SEMISIMPLE FROBENIUS (SUPER)MANIFOLDS AND QUANTUM COHOMOLOGY OF $\mathbf{P}^r$

Yu. I. Manin

Max–Planck–Institut für Mathematik, Bonn, Germany

S. A. Merkulov

Department of Mathematics, University of Glasgow, UK

Посвящается О. А. Ладыженской.

Abstract. We introduce and study a superversion of Dubrovin's notion of semisimple Frobenius manifolds. We establish a correspondence between semisimple Frobenius (super)manifolds and special solutions to the (supersymmetric) Schlesinger equations. Finally, we calculate the Schlesinger initial conditions for solutions describing quantum cohomology of projective spaces.

0. Introduction

B. Dubrovin introduced the notion of Frobenius manifold and made an extensive study of it in [D]. Roughly speaking, it is a triple $(M, g, \circ)$ where M is a manifold, g is a flat Riemannian metric on it, and $\circ$ is a $\mathcal{O}_M$–linear commutative and associative multiplication on the tangent sheaf $\mathcal{T}_M$, with compatibility conditions (see 1.1.1 below). An important class of examples is supplied by *quantum cohomology* (see [KM].) Actually, quantum cohomology furnishes versions of Dubrovin's definition in which M may be supermanifold, or even a formal supermanifold.

An important subclass of Frobenius manifolds consists of semisimple ones. This means that tangent spaces with $\circ$–multiplication are semisimple algebras. This is possible only if M has no odd coordinates, by purely formal reasons.

In [Ma1], p. 41, one of the authors suggested that it would be interesting to construct a natural superization of the notion of semisimple Frobenius manifolds. This is one of the goals of this paper. To avoid any misunderstanding, we must stress that semisimple Frobenius supermanifolds in the sense of this work *are not* Frobenius manifolds in the category of supermanifolds in the sense of [KM] or [Ma1].

Our extension is based upon Dubrovin's theory of semisimple Frobenius manifolds reducing their classification to that of isomonodromic deformations: see [D]. This reduction exists in two versions. The first version leads to the deformation of a flat connection on a vector bundle on $\mathbf{P}^1$ having two singular points, a regular and an irregular one (see [D] and [S].) The second one deals with connections having only regular singularities. The two versions are related by the formal Laplace transform, as was explained in [KM].

In the classical paper [Sch], L. Schlesinger constructed the universal deformation space of the connections with regular singularities on $\mathbf{P}^1$ (see [Mal3] for a modern treatment.) Schlesinger's equations govern the dependence of the universal connection on the deformation parameters. Semisimple Frobenius manifolds correspond to some solutions to Schlesinger's equations, with the structure group reduced to the orthogonal one, and supplied with an additional piece of data. These solutions are called here "strict special" ones.

Our supeversion of the Dubrovin theory includes a superization of Schlesinger's equations, of the notion of strict special solutions, and of the correspondence between them and semisimple Frobenius manifolds briefly described above. This is one of the arguments for the naturality of our definition. An additional detail is that the (rather mysterious) structural action of the Virasoro algebra on any semisimple Frobenius manifold is now replaced by that of the Neveu–Schwarz superalgebra. We hope that super–Schlesinger equations may present an independent interest.

Since the structure of the superversion is closely parallel to that of the pure even one, we start with a report on the theory of semisimple Frobenius manifolds and with its application to the quantum cohomology of projective spaces.

Quantum cohomology of a projective algebraic manifold V is the pair $(H^*(V),\Phi)$ consisting of the usual cohomology space, say, with complex coefficients, and *the potential* Φ, a formal function on the cohomology space whose Taylor coefficients are numerical invariants of V counting the number of parametrized rational curves subject to certain incidence conditions: cf. e. g. [KM], [M1] and [BM]. Its third derivatives form the structure constants of *the quantum cohomology algebra of* V.

For projective spaces, it turns out to be semisimple, and we characterize the relevant special solutions by their initial conditions.

The paper is structured as follows. §1 contains an overview of the basic facts about Frobenius manifolds. Omitted proofs can be found in [D] and [Ma1]. In §2, we discuss the Schlesinger equations and interrelations between them and Frobenius manifolds. The version we explain here is taken from [Ma1]; for a closely related treatment see [H]. An important amelioration is the Theorem 1.14 and the related notion of strict speciality. §3 is devoted to the quantum cohomology of projective spaces; see [Ma2] and [Ma1], §III.5 for the special case of $\mathbf{P}^2$. In §4, we supersymmetrize the notion of semisimple Frobenius manifolds. Finally, in §5 we discuss a correspondence between semisimple Frobenius supermanifolds and special solutions of supersymmetric Schlesinger equations.

§1. Frobenius manifolds

1.1. Frobenius manifolds. Throughout this paper, we work in the category of complex manifolds M. A metric on M is an even symmetric pairing $g : S^2(\mathcal{T}_M) \to \mathcal{O}_M$, inducing an isomorphism $g' : \mathcal{T}_M \to \mathcal{T}_M^*$. Here $\mathcal{O}_M$ is the structure sheaf, and $\mathcal{T}_M$ is the tangent sheaf.

An affine flat structure on M is a subsheaf $\mathcal{T}_M^f \subset \mathcal{T}_M$ of linear spaces of pairwise commuting vector fields, such that $\mathcal{T}_M = \mathcal{O}_M \otimes_{\mathbf{C}} \mathcal{T}_M^f$. Sections of $\mathcal{T}_M^f$ are called flat vector fields. The metric g is compatible with the structure $\mathcal{T}_M^f$, if $g(X, Y)$ is constant for flat X, Y.

An affine flat structure can be equivalently described by a complete atlas whose transition functions are affine linear, because for a maximal commuting set of linearly independent vector fields (X_a) one can find local coordinates such that $X_a = \partial/\partial x^a$, and they are defined up to a constant shift. If a metric g is compatible with an affine flat structure, it is flat in the sense of the formalism of Riemannian geometry. The parallel transport endows $\mathcal{T}_M^f$ with the structure of local system.

1.1.1. Definition. *Let M be a manifold. Consider a triple $(\mathcal{T}_M^f, g, A)$ consisting of an affine flat structure, a compatible metric, and an even symmetric tensor $A : S^3(\mathcal{T}_M) \to \mathcal{O}_M$.*

Define an $\mathcal{O}_M$–bilinear symmetric multiplication $\circ = \circ_{A,g}$ on $\mathcal{T}_M$:

$$\mathcal{T}_M \otimes \mathcal{T}_M \to S^2(\mathcal{T}_M) \xrightarrow{A'} \mathcal{T}_M^* \xrightarrow{g'} \mathcal{T}_M : \ X \otimes Y \to X \circ Y \tag{1.1}$$

where prime denotes a partial dualization, or equivalently,

$$A(X, Y, Z) = g(X \circ Y, Z) = g(X, Y \circ Z). \tag{1.2}$$

This means that the metric is invariant with respect to the multiplication.

a). M endowed with this structure is called a pre–Frobenius manifold.

b). A local potential Φ for $(\mathcal{T}_M^f, A)$ is a local even function such that for any flat local tangent fields X, Y, Z

$$A(X, Y, Z) = (XYZ)\Phi. \tag{1.3}$$

A pre–Frobenius manifold is called potential one, if A everywhere locally admits a potential.

c). A pre–Frobenius manifold is called associative, if the multiplication $\circ$ is associative.

d). A pre–Frobenius manifold is called Frobenius, if it is simultaneously potential and associative.

If a potential Φ exists, it is unique up to adding a quadratic polynomial in flat local coordinates.

In flat local coordinates (x^a) (1.3) becomes $A_{abc} = \partial_a\partial_b\partial_c\Phi$, and (1.2) can be rewritten as

$$\partial_a \circ \partial_b = \sum_c A_{ab}{}^c \partial_c, \tag{1.4}$$

where

$$A_{ab}{}^c := \sum_e A_{abe} g^{ec}, \ (g^{ab}) := (g_{ab})^{-1}.$$

Furthermore,

$$(\partial_a \circ \partial_b) \circ \partial_c = \left(\sum_e A_{bc}{}^e \partial_e\right) \circ \partial_c = \sum_{ef} A_{ab}{}^e A_{ec}{}^f \partial_f,$$

$$\partial_a \circ (\partial_b \circ \partial_c) = \partial_a \circ \sum_e A_{bc}{}^e \partial_e = \sum_{ef} A_{bc}{}^e A_{ae}{}^f \partial_f =$$

$$= \sum_{ef} A_{bc}{}^e A_{ea}{}^f \partial_f. \tag{1.5}$$

Comparing the coefficients of ∂_f in (1.5), lowering the superscripts and expressing A_{abc} through a potential, we finally see that the notion of the Frobenius manifold is a geometrization of the following highly non–linear and overdetermined system of PDE:

$$\forall a,b,c,d: \quad \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad}. \tag{1.6}$$

They are called Associativity Equations, or WDVV (Witten–Dijkgraaf–Verlinde–Verlinde) equations.

Following B. Dubrovin, we will now express (1.6) as a flatness condition.

1.2. The first structure connection. Let (M, g, A) be a pre–Frobenius manifold (we omit $\mathcal{T}_M^f$ in the notation, since it can be reconstructed from g.) Define the following objects:

a). The connection $\nabla_0 : \mathcal{T}_M \to \Omega_M^1 \otimes \mathcal{T}_M$ well determined by the condition that flat vector fields are ∇_0–horizontal.

Denote its covariant derivative along a vector field X by

$$\nabla_{0,X}(Y) = i_X(\nabla_0(Y)), \ i_X(df \otimes Z) = Xf \otimes Z.$$

b). A pencil of connections depending on an even parameter λ:

$$\nabla_\lambda : \mathcal{T}_M \to \Omega_M^1 \otimes \mathcal{T}_M : \ \nabla_{\lambda,X}(Y) := \nabla_{0,X}(Y) + \lambda X \circ Y. \tag{1.7}$$

We will call ∇_λ the first structure connection of (M, g, A).

In flat coordinates (1.7) reads:

$$\nabla_{\lambda,\partial_a}(\partial_b) = \lambda \sum_c A_{ab}{}^c \partial_c = \lambda\, \partial_a \circ \partial_b = (-1)^{ab} \lambda \partial_b \circ \partial_a = (-1)^{ab} \nabla_{\lambda,\partial_b}(\partial_a).$$

Therefore ∇_λ has vanishing torsion for any λ. In particular, ∇_0 is the Levi–Civita connection for g.

1.2.1. Theorem. *Let ∇_λ be the structure connection of the pre–Frobenius manifold (M, g, A). Put $\nabla_\lambda^2 = \lambda^2 R_2 + \lambda R_1$ (there is no constant term since $\nabla_0^2 = 0$.) Then*

a). $R_1 = 0 \iff (M, g, A)$ is potential.

b). $R_2 = 0 \iff (M, g, A)$ is associative.

Therefore (M, g, A) is Frobenius, iff ∇_λ is flat.

This can be proved by the direct computation.

1.3. Identity. Let (M, g, A) be a pre–Frobenius manifold. A vector field e on M is called identity, if $e \circ X = X$ for all X.

If e exists at all, it is uniquely defined by $\circ$, hence by g and A.

Conversely, given A and e, there can exist at most one metric g making (M, g, A) a pre–Frobenius manifold with this identity:

$$g(X, Y) = A(e, X, Y).$$

This follows from (1.2). If A has a potential Φ, this translates into a non–homogeneous linear differential equation for Φ supplementing the Associativity Equations (1.6):

$$\forall \text{ flat } X, Y, \quad eXY\Phi = g(X, Y). \tag{1.8}$$

In most (although not all) important examples e itself is *flat.* If this is the case, one can everywhere locally find a flat coordinate system $(x^0, \dots, x^n)$ such that $e = \partial/\partial x^0 = \partial_0$, and (1.8) becomes

$$\forall\, a, b, \quad \Phi_{0ab} = g_{ab}. \tag{1.9}$$

Since all g_{ab} are constants, we get the following result.

On a potential pre–Frobenius manifold with flat identity $e = \partial_0$ (in a flat coordinate system) we have modulo terms of degree ≤ 2:

$$\Phi(x^0, \dots, x^n) = \frac{1}{2} x^0 \left(\sum_{a,b \ne 0} g_{ab} x^a x^b + \sum_{a \ne 0} g_{0a} x^0 x^a + \frac{1}{3}\, g_{00} (x^0)^2 \right) + \Psi(x^1, \dots, x^n). \tag{1.10}$$

The metric g identifies $\mathcal{T}_M$ and $\mathcal{T}_M^*$. We will call *the co–identity* and denote ε the 1–form which is the image of e. ε is defined by

$$\forall X \in \mathcal{T}_M, \quad i_X(\varepsilon) = g(X, e).$$

If (x^a) is a local coordinate system,then

$$\varepsilon = \sum_a dx^a g(\partial_a, e).$$

Finally, if e and (x^a) are flat, then $g(\partial_a, e)$ are constant, and

$$\varepsilon = d\eta,\ \eta = \sum_a x^a g(\partial_a, e). \tag{1.11}$$

1.4. Euler field. We will say that a vector field E on a manifold with flat metric (M, g) is *conformal*, if $\mathrm{Lie}_E(g) = Dg$ for some constant D. In other words, for all vector fields X, Y we have

$$E(g(X,Y)) - g([E,X],Y) - g(X,[E,Y]) = Dg(X,Y). \tag{1.12}$$

It follows that in flat coordinates we have $E = \sum_a E^a(x)\partial_a$ where $E^a(x)$ are polynomials of degree ≤ 1. In fact, E is a sum of infinitesimal rotation, dilation and constant shift. Hence $[E, \mathcal{T}_M^f] \subset \mathcal{T}_M^f$. Moreover, the operator

$$\mathcal{V}:\ \mathcal{T}_M^f \to \mathcal{T}_M^f,\ \mathcal{V}(X) := [X,E] - \frac{D}{2}X$$

is skewsymmetric:

$$\forall \text{ flat } X, Y:\ g(\mathcal{V}(X), Y) + g(X, \mathcal{V}(Y)) = 0.$$

1.4.1. Definition. *Let E be a vector field on a pre–Frobenius manifold (M, g, A). It is called an Euler field, if it is conformal, and* $\mathrm{Lie}_E(\circ) = d_0\circ$ *for some constant d_0, that is, for all vector fields X, Y,*

$$[E, X\circ Y] - [E,X]\circ Y - X\circ [E,Y] = d_0 X\circ Y. \tag{1.13}$$

Clearly, any scalar multiple of an Euler field is also an Euler field. One can use this remark in order to normalize E by requiring that some non–vanishing eigenvalue becomes one. A convenient choice is often $d_0 = 1$, if we have reasons to restrict ourselves to the $d_0 \ne 0$ case.

1.4.2. Proposition. *Let E be a conformal vector field on a Frobenius manifold (M, g, Φ). Then E is Euler, iff*

$$E\Phi = (d_0 + D)\Phi + \text{a quadratic polynomial in flat coordinates.}$$

1.4.3. Case of semisimple $\mathrm{ad}\, E$. We will call the set of eigenvalues of $-\mathrm{ad}\, E$ on $\mathcal{T}_M^f$, together with d_0 and D, *the spectrum* of E. We will say that E is *semi–simple*, if $\mathrm{ad}\, E$, acting on flat fields, is. For semisimple E we can construct many homogeneous elements of $\mathcal{O}_M(*)$ and $\mathcal{T}_M(*)$ explicitly.

Let (∂_a) be a local basis of $\mathcal{T}_M^f$ such that

$$[\partial_a, E] = d_a \partial_a \tag{1.14}$$

where (d_a) form a part of the spectrum of E. Putting $E = \sum E^a(x)\partial_a$, we find from (1.14) that $\partial_a E^b = \delta_a^b\, d_a$. Hence if $\partial_a = \partial/\partial x^a$, we have

$$E = \sum_{a:\, d_a \neq 0} (d_a x^a + r^a)\partial_a + \sum_{b:\, d_b = 0} r^b \partial_b.$$

By shifting x^a, we can make $r^a = 0$ for $d_a \neq 0$. Multiplying x^b by a constant, we can make $r^b = 0$ or 1 for $d_b = 0$. So finally we can choose local flat coordinates in such a way that

$$E = \sum_{a:\, d_a \neq 0} d_a x^a \partial_a + \sum_{\text{some } b:\, d_b = 0} \partial_b. \tag{1.15}$$

Clearly, E assigns definite degrees to the following local functions:

$$Ex^a = d_a x^a \text{ for } d_a \neq 0;\ E \exp x^b = \exp x^b \text{ or } 0 \text{ for } d_b = 0. \tag{1.16}$$

Assume now that M has an identity e. From (1.13) we get

$$[e, E] = d_0 e. \tag{1.17}$$

Hence our notation for the spectrum will be consistent, if in the case of flat e we put $e = \partial_0$, and otherwise do not use 0 as one of the subscripts in (1.14).

Formula (1.12) in the basis (1.13) becomes

$$\forall\, a, b:\ g(d_a\partial_a, \partial_b) + g(\partial_a, d_b\partial_b) = Dg_{ab}$$

that is,

$$(d_a + d_b - D)g_{ab} = 0. \tag{1.18}$$

In particular, $g(e, e) = 0$ unless $D = 2d_0$.

1.5. Extended structure connection. Let M be a pre–Frobenius manifold with a conformal vector field E. Put $\widehat{M} := M \times (\mathbf{P}^1_\lambda \setminus \{0, \infty\})$, where $\mathbf{P}^1_\lambda$ is the completion of Spec $\mathbf{C}[\lambda, \lambda^{-1}]$. Furthermore, put $\widehat{\mathcal{T}} = \mathrm{pr}_M^*(\mathcal{T}_M)$. If X is a vector field on M, it may be lifted to $\widehat{M}$ in two different guises: as a vector field annihilating λ, denoted again X, and as a section of $\widehat{\mathcal{T}}$, then denoted $\widehat{X}$.

Choose a constant d_0 and put $\mathcal{E} := E - d_0\lambda\dfrac{\partial}{\partial\lambda} \in \mathcal{T}_{\widehat{M}}$. Clearly, $\widehat{X}$ for flat X span $\widehat{\mathcal{T}}$, whereas flat X and $\mathcal{E}$ span $\mathcal{T}_{\widehat{M}}$, *provided* $d_0 \neq 0$, which we will assume.

1.5.1. Definition. *Let M be a pre–Frobenius manifold with a conformal field E, and d_0 a non–zero constant. The extended structure connection for M is the connection $\widehat{\nabla}$ on the sheaf $\widehat{\mathcal{T}}$ on $\widehat{M}$, defined by the following formulas for its covariant derivatives: for any local vector fields $X \in \mathcal{T}_M$, $Y \in \mathcal{T}_M^f$,*

$$\widehat{\nabla}_X(\widehat{Y}) := \lambda\widehat{X \circ Y}, \tag{1.19}$$

$$\widehat{\nabla}_{\mathcal{E}}(\widehat{Y}) := \widehat{[E, Y]}. \tag{1.20}$$

1.5.2. Theorem. *The extended structure connection is flat iff M is Frobenius and E is Euler with* $\mathrm{Lie}_E\,(\circ) = d_0\circ$.

From (1.19) and (1.20) one can derive a formula for the covariant derivative in the λ–direction: if Y is flat, we have

$$\widehat{[E,Y]} = \widehat{\nabla}_{E-d_0\lambda\partial/\partial\lambda}(\widehat{Y}) = \widehat{\nabla}_E(\widehat{Y}) - d_0\lambda\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y}) = \lambda\widehat{E\circ Y} - d_0\lambda\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y})$$

so that

$$d_0\widehat{\nabla}_{\partial/\partial\lambda}(\widehat{Y}) = \widehat{E\circ Y} - \frac{1}{\lambda}\widehat{[E,Y]}. \tag{1.21}$$

1.6. Semisimple Frobenius manifolds. Let (M,g,A) be an associative pre–Frobenius manifold of dimension n.

1.6.1. Definition. *M is called semisimple (resp. split semisimple) if an isomorphism of the sheaves of $\mathcal{O}_M$–algebras*

$$(\mathcal{T}_M,\circ) \,\widetilde{\to}\, (\mathcal{O}_M^n, \text{componentwise multiplication}) \tag{1.22}$$

exists everywhere locally (resp. globally.)

This means that in a local (resp. global) basis $(e_1,\dots,e_n)$ of $\mathcal{T}_M$ the multiplication takes form

$$(\sum f_ie_i)\circ(\sum g_je_j) = \sum f_ig_ie_i,$$

and in particular,

$$e_i\circ e_j = \delta_{ij}e_j. \tag{1.23}$$

Such a family of idempotents is well defined up to renumbering. Another way of saying this is that a semisimple manifold comes with the structure group of $\mathcal{T}_M$ reduced to S_n. Notice that e_i are generally not flat, so that this reduction is not compatible with that induced by $\mathcal{T}_M^f$, with the structure group $GL(n)$.

Hence if M is semisimple, there exists an unramified covering of degree $\le n!$, upon which the induced pre–Frobenius structure is split.

Denote by (ν^i) the basis of 1–forms dual to (e_i). ¿From (1.2) and (1.23) we find

$$g(e_i,e_k) = g(e_i\circ e_i, e_k) = g(e_i, e_i\circ e_k) = \delta_{ik}g_{ii}.$$

We will denote g_{ii} by η_i. We see that the symmetric 2-form representing g is diagonal in the basis (ν^i):

$$g = \sum_i \eta_i(\nu^i)^2. \tag{1.24}$$

Moreover, according to (1.2), $A(e_i,e_j,e_k) = \delta_{ij}\delta_{ik}\eta_i$, so that the symmetric 3-form representing A, is diagonal with the same coefficients:

$$A = \sum_i \eta_i(\nu^i)^3. \tag{1.25}$$

Finally, $e := \sum_i e_i$ is the identity in $(\mathcal{T}_M,\circ)$, and the co–identity, defined in 2.1.2, nicely complements (1.24) and (1.25):

$$\varepsilon = \sum_i \eta_i\nu^i. \tag{1.26}$$

Thus the Definition 1.61 can be restated as follows:

1.6.2. Definition. *The structure of the semisimple pre–Frobenius manifold on M is determined by the following data:*

a). A reduction of the structure group of $\mathcal{T}_M$ to S_n, specified by a choice of local bases (e_i) and dual bases (ν^i).

b). A flat metric g, diagonal in (e_i), (ν^i).

c). A diagonal cubic tensor A with the same coefficients as g.

Associativity of $(\mathcal{T}_M, \circ)$ is automatic in both descriptions. However, potentiality (and the flatness of g which we postulated) are non–trivial conditions.

1.7. Theorem. *The structure described in the Definition 3.2 is Frobenius iff the following conditions are satisfied:*

a). $[e_i, e_j] = 0$, or equivalently, $e_i = \partial/\partial u^i$, $\nu^i = du^i$ for a local coordinate system (u^i) called canonical one.

b). $\eta_i = e_i \eta$ for a local function η defined up to addition of a constant. Equivalently, ε is closed.

We will call η *the metric potential* of this structure. (Sometimes this term refers to h such that $g_{ab} = \partial_a \partial_b h$; our meaning is different.)

Canonical coordinates are defined up to renumbering and constant shifts.

1.8. The Darboux–Egoroff equations. The Theorem 1.7 establishes a (not very explicit) equivalence between the following functional spaces on M (modulo self–evident equivalence):

a). Flat coordinates $(x^1, \dots, x^n)$, flat metric g_{ab}, function $\Phi(x)$ satisfying the Associativity Equations (1.6) and semisimplicity.

b). Canonical coordinates $(u^1, \dots, u^n)$, function $\eta(u)$ such that the metric $g = \sum e_i \eta (du^i)^2$ is flat, where $e_i = \partial/\partial u^i$.

