphi \wedge \psi$ be the Poincaré pairing on V. Leaving aside problems with pairings and duals for infinite dimensional vector spaces and replacing singular cochains with differential forms once again, it suffices to construct mappings

$$f_n : V^{\otimes n+1} \to \Omega^\bullet \underline{\mathcal{M}}(n) \tag{11}$$

which will define the structure of an algebra over an operad

$$C_\bullet \underline{\mathcal{M}}(n) \to \operatorname{Hom}(V^{\otimes n}, V)$$

after dualizing V with the help of the Poincaré pairing. (Honestly speaking, we would have to replace singular chains with currents then). We define the mapping (11) by the formula

$$f_n(\varphi_1, \dots, \varphi_{n+1}) := \sum_{\beta \in H_2(X,\mathbf{Z})} \exp(-\int_\beta \omega)\, \pi_*(s_1^* \Phi^* \varphi_1 \wedge \dots \wedge s_{n+1}^* \Phi^* \varphi_{n+1}),$$

where Φ^* and s_i^*'s are pull-backs and π_* is a push-forward (fiberwise integration). In fact, because of the summation over the lattice $H_2(X, \mathbb{Z})$, we have to replace the ground field $\mathbb{C}$ with formal power series in $\beta \in H_2(X, \mathbb{Z})$.

Claim 11. *The maps* (11) *define a morphism of operads, that is, the structure of an algebra over the chain operad* $C_\bullet \underline{\mathcal{M}}(n)$ *on the de Rham complex* $V = \Omega^\bullet X$.

As soon as some hard problems with the construction, such as the smoothness of the stack of stable maps, are solved, the verification of the operad properties of this claim is automatic. So, we postpone the proof of it until better times.

REFERENCES

1. S. Axelrod and I. M. Singer, *Chern-Simons perturbation theory* II, J. Diff. Geom. **39** (1994), 173–213, hep-th/9304087.
2. J. H. Baues, *The double bar and cobar constructions*, Compos. Math. **43** (1981), 331–341.
3. F. R. Cohen, *The homology of C_{n+1}-spaces, $n \geq 0$*, The homology of iterated loop spaces, Lecture Notes in Math., vol. 533, Springer-Verlag, 1976, pp. 207–351.
4. P. Deligne, *Letter to Stasheff, Gerstenhaber, May, Schechtman, Drinfeld*, May 17, 1993.
5. M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267–288.
6. M. Gerstenhaber and A. A. Voronov, *Higher operations on Hochschild complex*, Preprint, University of Pennsylvania, April 1994.
7. E. Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Israel Math. Conf. Proc. **7** (1993), 65–78.
8. E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, Preprint, Department of Mathematics, MIT, March 1994, hep-th/9403055.
9. V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Preprint, Northwestern University, 1993.
10. T. Kimura, J. Stasheff, and A. A. Voronov, *On operad structures of moduli spaces and string theory*, Preprint 936, RIMS, Kyoto University, Kyoto, Japan, July 1993, hep-th/9307114.
11. M. Kontsevich, *Feynman diagrams and low-dimensional topology*, Preprint, Max-Planck-Institut für Mathematik, Bonn, 1993.
12. ———, *Enumeration of rational curves via torus actions*, Preprint, Max-Planck-Institut für Mathematik, Bonn, May 1994, hep-th/9405035.
13. M. Kontsevich and Yu. I Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Preprint MPI/93, Max-Planck-Institut für Mathematik, Bonn, February 1994, hep-th/9402147.
14. J. D. Stasheff, *On the homotopy associativity of H-spaces*, II, Trans. Amer. Math. Soc. **108** (1963), 293–312.

UNIVERSITY OF PENNSYLVANIA
E-mail address: murray@math.upenn.edu

MAX-PLANCK-INSTITUT FÜR MATHEMATIK IN BONN AND UNIVERSITY OF PENNSYLVANIA
E-mail address: voronov@math.upenn.edu