The constraints on η, implicit in b), are called the Darboux–Egoroff equations. In order to write them down explicitly, let us introduce the rotation coefficients of the potential metric:

$$\gamma_{ij} := \frac{1}{2} \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}} \tag{1.27}$$

where as before, $\eta_i = e_i \eta$, $\eta_{ij} = e_i e_j \eta$.

1.8.1. Proposition. *The diagonal potential metric $g = \sum e_i \eta (du^i)^2$ is flat iff $\forall k \neq i \neq j \neq k$:*

$$e_k \gamma_{ij} = \gamma_{ik} \gamma_{kj} \tag{1.28}$$

and

$$e \gamma_{ij} = 0. \tag{1.29}$$

1.8.2. Proposition. *Let e be the identity, and ε the co–identity of the semisimple Frobenius manifold. Then*

a). $\varepsilon = d\eta$, where η is the metric potential.

b). e is flat iff for all i, $e\eta_i = 0$, or equivalently, $e\eta = g(e,e) = \text{const}$. This condition is satisfied in the presence of an Euler field with $D \neq 2d_0$ (see (1.12), (1.13), (1.14).)

c). If e is flat, and (x^a) is a flat coordinate system, then

$$\eta = \sum_a x^a g(\partial_a, e) + \text{const}. \tag{1.30}$$

The formula (1.30) shows that in the passage from the (x^a, Φ)–description to the (u^i, η)–description the main information is encoded in the transition formulas $u^i = u^i(x)$, at least in the presence of flat identity.

Like the identity, the Euler field is almost uniquely defined by the canonical coordinates, if it exists at all.

1.9. Theorem. *Let E be a vector field on the semisimple Frobenius manifold M, d_0 a constant.*

a). We have $\mathrm{Lie}\,(\circ) = d_0(\circ)$, *iff*

$$E = d_0 \sum_i (u^i + c^i) e_i, \tag{1.31}$$

where c^i are some constants.

b). For the field of the form (3.17) and a constant D, we have $\mathrm{Lie}_E(g) = Dg$ *iff for all i, $E\eta_i = (D - 2d_0)\eta_i$, or equivalently*

$$E\eta = (D - d_0)\eta + \text{const}. \tag{1.32}$$

Thus in the presence of a non–vanishing Euler field we may and will normalize the canonical coordinates so that $E = d_0 \sum u^i e_i$.

1.10. A pencil of flat metrics. Equations (1.28) are stable with respect to a semigroup of coordinate changes. Namely, let f_i be arbitrary functions of one variable such that $\check{u}^i := f_i(u^i)$ form a local coordinate system, $\check{e}_i = \partial/\partial \check{u}^i, \check{\eta}_i = \check{e}_i \eta$ etc.

1.10.1. Proposition. *If (e_i, γ_{ij}) satisfy (1.28), then $(\check{e}_i, \check{\gamma}_{ij})$ satisfy (1.28) as well.*

In order to satisfy (1.29) as well, we will have to restrict ourselves to the one–parameter family of local coordinate changes

$$\check{u}^i = \log\,(u^i - \lambda),\ \ \check{e}_i = (u^i - \lambda) e_i,\ \ \check{g}_\lambda = \sum_i (u^i - \lambda)^{-1} e_i \eta (du^i)^2 \tag{1.33}$$

which make sense on $M_\lambda := \{x \in M \,|\, \forall i,\ u^i \neq \lambda\}$.

1.10.2. Theorem. *Let M be a semisimple Frobenius manifold with canonical coordinates (u^i) and metric potential η. Then the following statements are equivalent.*

a). For all λ, the structure (1.33) is semisimple Frobenius on M_λ.

b). The same for a particular value of λ.

c). For all $i \neq j$,

$$\sum_k u^k e_k \gamma_{ij} = -\gamma_{ij}. \tag{1.34}$$

Moreover, (1.34) is satisfied if $E = \sum_k u^k e_k$ is the Euler field on M with $d_0 = 1$.

Notice that generally $\check{e} = \sum \check{e}_k$ is not flat for $\check{g}_\lambda$ and $\check{E} = \sum \check{u}^k \check{e}_k$ is not an Euler field.

If E is Euler, the metric $\check{g}_\lambda$ in (1.33) can be written in coordinate free form:

$$\check{g}_\lambda(X, Y) = g((E - \lambda)^{-1} \circ X, Y). \tag{1.35}$$

In fact (1.35) is flat on any Frobenius manifold with semisimple Euler field on it: cf. [D].

1.11. The second structure connection. From now on, we will restrict ourselves to the case of semisimple complex Frobenius manifolds carrying an Euler field with $d_0 = 1$ and admitting a global system of canonical coordinates (u^i). We will call *the second structure connection* $\check{\nabla}_\lambda$ the Levi–Civita connection of the flat metric (1.35), depending on a parameter λ and defined on the open subset $M_\lambda \subset M$ where $u^i \neq \lambda$ for all i. Put $\check{M} := \cup_\lambda (M_\lambda \times \{\lambda\}) \subset M \times \mathbf{P}^1_\lambda$ and denote by $\check{\mathcal{T}}$ the restriction of $\mathrm{pr}^*_M(\mathcal{T}_M)$ to $\check{M}$.

We will construct a flat extension of $\check{\nabla}$ of $\check{\nabla}_\lambda$ to $\check{\mathcal{T}}$ which will also be referred to as the second structure connection. Both extensions $\widehat{\nabla}$ and $\check{\nabla}$ will be further studied as isomonodromic deformations of their restrictions to the λ–direction parametrized by M.

More precisely, assume that $\mathcal{T}^f_M$ is a trivial local system (for instance, because M is simply connected.) Put $T := \Gamma(M, \mathcal{T}^f_M)$. Then $\widehat{\nabla}$ (resp. $\check{\nabla}$) induces an integrable family of connections with singularities on the trivial bundle on $\mathbf{P}^1_\lambda$ with the fiber T. The first connection $\widehat{\nabla}$ is singular only at $\lambda = 0$ and $\lambda = \infty$ but whereas 0 is a regular (Fuchsian) singularity, ∞ is irregular one, so that $\widehat{\nabla}$ cannot be an algebraic geometric Gauss–Manin connection, and its monodromy involves the Stokes phenomenon. To the contrary, the second connection $\check{\nabla}$ generally has only regular singularities at infinity and at $\lambda = u^i$ whose positions thus depend on the parameters. It is determined by the conventional monodromy representation and has a chance to define a variation of Hodge structure. For more details, see the next section.

It turns out that both deformations have a common moduli space and deserve to be studied together. In fact, fiberwise they are more or less formal Laplace transforms of each other. More to the point, they form a dual pair in the sense of J. Harnad.

In our calculations the key role will be played by the $\mathcal{O}_M$–linear skew symmetric operator $\mathcal{V}:\,\mathcal{T}_M\to\mathcal{T}_M$ which is the unique extension of the operator defined in 1.4 on flat vector fields by the formula

$$\mathcal{V}(X)=[X,E]-\frac{D}{2}X \text{ for } X\in\mathcal{T}_M^f. \tag{1.36}$$

1.11.1. Proposition. *a). We have for arbitrary $X\in\mathcal{T}_M$:*

$$\mathcal{V}(X)=\nabla_{0,X}(E)-\frac{D}{2}X. \tag{1.37}$$

b). Let $e_j=\partial/\partial u^j$, $f_j=e_j/\sqrt{\eta_j}$. Then

$$\mathcal{V}(f_i)=\sum_{j\ne i}(u^j-u^i)\gamma_{ij}f_j. \tag{1.38}$$

Formula (1.37) defines an $\mathcal{O}_M$–linear endomorphism of $\mathcal{T}_M$ which coincides with (1.36) on the flat fields, as a calculation in flat coordinates shows. To check (1.38), we can use (1.37) and the classical explicit expressions for the Levi–Civita connection, cf. below.

We can now state the main result of this section. In addition to (1.36), define the operator $\mathcal{U}:\,\mathcal{T}_M\to\mathcal{T}_M$:

$$\mathcal{U}(X):=E\circ X, \tag{1.39}$$

so that $\mathcal{U}(f_i)=u^if_i$.

1.12. Theorem. *For $X,Y\in \mathrm{pr}_M^{-1}(\mathcal{T}_M)\subset\mathcal{T}_{\check{M}}$ (meromorphic vector fields on $\mathcal{T}_{M\times\mathbf{P}^1_\lambda}$ independent on λ) put*

$$\check\nabla_X(Y)=\nabla_{0,X}(Y)-(\mathcal{V}+\frac{1}{2}\,\mathrm{Id})\,(\mathcal{U}-\lambda)^{-1}(X\circ Y), \tag{1.40}$$

$$\check\nabla_{\partial/\partial\lambda}(Y)=(\mathcal{V}+\frac{1}{2}\,\mathrm{Id})\,(\mathcal{U}-\lambda)^{-1}(Y). \tag{1.41}$$

Then $\check\nabla$ is a flat connection on $\check{\mathcal{T}}$ whose restriction on $M\times\{\lambda\}$ defined by (1.40) is the Levi–Civita connection for $\check g_\lambda$.

Remark. Rewriting $\widehat\nabla$ in the same notation, we get

$$\widehat\nabla_X(Y)=\nabla_{0,X}(Y)+\lambda X\circ Y, \tag{1.42}$$

$$\widehat\nabla_{\partial/\partial\lambda}(Y)=\left[\mathcal{U}+\frac{1}{\lambda}(\mathcal{V}+\frac{D}{2}\,\mathrm{Id})\right](Y). \tag{1.43}$$

Proof. We will first calculate the Levi–Civita connection for $\check g_\lambda$ in coordinates $\check u^i=\log{(u^i-\lambda)}$:

$$\check e_i=\frac{\partial}{\partial\check u^i}=(u^i-\lambda)e_i,\;\; \check\eta_i=(u^i-\lambda)\eta_i,\;\; \check\eta_{ij}=(u^i-\lambda)(u^j-\lambda)\eta_{ij}+\delta_{ij}(u^i-\lambda)\eta_i,$$

$$\check{\gamma}_{ij} = \gamma_{ij}(u^i - \lambda)^{1/2}(u^j - \lambda)^{1/2}.$$

Then for $i \neq j$

$$\check{\nabla}_{\check{e}_i}(\check{e}_j) = \frac{1}{2}\frac{\check{\eta}_{ij}}{\check{\eta}_i}\check{e}_i + \frac{1}{2}\frac{\check{\eta}_{ij}}{\check{\eta}_j}\check{e}_j = \frac{1}{2}(u^i - \lambda)(u^j - \lambda)\left(\frac{\eta_{ij}}{\eta_i}e_i + \frac{\eta_{ij}}{\eta_j}e_j\right)$$

so that

$$\check{\nabla}_{e_i}(e_j) = \frac{1}{2}\frac{\eta_{ij}}{\eta_i}e_i + \frac{1}{2}\frac{\eta_{ij}}{\eta_j}e_j = \nabla_{0,e_i}(e_j). \tag{1.44}$$

Similarly,

$$\check{\nabla}_{\check{e}_i}(\check{e}_i) = \frac{1}{2}\frac{\check{\eta}_{ii}}{\check{\eta}_i}\check{e}_i - \frac{1}{2}\sum_{j\neq i}\frac{\check{\eta}_{ij}}{\check{\eta}_j}\check{e}_j =$$

$$\frac{1}{2}(u^i - \lambda)^2\left[\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i - \lambda}\right]e_i - \frac{1}{2}\sum_{j\neq i}(u^i - \lambda)\,(u^j - \lambda)\,\frac{\eta_{ij}}{\eta_j}\,e_j$$

so that

$$\check{\nabla}_{e_i}(e_i) = \frac{1}{2}\left[\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i - \lambda}\right]e_i - \frac{1}{2}\sum_{j\neq i}\frac{u^j - \lambda}{u^i - \lambda}\,\frac{\eta_{ij}}{\eta_j}\,e_j. \tag{1.45}$$

Subtracting from this the Levi–Civita covariant derivative, we get

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(e_i) = -\frac{1}{2}\,\frac{1}{u^i - \lambda}\,e_i - \frac{1}{2}\sum_{j:\,j\neq i}\frac{u^j - u^i}{u^i - \lambda}\,\frac{\eta_{ij}}{\eta_j}\,e_j \tag{1.46}$$

and

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(f_i) = -\frac{1}{2}\,\frac{1}{u^i - \lambda}\,f_i - \sum_{j\neq i}\frac{u^j - u^i}{u^i - \lambda}\,\gamma_{ij}\,f_j. \tag{1.47}$$

In view of (1.38), we can write (1.44) and (1.45) together as

$$(\check{\nabla}_{e_i} - \nabla_{0,e_i})(f_j) = -\left(\mathcal{V} + \frac{1}{2}\,\mathrm{Id}\right)(\mathcal{U} - \lambda)^{-1}(e_i \circ f_j) \tag{1.48}$$

because $e_i \circ f_j = \delta_{ij} f_j$. This family of formulas is equivalent to (1.40) so that (1.40) is the Levi–Civita connection for $\check{g}_\lambda$. In particular, it is flat for each fixed λ.

Since $[X, \partial/\partial\lambda] = 0$ for $X \in \mathrm{pr}_M^{-1}(\mathcal{T}_M)$, it remains to show that the covariant derivatives (1.40) and (1.41) commute on $\check{M}$ i. e. , that for all i, j

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}(e_j) = \check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}(e_j). \tag{1.49}$$

First of all, from (1.41) and (1.42) we find

$$\check{\nabla}_{\partial/\partial\lambda}(e_j) = \frac{1}{2}\,\frac{1}{u^j - \lambda}\,e_j + \frac{1}{2}\sum_{k\neq j}\frac{u^k - u^j}{u^j - \lambda}\,\frac{\eta_{jk}}{\eta_k}\,e_k. \tag{1.50}$$

Together with (1.44) and (1.45) this gives for $i \neq j$:

$$\check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}(e_j) = \frac{1}{2}\frac{\eta_{ij}}{\eta_j}\left[\frac{1}{2}\frac{1}{u^j-\lambda}\,e_i + \frac{1}{2}\sum_{k\neq i}\frac{u^k-u^i}{u^i-\lambda}\,\frac{\eta_{ik}}{\eta_k}\,e_k\right] + (i \leftrightarrow j), \tag{1.51}$$

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}(e_j) = \frac{1}{2}\frac{1}{u^j-\lambda}\left(\frac{1}{2}\frac{\eta_{ij}}{\eta_i}e_i + \frac{1}{2}\frac{\eta_{ij}}{\eta_j}e_j\right) +$$

$$+\frac{1}{2}\sum_{k\neq j} e_i\left(\frac{u^k-u^j}{u^j-\lambda}\,\frac{\eta_{jk}}{\eta_k}\right)e_k + \frac{1}{2}\sum_{k\neq j,i}\frac{u^k-u^j}{u^j-\lambda}\,\frac{\eta_{jk}}{\eta_k}\left(\frac{1}{2}\,\frac{\eta_{ik}}{\eta_i}\,e_i + \frac{1}{2}\,\frac{\eta_{ik}}{\eta_k}\,e_k\right) +$$

$$\frac{1}{2}\,\frac{u^i-u^j}{u^j-\lambda}\,\frac{\eta_{ik}}{\eta_k}\left[\frac{1}{2}\left(\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i-\lambda}\right)e_i - \frac{1}{2}\sum_{j\neq i}\frac{u^j-\lambda}{u^i-\lambda}\,\frac{\eta_{ij}}{\eta_j}\,e_j\right]. \tag{1.52}$$

The coincidence of coefficients of e_k in (1.51) and (1.52) for $i \neq j \neq k \neq i$ can be checked with the help of the following identities which are equivalent to the Darboux–Egoroff equations:

$$\eta_{ijk} = \frac{1}{2}\left(\frac{\eta_{ik}\eta_{jk}}{\eta_k} + \frac{\eta_{ij}\eta_{ik}}{\eta_i} + \frac{\eta_{ij}\eta_{jk}}{\eta_j}\right).$$

The coincidence of the coefficients of e_i requires a little more work, and we will give some details, again for the case $i \neq j$.

In (1.51) the coefficient of e_i is

$$\frac{1}{4}\,\frac{1}{u^i-\lambda}\,\frac{\eta_{ij}}{\eta_i} + \frac{1}{4}\,\frac{u^k-u^j}{u^j-\lambda}\,\frac{\eta_{ij}^2}{\eta_i\eta_j}, \tag{1.53}$$

whereas in (1.52) we get

$$\frac{1}{4}\,\frac{1}{u^j-\lambda}\,\frac{\eta_{ij}}{\eta_i} + \frac{1}{2}\,e_i\left(\frac{u^i-u^j}{u^j-\lambda}\,\frac{\eta_{ij}}{\eta_i}\right) +$$

$$+\frac{1}{4}\sum_{k\neq i,j}\frac{u^k-u^j}{u^j-\lambda}\,\frac{\eta_{ik}\eta_{jk}}{\eta_i\eta_k} + \frac{1}{2}\left(\frac{\eta_{ii}}{\eta_i} - \frac{1}{u^i-\lambda}\right). \tag{1.54}$$

To identify (1.53) and (1.54) we have to get rid of the sum $\sum_k$ in (1.54). This can be done with the help of (1.28), (1.29) and (1.34):

$$\frac{1}{4}\sum_{k\neq i,j}\frac{u^k-u^j}{u^j-\lambda}\,\frac{\eta_{ik}\eta_{jk}}{\eta_i\eta_k} = \frac{1}{u^j-\lambda}\,\frac{\eta_j^{1/2}}{\eta_i^{1/2}}\left[\sum_{k\neq i,j}u^k\gamma_{ik}\gamma_{kj} - u^j\sum_{k\neq i,j}\gamma_{ik}\gamma_{kj}\right] =$$

$$= \frac{1}{u^j-\lambda}\,\frac{\eta_j^{1/2}}{\eta_i^{1/2}}\,\left[-\gamma_{ij} - u^i e_i\gamma_{ij} - u^j e_j\gamma_{ij} + u^j(e_i+e_j)\gamma_{ij}\right] =$$

$$= \frac{1}{2} \frac{1}{u^j - \lambda} \left[-\frac{\eta_{ij}}{\eta_i} + (u^j - u^i) \left(\frac{\eta_{iij}}{\eta_i} - \frac{1}{2} \frac{\eta_{ij}\eta_{ii}}{\eta_i^2} - \frac{1}{2} \frac{\eta_{ij}^2}{\eta_i \eta_j} \right) \right].$$

The remaining part of the calculation is straightforward, and we leave it to the reader, as well as the case $i = j$ which is treated similarly.

1.13. Formal Laplace transform. Assume now that $\mathcal{T}_M^f$ is a trivial local system. This means that if we put $T := \Gamma(M, \mathcal{T}_M^f)$, there is a natural isomorphism $\mathcal{O}_M \otimes T \to \mathcal{T}_M$.

Formulas (1.41) (resp. (1.43)) define two families of connections with singularities on the trivial vector bundle on $\mathbf{P}_\lambda^1$ with fiber T, parametrized by M. Namely, denote by ∂_λ the covariant derivative along $\partial/\partial\lambda$ on this bundle for which the constant sections are horizontal. Then the two connections are

$$\check{\nabla}_{\partial/\partial\lambda} = \partial_\lambda + \left(\mathcal{V} + \frac{1}{2}\,\mathrm{Id} \right) (\mathcal{U} - \lambda)^{-1}, \tag{1.55}$$

$$\widehat{\nabla}_{\partial/\partial\lambda} = \partial_\lambda + \mathcal{U} + \frac{1}{\lambda} \left(\mathcal{V} + \frac{D}{2}\,\mathrm{Id} \right). \tag{1.56}$$

Let M, N be two $\mathbf{C}[\lambda, \partial_\lambda]$–modules. *A formal Laplace transform $M \to N : Y \mapsto Y^t$* is a $\mathbf{C}$–linear map for which

$$(-\lambda Y)^t = \partial_\lambda(Y^t),\ (\partial_\lambda Y)^t = \lambda Y^t. \tag{1.57}$$

The archetypal Laplace transform is the Laplace integral

$$Y^t(\mu) = \int e^{-\lambda\mu} Y(\lambda) d\lambda \tag{1.58}$$

taken along a contour (not necessarily closed) in $\mathbf{P}^1(\mathbf{C})$. In an analytical setting we have to secure the convergence of (1.58), the possibility to derivate under the integral sign and the identity

$$\int \partial_\lambda (e^{-\lambda\mu} Y(\lambda)) d\lambda = 0.$$

However, (1.58) may admit other interpretations, for instance, in terms of asymptotic series.

Let now M (resp. N) be two $\mathbf{C}[\lambda, \partial_\lambda]$–modules of local (or formal, or distribution) sections of $\mathbf{P}_\lambda^1 \times T$ so that the operators $\check{\nabla} \cdot (\mathcal{U} - \lambda)$ (resp. $\lambda\widehat{\nabla}$) make sense in M (resp. N) (cf. (1.55), resp. (1.56)), and assume that we are given a formal Laplace transform $M \to N$.

1.13.1. Proposition. *We have:*

$$[\check{\nabla}_{\partial/\partial\lambda}((\mathcal{U} - \lambda)Y)]^t = (\lambda\widehat{\nabla}_{\partial/\partial\lambda} + \frac{1 - D}{2})\, Y^t = \lambda^{\frac{D+1}{2}} \widehat{\nabla}_{\partial/\partial\lambda}(\lambda^{\frac{1-D}{2}} Y^t).$$

In particular, $\lambda^{\frac{1-D}{2}} Y^t$ *is* $\widehat{\nabla}$*–horizontal, if* $(\mathcal{U}-\lambda)Y$ *is* $\check{\nabla}$*–horizontal.*

Proof. Using (1.55)–(1.57), we find:

$$[\check{\nabla}_{\partial/\partial\lambda}((\mathcal{U}-\lambda)Y)]^t = \left[(\partial_\lambda \cdot (\mathcal{U}-\lambda) + \mathcal{V} + \frac{1}{2}\,\mathrm{Id})Y\right]^t =$$

$$= \left[\lambda\,(\mathcal{U}+\partial_\lambda) + \mathcal{V} + \frac{1}{2}\,\mathrm{Id}\right] Y^t =$$

$$= \left[\lambda\widehat{\nabla}_{\partial/\partial\lambda} + \frac{1-D}{2}\,\mathrm{Id}\right] Y^t = \lambda^{\frac{D+1}{2}}\widehat{\nabla}_{\partial/\partial\lambda}(\lambda^{\frac{1-D}{2}}Y^t).$$

For a more detailed discussion of the formal Laplace transform, see [S], 1.6.

For later use we note that the connection $\check{\nabla}$ defined by (1.40), (1.41) can be further deformed. Namely, for any constant s put

$$\check{\nabla}^{(s)}_X(Y) = \check{\nabla}_X(Y) - s(\mathcal{U}-\lambda)^{-1}(X\circ Y), \tag{1.59}$$

$$\check{\nabla}^{(s)}_{\partial/\partial\lambda}(Y) = \check{\nabla}_{\partial/\partial\lambda}(Y) + s(\mathcal{U}-\lambda)^{-1}(Y). \tag{1.60}$$

1.14. Theorem. $\check{\nabla}^{(s)}_X$ *is a flat connection on* $\check{\mathcal{T}}$.

This can be checked by a direct calculation similar to that in the proof of the Theorem 1.12. The formal Laplace transform of $\check{\nabla}^{(s)}_{\partial/\partial\lambda}$ is given by

$$[\check{\nabla}^{(s)}_{\partial/\partial\lambda}((\mathcal{U}-\lambda)Y)]^t = (\lambda\widehat{\nabla}_{\partial/\partial\lambda} + \frac{1-D+2s}{2})\,Y^t = \lambda^{\frac{D+1-2s}{2}}\widehat{\nabla}_{\partial/\partial\lambda}(\lambda^{\frac{1-D+2s}{2}}Y^t).$$

In particular, $\lambda^{\frac{1-D+2s}{2}}Y^t$ is $\widehat{\nabla}$–horizontal, if $(\mathcal{U}-\lambda)Y$ is $\check{\nabla}^{(s)}$–horizontal.

§2. Schlesinger equations

2.1. Singularities of meromorphic connections. Let N be a complex manifold, $D \subset N$ a closed complex submanifold of codimension one, $\mathcal{F}$ a locally free sheaf of finite rank on N. A meromorphic connection with singularities on D is given by a covariant differential $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_N((r+1)D)$ for some $r \geq 0$. It is called flat (or integrable) if it is flat outside D. We start with a list of elementary notions and constructions that will be needed later. They depend only on the local behaviour of $\mathcal{F}$ and ∇ in a neighborhood of D, so we will assume D irreducible.

i) Order of singularity. We will say that ∇ as above is of order $\leq r+1$ on D if $\nabla_X(\mathcal{F}) \subset \mathcal{F}(rD)$ for any vector field X tangent to D (i. e. satisfying $XJ_D \subset J_D$ where J_D is the ideal of D), and $\nabla_X(\mathcal{F}) \subset \mathcal{F}((r+1)D)$ in general. Locally, if $(t^0, t^1, \dots, t^n)$ is a coordinate system on N such that $t^0 = 0$ is the equation of D, the connection matrix of ∇ in a basis of $\mathcal{F}$ can be written as

$$G_0 \frac{dt^0}{(t^0)^{r+1}} + \sum_{i=1}^{n} G_i \frac{dt^i}{(t^0)^r} \tag{2.1}$$

where $G_i = G_i(t^0, t^1, \dots, t^n)$ are holomorphic matrix functions.

Note that $G_0(0, t^1, \dots, t^n) \in H^0(D, \mathrm{End}\mathcal{F})$ is well-defined, i.e. it does not depend on the choice of local coordinates. It is called the residue of ∇ at D and is denoted by $\mathrm{res}_D(\nabla)$.

ii) Restriction to a transversal submanifold. Let $i : N' \to N$ be a closed embedding of a submanifold transversal to D, $D' = N' \cap D$, $\mathcal{F}' = i^*(\mathcal{F})$. Then the induced connection $\nabla' = i^*(\nabla)$ on $\mathcal{F}'$ is flat and of order $\leq r+1$ on D' if ∇ has these properties.

iii) Residual connection. Assume that ∇ is of order ≤ 1 on D. For any given local trivialization f of $[D]$, one can define a connection without singularities $\nabla^{D,f}$ on $j^*(\mathcal{F})$ where j is the embedding of D in N. Namely, to define $\nabla^{D,f}_{X'}(s')$ where $s' \in j^*(\mathcal{F})$, $X' \in \mathcal{T}_D$, we extend locally s' to a section s of $\mathcal{F}$, X' to a vector field X on N, $\mathrm{res}_D(\nabla)$ to a section $\mathrm{res}(\nabla)$ of $\mathcal{F} \otimes \mathcal{F}^*$ on N, calculate $(\nabla_X - \frac{Xf}{f}\mathrm{res}(\nabla))(s)$ and restrict it to D. One checks that the result does not depend on the choices made. In the notation of (2.1), the matrix of the residual connection can be written as $(r = 0)$:

$$\sum_{i=1}^{n} G_i(0, t^1, \dots, t^n) dt^i. \tag{2.2}$$

If ∇ is flat, $\nabla^{D,f}$ is flat for any local trivialization f of $[D]$.

iv) Principal part of order $r+1$. Similarly to (2.2), we can consider the matrix function on D

$$G_0(0, t^1, \dots, t^n) \tag{2.3}$$

which we will call the principal part of order $r+1$ of ∇. In more invariant terms, it is the $\mathcal{O}_D$–linear map $j^*(\mathcal{F}) \to j^*(\mathcal{F})$ induced by $\mathcal{F} \to j^*(\mathcal{F}) : \, s \mapsto (t^0)^{r+1} \nabla_{\partial/\partial t^0}(s) \mid_D$. For $r \geq 1$ it depends on the choice of local coordinates, and is multiplied by an invertible local function on D when this choice is changed. Hence its spectrum is well defined globally on D for $r = 0$, and the simplicity of the spectrum makes sense for any r.

v) Tameness and resonance. Two general position conditions are important in the study of meromorphic singularities of order $\leq r + 1$.

If $r \geq 1$ (irregular case), the singularity is called *tame*, if the spectrum of its principal part at any point of D is simple.

If $r = 0$ (regular case), the singularity is called *non–resonant*, if it is tame and moreover, the difference of any two eigenvalues never takes an integer value on D.

2.1.1. Example: the structure connections of Frobenius manifolds. As above, we will assume that $\mathcal{T}_M^f$ is trivial, and its fibers are identified with the space T of global flat vector fields.

Put $N = M \times \mathbf{P}^1_\lambda$, $\mathcal{F} = \mathcal{O}_N \otimes T$. We can apply the previous considerations to $\widehat{\nabla}$ and $\check{\nabla}$.

Analysis of $\widehat{\nabla}$. Clearly, $\widehat{\nabla}$ has singularity of order 1 at $\lambda = 0$ (i. e. on $D_0 = M \times \{0\}$) and of order 2 at $\lambda = \infty$ (i. e. on $D_\infty = M \times \{\infty\}$): cf. (1.42) and (1.43). Restricting $\widehat{\nabla}$ to $\{y\} \times \mathbf{P}^1_\lambda$ for various $y \in M$ we get a family of meromorphic connections on $\mathbf{P}^1_\lambda$ parametrized by M.

The residual connection is defined on $D_0 = M$ and it coincides with the Levi–Civita connection of g. The principal part of order 1 on D_0 is $\mathcal{V} + \dfrac{D}{2}\,\mathrm{Id}$. The eigenvalues of this operator do not depend on $y \in D_0$: in 1.5 they were denoted (d_a). In the case of quantum cohomology the principal part *is always resonant.*

The principal part of order 2 on $D_\infty = M$ is (proportional to) $\mathcal{U}$ (cf. (1.43), use the local equation $\mu = \lambda^{-1} = 0$ for D_∞.) Its eigenvalues now depend on $y \in M$: they are just the canonical coordinates $u^i(y)$. We will call the point y *tame* if $u^i(y) \neq u^j(y)$ for $i \neq j$. We will call M tame, if all its points are tame. Every M contains the maximum tame subset which is open and dense.

Analysis of $\check{\nabla}$. According to (1.40), (1.41), $\check{\nabla}$ has singularities of order 1 at the divisors $\lambda = u^i$ and $\lambda = \infty$. These divisors do not intersect pairwise iff M is tame. The principal part of order 1 at $\lambda = u^i$ is $-(\mathcal{V} + \dfrac{1}{2}\,\mathrm{Id}) \cdot (e_i \circ)$.

The residual connection of $\check{\nabla}$ on $\lambda = \infty$ is again the Levi–Civita connection ∇_0 of g. In fact, using (1.48) we find

$$\check{\nabla} = d\lambda \check{\nabla}_{\partial/\partial\lambda} + \sum_i du^i \check{\nabla}_{e_i} =$$

$$= d\lambda \check{\nabla}_{\partial/\partial\lambda} + \sum_i du^i [\nabla_{0,e_i} - (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mathcal{U} - \lambda)^{-1}(e_i \circ)].$$

Replacing λ by the local parameter $\mu = \lambda^{-1}$ at infinity, we have

$$\check{\nabla} = d\mu \check{\nabla}_{\partial/\partial\mu} + \sum_i du^i [\nabla_{0,e_i} - \mu\,(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,(\mu\mathcal{U} - \mathrm{Id})^{-1}(e_i \circ)]$$

so that the expression (2.2) (with $(\mu, u^1, \dots, u^m)$ in lieu of $(t^0, t^1, \dots, t^n)$) becomes $\sum_i du^i \nabla_{0,e_i} = \nabla_0$.

2.2. Versal deformation. We will now review the basic results on the deformation of meromorphic connections on $\mathbf{P}^1_\lambda$, restricting ourselves to the case of singularities of order ≤ 2. This suffices for applications to both structure connections, on the other hand, this is precisely the case treated in full detail by B. Malgrange in [Mal4], Theorem 3.1. It says that the positions of finite poles and the spectra of the principal parts of order 2 form coordinates on the coarse moduli space with tame singularities. To be more precise, one has to rigidify the data slightly.

Let ∇^0 be a meromorphic connection on a locally free sheaf $\mathcal{F}^0$ on $\mathbf{P}^1_\lambda$ of rank p, with $m+1 \ge 2$ tame singularities (including $\lambda = \infty$) of order ≤ 2. Call *the rigidity* for ∇^0 the following data:

a). A numbering of singular points: $a_0^1, \dots, a_0^m$, $a^{m+1} = \infty$.

b). The subset $I \subset \{1, \dots, m+1\}$ such that a_0^j is of order 2 exactly when $j \in I$.

c). For each $j \in I$, a numbering $(b_0^{j1}, \dots, b_0^{jp})$ of the eigenvalues of the principal part at a_0^j.

Construct the space $B = B(m, p, S)$ as the universal covering of

$$(\mathbf{C}^m \setminus \text{diagonals}) \times \prod_{j \in I} (\mathbf{C}^p \setminus \text{diagonals})$$

with the base point $(a_0^i; b_0^{jk})$, let $b_0 \in B$ be its lift. We denote by a^i, b^{jk} the coordinate functions lifted to B. Let $i: \mathbf{P}^1_\lambda \to B \times \mathbf{P}^1_\lambda$ be the embedding $\lambda \mapsto (b_0, \lambda)$, and D_j the divisor $\lambda = a^j$ in $B \times \mathbf{P}^1_\lambda$.

2.2.1. Theorem ([Mal4], Th. 3.1). *For a given $(\nabla^0, \mathcal{F}^0)$ with rigidity, there exists a locally free sheaf $\mathcal{F}$ of rank p on $\mathbf{P}^1_\lambda \times B$, a flat meromorphic connection ∇ on it, and an isomorphism $i^0 : i^*(\mathcal{F}, \nabla) \to (\mathcal{F}^0, \nabla^0)$ with the following properties:*

D_j, $j = 1, \dots, m+1$, are all the poles of ∇, of order 1 (resp. 2) if $j \neq I$ (resp. $j \in I$.) If $j \in I$, then $(b^{j1}, \dots, b^{jp})$ (as functions on D_j) form the spectrum of the principal part of order 2 of ∇ at D_j.

It follows that the restrictions of ∇ to the fibers $\{b\} \times \mathbf{P}^1_\lambda$ are endowed with the induced rigidity, and i^0 is compatible with it.

The data $(\mathcal{F}, \nabla, i^0)$ are unique up to unique isomorphism.

2.2.2. Comments on the proof. a). The case when all singularities are of order 1 is easier. It is treated separately in [Mal3], Th. 2.1. Since the second structure connection satisfies this condition, we sketch Malgrange's argument in this case.

Choose base points $a \in U := \mathbf{P}^1_\lambda \setminus \cup_{j=1}^{m+1} \{a_0^j\}$ and $(b_0, a) \in B \times \mathbf{P}^1_\lambda$. Notice that (b_0, a) belongs to $V := B \times \mathbf{P}^1_\lambda \setminus \cup_{j=1}^m D_j$.

The restriction of $(\mathcal{F}^0, \nabla^0)$ to U is determined uniquely up to unique isomorphism by the monodromy action of $\pi_1(U, a)$ on the space F, the geometric fiber $\mathcal{F}^0(a)$ at a, which can be arbitrary. Similarly, there is a bijection between flat connections $(\mathcal{F},\nabla)$ on V with fixed identification $\mathcal{F}^0(a)\to\mathcal{F}(a)=F$ and actions of $\pi_1(V,(a,b))$ on F. Hence to construct an extension $(\mathcal{F},\nabla)$ to V together with an isomorphism of its restriction to U with $(\mathcal{F}^0,\nabla^0)$, it suffices to check that i induces an isomorphism $\pi_1(U,a)\to\pi_1(V,(a,b))$, which follows from the homotopy exact sequence and the fact that B is contractible.

This argument explains the term "isomonodromic deformation."

Next, we must extend $(\mathcal{F},\nabla)$ to $B\times\mathbf{P}^1_\lambda$. It suffices to do this separately in a tubular neighborhood of each D_j disjoint from other D_k. The coordinate change $\lambda\mapsto\lambda-a^j$ (or $\lambda\mapsto\lambda^{-1}$) allows us to assume that the equation of D_j is $\lambda=0$. Take a neighborhood W of 0 in which $\mathcal{F}^0$ can be trivialized, describe ∇^0 by its connection matrix, lift $(\mathcal{F}^0,\nabla^0)$ to $B\times W$ and restrict to a tubular neighborhood of D_j. On the complement to D_j, this lifting can be canonically identified with $(\mathcal{F},\nabla)$ through their horizontal sections. Clearly, it is of order ≤ 1 at D_j.

It remains to establish that any two extensions are canonically isomorphic. Outside singularities, an isomorphism exists and is unique. An additional argument which we omit shows that it extends holomorphically to $B\times\mathbf{P}^1_\lambda$.

b). When ∇ admits singularity of order 2, this argument must be completed. The extension of $(\mathcal{F}^0,\nabla^0)$ first to V and then to the singular divisors of order ≤ 1 can be done exactly as before. But both the existence and the uniqueness of the extension to the irregular singularities requires an additional local analysis in order to show that the simple spectrum of the principal polar part determines the singularity. When formulated in terms of the asymptotic behaviour of horizontal sections, this analysis introduces the Stokes data as a version of irregular monodromy, which also proves to be deformation invariant.

2.3. The theta divisor and Schlesinger's equations. In this subsection we will assume that $\mathcal{F}^0=T\otimes\mathcal{O}_{\mathbf{P}^1_\lambda}$ where T is a finite dimensional vector space which can be identified with the space of global sections of $\mathcal{F}^0$. This is the case of the two structure connections, when the local system $\mathcal{T}^f_M$ is trivial.

Then there exists a divisor Θ, eventually empty, such that the restriction of $\mathcal{F}$ to all fibers $\{b\}\times\mathbf{P}^1_\lambda$, $b\notin\Theta$, is free. This can be proved using the fact that a locally free sheaf $\mathcal{E}$ on $\mathbf{P}^1$ is free iff $H^0(\mathbf{P}^1,\mathcal{E}(-1))=H^1(\mathbf{P}^1,\mathcal{E}(-1))=0$, and that the cohomology of fibers is semi–continuous. For an analytic treatment, see [Mal4], sec. 4 and 5.

Moreover, assume that $\lambda=\infty$ is a singularity of order 1 (to achieve this for the first structure connection, we must replace λ by λ^{-1}.) Then we can identify the inverse image of $\mathcal{F}$ on $B\setminus\Theta\times\mathbf{P}^1_\lambda$ with $T\otimes\mathcal{O}_{B\setminus\Theta\times\mathbf{P}^1_\lambda}$ compatibly with the respective trivialization of $\mathcal{F}^0$. To this end trivialize $\mathcal{F}$ along $\lambda=\infty$ using the residual connection (see 2.1 iii))and then take the constant extension of each residually horizontal section along $\mathbf{P}^1_\lambda$. (If there are no poles of order 1, one can extend this argument using a different version of the residual connection, see [Mal4], p.430, Remarque 1.4.)

Using this trivialization, we can define a meromorphic integrable connection ∂ on $\mathcal{F}$ with the space of horizontal sections T on $B\setminus\Theta\times\mathbf{P}^1_\lambda$. As sections of $\mathcal{F}$, they develop a singularity at Θ. Therefore, the respective connection form $\nabla-\partial$ is a meromorphic matrix one–form with eventual pole at Θ.

The following classical result clarifies the structure of this form in the case *when all poles of* ∇ *are of order 1.*

2.3.1. Theorem. *a). Let* $(a^1,\dots,a^m)$ *be the functions on* B *describing the* λ*–coordinates of finite poles of* ∇ *(with given rigidity.) Then*

$$\nabla = \partial + \sum_{i=1}^{m} A_i(a^1,\dots,a^m)\,\frac{d(\lambda-a^i)}{\lambda-a^i} \tag{2.4}$$

where A_i *are meromorphic functions* $B\to \mathrm{End}\,(T)$ *which can be considered as multivalued meromorphic functions of* a_i.

b). The connection (2.4) is flat iff A_i *satisfy the Schlesinger equations*

$$\forall j, \qquad dA_j = \sum_{i\ne j}[A_i,A_j]\,\frac{d(a^i-a^j)}{a^i-a^j}. \tag{2.5}$$

c). Fix a tame point $a_0=(a_0^1,\dots,a_0^m)$*. Then arbitrary initial conditions* $A_i^0 = A_i(a_0)$ *define a solution of (2.5) holomorphic on* $B\setminus\Theta$*, with eventual pole at* Θ *of order 1.*

d). For any such solution ∇ *of (2.5), define the meromorphic 1-form on* B*:*

$$\omega_\nabla := \sum_{i<j} \mathrm{Tr}\,(A_iA_j)\,\frac{d(a^i-a^j)}{a^i-a^j}. \tag{2.6}$$

This form is closed, and for any local equation $t=0$ *of* Θ *the form* $\omega_\nabla - \dfrac{dt}{t}$ *is locally holomorphic.*

2.3.2. Corollary. *For any solution* ∇ *of (2.5), there exists a holomorphic function* τ_∇ *on* B *such that* $\omega_\nabla = d\log\tau_\nabla$*. It is defined uniquely up to a multiplication by a constant.*

In fact, B is simply connected.

For a proof of Theorem 2.3.1, we refer to [Mal4]: a), b), and c) are proved on pp. 406–410, d) on pp. 420–425.

2.4. Special solutions. Slightly generalizing (2.5), we will call *a solution to Schlesinger's equations* any data $(M,(u^i),T,(A_i))$ where M is a complex manifold of dimension $m\ge 2$; $(u^1,\dots,u^m)$ a system of holomorphic functions on M such that du^i freely generate Ω^1_M and for any $i\ne j$, $x\in M$, we have $u^i(x)\ne u^j(x)$; T a finite dimensional complex vector space; $A_j : M\to \mathrm{End}\,T$, $j=1,\dots,m$, a family of holomorphic matrix functions such that

$$\forall j: \qquad dA_j = \sum_{i:\,i\ne j}[A_i,A_j]\,\frac{d(u^i-u^j)}{u^i-u^j}. \tag{2.7}$$

Let such a solution be given. Summing (3.1) over all j, we find $d(\sum_j A_j)=0$. Hence $\sum_j A_j$ is a constant matrix function; denote its value by $\mathcal{W}$.

2.4.1. Definition. *A solution to Schlesinger's equations as above is called special, if* $\dim T = m = \dim M$*;* T *is endowed with a complex nondegenerate quadratic form* g*;* $\mathcal{W} = -\mathcal{V} - \frac{1}{2}\,\mathrm{Id}$, *where* $\mathcal{V} \in \mathrm{End}\,T$ *is a skew symmetric operator with respect to* g*, and finally*

$$\forall j: \qquad A_j = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})P_j \tag{2.8}$$

where $P_j : M \to \mathrm{End}\,T$ *is a family of holomorphic matrix functions whose values at any point of* M *constitute a complete system of orthogonal projectors of rank one with respect to* g*:*

$$P_i P_k = \delta_{ik} P_i, \quad \sum_{i=1}^{m} P_i = \mathrm{Id}_T, \quad g(\mathrm{Im}\,P_i, \mathrm{Im}\,P_j) = 0 \tag{2.9}$$

if $i \neq j$*. Moreover, we require that* A_j *do not vanish at any point of* M*.*

2.4.2. Comment. We committed a slight abuse of language: the notion of special solution involves a choice of additional data, the metric g. However, when it is chosen, the rest of the data is defined unambiguously if it exists at all.

In fact, assume that $A_j = \mathcal{W}P_j$ as above do not vanish anywhere. Then they have constant rank one. Hence at any point of M we have

$$\mathrm{Ker}\,A_j = \mathrm{Ker}\,\mathcal{W}P_j = \mathrm{Ker}\,P_j = \oplus_{i:\,i\neq j}\mathrm{Im}\,P_i,$$

so that

$$\mathrm{Im}\,P_i = \cap_{j:\,j\neq i} \oplus_{k:\,k\neq j} \mathrm{Im}\,P_k = \cap_{j:\,j\neq i}\mathrm{Ker}\,A_j.$$

This means that P_j can exist for given A_j only if the spaces $\mathcal{T}_j = \cap_{j:\,j\neq i}\mathrm{Ker}\,A_j$ are one–dimensional and pairwise orthogonal at any point of M.

Conversely, assume that this condition is satisfied. Define P_j as the orthogonal projector onto $\mathcal{T}_j$. Then $A_i P_j = 0$ for $i \neq j$ because $\mathcal{T}_j = \mathrm{Im}\,P_j \subset \mathrm{Ker}\,A_i$. Hence

$$A_j = A_j(\sum_{i=1}^{m} P_i) = A_j P_j = (\sum_{i=1}^{m} A_i)P_j = \mathcal{W}P_j. \tag{2.10}$$

Notice that all A_j are conjugate to $\mathrm{diag}\,(-\frac{1}{2}, 0, \dots, 0)$ and satisfy $A_j^2 + \frac{1}{2}\,A_j = 0$. These conditions, as well as $\sum_j A_j = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})$, are compatible with the equations (2.8) and so must be checked at one point only.

2.4.3. Strictly special solutions. A special solution to Schlesinger's equations as above is called *strictly special* if the operators

$$A_j^{(t)} := A_j + tP_j$$

also satisfy Schlesinger's equations for any $t \in \mathbf{C}$.

2.4.4. Lemma. *If $\mathcal{W}$ is invertible, then any special solution with given $\mathcal{W}$ is strictly special.*

Proof. Inserting $A_j^{(t)}$ into (2.7) one sees that the solution is strictly special iff

$$\forall j : \qquad dP_j = \sum_{i:\, i\neq j} (P_i \mathcal{W} P_j - P_j \mathcal{W} P_i)\, \frac{d(u^i - u^j)}{u^i - u^j}.$$

On the other hand, replacing A_k by $\mathcal{W}P_k$ in (2.7), one sees that after left multiplication by $\mathcal{W}$ this becomes a consequence of (2.7).

2.5. From Frobenius manifolds to special solutions. Given a semisimple Frobenius manifold with flat identity and an Euler field E with $d_0 = 1$, we can produce a special solution to the Schlesinger equations rephrasing the results of the previous two sections.

Namely, we first pass to a covering M of the subspace of tame points of the initial manifold such that $\mathcal{T}_M^f$ is trivial and a global splitting can be chosen, represented by the canonical coordinates (u^i). Then we put $T = \Gamma(M, \mathcal{T}_M^f)$ and $A_i =$ the coefficients of the second structure connection written as in (2.4).

Since this connection is flat, $(M, (u^i), T, (A_i))$ form a solution of (2.7).

Moreover, this solution is special. In fact, T comes equipped with the metric g. The operator A_i is the principal part of order 1 of $\check{\nabla}$ at $\lambda = u^i$ which is of the form (2.8), with $P_j = e_j\circ$.

Finally, this special solution comes with one more piece of data, the identity $e \in T$. We will axiomatize its properties in the following definition.

2.5.1. Definition. *Consider a special solution to Schlesinger's equations as in the Definition 2.4.1. A vector $e \in T$ is called an identity of weight D for this solution, if*

a). $\mathcal{V}(e) = (1 - \frac{D}{2})\, e.$

b). $e_j := P_j(e)$ *do not vanish at any point of* M.

For Frobenius manifolds with $d_0 = 1$, a) is satisfied in view of (1.17) and (1.36).

Theorem 1.14 shows moreover that in this way we always obtain strictly special solutions, although the operator $\mathcal{W}$ need not be invertible. For example, for quantum cohomology of $\mathbf{P}^r$ (which is semisimple, cf. below) the spectrum of $\mathcal{W}$ is $\{a - \frac{r+1}{2} \mid a = 0, \dots, r\}$. It contains 0 if r is odd.

2.6. From special solutions to Frobenius manifolds. Let $(M, (u^i), T, g, (A_i))$ be a strictly special solution, and $e \in T$ an identity of weight D for it.

2.6.1. Theorem. *These data come from the unique structure of semisimple split Frobenius manifold on M, with flat identity and Euler field, as it was described in 2.5.*

Proof. Proceeding as in 2.5, but in the reverse direction, we are bound to make the following choices.

Put $e_j = P_j(e) \subset \mathcal{O}_M \otimes T$, $j = 1, \dots, m$. Identify $\mathcal{O}_M \otimes T$ with $\mathcal{T}_M$ by setting $e_j = \partial/\partial u^j$. Transfer the metric g from T to $\mathcal{T}_M$. Because of 2.5.1 c), it will be flat, with T as the space of flat vector fields. Define the multiplication on $\mathcal{T}_M$ for which $e_i \circ e_j = \delta_{ij} e_j$. Put $\eta_i := g(e_i, e_i)$.

Let $\mathcal{T}_M^f$ be the image of T under this identification. We will first check that it is an abelian Lie subalgebra of $\mathcal{T}_M$. It will then follow that g is flat, so that we get a structure of semisimple pre–Frobenius manifold in the sense of the Definition 1.6.2.

Choose $t \in \mathbf{C}$ in such a way that

$$\mathcal{W}^{(t)} := \sum_j A_j^{(t)} = \mathcal{W} + t\,\mathrm{Id} \in \mathrm{End}\,T$$

is invertible. The section $X = \sum_j f_j e_j$ of $\mathcal{O}_M \otimes T$ lands in $\mathcal{T}_M^f$ iff

$$\mathcal{W}^{(t)} X = \sum_j f_j \mathcal{W}^{(t)} P_j(e) = \left(\sum_j f_j A_j^{(t)} \right) (e) \in T.$$

Let ∇ be the connection on $\mathcal{T}_M$ for which $\mathcal{T}_M^f$ is horizontal. Applying it to $\left(\sum_j f_j A_j^{(t)}\right)(e)$ we see that the last condition is in turn equivalent to

$$\forall k : \qquad \sum_j \frac{\partial f_j}{\partial u^k}\, A_j^{(t)}(e) = -\sum_j f_j\, \frac{\partial A_j^{(t)}}{\partial u^k}\,(e). \qquad (*)$$

We can similarly rewrite the condition $Y := \sum_j g_j e_j \in \mathcal{T}_M^f$.

Commutator of vector fields induces on $\mathcal{O}_M \otimes T$ the bracket

$$[X, Y] = \sum_{j,k} \left(f_j\, \frac{\partial g_k}{\partial u^j} - g_j\, \frac{\partial f_k}{\partial u^j} \right) e_k.$$

From (*) for Y and X we find:

$$\sum_{j,k} f_j\, \frac{\partial g_k}{\partial u^j}\, A_k^{(t)}(e) = -\sum_j f_j \sum_k g_k\, \frac{\partial A_k^{(t)}}{\partial u^j}\,(e),$$

$$\sum_{j,k} g_j\, \frac{\partial f_k}{\partial u^j}\, A_k^{(t)}(e) = -\sum_j g_j \sum_k f_k\, \frac{\partial A_k^{(t)}}{\partial u^j}\,(e).$$

The terms $j = k$ in the right hand sides are the same. For $j \neq k$, using the strict speciality of our solution, we find

$$f_j g_k\, \frac{\partial A_k^{(t)}}{\partial u^j} = f_j g_k\, \frac{[A_j^{(t)}, A_k^{(t)}]}{u^j - u^k}$$

so that the (j,k)–term of the first identity cancels with the (k,j)–term of the second one.

To establish that this structure is Frobenius, it suffices to prove that $e_i\eta_j = e_j\eta_i$ for all i,j: see Theorem 1.7.

We have $\eta_j = g(e, e_j)$. Therefore

$$g(e, A_j(e)) = -g(e, (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,P_j e) = g(\mathcal{V}e, e_j) - \frac{1}{2}\,g(e, e_j) = \frac{1-D}{2}\,\eta_j \qquad (2.11)$$

since $\mathcal{V}$ is skewsymmetric, and e is an eigenvector of $\mathcal{V}$. Furthermore, let ∇ be the Levi–Civita connection of the flat metric g. Then derivating (2.11) we find for every i,j:

$$\frac{1-D}{2}\,\frac{\partial}{\partial u^i}\,\eta_j = g(\nabla_{e_i}(e), A_j(e)) + g(e, \nabla_{e_i}(A_j(e))) =$$

$$= g(e, \frac{\partial A_j}{\partial u^i}\,(e)), \qquad (2.12)$$

because $e \in T$ so that $\nabla(e) = 0$. If $i \neq j$, we find from (2.5)

$$\frac{\partial A_j}{\partial u^i} = \frac{[A_i, A_j]}{u^i - u^j} = \frac{\partial A_i}{\partial u^j}. \qquad (2.13)$$

This shows that if $D \neq 1$, $e_i\eta_j = e_j\eta_i$.

To see that $D = 1$ is not exceptional, one can replace in this argument A_j by $A_j^{(t)}$ for any $t \neq 0$, so that $\frac{1-D}{2}$ in (2.11) will become $\frac{1-D}{2} + t$.

It remains to check that $E = \sum_i u^i e_i$ is the Euler field. According to the Theorem 1.9, we must prove that $E\eta_j = (D-2)\eta_j$ for all j. Insert (2.13) into (2.12) and sum over $i \neq j$. We obtain:

$$\frac{1-D}{2}\,E\eta_j = \frac{1-D}{2}\sum_{i:\, i\neq j} u^i\,\frac{\partial \eta_j}{\partial u^i} + \frac{1-D}{2}\,u^j\,\frac{\partial \eta_j}{\partial u^j} =$$

$$= \sum_{i:\, i\neq j} g\left(e, u^i\frac{[A_i, A_j]}{u^i - u^j}\,(e)\right) + u^j g(e, \frac{\partial A_j}{\partial u^j}\,(e)). \qquad (2.14)$$

From (2.8) it follows that

$$\frac{\partial A_j}{\partial u^j} = -\sum_{i:\, i\neq j} \frac{[A_i, A_j]}{u^i - u^j}. \qquad (2.15)$$

On the other hand,

$$u^i\,\frac{[A_i, A_j]}{u^i - u^j} = [A_i, A_j] + u^j\,\frac{[A_i, A_j]}{u^i - u^j}. \qquad (2.16)$$

Inserting (2.15) and (2.16) into (2.14), we find

$$\frac{1-D}{2}\,E\eta_j = \sum_{i:\, i\neq j} g(e, [A_i, A_j]\,(e)) + u^j \sum_{i:\, i\neq j} g\left(e, \frac{[A_i, A_j]}{u^i - u^j}\,(e)\right) +$$

$$+u^j g(e, \frac{\partial A_j}{\partial u^j}(e)) = g(e, [\sum_{i:\, i\neq j} A_i, A_j]\,(e)) =$$

$$= -g(e, [\mathcal{V} + \frac{1}{2}\,\mathrm{Id}, (\mathcal{V} + \frac{1}{2}\,\mathrm{Id})P_j]\,(e)). \tag{2.17}$$

Using the skew symmetry of $\mathcal{V}$, we see that the last expression in (2.17) equals $\dfrac{1-D}{2}\,(D-2)\eta_j$. Hence $E\eta_j = (D-2)\eta_j$ if $D \neq 1$.

Again, replacing in this argument A_j by $A_j^{(t)}$ we see that the restriction $D \neq 1$ is irrelevant.

2.7. Special initial conditions. Theorem 2.3.1 shows that arbitrary initial conditions for the Schlesinger equations determine a global meromorphic solution on the universal covering $B(m)$ of $\mathbf{C}^m \setminus \{\text{diagonals}\}, m \geq 2$.

Fix a base point $b_0 \in B(m)$. Studying the special solutions, we may and will identify T with the tangent space at b_0 thus eliminating the gauge freedom. This tangent space is already coordinatized: we have e_i and e.

We will call a family of matrices $A_1^0, \dots, A_m^0 \in \mathrm{End}\, T$ *special initial conditions* if we can find a diagonal metric g and a skew symmetric operator $\mathcal{V}$ such that $A_j^0 = -(\mathcal{V} + \frac{1}{2}\,\mathrm{Id})\,P_j$ where P_j is the projector onto $\mathbf{C}\, e_j$.

We will describe explicitly the space $I(m)$ of the special initial conditions.

2.7.1. Notation. Let R be any equivalence relation on $\{1, \dots, m\}$, $|R|$ the number of its classes. Put $F(m) = (\mathrm{End}\, \mathbf{C}^m)^m$, Furthermore, denote $F_R(m)$ the subset of families $(A_1, \dots, A_m)$ in $F(m)$ such that R coincides with the minimal equivalence relation for which iRj if $\mathrm{Tr}\, A_i A_j \neq 0$, and put $I_R(m) = F_R(m) \cap I(m)$.

2.7.2. Construction. Denote by $\overline{I}(m) \subset \mathbf{C}^m \times \mathbf{C}^{m(m-1)/2}$ the locally closed subset defined by the equations:

$$\sum_{i=1}^m \eta_i = 0, \quad \eta_i \neq 0 \quad \text{for all } i; \tag{2.18}$$

$$v_{ij}\eta_j = -v_{ji}\eta_i \quad \text{for all } i, j; \tag{2.19}$$

$$\sum_{i=1}^m v_{ij} := 1 - \frac{D}{2} \quad \text{does not depend on } j. \tag{2.20}$$

Each point of $\overline{I}(m)$ determines the diagonal metric $g(e_i, e_i) = \eta_i$ and the operator $\mathcal{V}:\ e_i \mapsto \sum_i v_{ij} e_j$ which is skew symmetric with respect to g and for which e is an eigenvector. Setting $A_i = -(\mathcal{V} + \dfrac{1}{2}\,\mathrm{Id})\,P_i$ we get a point in $I(m)$.

This amounts to forgetting (η_i) which furnishes the surjective map $\overline{I}(m) \to I(m)$ because

$$A_i(e_j) = 0 \text{ for } i \neq j, \quad A_i(e_i) = -\frac{1}{2}\, e_i - \sum_{j=1}^m v_{ij} e_j.$$

2.7.3. Theorem. *a). The space $\overline{I}(m)$ can be realized as a Zariski open dense subset in $\mathbf{C}^{m+(m-1)(m-2)/2}$.*

b). Inverse image in $\overline{I}(m)$ of any point in $I_R(m)$ is a manifold of dimension 1 *for* $|R|=1$, $|R|-1$ *for* $|R|\geq 2$.

Proof. Fixing η_i, we can solve (2.19) and (2.20) explicitly. Put $w_{ij}=v_{ij}\eta_j$ so that $w_{ij}=-w_{ji}$ and (2.20) becomes

$$\forall\, j: \qquad \sum_{i=1}^{m} w_{ij}=\eta_j(1-\frac{D}{2}). \tag{2.21}$$

If we choose arbitrarily the values (w_{ij}) for all $1\leq i<j\leq m-1$, we can find w_{mj} from the first $m-1$ equations (2.21), and then the last equations will hold automatically:

$$w_{mk}=\eta_k(1-\frac{D}{2})-\sum_{i=1}^{m-1} w_{ik},$$

$$\sum_{i=1}^{m} w_{im}=-\sum_{k=1}^{m} w_{mk}=-\sum_{k=1}^{m-1}\eta_k\,(1-\frac{D}{2})+\sum_{i,k=1}^{m-1} w_{ik}=\eta_m\,(1-\frac{D}{2})$$

because of (2.18).

It remains to determine the fiber of the projection onto $I(m)$.

We have for $i\neq j$: $\mathrm{Tr}\,A_iAj=v_{ij}v_{ji}$. Hence in the generic case when all these traces do not vanish, we can reconstruct η_i compatible with given v_{ij} from (2.19) uniquely up to a common factor. Generally, for i,j in the same R–equivalence class, (3.12) allows us to determine the value η_i/η_j so that we have $|R|$ overall arbitrary factors constrained by (3.11).

2.7.4. Question. If we choose a special initial condition for the Schlesinger equation, does the solution remain special at every point?

Generically, the answer is positive. If this is the case, we obtain the action of the braid group Bd_m as the group of deck transformations on the space $I(m)$.

2.8. Analytic continuation of the potential. The picture described in this section gives a good grip on the analytic continuation of a germ of semisimple Frobenius manifold (M_0,m_0) in terms of its canonical coordinates. Namely, construct the universal covering M of the subset of the tame points of M_0, then fix at the point $b_0=(u^i(m_0))\in B(m)$ the initial conditions of M at m_0. This provides an open embedding $(M,m_0)\subset(B(m),b_0)$. Loosely speaking, we find in this way a maximal tame analytic continuation of the initial germ.

Now construct some global flat coordinates (x^a) on $B(m)$ corresponding to a given Frobenius structure. They map $B(m)$ to a subdomain in $\mathbf{C}^m$. This is the natural domain of the analytic continuation of the potential Φ of this Frobenius structure, which is the most important object for Quantum Cohomology. Unfortunately, its properties are not clear from this description.

§3. Quantum cohomology of projective spaces

In this section we will apply the developed formalism to the study of the quantum cohomology of projective spaces $\mathbf{P}^r$, $r \geq 2$. Our main goal is the calculation of the initial conditions of the relevant solutions to the Schlesinger's equations.

3.1. Notation. We start with introducing the basic notation. Put $H = H^*(\mathbf{P}^r, \mathbf{C}) = \sum_{a=0}^r \mathbf{C}\Delta_a$, Δ_a = the dual class of $\mathbf{P}^{r-a} \subset \mathbf{P}^r$. Denote the dual coordinates on H by $x_0, \dots, x_r$ (lowering indices for visual convenience), $\partial_a = \partial/\partial x_a$. The Poincaré form is $(g_{ab}) = (g^{ab}) = (\delta_{a+b,r})$. The classical (cubic) part of the Frobenius potential is

$$\Phi_{\mathrm{cl}}(x) := \frac{1}{6} \sum_{a_1+a_2+a_3=r} x_{a_1} x_{a_2} x_{a_3}. \tag{3.1}$$

The remaining part of the potential is the sum of physicists' instanton corrections to the self–intersection form:

$$\Phi_{\mathrm{inst}}(x) := \sum_{d=1}^{\infty} \Phi_d(x_2, \dots, x_r) e^{dx_1}, \tag{3.2}$$

where we will write Φ_d as

$$\Phi_d(x_2, \dots, x_r) = \sum_{n=2}^{\infty} \sum_{\substack{a_1+\dots+a_n= \\ r(d+1)+d-3+n}} I(d; a_1, \dots, a_n) \frac{x_{a_1} \dots x_{a_n}}{n!}. \tag{3.3}$$

This means that if we assign the weight $a-1$ to x_a, $a = 2, \dots, n$, Φ_d becomes the weighted homogeneous polynomial of weight $(r+1)d+r-3$. Moreover, if we assign to e^{dx_1} the weight $-(r+1)$, Φ_{cl} and Φ become weighted homogeneous formal series of weight $r-3$. (Notice that e in the expressions e^{dx_1} and alike is $2,71828\dots$, whereas in other contexts e means the identity vector field. This cannot lead to confusion.)

The starting point of our study in this section will be the following result.

3.2. Theorem. *a). For each $r \geq 2$, there exists a unique formal solution of the Associativity Equations (1.6) of the form*

$$\Phi(x) = \Phi_{\mathrm{cl}}(x) + \Phi_{\mathrm{inst}}(x) \tag{3.4}$$

for which $I(1; r, r) = 1$.

b). This solution has a non–empty convergence domain in H on which it defines the structure of semisimple Frobenius manifold $H_{\mathrm{quant}}(\mathbf{P}^r)$ with flat identity $e = \partial_0$ and Euler field

$$E = \sum_{a=0}^{r} (1-a) x^a \partial_a + (r+1)\partial_1 \tag{3.5}$$

with $d_0 = 1, D = 2-r$.

c). The coefficient $I(d;a_1,\dots,a_n)$ is the number of rational curves of degree d in $\mathbf{P}^r$ intersecting n projective subspaces of codimensions $a_1,\dots a_n\ge 2$ in general position.

Uniqueness of the formal solution can be established by showing that the Associativity Equations imply recursive relations for the coefficients of Φ which allow one to express all of them through $I(1;r,r)$. This is an elementary exercise for $r=2$. A more general result (stated in the language of Gromov–Witten invariants but of essentially combinatorial nature) is proved in [KM], Theorem 3.1, and applied to the projective spaces in [KM], Claim 5.2.2.

Existence is a subtler fact. The algebraic geometric (or symplectic) theory of the Gromov–Witten invariants provides the numbers $I(d;a_1,\dots a_n)$ satisfying the necessary relations, together with their numerical interpretation: see [KM], [BM]. Another approach consists in calculating *ad hoc* the "special initial conditions" for the semisimple Frobenius manifold $H_{\mathrm{quant}}(\mathbf{P}^r)$ in the sense of the previous section and identifying the appropriate special solution to the Schlesinger equations with this manifold. For $r=2$, direct estimates of the coefficients showing convergence can be found in [D], p. 185. Probably, they can be generalized to all r.

Our approach in this section consists in taking Theorem 3.2 for granted and investigating the passage to the Darboux–Egoroff picture as a concrete illustration of the general theory. The net outcome are formulas (3.18) and (3.19) for the special initial conditions.

Conversely, starting with them, we can construct the Frobenius structure on the space $B(r+1)$ as was explained in 2.8 above. Expressing the E–homogeneous flat coordinates $(x_0,\dots,x_r)$ on this space satisfying (3.17) in terms of the canonical coordinates and then calculating the multiplication table of the flat vector fields, we can reconstruct the potential which now will be a germ of holomorphic function of (x_a). Because of the unicity, it must have the Taylor series (3.4). So the Theorem 3.2 a),b) can be proved essentially by reading this section in the reverse order. Of course, the last statement is of different nature.

3.3. Tensor of the third derivatives. Most of our calculations in $(\mathcal{T},\circ)$ will be restricted to the first infinitesimal neighborhood of the plane $x_2=\dots=x_r=0$ in H. This just suffices for the calculation of the Schlesinger initial conditions. We denote by J the ideal $(x_2,\dots,x_r)$.

Multiplication by the identity $e=\partial_0$ is described by the components ${\Phi_{0a}}^b=\delta_{ab}$ of the structure tensor. Of the remaining components, we will need only ${\Phi_{1a}}^b$ which allow us to calculate multiplication by ∂_1, and proceed inductively. This is where the Associativity Equations are implicitly used.

Obviously, ${\Phi_{10}}^b=\delta_{1b}$.

3.3.1. Claim. *We have*

$$\text{for } 1\le a\le r-1: \qquad {\Phi_{1a}}^b=\delta_{a+1,b}+x_{r+1-a+b}e^{x_1}+O(J^2), \tag{3.6}$$

$${\Phi_{1r}}^b=\delta_{b0}e^{x_1}+x_{b+1}e^{x_1}+O(J^2). \tag{3.7}$$

(Here and below we agree that $x_c = 0$ for $c > r$.

Proof. The term $\delta_{a+1,b}$ in (3.6) comes from $\Phi_{\rm cl}$. The remaining terms are provided by the summands of total degree ≤ 3 in $x_2, \dots, x_r$ in

$$\partial_1 \Phi_{\rm inst} = \sum_{d\ge 1} d e^{dx_1} \Big(\sum I(d;a_1,a_2)\frac{x_{a_1}x_{a_2}}{2} + \sum I(d;a_1,a_2,a_3)\frac{x_{a_1}x_{a_2}x_{a_3}}{6}\Big) + O(J^4).$$

For $n = 2$, the grading condition means that $d = 1$, $a_1 = a_2 = r$. For $n = 3$, it means that $d = 1$, $a_1 + a_2 + a_3 = 2r + 1$. We know that $I(1;r,r) = 1$. Similarly, $I(1;a_1,a_2,a_3) = 1$ in this range. This can be deduced formally from the Associativity Equations. A nice exercise is to check that this agrees also with the geometric description (for instance, only one line intersects two given generic lines and passes through a given point in the three space.) So finally

$$\partial_1 \Phi_{\rm inst} = \left(\frac{x_r^2}{2} + \frac{1}{6}\sum_{a_1+a_2+a_3=2r+1} x_{a_1}x_{a_2}x_{a_3}\right) e^{x_1} + O(J^4).$$

The term $\delta_{b0}e^{x_1}$ in (3.7) comes from $\dfrac{x_r^2}{2}$. Furthermore,

$$\Phi_{{\rm inst};1ab} = x_{2r+1-a-b}e^{x_1} + O(J^2)$$

and

$$\Phi_{{\rm inst};1a}{}^{b} = \Phi_{{\rm inst};1,a,r-b} = x_{r+1-a+b}e^{x_1} + O(J^2).$$

3.4. Multiplication table. The main formula of this subsection is

$$\partial_1^{\circ(r+1)} = e^{x_1}\left(\partial_0 + \sum_{b=1}^{r-1}(b+1)x_{b+1}\partial_b\right) + O(J^2). \tag{3.8}$$

We will prove it by consecutively calculating the powers $\partial_1^{\circ a}$. The intermediate results will also be used later. (Notice that $O(J^2)$ in (3.8) now means $O(\sum_i J^2\partial_i)$.)

First, we find from (3.6) and (3.7) for $1 \le a \le r-1$:

$$\partial_1 \circ \partial_a = \sum_{b=0}^{r} \Phi_{1a}{}^{b}\partial_b = \partial_{a+1} + e^{x_1}\sum_{b=0}^{a-1} x_{r+1-a+b}\partial_b + O(J^2), \tag{3.9}$$

$$\partial_1 \circ \partial_r = \sum_{b=0}^{r} \Phi_{1r}{}^{b}\partial_b = e^{x_1}\left(\partial_0 + \sum_{b=1}^{r-1} x_{b+1}\partial_b\right) + O(J^2). \tag{3.10}$$

Then using (3.9) and induction, we obtain

$$\text{for } 1 \le a \le r: \qquad \partial_1^{\circ a} = \partial_a + e^{x_1}\sum_{b=0}^{a-2}(b+1)x_{r+2-a+b}\partial_b + O(J^2). \tag{3.11}$$

Multiplying this formula for $a=r$ by ∂_1 and using (3.10), we finally find (3.8).

¿From (3.11) it follows that $\partial_1^{\circ a}$ for $0\le a\le r$ freely span the tangent sheaf.

3.5. Idempotents. Formula (3.8) allows us to calculate all $e_i \bmod J^2$ thus demonstrating semisimplicity. Namely, denote by q the $(r+1)-$th root of the right hand side of (3.8) congruent to $e^{\frac{x_1}{r+1}} \bmod J$ and put $\zeta=\exp\left(\frac{2\pi i}{r+1}\right)$. Then

$$e_i=\frac{1}{r+1}\sum_{j=0}^{r}\zeta^{-ij}(\partial_1\circ q^{-1})^{\circ j} \tag{3.12}$$

satisfy

$$e_i\circ e_j=\delta_{ij}e_i,\ \sum_i e_i=\partial_0$$

for all $i=0,\dots r$. A straightforward check shows this.

3.5.1. Proposition. *We have*

$$e_i=\frac{1}{r+1}\sum_{j=0}^{r}\zeta^{-ij}e^{-x_1\frac{j}{r+1}}\left(e^{x_1}\sum_{b=0}^{j-2}\frac{(b+1-j)(r+1-j)}{r+1}x_{r+b+2-j}\partial_b+\right.$$
$$\left.+\,\partial_j-\sum_{b=j+1}^{r}\frac{(b+1-j)j}{r+1}x_{b+1-j}\partial_b\right)+O(J^2). \tag{3.13}$$

Proof. We have

$$q^{-1}=e^{-\frac{x_1}{r+1}}\left(\partial_0-\sum_{b=1}^{r-1}\frac{b+1}{r+1}x_{b+1}\partial_b\right)+O(J^2).$$

Together with (3.9) this gives

$$\partial_1\circ q^{-1}=e^{-\frac{x_1}{r+1}}\left(\partial_1-\sum_{b=1}^{r-1}\frac{b+1}{r+1}x_{b+1}\partial_{b+1}\right)+O(J^2).$$

Hence

$$(\partial_1\circ q^{-1})^j=e^{-\frac{jx_1}{r+1}}\left(\partial_1^{\circ j}-j\partial_1^{\circ(j-1)}\circ\sum_{b=1}^{r-1}\frac{b+1}{r+1}x_{b+1}\partial_{b+1}\right)+O(J^2).$$

Inserting this into (3.12) and using (3.9)–(3.11) once again, we finally obtain (3.13).

3.6. Metric coefficients in canonical coordinates. The metric potential η is simply x_r (see (1.11).) Hence we can easily calculate $\eta_i=e_ix_r$. The answer is

$$\eta_i=\frac{\zeta^i}{r+1}e^{-x_1\frac{r}{r+1}}-\sum_{b=2}^{r}\frac{\zeta^{ib}}{(r+1)^2}\,b(r+1-b)\,e^{-x_1\frac{r+1-b}{r+1}}x_b+O(J^2). \tag{3.14}$$

As an exercise, the reader can check that the same answer results from the (longer) calculation of $\eta_i=g(e_i,e_i)$.

3.7. Derivatives of the metric coefficients. We now see that the chosen precision just suffices to calculate the restriction of η_{ij}, γ_{ij} and the matrix elements of A_j to the plane $x_2=\dots=x_r=0$ any point of which can be taken as initial one.

3.7.1. Claim. *We have*

$$\eta_{ki} = e_k \eta_i = -2\,\frac{\zeta^{i-k}}{(\zeta^{i-k}-1)^2}\,\frac{e^{-x_1}}{(r+1)^2} + O(J). \tag{3.15}$$

Notice that (3.15) is symmetric in i, k as it should be.

This is obtained by a straightforward calculation from (3.13) and (3.14). The numerical coefficient in (3.16) comes as a combination of $\sum_{j=1}^r j\zeta^j$ and $\sum_{j=1}^r j^2\zeta^j$ which are then summed by standard tricks.

3.8. Canonical coordinates. We find u^i from the formula $E \circ e_i = u^i e_i$. To calculate $E \circ e_i$, use (3.5), (3.13) and (3.9)–(3.11). We omit the details. The result is:

3.8.1. Claim. *We have*

$$u^i = x_0 + \zeta^i (r+1) e^{\frac{x_1}{r+1}} + \sum_{a=2}^{r} \zeta^{ai} e^{\frac{a x_1}{r+1}} x_a + O(J^2). \tag{3.16}$$

The reader can check that $e_i u^j = \delta_{ij} + O(j)$.

3.9. Schlesinger's initial conditions. Recall that the matrix residues A_i of Schlesinger's equations for Frobenius manifolds are

$$A_j(e_i) = 0 \text{ for } i \neq j,$$

$$A_j(e_j) = -\frac{1}{2} e_j - \frac{1}{2} \sum_k (u^k - u^j) \frac{\eta_{jk}}{\eta_k} e_k \tag{3.17}$$

(cf (1.46).) Substituting here (3.14), (3.15) and (3.16), we finally get the main result of this section.

3.9.1. Theorem. *The point* $(x_0, x_1, 0, \dots, 0)$ *has canonical coordinates* $u^i = x_0 + \zeta^i (r+1) e^{\frac{x_1}{r+1}}$.

The special initial conditions at this point (in the sense of 2.7) corresponding to $H_{\mathrm{quant}}(\mathbf{P}^r)$ *are given by*

$$v_{jk} = -\frac{\zeta^{j-k}}{1-\zeta^{j-k}} \tag{3.18}$$

and

$$\eta_i = \frac{\zeta^i}{r+1}\, e^{-x_1 \frac{r}{r+1}}. \tag{3.19}$$

As an exercise, the reader can check that

$$-\sum_{k:\, k\neq j} \frac{\zeta^{j-k}}{1-\zeta^{j-k}} = 1 - \frac{D}{2} = \frac{r}{2}.$$

§4. Semisimple Frobenius supermanifolds

4.1. Supermanifolds and SUSY-structures. A (smooth, analytic, etc.) *supermanifold* of dimension $(m|n)$ is a locally ringed space $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ with the following properties [Ma3]: (i) the structure sheaf $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$ is a sheaf of $\mathbb{Z}_2$-graded supercommutative rings; (ii) $\mathcal{M}_{\rm red} = (\mathcal{M}, \mathcal{O}_{\mathcal{M},\rm red} := \mathcal{O}_{\mathcal{M}}/[\mathcal{O}_{\mathcal{M},1} + \mathcal{O}^2_{\mathcal{M},1}])$ is a (smooth, analytic, etc.) classical manifold of dimension m; (iii) $\mathcal{O}_{\mathcal{M}}$ is locally isomorphic to the exterior algebra $\Lambda(E)$ of a locally free $\mathcal{O}_{\mathcal{M},\rm red}$-module E of rank n. If $\phi : \Lambda(E) \to \mathcal{O}_{\mathcal{M}}$ is any such local isomorphism, $\bar{x}^1, \dots, \bar{x}^m$ are local coordinates on $\mathcal{M}_{\rm red}$ and $\bar{\theta}^1, \dots, \bar{\theta}^n$ are free local generators of E, then the set of $m + n$ sections

$$x^1 = \phi(\bar{x}^1), \dots, x^m = \phi(\bar{x}^m), \ \theta^1 = \phi(\bar{\theta}^1), \dots, \theta^n = \phi(\bar{\theta}^n)$$

of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ form a local coordinate system on $\mathcal{M}$. Any local function f on $\mathcal{M}$ can be expressed as a polynomial in anticommuting odd coordinates θ^α, $\alpha = 1, \dots, n$,

$$f(x,\theta) = \sum_{k=0}^{n} \sum_{\alpha_1,\dots,\alpha_k=1}^{n} f_{\alpha_1\dots\alpha_k}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_k}$$

whose coefficients $f_{\alpha_1\dots\alpha_k}(x)$ are classical (smooth, analytic, etc.) functions of the commuting variables x^a, $a = 1, \dots, m$.

When a need arises to use odd constants in the structure sheaf of a supermanifold M, one simply replaces $\mathcal{M}$ by its relative version, i.e. by a submersion of supermanifolds $\pi : M \to S$ whose typical fibre is $\mathcal{M}$. Then (odd) constants are just (odd) elements of $\pi^{-1}(\mathcal{O}_S)$. The necessary changes are routine, see [Ma3] and [Ma4].

4.1.1. Definition. *Let $\mathcal{M}$ be an $(m|n)$-dimensional supermanifold. A SUSY-structure on $\mathcal{M}$ is a rank $0|n$ locally split subsheaf $\mathcal{T}_1 \subset \mathcal{T}\mathcal{M}$ such that the associated Frobenius form*

$$\begin{array}{cccc} \Phi: & \Lambda^2\mathcal{T}_1 & \longrightarrow & \mathcal{T}_0 := \mathcal{T}\mathcal{M}/\mathcal{T}_1 \\ & X\otimes Y & \longrightarrow & \frac{1}{2}[X,Y] \bmod \mathcal{T}_1 \end{array}$$

is surjective.

With any SUSY-structure on $\mathcal{M}$ there is canonically associated an extension

$$0 \longrightarrow \mathcal{T}_1 \xrightarrow{i} \mathcal{T}M \xrightarrow{p} \mathcal{T}_0 \longrightarrow 0, \qquad (4.1)$$

i.e. an element $\mathfrak{t} \in \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_0, \mathcal{T}_1) \simeq H^1(\mathcal{M}, \mathcal{T}_1 \otimes \mathcal{T}_0^*)$.

4.1.2. Examples. 1). A $(1|1)$-dimensional supermanifold with a SUSY-structure is called a SUSY$_1$-curve [Ma4]. 2). A SUSY-structure on a $(3|2)$-dimensional is equivalent to a simple conformal supergravity in 3 dimensions [Ma3]. 3). A SUSY-structure on a $(4|4)$-dimensional supermanifold with $\mathcal{T}_1$ being a direct sum of two integrable rank $(0|2)$ distributions $\mathcal{T}_l$ and $\mathcal{T}_r$ is the same as a simple superconformal supergravity in 4 dimensions [Ma3].

4.2. Pre-Frobenius supermanifolds. Let S be a module over a supercommutative ring R. A *left odd involution* on S is by definition a map

$$\begin{array}{rccc} \Pi_l : & S & \longrightarrow & S \\ & X & \longrightarrow & \Pi_l X \end{array}$$

such that $\Pi_l^2 = \mathrm{Id}$ and $\Pi_l(aX) = (-1)^a a \Pi_l(X)$, $\Pi_l(Xa) = \Pi_l(X)a$ for any $X \in S$, $a \in R$.

A *right odd involution*

$$\begin{array}{rccc} \Pi_r : & S & \longrightarrow & S \\ & X & \longrightarrow & X\Pi_r \end{array}$$

also satisfies, by definition, $\Pi_r^2 = \mathrm{Id}$ but has different linearity properties: $(aX)\Pi_r = a(X\Pi_r)$, $(Xa)\Pi_r = (-1)^a (X\Pi_r)a$.

4.2.1. Definition. *A pre-Frobenius structure on an $(n|n)$-dimensional supermanifold $\mathcal{M}$ is a quaternary $(\mathcal{T}_1, s, \Pi_l, \Pi_r)$ consisting of a SUSY-structure (4.1), an even isomorphism $s : \mathcal{T}\mathcal{M} \longrightarrow \mathcal{T}_1 \oplus \mathcal{T}_0$ and a pair of left and right odd involutions*

$$\Pi_{l,r} : \mathcal{T}_1 \oplus \mathcal{T}_0 \longrightarrow \mathcal{T}_1 \oplus \mathcal{T}_0$$

such that

(i) $\Pi_l(\mathcal{T}_0) = (\mathcal{T}_0)\Pi_r = \mathcal{T}_1$, $\Pi_l(\mathcal{T}_1) = (\mathcal{T}_1)\Pi_r = \mathcal{T}_0$;

(ii) a product defined by

$$X \circ Y = \begin{cases} \Phi(X,Y), & \text{for any } X \in \mathcal{T}_1, Y \in \mathcal{T}_1, \\ \Pi_l \Phi(\Pi_l X, Y), & \text{for any } X \in \mathcal{T}_0, Y \in \mathcal{T}_1, \\ \Phi(X, Y\Pi_r)\Pi_r, & \text{for any } X \in \mathcal{T}_1, Y \in \mathcal{T}_0, \\ \Phi(X\Pi_r, \Pi_l Y), & \text{for any } X \in \mathcal{T}_0, Y \in \mathcal{T}_0 \end{cases}$$

makes $\mathcal{T} := \mathcal{T}_1 \oplus \mathcal{T}_0$ a sheaf of associative algebras;

(iii) $s\,|_{\mathcal{T}_1} = i$ *and* $s|_{\mathcal{T}_0}$ *is a splitting of the extension (4.1), i.e.* $p \circ s|_{\mathcal{T}_0} = \mathrm{Id}_{\mathcal{T}_0}$.

4.2.2. Lemma. *There is an even morphism of $\mathcal{O}_\mathcal{M}$-modules, $c : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$, such that the product $\circ : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ factors through the map*

$$\mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T} \otimes \mathcal{T} \xrightarrow{c} \mathcal{T}.$$

Proof. The product $\circ$ obviously satisfies $(aX) \circ Y = a(X \circ Y)$, $X \circ (aY) = (Xa) \circ Y$ and $X \circ (Ya) = (X \circ Y)a$.

4.3. Semisimple pre-Frobenius supermanifolds. Let $\mathcal{M}$ be a pre-Frobenius supermanifold.

4.3.1. Definition. *A pre-Frobenius structure is called (split) almost semisimple if there exists a local (global) basis $\{e_{\dot\alpha}\}$, $\dot\alpha = \dot 1, \dots, \dot n$, of $\mathcal{T}_1$ called canonical, such that* $\Phi(e_{\dot\alpha}, e_{\dot\beta}) = \delta_{\dot\alpha\dot\beta} \Pi_l(e_{\dot\alpha}) = \delta_{\dot\alpha\dot\beta}(e_{\dot\alpha})\Pi_r$.

Since in the almost semisimple case Π_l completely determines Π_r and vice versa, we can and will omit the subscripts l, r. Note that the definition 4.3.1 makes sense, because the extra condition $\Phi(e_{\dot\alpha}, e_{\dot\beta}) = \delta_{\dot\alpha\dot\beta}\Pi(e_{\dot\alpha})$ is consistent with 4.2.1(ii) (and in fact implies the latter). Indeed, in the basis $\{e_{\dot\alpha}, e_\alpha := \Pi(e_{\dot\alpha})\}$ of $\mathcal{T}_1 \oplus \mathcal{T}_0$ the multiplication $\circ$ takes the form

$$\begin{aligned} e_{\dot\alpha} \circ e_{\dot\beta} &= \delta_{\dot\alpha\dot\beta} e_\alpha, \\ e_{\dot\alpha} \circ e_\beta &= e_\alpha \circ e_{\dot\beta} = \delta_{\alpha\beta} e_{\dot\alpha}, \\ e_\alpha \circ e_\beta &= \delta_{\alpha\beta} e_\alpha. \end{aligned}$$

This algebra structure is evidently associative. This table together with Lemma 4.2.2 immediately imply that

$$e := \sum_{\alpha=1}^{n} e_\alpha$$

is the identity, i.e. $e \circ X = X \circ e = X$ for any $X \in \mathcal{T}$. It also follows that

$$\varepsilon := \sum_{\dot\alpha=1}^{n} e_{\dot\alpha} = \Pi e$$

satisfies $\varepsilon \circ X = \Pi X$, $X \circ \varepsilon = X\Pi$ for any $X \in \mathcal{T}$. We call ε the Π-*identity.*

4.3.2. Odd multiplication. If $\mathcal{M}$ is an almost semisimple pre-Frobenius supermanifold, then the formulae

$$X \bullet Y := X \circ (\Pi Y) = \begin{cases} \Phi(X, \Pi Y \Pi)\Pi, & \text{for any } X \in \mathcal{T}_1, Y \in \mathcal{T}_1, \\ \Phi(X\Pi, Y), & \text{for any } X \in \mathcal{T}_0, Y \in \mathcal{T}_1, \\ \Phi(X, \Pi Y), & \text{for any } X \in \mathcal{T}_1, Y \in \mathcal{T}_0, \\ \Pi\Phi(\Pi X, \Pi Y), & \text{for any } X \in \mathcal{T}_0, Y \in \mathcal{T}_0 \end{cases}$$

define an odd associative multiplication in $\mathcal{T}$. Indeed, in the basis $\{e_{\dot\alpha}, e_\alpha\}$ one has

$$\begin{aligned} e_{\dot\alpha} \bullet e_{\dot\beta} &= \delta_{\dot\alpha\dot\beta} e_{\dot\alpha}, \\ e_{\dot\alpha} \bullet e_\beta &= e_\alpha \bullet e_{\dot\beta} = \delta_{\alpha\beta} e_\alpha, \\ e_\alpha \bullet e_\beta &= \delta_{\alpha\beta} e_{\dot\alpha}. \end{aligned}$$

The roles of e and ε get interchanged: ε is the identity, that is $\varepsilon \bullet X = X \bullet \varepsilon = X$, while e is the Π-identity, that is $e \bullet X = \Pi X$, $X \bullet e = X\Pi$ for any $X \in \mathcal{T}$.

4.3.3. Definition. *A pre-Frobenius structure on $\mathcal{M}$ is called (split) semisimple if*

(i) it is (split) almost semisimple;

(ii) there is a local (global) coordinate system $\{u^\alpha, \theta^{\dot\alpha}\}$ called canonical, such that the isomorphism $s : \mathcal{T}_1 \oplus \mathcal{T}_0 \longrightarrow \mathcal{T}\mathcal{M}$ is given by

$$s(e_{\dot\alpha}) = \partial_{\dot\alpha} + \theta^{\dot\alpha}\partial_\alpha, \quad s(e_\alpha) = \partial_\alpha,$$

where $(e_\alpha, e_{\dot\alpha})$ is the canonical basis, $\partial_{\dot\alpha} = \partial/\partial\theta^{\dot\alpha}$ and $\partial_\alpha = \partial/\partial u^\alpha$.

(iii) there is an odd metric g on $\mathcal{TM}$ such that $\mathcal{T}_1 \subset \mathcal{T}M$ is isotropic and

$$g(\partial_\alpha, \partial_\beta) = -\delta_{\alpha\beta}\eta_\beta, \qquad g(e_\alpha, s(e_{\dot\beta})) = \delta_{\alpha\dot\beta}\eta_{\dot\alpha}, \tag{4.2}$$

where $\eta_\alpha = \partial_\alpha \Psi$, $\eta_{\dot\alpha} = (\partial_{\dot\alpha} + \theta^{\dot\alpha}\partial_\alpha)\Psi$ and Ψ is an odd function. Such a metric is called an Egoroff metric.

Since the restriction of $s : \mathcal{T}_1 \oplus \mathcal{T}_0 \longrightarrow \mathcal{TM}$ to $\mathcal{T}_1$ coincides with i and hence is rigidly fixed by the choice of a SUSY structure on $\mathcal{M}$, we identify from now on X and $s(X)$ for any $X \in \mathcal{T}_1$. In particular, whenever the pre-Frobenius structure is semisimple, we always assume that $e_{\dot\alpha} = \partial_{\dot\alpha} + \theta^{\dot\alpha}\partial_\alpha$.

Note that canonical coordinates are defined up to a transformation

$$\begin{aligned} \theta^{\dot\alpha} &\longrightarrow \hat\theta^{\dot\alpha} = \theta^{\dot\alpha} + c^{\dot\alpha} \\ u^\alpha &\longrightarrow \hat u^\alpha = u^\alpha + c^\alpha + \theta^{\dot\alpha} c^{\dot\alpha} \end{aligned}$$

which satisfy $\hat\partial_\alpha = \partial_\alpha$, $\hat e_{\dot\alpha} = e_{\dot\alpha}$ and hence leave all the defining relations invariant.

4.3.4. Example. If $\mathcal{M}$ is a SUSY$_1$-curve, then any odd isomorphism $\mathcal{T}_1 \to \mathcal{T}_0$ together with a global nowhere vanishing section of $\mathcal{T}_1$ (if any) and an odd function Ψ equips $\mathcal{M}$ with a semisimple pre-Frobenius structure.

4.3.5. Algebra structure in $\mathcal{TM}$. The isomorphism $s : \mathcal{T} \to \mathcal{TM}$ translates the product $\circ$ from $\mathcal{T}$ to $\mathcal{TM}$ which we denote by the same symbol. The element $\bar e := s(e) = \sum_\alpha \partial_\alpha$ is the identity in $(\mathcal{TM}, \circ)$.

It is easy to check that in the basis $\{\partial_{\dot\alpha}, \partial_\alpha\}$ the induced multiplication $\circ$ takes the form

$$\begin{aligned} &\partial_{\dot\alpha} \circ \partial_{\dot\beta} = \delta_{\dot\alpha\dot\beta}\partial_\alpha, \\ &\partial_{\dot\alpha} \circ \partial_\beta = \partial_\alpha \circ \partial_{\dot\beta} = \delta_{\alpha\beta}\partial_{\dot\alpha}, \\ &\partial_\alpha \circ \partial_\beta = \delta_{\alpha\beta}\partial_\alpha, \end{aligned}$$

implying that the element $\bar\varepsilon = \sum_{\dot\alpha=1}^n \partial_{\dot\alpha}$ is the $\bar\Pi$-identity in $(\mathcal{TM}, \circ)$, where the odd automorphism $\bar\Pi : \mathcal{TM} \to \mathcal{TM}$ is defined by

$$\bar\Pi(\partial_{\dot\alpha}) = \partial_\alpha, \quad \bar\Pi(\partial_\alpha) = \partial_{\dot\alpha}.$$

4.3.6. Representation of the Neveu-Schwarz Lie superalgebra in $\mathcal{T}M$. Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. Consider vector fields

$$\begin{aligned} E &= \sum_{\alpha=1}^n (u^\alpha \partial_\alpha + \frac{1}{2}\theta^{\dot\alpha} e_{\dot\alpha}) \\ F &= \sum_{\alpha=1}^n u^\alpha e_{\dot\alpha} \end{aligned}$$

on $\mathcal{M}$. A direct calculation shows that the vector fields

$$\mathfrak{e}_a := E^{\circ(a+1)} = \sum_{\alpha=1}^{n} \left[(u^\alpha)^{a+1}\partial_\alpha + \frac{a+1}{2}(u^\alpha)^a \theta^{\dot\alpha} e_{\dot\alpha} \right],\ a = 0, 1, 2, \ldots$$

$$\mathfrak{f}_{i+1/2} := \Pi^i F^{\circ(i+1)} = \sum_{\alpha=1}^{n} (u^\alpha)^{i+1} e_{\dot\alpha},\ i = 0, 1, 2, \ldots$$

satisfy the following commutation relations

$$\begin{aligned} [\mathfrak{e}_a, \mathfrak{e}_b] &= (b-a)\mathfrak{e}_{a+b}, \\ [\mathfrak{e}_a, \mathfrak{f}_i] &= (i - \frac{a}{2})\mathfrak{f}_{i+a}, \\ [\mathfrak{f}_i, \mathfrak{f}_j] &= 2\mathfrak{e}_{i+j}, \end{aligned}$$

where $a, b = 0, 1, 2, \ldots$ and $i, j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

4.4. Flat connections on pre-Frobenius supermanifolds. On a supermanifold $\mathcal{M}$ with a SUSY-structure one may define the differential operator $\delta : \mathcal{O}_\mathcal{M} \to \mathcal{T}_1^*$ as the composition

$$\delta : \mathcal{O}_\mathcal{M} \longrightarrow \Omega^1\mathcal{M} \xrightarrow{i^*} \mathcal{T}_1^*.$$

Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold with the associated commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \mathcal{T}_0^* \arrow[d, "p^*"'] & \\
0 \arrow[r] & \mathcal{O}_\mathcal{M} \arrow[r, "d"] & \Omega^1\mathcal{M} \arrow[r, "d"] & \Omega^2\mathcal{M} \arrow[d, "\gamma"] \\
 & & \mathcal{T}_1^* \arrow[u, "s"] \arrow[r, "\delta'"] & \Lambda^2\mathcal{T}_1^* \arrow[d] \\
 & & 0 \arrow[u] & 0
\end{tikzcd}$$

and let $\Phi^* : \mathcal{T}_0^* \xrightarrow{\gamma\circ d\circ p^*} \Lambda^2\mathcal{T}_1^*$ be the (dual) Frobenius form of the SUSY-structure. Then one has

4.4.1. Proposition. *There is a one-to-one correspondence between locally flat connections in a locally free sheaf $\mathcal{F}$ on $\mathcal{M}$ and covariant differentials*

$$\nabla : \mathcal{F} \longrightarrow \mathcal{T}_1^* \otimes \mathcal{F}$$

such that

$$\nabla(fv) = \delta(f)v + f\nabla(v), \qquad \forall f \in \mathcal{O}_{\mathcal{M}}, v \in \mathcal{F},$$

and the composition

$$C(\nabla^2) : \mathcal{F} \xrightarrow{\nabla} \mathcal{T}_1^* \otimes \mathcal{F} \xrightarrow{\nabla} \Lambda^2\mathcal{T}_1^* \otimes \mathcal{F} \longrightarrow \Lambda^2\mathcal{T}_1^* \otimes \mathcal{F}/\Phi^*(\mathcal{T}_0) \otimes \mathcal{F}$$

(which is $\mathcal{O}_M$-linear) is zero, where the action of ∇ on $\mathcal{T}_1^* \otimes \mathcal{F}$ is defined as follows

$$\nabla(t \otimes v) = (-1)^{\tilde{t}}\gamma \otimes \mathrm{Id}_{\mathcal{F}}(t \otimes \nabla(v)) + \delta'(t) \otimes v, \qquad t \in \mathcal{T}_1^*, v \in \mathcal{F}.$$

Proof. Given a linear connection $D : \mathcal{F} \to \Omega^1\mathcal{M} \otimes \mathcal{F}$, one defines ∇ as the composition

$$\nabla : \mathcal{F} \xrightarrow{D} \Omega^1\mathcal{M} \otimes \mathcal{F} \xrightarrow{i^*\otimes \mathrm{Id}} \mathcal{T}_1^* \otimes \mathcal{F}.$$

Since the curvature tensor, $F \in \Omega^2\mathcal{M} \otimes \mathcal{F} \otimes \mathcal{F}^*$, of D satisfies

$$F(X,Y)v = [D_X, D_Y]v - D_{[X,Y]}v$$

for any $X, Y \in \mathcal{T}\mathcal{M}$ and $v \in \mathcal{F}$, the composition $\nabla^2 := \nabla\circ\nabla$, viewed as a morphism $\Lambda^2\mathcal{T}_1 \otimes \mathcal{F} \to \mathcal{F}$, can be written explicitly as

$$\begin{array}{rccc} \nabla^2 : & \Lambda^2\mathcal{T}_1 \otimes \mathcal{F} & \longrightarrow & \mathcal{F} \\ & X \otimes Y \otimes v & \longrightarrow & D_{2\Phi(X,Y)}v + F(X,Y)v \end{array}$$

implying that $C(\nabla^2)$ is essentially $\gamma \otimes \mathrm{Id}_{\mathcal{F}\otimes\mathcal{F}^*}(F)$. Hence $C(\nabla^2)$ is always $\mathcal{O}_{\mathcal{M}}$-linear and vanishes when D is flat.

In the other direction, let $\nabla : \mathcal{F} \to \mathcal{T}_1^* \otimes \mathcal{F}$ be a covariant differential such that $C(\nabla^2) = 0$. Then ∇^2 factors through the composition

$$\nabla^2 : \Lambda^2\mathcal{T}_1 \otimes \mathcal{F} \xrightarrow{\Phi\otimes\mathrm{Id}} \mathcal{T}_0 \otimes \mathcal{F} \xrightarrow{\nabla^0} \mathcal{F},$$

for some covariant differential operator $\nabla^0 : \mathcal{F} \to \mathcal{T}_0^* \otimes \mathcal{F}$. Define

$$D : \mathcal{F} \to \Omega^1\mathcal{M} \otimes \mathcal{F} \simeq \mathcal{T}_1^* \otimes \mathcal{F} \ \oplus \ \mathcal{T}_0^* \otimes \mathcal{F}$$

as $\nabla\oplus\nabla^0$. A simple calculation in the canonical coordinates (which we omit) shows that D is flat. This completes the proof.

Note that in the presence of a SUSY-structure with invertible Frobenius form *any* covariant differential $\nabla : \mathcal{F} \to \mathcal{T}_1^* \otimes \mathcal{F}$ can be canonically extended to a linear connection $D : \mathcal{F} \to \Omega^1\mathcal{M} \otimes \mathcal{F}$ as $\nabla \oplus \nabla^0$, where

$$\nabla^0 : \mathcal{F} \xrightarrow{\nabla^2} \Lambda^2\mathcal{T}_1^* \otimes \mathcal{F} \xrightarrow{\Phi^{*-1}} \mathcal{T}_0^* \otimes \mathcal{F}.$$

More generally, such an extension is possible whenever $\mathcal{M}$ comes equipped with a monomorphism $\Theta : \mathcal{T}_0 \to \Lambda^2\mathcal{T}_1$ satisfying $\Theta \circ \Phi = \mathrm{Id}$. For example, if $\mathcal{M}$ is almost semisimple pre-Frobenius, then

$$\begin{array}{rccc} \Theta : & \mathcal{T}_0 & \longrightarrow & \Lambda^2\mathcal{T}_1 \\ & e_\alpha & \longrightarrow & e_{\dot\alpha} \otimes e_{\dot\alpha} \end{array}$$

does have this property. When we call a covariant differential $\nabla : \mathcal{F} \to \mathcal{T}_1^* \otimes \mathcal{F}$ a connection on $\mathcal{F} \to \mathcal{M}$, we mean precisely this SUSY extension.

4.5. Semisimple Frobenius structures. Let $(\mathcal{M}, \mathcal{T}_1, s, \Pi, g)$ be a semisimple pre-Frobenius supermanifold.

4.5.1. Definition. *$\mathcal{M}$ is called semisimple Frobenius if g is flat.*

4.5.2. Levi-Civita connection. If g is an odd metric on a supermanifold $\mathcal{M}$, $g_{AB} := (-1)^{\tilde{A}} g(e_A, e_B)$ are the components of g in a basis e_A of $\mathcal{TM}$, then the Christoffel symbols, $\nabla_{e_A} e_B = \sum_C \Gamma^C_{AB} e_C$, of the associated Levi-Civita connection ∇ are given by

$$\Gamma^C_{AB} = \frac{1}{2} \sum_D \Big[e_A g_{BD} + (-1)^{AB} e_B g_{AD} - (-1)^{D(A+B+1)+B} e_D g_{AB} \\ + \sum_M (C^M_{AB} g_{MD} - (-1)^{BD+B+D} C^M_{AD} g_{MB} - (-1)^{A(B+D+1)+D} C^M_{BD} g_{MA}) \Big] g^{DC},$$

where C^M_{AB} are defined by

$$[e_A, e_B] = \sum_M C^M_{AB} e_M.$$

Consider now a special case when g is an Egoroff metric on a semisimple pre-Frobenius supermanifold $\mathcal{M}$.

4.5.3. Proposition-definition (Darboux-Egoroff equations). *The metric g is flat if and only if Ψ satisfies the equations*

$$e_{\dot\mu} \gamma_{\dot\alpha\dot\beta} = \gamma_{\dot\mu\dot\alpha} \gamma_{\dot\mu\dot\beta} \quad \textit{for all } \dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu, \tag{4.3}$$

$$\bar{e} \gamma_{\dot\alpha\dot\beta} = 0 \qquad \textit{for all } \dot\alpha \neq \dot\beta, \tag{4.4}$$

where

$$\gamma_{\dot\alpha\dot\beta} = \frac{e_{\dot\alpha} \eta_{\dot\beta}}{2\sqrt{\eta_{\dot\alpha}\eta_{\dot\beta}}}.$$

These equations are called Darboux-Egoroff equations.

A sketch of the proof. The only non-trivial commutator of basis vector fields $(e_{\dot\alpha}, \partial_\alpha)$ is $[e_{\dot\alpha}, e_{\dot\beta}] = 2\delta_{\dot\alpha\dot\beta} \partial_\alpha$ so that the only non-vanishing components of C^M_{AB} are $C^\gamma_{\dot\alpha\dot\beta} = 2\delta_{\dot\alpha\dot\beta}\delta_{\dot\alpha\gamma}$. Then, using the above formulae for Γ^C_{AB}, one obtains the following Cristoffel symbols of the Levi-Civita connection of g:

$$\nabla_{e_{\dot\mu}} e_{\dot\alpha} = \delta_{\dot\mu\dot\alpha} \partial_\alpha + \frac{e_{\dot\mu} \eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_{\dot\alpha} - \frac{e_{\dot\alpha} \eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu}, \tag{4.5}$$

$$\nabla_{e_{\dot\mu}} \partial_\alpha = e_{\dot\mu} \left(\frac{\eta_\alpha}{2\eta_{\dot\alpha}} \right) e_{\dot\alpha} + \frac{e_\alpha \eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu} - \delta_{\mu\alpha} \sum_{\dot\beta} e_{\dot\beta} \left(\frac{e_{\dot\beta} \eta_{\dot\mu}}{2\eta_{\dot\beta}} \right) e_{\dot\beta} \\ + \frac{e_{\dot\mu} \eta_{\dot\alpha}}{2\eta_{\dot\alpha}} \partial_\alpha - \delta_{\mu\alpha} \frac{\eta_\alpha}{\eta_{\dot\alpha}} \partial_\alpha + \delta_{\mu\alpha} \sum_{\dot\beta} \frac{e_{\dot\beta} \eta_{\dot\alpha}}{2\eta_{\dot\beta}} \partial_\beta. \tag{4.6}$$

By Proposition 4.4.1, the connection ∇ is flat if and only if

$$[\nabla_{e_{\dot\mu}}, \nabla_{e_{\dot\nu}}] = 0 \qquad \text{for all } \dot\mu \neq \dot\nu.$$

A straightforward but very tedious calculation shows that the latter equations are equivalent to the Darboux-Egoroff equations (4.3) and (4.4).

4.6. Flat identity. We say that the $\circ$-identity e on a semisimple pre-Frobenius supermanifold $\mathcal{M}$ is *flat* if $\nabla\bar e = 0$, where ∇ is the Levi-Civita covariant differential (4.5)-(4.6).

4.6.1. Proposition. *The identity e is flat if and only if the potential Ψ satisfies the equations*

$$\bar e\eta_{\dot\alpha} = 0, \tag{4.7}$$

or, equivalently,

$$\sum_\alpha \eta_\alpha = \text{const}. \tag{4.8}$$

Proof. It follows from (4.6) that

$$\begin{aligned}
\nabla_{e_{\dot\mu}}\bar e &= \sum_\beta \nabla_{e_{\dot\mu}} e_\beta \\
&= \sum_{\dot\beta} e_{\dot\mu}\left(\frac{\eta_\beta}{2\eta_{\dot\beta}}\right) e_{\dot\beta} + \sum_\beta \frac{e_\beta\eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu} - \sum_{\dot\beta} e_{\dot\beta}\left(\frac{e_{\dot\beta}\eta_{\dot\mu}}{2\eta_{\dot\beta}}\right) e_{\dot\beta} \\
&\quad + \sum_\beta \frac{e_{\dot\mu}\eta_{\dot\beta}}{2\eta_{\dot\beta}}\partial_\beta - \frac{\eta_\mu}{\eta_{\dot\mu}}\partial_\mu + \sum_\beta \frac{e_{\dot\beta}\eta_{\dot\mu}}{2\eta_{\dot\beta}}\partial_\beta \\
&= \frac{\sum_\beta e_\beta\eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu},
\end{aligned}$$

where we used the fact that $e_{\dot\mu}\eta_{\dot\alpha} + e_{\dot\alpha}\eta_{\dot\mu} = 2\delta_{\dot\mu\dot\alpha}\eta_\alpha$.

4.6.2. Proposition. *Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. The $\bar\Pi$-identity $\bar\varepsilon$ is flat, i.e. $\nabla\bar\varepsilon = 0$ for ∇ being the Levi-Civita covariant differential, if and only if the potential Ψ satisfies the equations*

$$(\theta^{\dot\mu} - \theta^{\dot\alpha})e_{\dot\mu}\eta_{\dot\alpha} = 0, \tag{4.9}$$

$$\bar\varepsilon\eta_{\dot\alpha} = \eta_\alpha. \tag{4.10}$$

Proof is a straightforward calculation.

4.7 Euler field. We want to introduce the notion of homogeneity of a Frobenius structure by assigning the scaling degrees 1 and 1/2 to the canonical coordinates u^α and $\theta^{\dot\alpha}$ respectively (reflecting the fact that $\partial_\alpha = e_{\dot\alpha}e_{\dot\alpha}$). With this motivation, we define a *scaling field* on a Frobenius supermanifold $\mathcal{M}$ as an even vector field E satisfying

$$[E, \partial_\alpha] = -\partial_\alpha, \qquad [E, e_{\dot\alpha}] = -\frac{1}{2}e_{\dot\alpha}.$$

4.7.1. Proposition. *If E is a scaling field, then*

$$E = \sum_{\alpha} \left[(u^{\alpha} + c^{\alpha} + \theta^{\dot\alpha} c^{\dot\alpha})\partial_{\alpha} + \frac{1}{2}(\theta^{\dot\alpha} + c^{\dot\alpha}) e_{\dot\alpha} \right]$$

for some even constants c^{α} and odd constants $c^{\dot\alpha}$.

Proof. Putting $E = \sum_{\alpha} E^{\alpha}\partial_{\alpha} + E^{\dot\alpha} e_{\dot\alpha}$, one obtains

$$[E, \partial_{\alpha}] = -e_{\alpha} \Longleftrightarrow \sum_{\beta} \left[(\partial_{\alpha} E^{\beta})\partial_{\beta} + (\partial_{\alpha} E^{\dot\beta}) e_{\dot\beta} \right] = \partial_{\alpha},$$

$$[E, e_{\dot\alpha}] = -\frac{1}{2} e_{\dot\alpha} \Longleftrightarrow \sum_{\beta} \left[(e_{\dot\alpha} E^{\beta})\partial_{\beta} + (e_{\dot\alpha} E^{\dot\beta}) e_{\dot\beta} \right] - 2E^{\dot\alpha}\partial_{\alpha} = \frac{1}{2} e_{\dot\alpha},$$

implying $\partial_{\alpha} E^{\beta} = \partial_{\alpha} E^{\dot\beta} = e_{\dot\alpha} E^{\beta} = e_{\dot\alpha} E^{\dot\beta} = 0$ for all $\alpha \neq \beta$ as well as $\partial_{\alpha} E^{\dot\alpha} = 0$, $e_{\dot\alpha} E^{\alpha} = 2E^{\dot\alpha}$ and $e_{\dot\alpha} E^{\dot\alpha} = 1/2$. Hence $E^{\dot\alpha} = \frac{1}{2}(\theta^{\dot\alpha} + c^{\dot\alpha})$ and $E^{\alpha} = u^{\alpha} + c^{\alpha} + \theta^{\dot\alpha} c^{\dot\alpha}$ for some even constants c^{α} and odd constants $c^{\dot\alpha}$.

Given a scaling field E, we can and will normalize the canonical coordinates so that $E = \sum_{\alpha} \left[u^{\alpha}\partial_{\alpha} + \frac{1}{2}\theta^{\dot\alpha} e_{\dot\alpha} \right]$.

4.7.2. Definition. *A scaling vector field E on a semisimple pre-Frobenius supermanifold $\mathcal{M}$ is called an Euler field if $Lie_E(g) = Dg$ for some constant D, that is,*

$$E(g(X, Y)) - g([E, X], Y) - g(X, [E, Y]) = Dg(X, Y), \tag{4.11}$$

for all vector fields X, Y.

4.7.3. Proposition. *If E is an Euler field on a semisimple pre-Frobenius supermanifold $\mathcal{M}$, then the potential Ψ satisfies*

$$E\Psi = (D - 1)\Psi + \text{ const},$$

or, equivalently

$$E\eta_{\dot\alpha} = (D - \frac{3}{2})\eta_{\dot\alpha}. \tag{4.12}$$

Proof. Write (4.11) for (i) $X = \partial_{\alpha}$, $Y = \partial_{\beta}$, (ii) $X = e_{\dot\alpha}$, $Y = \partial_{\beta}$ and (iii) $X = e_{\dot\alpha}$, $Y = e_{\dot\beta}$ to find that

$$\begin{aligned} (i) &\Leftrightarrow \; Eg(\partial_{\alpha}, \partial_{\beta}) + 2g(\partial_{\alpha}, \partial_{\beta}) = Dg(\partial_{\alpha}, \partial_{\beta}), \\ (ii) &\Leftrightarrow \; Eg(e_{\dot\alpha}, \partial_{\beta}) + \frac{3}{2} g(e_{\dot\alpha}, \partial_{\beta}) = Dg(e_{\dot\alpha}, \partial_{\beta}), \\ (iii) &\Leftrightarrow \; 0 = 0, \end{aligned}$$

which imply equation (4.12) and

$$E\eta_{\alpha} = (D - 2)\eta_{\alpha}. \tag{4.13}$$

However, the latter equation is not independent — applying $e_{\dot\beta}$ to both sides of (4.12), one easily obtains $E(e_{\dot\beta}\eta_{\dot\alpha}) = (D - 2)e_{\dot\beta}\eta_{\dot\alpha}$ implying (4.13).

4.7.4. Corollary. *If E is an Euler field on a semisimple pre-Frobenius supermanifold $\mathcal{M}$, then*

$$E\gamma_{\dot\mu\dot\nu} = -\frac{1}{2}\gamma_{\dot\mu\dot\nu}. \tag{4.14}$$

Proof.

$$\begin{aligned}
E\gamma_{\dot\mu\dot\nu} &= E\left(\frac{e_{\dot\mu}\eta_{\dot\nu}}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}}\right)\\
&= \frac{(Ee_{\dot\mu}\eta_{\dot\nu})}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{(E\eta_{\dot\mu})(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\mu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{(E\eta_{\dot\nu})(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\nu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}}\\
&= (D-2)\gamma_{\dot\mu\dot\nu} - \frac{1}{2}(D-\frac{3}{2})\gamma_{\dot\mu\dot\nu} - \frac{1}{2}(D-\frac{3}{2})\gamma_{\dot\mu\dot\nu}\\
&= -\frac{1}{2}\gamma_{\dot\mu\dot\nu}.
\end{aligned}$$

4.7.5. Nullness of the $\circ$-identity. In the presence of an Euler field, the flatness of e implies that either $D = 2$ or $g(\bar e, \bar e)$ is identically zero, i.e. the identity $\bar e \in (\mathcal{TM}, \circ)$ is everywhere a null vector. Indeed,

$$g(\bar e, \bar e) = g(\sum_\alpha \partial_\alpha, \sum_\beta \partial_\beta) = \sum_\alpha \eta_\alpha,$$

while equations (4.8) and (4.13) imply $(D-2)\sum_\alpha \eta_\alpha = 0$.

4.8. Geometry on $\mathcal{T}_1$. Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. Define a new splitting of the SUSY-extension (4.1) as follows

$$\begin{array}{cccc}
\tilde s: & \mathcal{T}_0 & \longrightarrow & \mathcal{TM},\\
& \partial_\alpha \bmod \mathcal{T}_1 & \longrightarrow & \tilde e_\alpha := \partial_\alpha - \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_{\dot\beta}.
\end{array}$$

4.8.1. Lemma. *The splitting $\tilde s$ decomposes $\mathcal{TM}$ into a direct sum $\mathcal{T}_1 \oplus \tilde s(\mathcal{T}_0)$ of isotropic submodules.*

Proof. The restriction of the Egoroff metric to $\tilde s(\mathcal{T}_0)$ is

$$\begin{aligned}
g(\tilde e_\alpha, \tilde e_\beta) &= g(\partial_\alpha - \sum_{\dot\mu} \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\mu}} e_{\dot\mu}, \partial_\beta - \sum_{\dot\nu} \frac{e_{\dot\nu}\eta_{\dot\alpha}}{2\eta_{\dot\nu}} e_{\dot\nu})\\
&= -\delta_{\alpha\beta}\eta_\alpha + \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2} + \frac{e_{\dot\alpha}\eta_{\dot\beta}}{2}\\
&= 0.
\end{aligned}$$

Note for future reference that $g(\tilde s(e_\alpha), e_{\dot\beta}) = g(s(e_\alpha), e_{\dot\beta}) = \delta_{\alpha\beta}\eta_{\dot\alpha}$.

4.8.2. Distinguished connections on $\mathcal{T}_1$ and $\mathcal{T}_0$. The decomposition $\mathcal{TM} = \mathcal{T}_1 \oplus \tilde s(\mathcal{T}_0)$ induces a projection

$$p_1 : \mathcal{TM} \to \mathcal{T}_1.$$

Then, if $\nabla : \mathcal{TM} \to \mathcal{T}_1^* \otimes \mathcal{TM}$ is the Levi-Civita covariant differential of the Egoroff metric, one may define the operators

$$\tilde{\nabla}^1 : \mathcal{T}_1 \to \mathcal{T}_1^* \otimes \mathcal{T}_1, \qquad \tilde{\nabla}^0 : \mathcal{T}_0 \to \mathcal{T}_1^* \otimes \mathcal{T}_0$$

as the compositions

$$\begin{aligned}
\tilde{\nabla}^1 &: \mathcal{T}_1 \xrightarrow{i} i(\mathcal{T}_1) \xrightarrow{\nabla} \mathcal{T}_1^* \otimes \mathcal{TM} \xrightarrow{\mathrm{Id}\otimes p_1} \mathcal{T}_1^* \otimes \mathcal{T}_1, \\
\tilde{\nabla}^0 &: \mathcal{T}_0 \xrightarrow{\tilde{s}} \tilde{s}(\mathcal{T}_0) \xrightarrow{\nabla} \mathcal{T}_1^* \otimes \mathcal{TM} \xrightarrow{\mathrm{Id}\otimes p} \mathcal{T}_1^* \otimes \mathcal{T}_0,
\end{aligned}$$

where i and p are defined in (4.1).

Remarkably, the connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^0$ are essentially one and the same thing:

4.8.3. Lemma. *$\tilde{\nabla}^1(X\Pi) = (\tilde{\nabla}^0 X)\Pi$, $\tilde{\nabla}^0(Y\Pi) = (\tilde{\nabla}^1 Y)\Pi$, for any $X \in \mathcal{T}_0$, $Y \in \mathcal{T}_1$.*

Proof. Since

$$p_1(e_\alpha) = \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_{\dot\beta},$$

one obtains from (4.5)

$$\tilde{\nabla}_{e_{\dot\mu}} e_{\dot\alpha} = \delta_{\dot\mu\dot\alpha} \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_{\dot\beta} + \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_{\dot\alpha} - \frac{e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu}. \tag{4.15}$$

Analogously,

$$\begin{aligned}
\tilde{\nabla}_{e_{\dot\mu}} e_\alpha &= p\left(\nabla_{e_{\dot\mu}}(\partial_\alpha - \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_{\dot\beta})\right) \\
&= p(\nabla_{e_{\dot\mu}}\partial_\alpha) + \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} p(\nabla_{e_{\dot\mu}} e_{\dot\beta}) \\
&= \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_\alpha + \delta_{\dot\mu\dot\alpha} \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_\beta - \delta_{\dot\mu\dot\alpha} \frac{\eta_\mu}{\eta_{\dot\mu}} e_\mu + \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\mu}} e_\mu \\
&= \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_\alpha + \delta_{\dot\mu\dot\alpha} \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_\beta - \frac{e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}} e_\mu.
\end{aligned}$$

Then the required statement follows.

From now on we use one symbol $\tilde{\nabla}$ to denote covariant differentials $\tilde{\nabla}^1$ on $\mathcal{T}_1$, $\tilde{\nabla}^0$ on $\mathcal{T}_0$ and $\tilde{\nabla}^1 \oplus \tilde{\nabla}^0$ on $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_0$.

4.8.4 Metrics on $\mathcal{T}$. The Egoroff metric g gives rise to

(i) an even metric h on $\mathcal{T}_0$, $h(X, Y) := g(\Pi X, \tilde{s}(Y))$, for any $X, Y \in \mathcal{T}_0$ (note that $g(\Pi X, \tilde{s}(Y)) = g(\Pi X, s(Y))$);

(ii) an odd metric $\tilde{g}$ on $\mathcal{T}$ as the pullback of g relative to the isomorphism $\tilde{s}:\mathcal{T}\to\mathcal{TM}$. Note that that $\mathcal{T}_0$ and $\mathcal{T}_1$ are isotropic and $\tilde{g}(X,Y)=g(\tilde{s}(X),Y)=g(s(X),Y)$ for any $X\in\mathcal{T}_0$, $Y\in\mathcal{T}_1$.

Due to the isomorphism $\odot^2(\mathcal{T}_0^*)=\odot^2(\Pi\mathcal{T}_1^*)=\Lambda^2(\mathcal{T}_1^*)$, the metric h on $\mathcal{T}_0$ can also be viewed as an even non-degenerate skew-symmetric form on $\mathcal{T}_1$. Explicitly, h is given by

$$h(e_\alpha,e_\beta)=\delta_{\alpha\beta}\eta_{\dot\alpha},\quad h(e_\alpha,e_{\dot\beta})=0,\quad h(e_{\dot\alpha},e_{\dot\beta})=\delta_{\dot\alpha\dot\beta}\eta_{\dot\alpha},$$

while $\tilde{g}$ satisfies

$$\tilde{g}(e_\alpha,e_\beta)=0,\quad \tilde{g}(e_\alpha,e_{\dot\beta})=\delta_{\alpha\beta}\eta_{\dot\alpha},\quad \tilde{g}(e_{\dot\alpha},e_{\dot\beta})=0.$$

4.8.5. Frobenius property. The triple $(\mathcal{T},\circ,\tilde{g})$ obviously satisfies

$$\tilde{g}(X,Y)=\theta(X\circ Y)$$

for any $X,Y\in\mathcal{T}$, where the odd 1-form θ is defined by

$$\theta=\mathcal{P}\delta\Psi,$$

δ being the SUSY-differential and $\mathcal{P}$ the parity change functor [Ma3]. In particular, one has

$$\tilde{g}(X\circ Y,Z)=\tilde{g}(X,Y\circ Z)$$

for any $X,Y,Z\in\mathcal{T}$ (which is a defining property of the so-called *Frobenius algebras* [D], [H]).

4.8.6 Proposition. $\tilde\nabla h=0,\ \ \tilde\nabla\tilde{g}=0.$

Proof. Let us view h as, for example, a skew-form on $\mathcal{T}_1$. Then

$$\begin{aligned}(\tilde\nabla_{e_{\dot\mu}}h)(e_{\dot\alpha},e_{\dot\gamma}) &= e_{\dot\mu}h(e_{\dot\alpha},e_{\dot\gamma})-h(\tilde\nabla_{e_{\dot\mu}}e_{\dot\alpha},e_{\dot\gamma})+h(e_{\dot\alpha},\tilde\nabla_{e_{\dot\mu}}e_{\dot\gamma})\\ &= \delta_{\dot\alpha\dot\gamma}e_{\dot\mu}\eta_{\dot\alpha}-\frac{1}{2}\delta_{\dot\alpha\dot\gamma}e_{\dot\mu}\eta_{\dot\alpha}+\frac{1}{2}\delta_{\dot\mu\dot\gamma}e_{\dot\alpha}\eta_{\dot\mu}-\frac{1}{2}\delta_{\dot\alpha\dot\mu}e_{\dot\gamma}\eta_{\dot\alpha}\\ &\quad-\frac{1}{2}\delta_{\dot\alpha\dot\gamma}e_{\dot\mu}\eta_{\dot\alpha}+\frac{1}{2}\delta_{\dot\alpha\dot\mu}e_{\dot\gamma}\eta_{\dot\alpha}-\frac{1}{2}\delta_{\dot\mu\dot\gamma}e_{\dot\alpha}\eta_{\dot\mu}\\ &= 0.\end{aligned}$$

Analogously one checks other statements.

4.9. Odd identity. The $\bullet$-identity ε is said to be *flat* if $\tilde\nabla\varepsilon=0$.

4.9.1. Proposition. $\tilde\nabla\varepsilon=0\ \Leftrightarrow\ \sum_{\dot\alpha}\eta_{\dot\alpha}=\mathrm{const}.$

Proof.

$$\begin{aligned}\tilde\nabla_{e_{\dot\mu}}\left(\sum_\alpha e_{\dot\alpha}\right) &= \sum_{\dot\beta}\frac{e_{\dot\mu}\eta_{\dot\beta}}{2\eta_{\dot\beta}}e_{\dot\beta}-\frac{\sum_\alpha e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}}e_{\dot\mu}+\sum_{\dot\beta}\frac{e_{\dot\mu}\eta_{\dot\beta}}{2\eta_{\dot\beta}}e_{\dot\beta}\\ &= \frac{\eta_{\mu}}{2\eta_{\dot\mu}}-\frac{\sum_\alpha e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}}e_{\dot\mu}\\ &= \frac{e_{\dot\mu}(\sum_\alpha\eta_{\dot\alpha})}{2\eta_{\dot\mu}}e_{\dot\mu}\\ &= 0.\end{aligned}$$

4.9.2. Proposition. *Flatness of ε implies flatness of e, or, equivalently,*

$$\sum_{\dot\alpha} \eta_{\dot\alpha} = \text{const} \quad \Rightarrow \quad \sum_{\alpha} \eta_{\alpha} = \text{const}.$$

Proof.

$$\begin{aligned} \sum_{\dot\alpha} \eta_{\dot\alpha} = \text{const} \;&\Leftrightarrow\; e_{\dot\beta}\left(\sum_{\dot\alpha} \eta_{\dot\alpha}\right) = 0 \\ &\Leftrightarrow\; \eta_\beta = \sum_{\dot\alpha:\dot\alpha\neq\dot\beta} e_{\dot\alpha}\eta_{\dot\beta} \\ &\Rightarrow\; \sum_\beta \eta_\beta = \sum_{\dot\alpha,\dot\beta:\dot\alpha\neq\dot\beta} e_{\dot\alpha}\eta_{\dot\beta} = 0. \end{aligned}$$

4.9.3. Orthogonality of flat identities. In the presence of an Euler field, the flatness of the odd identity ε implies that either $D = 3/2$ or $g(\bar e, \varepsilon)$ is identically zero, i.e. the even and odd identities $\bar e, \varepsilon \in (\mathcal{TM}, \circ)$ are everywhere g-orthogonal. Indeed,

$$g(e, \varepsilon) = g(\sum_\alpha \partial_\alpha, \sum_{\dot\beta} e_{\dot\beta}) = \sum_{\dot\alpha} \eta_{\dot\alpha},$$

while Proposition 4.9.1 and equation (4.12) imply $(D - 3/2)\sum_{\dot\alpha} \eta_{\dot\alpha} = 0$.

Since $\tilde g(e,\varepsilon) = g(s(e),\varepsilon) = g(\bar e, \varepsilon)$, one may reformulate the above observation as the $\tilde g$-orthogonality of the identities $(e,\varepsilon) \in (\mathcal{T}, \circ)$.

4.10. Uniqueness and flatness of $\tilde\nabla$. The main justification for introducing the connection $\tilde\nabla$ comes from the following result.

4.10.1. Proposition. *Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. Then the associated connection $\tilde\nabla$ is flat if and only if Ψ satisfies the equation (4.3) and*

$$\sum_{\dot\beta:\dot\beta\neq\dot\mu,\dot\nu} e_{\dot\beta}\gamma_{\dot\mu\dot\nu} = e_{\dot\mu}\gamma_{\dot\mu\dot\nu} + e_{\dot\nu}\gamma_{\dot\mu\dot\nu} \tag{4.16}$$

for all $\dot\mu \neq \dot\nu$.

Proof is a straightforward but lengthy calculation.

4.10.2. Corollary. *Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. If the associated connection $\tilde\nabla$ is flat, then $\mathcal{M}$ is semisimple Frobenius.*

Proof. By Proposition 4.5.3, it will suffice to show that equation (4.16) implies equation (4.4). This is established by the following calculation:

$$\begin{aligned}
\sum_{\alpha} \partial_\alpha \gamma_{\dot\mu\dot\nu} &= \sum_{\dot\alpha,\dot\beta} e_{\dot\alpha} e_{\dot\beta} \gamma_{\dot\mu\dot\nu} \\
&= 2\sum_{\dot\alpha} e_{\dot\alpha} e_{\dot\mu} \gamma_{\dot\mu\dot\nu} + 2\sum_{\dot\alpha} e_{\dot\alpha} e_{\dot\nu} \gamma_{\dot\mu\dot\nu} \\
&= 4e_\mu \gamma_{\dot\mu\dot\nu} - 2\sum_{\dot\alpha} e_{\dot\mu}(e_{\dot\alpha}\gamma_{\dot\mu\dot\nu}) + 4e_\nu \gamma_{\dot\mu\dot\nu} - 2\sum_{\dot\alpha} e_{\dot\nu}(e_{\dot\alpha}\gamma_{\dot\mu\dot\nu}) \\
&= 4e_\mu \gamma_{\dot\mu\dot\nu} - 2e_{\dot\mu}(2e_{\dot\mu}\gamma_{\dot\mu\dot\nu} + 2e_{\dot\nu}\gamma_{\dot\mu\dot\nu}) + 4e_\nu \gamma_{\dot\mu\dot\nu} - 2e_{\dot\nu}(2e_{\dot\mu}\gamma_{\dot\mu\dot\nu} + 2e_{\dot\nu}\gamma_{\dot\mu\dot\nu}) \\
&= -4(e_{\dot\mu}e_{\dot\nu} + e_{\dot\nu}e_{\dot\mu})\gamma_{\dot\mu\dot\nu} \\
&= 0.
\end{aligned}$$

4.10.3. Corollary. *Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. If ε is flat and Ψ satisfies the equation (4.3), then $\tilde\nabla$ is flat as well (implying that $\mathcal{M}$ is Frobenius).*

Proof. Assume $e_{\dot\mu}(\sum_{\dot\beta} \eta_{\dot\beta}) = 0$, or, equivalently,

$$\eta_\mu = \sum_{\dot\beta:\dot\beta\neq\mu} e_{\dot\beta}\eta_{\dot\mu}.$$

Then

$$\begin{aligned}
\sum_{\dot\beta:\dot\beta\neq\dot\mu,\dot\nu} e_{\dot\beta}\gamma_{\dot\mu\dot\nu} &= \sum_{\dot\beta:\dot\beta\neq\dot\mu,\dot\nu} \frac{e_{\dot\beta}e_{\dot\mu}\eta_{\dot\nu}}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \sum_{\dot\beta:\dot\beta\neq\dot\mu,\dot\nu} \frac{(e_{\dot\beta}\eta_{\dot\mu})(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\mu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \sum_{\dot\beta:\dot\beta\neq\dot\nu} \frac{(e_{\dot\beta}\eta_{\dot\nu})(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\nu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} \\
&= -\sum_{\dot\beta:\dot\beta\neq\dot\nu} \frac{e_{\dot\mu}(e_{\dot\beta}\eta_{\dot\nu})}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} + \frac{e_\mu\eta_{\dot\nu}}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{\eta_\mu(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\mu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{\eta_\nu(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\nu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} \\
&= -\frac{e_\nu\eta_{\dot\mu}}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} + \frac{e_\mu\eta_{\dot\nu}}{2\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{\eta_\mu(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\mu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} - \frac{\eta_\nu(e_{\dot\mu}\eta_{\dot\nu})}{4\eta_{\dot\nu}\sqrt{\eta_{\dot\mu}\eta_{\dot\nu}}} \\
&= e_{\dot\mu}\gamma_{\dot\mu\dot\nu} + e_{\dot\nu}\gamma_{\dot\mu\dot\nu},
\end{aligned}$$

so that the equation (4.16) is satisfied.

For later use we give the following characterization of $\tilde\nabla$:

4.10.4. Proposition. *Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold. A linear connection $\nabla : \mathcal{T}_1 \to \mathcal{T}_1^* \otimes \mathcal{T}_1$ satisfies the conditions*

(a) $\nabla h = 0$,

(b) $\nabla_{e_{\dot\mu}} e_{\dot\alpha} + \nabla_{e_{\dot\alpha}} e_{\dot\mu} = 2\delta_{\dot\mu\dot\alpha}\nabla_{e_{\dot\mu}} e_{\dot\mu}$,

(c) $h(\nabla_{e_{\dot\mu}} e_{\dot\alpha}, e_{\dot\beta}) = 0$ for any $\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu$

if and only if it is given by (4.15).

Proof. Define $\Delta_{\dot\mu\dot\alpha\dot\beta}$ as

$$\nabla_{e_{\dot\mu}} e_{\dot\alpha} = \sum_{\dot\beta} \frac{\Delta_{\dot\mu\dot\alpha\dot\beta}}{2\eta_{\dot\beta}} e_{\dot\beta}.$$

Then

$$
\begin{aligned}
(a) &\;\Leftrightarrow\; e_{\dot\mu}\eta_{\dot\alpha}\delta_{\dot\alpha\dot\beta} = \frac{1}{2}(\Delta_{\dot\mu\dot\alpha\dot\beta} + \Delta_{\dot\mu\dot\beta\dot\alpha}),\\
(b) &\;\Leftrightarrow\; \Delta_{\dot\mu\dot\alpha\dot\beta} + \Delta_{\dot\alpha\dot\mu\dot\beta} = 2\delta_{\dot\alpha\dot\mu}\Delta_{\dot\mu\dot\mu\dot\beta},\\
(c) &\;\Leftrightarrow\; \Delta_{\dot\mu\dot\alpha\dot\beta} = 0 \text{ for all } \dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu,
\end{aligned}
$$

implying that the only non-vanishing components of $\Delta_{\dot\mu\dot\alpha\dot\beta}$ are

$$\Delta_{\dot\mu\dot\alpha\dot\alpha} = \Delta_{\dot\mu\dot\alpha\dot\mu} = \Delta_{\dot\alpha\dot\alpha\dot\mu} = e_{\dot\mu}\eta_{\dot\alpha}.$$

Hence

$$\nabla_{e_{\dot\mu}} e_{\dot\alpha} = \begin{cases} \sum_{\dot\beta} \frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}} e_{\dot\beta} & \text{for } \dot\mu = \dot\alpha,\\ \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_{\dot\alpha} + \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\mu}} e_{\dot\mu} = \frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} e_{\dot\alpha} - \frac{e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}} e_{\dot\mu} & \text{for } \dot\mu \neq \dot\alpha, \end{cases}$$

implying $\nabla = \tilde\nabla$.

§5. Supersymmetric Schlesinger equations

5.1. Meromorphic connections with logarithmic singularities. Let $\mathcal{M}$ be a complex supermanifold equipped with a SUSY structure (4.1) and let $\mathcal{F} \to \mathcal{M}$ be a locally free holomorphic sheaf on $\mathcal{M}$. Keeping in mind semisimple Frobenius structures, we also assume that $\mathcal{M}$ comes equipped with a monomorphism $\Theta : \mathcal{T}_0 \to \Lambda^2\mathcal{T}_1$ which, as explained in 4.4, canonically extends any covariant differential $\nabla : \mathcal{F} \to \mathcal{T}_1^* \otimes V$ to a linear connection on $\mathcal{F}$.

Let D be a complex super-submanifold of $\mathcal{M}$ of codimension $1|0$.

Assume first that D is irreducible and that the associated divisor line bundle $[D]$ is free. Let f be a global basis section of $[D]$. A holomorphic covariant differential

$$\nabla : \mathcal{F} \to \mathcal{T}_1^* \otimes \mathcal{F}$$

on $\mathcal{M} \setminus D$ is said to be a *meromorphic connection with logarithmic singularities* along D if there is a holomorphic covariant differential

$$\nabla' : \mathcal{F} \to \mathcal{T}_1^* \otimes \mathcal{F}$$

on $\mathcal{M}$ such that

$$\nabla - \nabla' = A\frac{\delta f}{f}$$

for some even holomorphic section $A \in H^0(\mathcal{M}, \mathcal{F} \otimes \mathcal{F}^*)$.

Note that

a) this definition does not depend on the choice of a particular trivialization f of $[D]$ and hence can be appropriately localized and generalized;

b) the section A restricted to D does not depend on the choices made and hence gives a well-defined element of $H^0(D, \mathcal{F} \otimes \mathcal{F}^*)$ which is called the *residue* of ∇ at D;

c) for any local trivialization f of $[D]$, the connection ∇ induces a holomorphic *residual connection* $\nabla^{D,f}$ on $\mathcal{F}|_D$ associated with the holomorphic covariant differential $\nabla'|_D = (\nabla - A\frac{\delta f}{f})\Big|_D$; if ∇ is flat, $\nabla^{D,f}$ is flat for any f.

5.2. Universal isomonodromic deformation. Consider $\mathbb{C}^{n|n}$ with its natural SUSY structure $\mathcal{T}_1 \subset \mathcal{T}\mathbb{C}^{n|n}$ spanned by the vector fields $e_{\dot\alpha} = \partial_{\dot\alpha} + \theta^{\dot\alpha}\partial_\alpha$, where $(u^\alpha, \theta^{\dot\alpha})$ are natural coordinates. Let B be the universal covering of $\mathbb{C}^{n|n} \setminus (u^\alpha - u^\beta - \theta^{\dot\alpha}\theta^{\dot\beta} = 0)$, where pairwise distinct integers α and β run over $1, \dots, n$.

Consider a supermanifold $B \times \mathbf{P}^{1|1}$ with the direct product SUSY-structure, and denote by $\tilde D_\alpha$ the inverse image in $B \times \mathbf{P}^{1|1}$ of the submanifold $\lambda - u^\alpha - \xi\theta^{\dot\alpha} = 0$ in $\mathbb{C}^{n|n} \times \mathbf{P}^{1|1}$, where (λ, ξ) are natural coordinates on a big cell of $\mathbf{P}^{1|1}$. Furthermore, define $\tilde D_\infty = B \times \infty \subset B \times \mathbf{P}^{1|1}$, where ∞ stands for the codimension $1|0$ submanifold of $\mathbf{P}^{1|1}$ given by $\hat\lambda = 0$, where $\hat\lambda = 1/\lambda$.

To any given point $(x_0^\alpha, \theta_0^\alpha) \in \mathbb{C}^{n|n}$ one may associate an embedding

$$\begin{array}{rccc} i_0 : & \mathbf{P}^{1|1} & \longrightarrow & B \times \mathbf{P}^{1|1} \\ & (\lambda, \xi) & \longrightarrow & (x_0^\alpha, \theta_0^\alpha, \lambda, \xi). \end{array}$$

Theorem 5.2.1. *Let $\mathcal{F}^0$ be a locally free sheaf of rank $p|q$ on $\mathbf{P}^{1|1}$ and let ∇^0 be a flat meromorphic connection on it with logarithmic singularities at $\cup_{\alpha=1}^n D_\alpha^0 \cup \infty$, where $D_\alpha^0 \subset \mathbf{P}^{1|1}$ is given by $\lambda - u_0^\alpha - \xi\theta_0^{\dot\alpha} = 0$.*

Then there exists a locally free sheaf $\mathcal{F}$ of rank $p|q$ on $B \times \mathbf{P}^{1|1}$ and a flat meromorphic connection ∇ on it such that

(a) ∇ has logarithmic singularities at $\tilde{D}_\alpha$, $\alpha = 1, \ldots, n$, and $\tilde{D}_\infty$;

(b) there is a canonical isomorphism $i : i_0^*(\mathcal{F}, \nabla) \to (\mathcal{F}^0, \nabla^0)$;

(c) the data $(\mathcal{F}, \nabla, i)$ are unique up to unique isomorphism.

Comment on the proof. According to Penkov [P], a pair $(\mathcal{E}, \nabla)$ consisting of a locally free sheaf $\mathcal{E}$ on a supermanifold $\mathcal{M}$ and a flat holomorphic connection ∇ on $\mathcal{E}$ is uniquely determined by the associated monodromy representation of $\pi_1(\mathcal{M}_{\mathrm{red}})$ on $\mathcal{E}_{\mathrm{red}}$. This together with the observation made in 2.2.2 about the isomorphism of the first homotopy groups of the underlying classical manifolds, immediately implies that there is a pair $(\mathcal{F}, \nabla)$ on $B \times \mathbf{P}^{1|1} \setminus (\cup_{\alpha=1}^n \tilde{D}_\alpha \cup \tilde{D}_\infty)$ such that the statements (b) and (c) are true outside singularities.

Using a straightforward generalization of the original Malgrange's arguments, one may extend $(\mathcal{F}, \nabla)$ to $B \times \mathbf{P}^{1|1}$ in such a way that (a)-(c) hold.

5.3. Supersymmetric Schlesinger equations. In this subsection we will assume that $\mathcal{F}^0 = T \otimes \mathcal{O}_{\mathbf{P}^{1|1}}$ where T is a vector superspace of dimension $p|q$.

Using the semicontinuity principle as in Section 2.3, one may show that there is an open subset $B' \subset B$ such that $\mathcal{F}$ is free on $B' \times \mathbf{P}^{1|1}$. Moreover, one may identify $\mathcal{F}$ on $B' \times \mathbf{P}^{1|1}$ with $T \otimes \mathcal{O}_{B' \times \mathbf{P}^{1|1}}$ compatibly with the respective trivialization of $\mathcal{F}^0$. Indeed, one may first trivialize $\mathcal{F}$ along $\tilde{D}_\infty$ using the residual connection, and then take the constant extension of each horizontal section along $\mathbf{P}^{1|1}$. Since

$$\delta(u^\nu - \lambda - \theta^{\dot\nu}\xi) = (\theta^{\dot\nu} - \xi) d(\theta^{\dot\nu} - \xi),$$

in the chosen trivialization $\mathcal{F}|_{B' \times \mathbf{P}^{1|1}} = T \otimes \mathcal{O}_{B' \times \mathbf{P}^{1|1}}$ the covariant differential ∇ must be of the form

$$\nabla = \delta + \sum_{\nu=1}^n \frac{A_\nu(\theta^{\dot\nu} - \xi)}{u^\nu - \lambda - \theta^{\dot\nu}\xi} d(\theta^{\dot\nu} - \xi) \tag{5.1}$$

for some even meromorphic sections $A_\nu \in H^0(B, \mathcal{F} \otimes \mathcal{F}^*)$.

Theorem 5.3.1. *The connection (5.1) is flat if and only if*

$$e_{\dot\mu} A_\nu = -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} [A_\mu, A_\nu] \tag{5.2}$$

$$e_{\dot\mu} A_\mu = \sum_{\nu:\nu\neq\mu} \frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} [A_\mu, A_\nu]. \tag{5.3}$$

or, equivalently,

$$dA_\mu = \sum_{\nu:\nu\neq\mu} \frac{d(u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu})}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}}[A_\mu, A_\nu] \tag{5.4}$$

where d is the usual exterior differential on B.

Proof is straightforward.

The equations (5.2) and (5.3) are called *supersymmetric Schlesinger's equations.*

5.4. From Frobenius supermanifolds to strict special solutions of Schlesinger's equations. Let $\mathcal{M}$ be a semisimple Frobenius supermanifold with an Euler field E and flat identities ε and $\bar\varepsilon$. Define an even linear operator $\mathcal{V} : \mathcal{T}_1 \to \mathcal{T}_1$ as follows,

$$\mathcal{V}(X) = p_1(\nabla_X E) - \frac{1}{2}(D - \frac{1}{2})X \quad \text{for any } X \in \mathcal{T}_1,$$

where ∇ is the Levi-Civita connection and $p_1 : \mathcal{TM} \to \mathcal{T}_1$ the projection defined in 4.8.2.

Theorem 5.4.1. *a). Let $f_{\dot\alpha} = e_{\dot\alpha}/\sqrt{\eta_{\dot\alpha}}$, then*

$$\mathcal{V}(f_{\dot\alpha}) = \sum_{\dot\beta:\dot\beta\neq\dot\alpha} \left[\theta^{\dot\alpha}\gamma_{\dot\beta\dot\alpha} + u^{\dot\beta}\partial_{\dot\beta}\gamma_{\dot\beta\dot\alpha} - u^{\dot\alpha}\partial_{\dot\alpha}\gamma_{\dot\alpha\dot\beta} + \sum_{\gamma:\gamma\neq\alpha,\beta} u^\gamma e_{\dot\gamma}\gamma_{\dot\alpha\dot\beta}\right] f_{\dot\beta}. \tag{5.5}$$

b). $\mathcal{V}$ is symmetric relative to h, i.e.

$$h(\mathcal{V}(X), Y) + h(X, \mathcal{V}(Y)) = 0 \quad \textit{for any } X, Y \in \mathcal{T}_1,$$

c). $\tilde\nabla\mathcal{V} = 0$.

d). $\mathcal{V}(\varepsilon) = \frac{3-2D}{4}\varepsilon$.

Proof. Items (a) and (c) follow from a straightforward calculation which we omit, while (b) immediately follows from (4.9) and (5.5). To check (c), note that

$$\begin{aligned}
\mathcal{V}(\varepsilon) &= p_1(\nabla_\varepsilon E) - \frac{1}{2}(D - \frac{1}{2})\varepsilon \\
&= p_1(\nabla_E \varepsilon + [\varepsilon, E]) - \frac{1}{2}(D - \frac{1}{2})\varepsilon \\
&= \frac{1}{2}\varepsilon - \frac{1}{2}(D - \frac{1}{2})\varepsilon \\
&= \frac{3 - 2D}{4}\varepsilon.
\end{aligned}$$

5.4.2. The structure connection. Let $\mathcal{M}$ be a semisimple pre-Frobenius supermanifold with canonical coordinates $(u^\alpha, \theta^{\dot\alpha})$, and let $\check{\mathcal{M}}$ be a $(n+1|n+1)$-supermanifold $\mathcal{M} \times \mathbf{P}^{1|1} \setminus \cup_\nu(u^\nu - \lambda - \theta^{\dot\nu}\xi = 0)$ equipped with a product SUSY structure

$$\check{\mathcal{T}}_1 := \mathrm{span}(e_{\dot\alpha}, e_\xi) \subset \mathcal{T}\check{\mathcal{M}},$$

where $e_\xi = \partial/\partial\xi + \xi\partial/\partial\lambda$. Let pr : $\check{\mathcal{M}} \to \mathcal{M}$ be a natural projection and denote $E_0 := \sum_\alpha u^\alpha e_{\dot\alpha}$ and $E_1 := \sum_\alpha \theta^{\dot\alpha} e_{\dot\alpha}$.

The following theorem introduces a so-called *structure connection*on $\mathrm{pr}^*(\mathcal{T}_1) \subset \check{\mathcal{T}}_1$ which associates with any semisimple Frobenius structure a 1-parameter solution of Schlesinger's equations.

5.4.3. Theorem. *For any $X, Y \in \mathrm{pr}^{-1}(\mathcal{T}_1) \subset \check{\mathcal{T}}_1$ put*

$$\check\nabla_X Y = \tilde\nabla_X Y - (\mathcal{V} + \kappa\mathrm{Id})(E_0 - \lambda + \xi E_1)^{-1} \bullet (E_1 - \xi) \bullet X \bullet Y \tag{5.6}$$

$$\check\nabla_{e_\xi} Y = (\mathcal{V} + \kappa\mathrm{Id})(E_0 - \lambda + \xi E_1)^{-1} \bullet (E_1 - \xi) \bullet Y. \tag{5.7}$$

where $\kappa \in \mathbb{C}$.

If $\mathcal{M}$ is a semisimple Frobenius supermanifold with an Euler field and flat identities ε and $\tilde\varepsilon$, then $\check\nabla$ is a flat connection on $\mathrm{pr}^*(\mathcal{T}_1) \subset \check{\mathcal{T}}_1$ for any κ.

Proof. If $X = e_{\dot\mu}$ and $Y = e_{\dot\alpha}$, the equations (5.6) and (5.7) take the form

$$\check\nabla_{e_{\dot\mu}} e_{\dot\alpha} = \tilde\nabla_{e_{\dot\mu}} e_{\dot\alpha} + \delta_{\dot\alpha\dot\mu}\frac{[\mathcal{V}(e_{\dot\alpha}) + \kappa e_{\dot\alpha}](\theta^{\dot\alpha} - \xi)}{u^\alpha - \lambda - \theta^{\dot\alpha}\xi}$$

$$\check\nabla_{e_\xi} e_{\dot\alpha} = -\frac{[\mathcal{V}(e_{\dot\alpha}) + \kappa e_{\dot\alpha}](\theta^{\dot\alpha} - \xi)}{u^\alpha - \lambda - \theta^{\dot\alpha}\xi}.$$

Then, by Proposition 5.3.1, it will suffice to show that under the conditions stated in 5.4.2 the matrix valued fields

$$A_{\nu\,\dot\alpha}^{\ \dot\beta} = -\delta_{\dot\nu\alpha}[\mathcal{V}_{\dot\alpha\dot\beta} + \kappa\delta_{\dot\alpha\dot\beta}]$$

satisfy, for any $\kappa \in \mathbb{C}$, the Schlesinger's equations

$$e_{\dot\mu} A_{\nu\,\dot\alpha}^{\ \dot\beta} + \Gamma^{\dot\beta}_{\dot\mu\dot\delta} A_{\nu\,\dot\alpha}^{\ \dot\delta} - \Gamma^{\dot\delta}_{\dot\mu\dot\alpha} A_{\nu\,\dot\delta}^{\ \dot\beta} = -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}}[A_\mu, A_\nu]^{\dot\alpha}_{\dot\beta}, \quad \mu \neq \nu \tag{5.8}$$

$$e_{\dot\mu} A_{\mu\,\dot\alpha}^{\ \dot\beta} + \Gamma^{\dot\beta}_{\dot\mu\dot\delta} A_{\mu\,\dot\alpha}^{\ \dot\delta} - \Gamma^{\dot\delta}_{\dot\mu\dot\alpha} A_{\nu\,\dot\delta}^{\ \dot\beta} = \sum_{\nu:\nu\neq\mu} \frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}}[A_\mu, A_\mu]^{\dot\beta}_{\dot\alpha}, \tag{5.9}$$

where

$$\Gamma^{\dot\beta}_{\dot\mu\dot\alpha} = \delta_{\dot\alpha\dot\beta}\frac{e_{\dot\mu}\eta_{\dot\alpha}}{2\eta_{\dot\alpha}} - \delta_{\dot\beta\dot\mu}\frac{e_{\dot\alpha}\eta_{\dot\mu}}{2\eta_{\dot\mu}} + \delta_{\dot\mu\dot\alpha}\frac{e_{\dot\beta}\eta_{\dot\alpha}}{2\eta_{\dot\beta}}$$

are the coefficients of the connection $\tilde\nabla$, and $\mathcal{V}_{\dot\alpha\dot\beta}$ are the coefficients of the operator $\mathcal{V}$ in the basis $e_{\dot\alpha}$.

It is not hard to show that equations (4.9) and (5.5) imply, for $\mu \neq \nu$,

$$\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}}\mathcal{V}_{\dot\mu\dot\nu} = \frac{e_{\dot\mu}\eta_{\dot\nu}}{2\eta_{\dot\nu}}$$

which in turn imply that the terms of order κ^0 in (5.8) and (5.9) are all equivalent to the equation $\tilde\nabla\mathcal{V} = 0$ which follows from 5.4.1(c).

Since there are no terms quadratic in κ in (5.8) and (5.9), it remains to show that the terms of order κ^1 cancel. Indeed, the terms linear in κ in the l.h.s. of (5.8) are

$$-\delta_{\nu\alpha}\Gamma^{\dot\beta}_{\dot\mu\dot\alpha}+\delta_{\nu\beta}\Gamma^{\dot\beta}_{\dot\mu\dot\alpha}=(\delta_{\nu\alpha}\delta_{\mu\beta}-\delta_{\mu\alpha}\delta_{\nu\beta})\frac{e_{\dot\alpha}\eta_{\dot\beta}}{2\eta_{\dot\beta}},$$

while the terms linear in κ in the r.h.s. of (5.8) are

$$-\frac{\theta^{\dot\mu}-\theta^{\dot\nu}}{u^\mu-u^\nu-\theta^{\dot\mu}\theta^{\dot\nu}}(-\delta_{\nu\alpha}\delta_{\mu\beta}\mathcal{V}_{\dot\nu\dot\mu}+\delta_{\mu\alpha}\delta_{\nu\beta}\mathcal{V}_{\dot\mu\dot\nu})=(\delta_{\nu\alpha}\delta_{\mu\beta}-\delta_{\mu\alpha}\delta_{\nu\beta})\frac{e_{\dot\alpha}\eta_{\dot\beta}}{2\eta_{\dot\beta}}.$$

Analogously one checks that the terms linear in κ in (5.9) also vanish identically.

5.5. Strict special solutions of Schlesinger's equations. Consider the supermanifold $\mathcal{M}=\mathbb{C}^{n|n}$ with canonical coordinates $(u^\alpha,\theta^{\dot\alpha})$, an $(p|q)$-dimensional vector space T and a set of even holomorphic matrix functions $A_\nu:\mathcal{M}\to\mathrm{End}(T)$, $\nu=1,\dots,n$ such that the Schlesinger equations (5.4) are satisfied. In particular, summing (5.4) over ν and denoting $\mathcal{W}=\sum_\nu A_\nu$, we find $d\mathcal{W}=0$, i.e. $\mathcal{W}\in\mathrm{End}(T)$.

Theorem 5.4.3 motivates the following definition (cf. Section 2.4):

5.5.1. Definition. *A solution to Schlesinger's equation as above is called* strict special *if*

a). rank $T=0|n$*;*

b). T is endowed with a complex non-degenerate skew-symmetric form $h\in\Lambda^2(T)$;

c). $\mathcal{W}=-\mathcal{V}-\kappa\mathrm{Id}$, *where $\kappa\in\mathbb{C}$ is an arbitrary parameter and* $\mathcal{V}\in\mathrm{End}(T)$ *is an even operator symmetric with respect to h;*

d). for any ν,

$$A_\nu=-(\mathcal{V}+\kappa\mathrm{Id})P_\nu, \tag{5.10}$$

where $P_\nu:\mathcal{M}\to\mathrm{End}(T)$ *is a set of even holomorphic matrix functions whose values at any point of $\mathcal{M}$ constitute a complete system of orthogonal projectors of rank 1 with respect to h:*

$$P_\nu P_\mu=\delta_{\mu\nu}P_\nu,\quad \sum_\nu P_\nu=\mathrm{Id}_T,\quad h(\mathrm{Im}P_\nu,\mathrm{Im}P_\mu)=0 \tag{5.11}$$

if $\mu\neq\nu$.

e). There is a vector $\varepsilon\in T$ such that

$$\mathcal{V}(\varepsilon)=\frac{3-2D}{4}\varepsilon \tag{5.12}$$

for some $D\in\mathbb{C}$ and $e_{\dot\nu}:=P_\nu(\varepsilon)$ are nowhere vanishing on $\mathcal{M}$.

Theorem 5.4.3 says that to any semisimple Frobenius supermanifold there is canonically associated a strict special solution of Schlesinger's equations.

5.6. From strict special solutions to Frobenius supermanifolds. Let $(\mathcal{M},T,h,A_\nu,\varepsilon)$ be a strict special solution of Schlesinger's equations.

5.6.1. Theorem. *These data come from the unique structure of semisimple split Frobenius supermanifold on $\mathcal{M}$, with Euler field and flat identities ε and $\bar{\varepsilon}$.*

Proof. Put $e_{\dot\nu} = P_\nu(\varepsilon) \in T \otimes \mathcal{O}_{\mathcal{M}}$ and identify $\mathcal{O}_{\mathcal{M}} \otimes T$ with $\mathcal{T}_1 \subset \mathcal{T}\mathcal{M}$ by setting $e_{\dot\nu} = \partial/\partial\theta^{\dot\nu} + \theta^{\dot\nu}\partial/\partial u^{\dot\nu}$. This transfers h from T to $\mathcal{T}_1$. Denote $\eta_{\dot\alpha} = h(e_{\dot\alpha}, e_{\dot\alpha})$. Define the multiplication in $\mathcal{T}\mathcal{M}$ by the formulae 4.3.2.

To prove Theorem 5.6.1 it will suffice to show that 1) $\eta_{\dot\alpha} = e_{\dot\alpha}\Psi$ for an odd function Ψ (potentiality); 2) Ψ satisfies the Darboux-Egoroff equations (4.3) and (4.4) (flatness of the Egoroff metric); 3) the equation (4.12) is satisfied (Euler property); 4) the equations (4.9) and (4.10) are satisfied (flatness of $\bar\varepsilon$). That the condition $\sum_{\dot\alpha} \eta_{\dot\alpha}$=const (flatness of ε) is satisfied follows immediately from the definition of $\eta_{\dot\alpha}$ and the fact that $\sum_{\dot\alpha} \eta_{\dot\alpha} = h(\varepsilon, \varepsilon)$.

Step 1 (potentiality). Since

$$\sum_{\dot\alpha} e_{\dot\alpha} = \sum_{\alpha} P_\alpha(\varepsilon) = \mathrm{Id}(\varepsilon) = \varepsilon$$

and $h(\mathrm{Im}P_\alpha, \mathrm{Im}P_\beta) = 0$ if $\alpha \neq \beta$, we have $h_{\dot\alpha} = h(\varepsilon, e_{\dot\alpha})$ and hence, in view of 5.5.1(c,d),

$$h(\varepsilon, A_\nu(\varepsilon)) = -h(\varepsilon, (\mathcal{V} + \kappa \mathrm{Id}) e_{\dot\nu}) = h(\mathcal{V}(\varepsilon), e_{\dot\nu}) - \kappa h(\varepsilon, e_{\dot\nu}) = \frac{3 - 2D - 4\kappa}{4}\eta_{\dot\nu}. \tag{5.13}$$

Let ∇ be the unique flat connection in $\mathcal{T}_1 \simeq T \times \mathcal{M}$ which makes constants sections of $T \times \mathcal{M}$ horizontal. Obviously, $\tilde\nabla$ preserves h and satisfies $\nabla\varepsilon = 0$. Then derivating (5.13) we find, for every $\mu \neq \nu$,

$$\frac{3 - 2D - 4\kappa}{4} e_{\dot\mu}\eta_{\dot\nu} = h(\nabla_{e_{\dot\mu}}\varepsilon, A_\nu\varepsilon) - h(\varepsilon, \nabla_{e_{\dot\mu}}(A_\nu\varepsilon)) = h(\varepsilon, (e_{\dot\mu}A_\nu)\varepsilon). \tag{5.14}$$

Since, in view of (5.2),

$$e_{\dot\mu}A_\nu = -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}}[A_\mu, A_\nu] = -\, e_{\dot\nu}A_{\dot\mu} \quad \forall\ \mu \neq \nu,$$

we find

$$e_{\dot\mu}\eta_{\dot\nu} + e_{\dot\nu}\eta_{\dot\mu} = 0 \qquad \forall\ \mu \neq \nu,$$

or, equivalently,

$$e_{\dot\mu}\eta_{\dot\nu} + e_{\dot\nu}\eta_{\dot\mu} = 2\delta_{\dot\mu\dot\nu}\eta_\mu \qquad \forall\ \mu, \nu,$$

where $\eta_\mu := e_{\dot\mu}\eta_{\dot\mu}$. Analogous calculations, involving Schlesinger's equations (5.2) and (5.3), show that

$$e_{\dot\mu}\eta_\nu - \partial_\nu\eta_{\dot\mu} = 0, \qquad \partial_\mu\eta_\nu - \partial_\nu\eta_\mu = 0.$$

Finally, defining the 1-form $\omega = \sum_\alpha [d\theta^{\dot\alpha}(\eta_{\dot\alpha} - \theta^{\dot\alpha}\eta_\alpha) + du^\alpha\eta_\alpha]$, it is straightforward to check that the latter three equations are equivalent to $d\omega = 0$. Hence $\omega = d\Psi$ for some odd function Ψ, i.e. $\eta_{\dot\alpha} = e_{\dot\alpha}\Psi$.

Step 2 (flatness of the Egoroff metric). Let Ψ be as above and g the associated Egoroff metric. Let us prove that g is flat.

By Corollary 4.10.2, it will suffice to show that the flat connection ∇ coincides with the connection (4.15), i.e. that ∇ satisfies all three conditions of Proposition 4.10.4.

The condition 4.10.4(a) is obviously satisfied.

Since κ is arbitrary, we may assume without loss of generality that $\mathcal{W}$ is invertible and rewrite Schlesinger's equation (5.2) in the form

$$e_{\dot\mu} P_\nu = -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} \mathcal{W}^{-1}[A_\mu, A_\nu].$$

Then for every $\dot\mu \neq \dot\nu$ we have

$$\begin{aligned} \nabla_{e_{\dot\mu}} e_{\dot\nu} + \nabla_{e_{\dot\nu}} e_{\dot\mu} &= \nabla_{e_{\dot\mu}}(P_\nu \varepsilon) + \nabla_{e_{\dot\nu}}(P_\mu \varepsilon) \\ &= (e_{\dot\mu} P_\nu)\varepsilon + (e_{\dot\nu} P_\mu)\varepsilon \\ &= -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} \mathcal{W}^{-1}[A_\mu, A_\nu]\varepsilon - \frac{\theta^{\dot\nu} - \theta^{\dot\mu}}{u^\nu - u^\mu - \theta^{\dot\nu}\theta^{\dot\mu}} \mathcal{W}^{-1}[A_\nu, A_\mu]\varepsilon \\ &= 0, \end{aligned}$$

or, equivalently,

$$\nabla_{e_{\dot\mu}} e_{\dot\nu} + \nabla_{e_{\dot\nu}} e_{\dot\mu} = 2\delta_{\dot\mu\dot\nu} \nabla_{e_{\dot\mu}} e_{\dot\mu} \qquad \forall\ \mu, \nu.$$

Thus, it remains to check the condition 4.10.4(c) for all $\dot\mu \neq \dot\nu \neq \dot\alpha \neq \dot\mu$:

$$\begin{aligned} h(\nabla_{e_{\dot\mu}} e_{\dot\nu}, e_{\dot\alpha}) &= h((e_{\dot\mu} P_\nu)\varepsilon, P_\alpha \varepsilon) \\ &= -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} h(\mathcal{W}^{-1}[A_\mu, A_\nu]\varepsilon, P_\alpha \varepsilon) \\ &= -\frac{\theta^{\dot\mu} - \theta^{\dot\nu}}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} h(P_\mu(\dots) + P_\nu(\dots), P_\alpha \varepsilon) \\ &= 0. \end{aligned}$$

This establishes the flatness of the Egoroff metric.

Step 3 (Euler property). Schlesinger's equations (5.2)-(5.3) imply

$$\begin{aligned} EA_\nu &= \sum_{\mu:\mu\neq\nu} \left(u^\mu \partial_\mu A_\nu + \frac{1}{2}\theta^{\dot\mu} e_{\dot\mu} A_\nu \right) + u^\nu \partial_\nu A_\nu + \frac{1}{2}\theta^{\dot\nu} e_{\dot\nu} A_{\dot\nu} \\ &= \sum_{\mu:\mu\neq\nu} \left(-\frac{u^\mu [A_\mu, A_\nu]}{u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu}} - \frac{\theta^{\dot\mu}(\theta^{\dot\mu} - \theta^{\dot\nu})[A_\mu, A_\nu]}{2(u^\mu - u^\nu - \theta^{\dot\mu}\theta^{\dot\nu})} \right. \\ &\quad \left. + \frac{u^\nu [A_\nu, A_\nu]}{u^\nu - u^\mu - \theta^{\dot\nu}\theta^{\dot\mu}} + \frac{\theta^{\dot\nu}(\theta^{\dot\nu} - \theta^{\dot\mu})[A_\nu, A_\mu]}{2(u^\nu - u^\mu - \theta^{\dot\nu}\theta^{\dot\mu})} \right) \\ &= -\sum_{\mu:\mu\neq\nu} [A_\mu, A_\nu]. \end{aligned}$$

Then, using (5.13), we find

$$\begin{aligned}\frac{3-2D-4\kappa}{4}E\eta_{\dot\nu} &= h(\varepsilon,(EA_\nu)\varepsilon)\\ &= -h(\varepsilon,\sum_{\mu:\mu\neq\nu}[A_\mu,A_\nu]\varepsilon)\\ &= -h(\varepsilon,[\mathcal{V}+\kappa\mathrm{Id},(\mathcal{V}+\kappa\mathrm{Id})P_\nu]\varepsilon)\\ &= -\left(\frac{3-2D}{2}\right)\left(\frac{3-2D-4\kappa}{4}\right)\eta_{\dot\nu}.\end{aligned}$$

Hence $E\eta_{\dot\nu}=(D-\frac{3}{2})\eta_{\dot\nu}$.

Step 4 (flatness of $\bar\varepsilon$). One finds from (5.14):

$$\begin{aligned}\frac{3-2D-4\kappa}{4}(\theta^{\dot\mu}-\theta^{\dot\nu})e_{\dot\mu}\eta_{\dot\nu} &= (\theta^{\dot\mu}-\theta^{\dot\nu})h(\varepsilon,(e_{\dot\mu}A_\nu)\varepsilon)\\ &= -(\theta^{\dot\mu}-\theta^{\dot\nu})h(\varepsilon,\frac{\theta^{\dot\mu}-\theta^{\dot\nu}}{u^\mu-u^\nu-\theta^{\dot\mu}\theta^{\dot\nu}}[A_\mu,A_\nu]\varepsilon)\\ &= 0.\end{aligned}$$

Hence the equation (4.9) is satisfied. Analogously one checks that (4.10) is valid as well.

This completes the proof of Theorem 5.6.1.

Bibliography

[BM] K. Behrend, Yu. Manin. *Stacks of stable maps and Gromov–Witten invariants.* Duke Math. J., 85:1 (1996), 1–60.

[D] B. Dubrovin. *Geometry of 2D topological field theories.* In: Springer LNM, 1620 (1996), 120–348.

[H] N. Hitchin. *Frobenius manifolds (notes by D. Calderbank.)* Preprint, 1996.

[KM] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Comm. Math. Phys., 164:3 (1994), 525–562.

[Mal1] B. Malgrange. *Déformations de systèmes différentielles et microdifférentielles.* In: Séminaire de l'ENS 1979–1982, Progress in Math. 37, Birkhäuser, Boston (1983), 353–379.

[Mal2] B. Malgrange. *La clasification des connections irrégulieres à une variable.* ibid, 381–399.

[Mal3] B. Malgrange. *Sur les déformations isomonodromiques. I. Singularités régulières.* ibid, 401–426.

[Mal4] B. Malgrange. *Sur les déformations isomonodromiques. II. Singularités irrégulières.* ibid, 427–438.

[Ma1] Yu. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces (Chapters I, II, III).* Preprint MPI 96–113, 111 pp.

[Ma2] Yu. Manin. *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbf{P}^2$. Preprint MPI 96–114 and alg–geom/9605010.

[Ma3] Yu. Manin. *Gauge field theory and complex geometry.* Second Edition, Springer, 1997.

[Ma4] Yu. Manin. *Topics in Noncommutative geometry.* Princeton University Press, 1991.

[P] I. Penkov. *$\mathcal{D}$-modules on supermanifolds.* Inv. Math. 71 (1983), 501-512.

[S] C. Sabbah. *Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings.* Preprint, 1996.

[Sch] L. Schlesinger. *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischer Punkten.* J. für die reine und angew. Math., 141 (1912), 96–145.