QUASIDETERMINANTS

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Abstract

The determinant is a main organizing tool in commutative linear algebra. In this review we present a theory of the quasideterminants defined for matrices over a division algebra. We believe that the notion of quasideterminants should be a main organizing tool in noncommutative algebra giving them the same role determinants play in commutative algebra.

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Introduction

The universal notion of a determinant has a long history, both visible and invisible. The determinant was a main organizing tool in commutative linear algebra and we cannot accept the point of view of a modern textbook [FIS] that "determinants ... are of much less importance than they once were".

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Attempts to define a determinant for matrices with noncommutative entries started more than 150 years ago and include several great names. For many years the most famous examples of noncommutative objects were quaternions and block-matrices. It is not suprising that the first noncommutative determinants or similar notions were defined for such structures.

A. Cayley [C] was the first to define, in 1845, the determinant of a matrix with noncommutative entries. He mentioned that for a quaternionic matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the expressions $a_{11}a_{22} - a_{12}a_{21}$ and $a_{11}a_{22} - a_{21}a_{12}$ are different and suggested to choose one of them as a determinant of matrix A. The analog of this construction for 3×3-matrices was also proposed in [C] and later developed in [J]. This "naive" approach works now for quantum determinants and other cases. Different forms of quaternionic determinants were considered later by E. Study [St], E.H. Moore [Mo] and F. Dyson [Dy].

There were no direct "determinantal" attacks on block-matrices (excluding evident cases) but important insights were given by G. Frobenius [Fr] and I. Schur [Schur].

A theory of determinants of matrices with general noncommutative entries was in fact originated by J.H.M. Wedderburn in 1913. In [W] he constructed a theory of noncommutative continued fractions or, in modern terms, "determinants" of noncommutative Jacobi matrices.

In 1926-1928 A. Heyting [H] and A. Richardson [Ri, Ri1] suggested analogs of a determinant for matrices over division algebras. A. Heyting is known as a founder of intuitionist logic and A. Richardson as a creator of the Littlewood-Richardson rule. Heyting tried to construct a noncommutative projective geometry. As a computational tool, he introduced the "designant" of a noncommutative matrix. The designant of a 2×2-matrix $A = (a_{ij})$ is defined as $a_{11} - a_{12}a_{22}^{-1}a_{21}$. The designant of an $n \times n$-matrix is defined then by a complicated inductive procedure. The inductive procedures used by Richardson were even more complicated. It is important to mention that determinants of Heyting and Richardson in general are rational functions (and not polynomials!) in matrix entries.

The idea to have non-polynomial determinants was strongly criticized by O. Ore [O]. In [O] he defined a polynomail determinant for matrices over an imporatnt class of noncommutative rings which were later named Ore rings.

The most famous and widely used noncommutaive determinant is the Dieudonne determinant. It was defined for matrices over a division algebra R by J. Dieudonne in 1943 [D]. His idea was to consider determinants with values in $R^*/[R^*, R^*]$ where R^* is the monoid of invertible elements in R. The properties of Dieudonne determinants are close to those of commutative ones but, evidently, Dieudonne determinants cannot be used for solving systems of linear equations.

An interesting generalization of commutative determinants belongs to F. Berezin

[B, Le]. He defined determinants for matrices over so called super-commutative algebras. In particular, Berezin also understood that it is impossible to avoid rational functions in matrix entries in his definition.

Other famous examples of noncommutative determinants developed for different special cases are: quantum determinants [KS, Ma], Capelli determinants [We], determinants introduced by Cartier-Foata [CF, F] and Birman-Williams [BW], etc. As we explain later by introducing another universal notion of quasideterminants, these determinants and determinants of Dieudonne, Study, Moore, etc., are related to each other much more than one would expect.

A notion of quasideterminants for matrices over a noncommutative skew-field was introduced in [GR, GR1, GR2]. Quasideterminants are defined in the "most noncommutative case", namely, for matrices over free skew-fields. We believe that quasideterminants should be one of main organizing tools in noncommutative algebra giving them the same role determinants play in commutative algebra. Quasideterminants have been effective in many areas including noncommutative symmetric functions [GKLLRT], noncommutative integrable systems [RS, EGR, EGR1], quantum algebras [GR, GR1, GR2, KL, Mol, Mol1], and so on [P, Sch, RSh, RRV].

The main property of quasideterminants is a "heredity principle": let A be a square matrix over a skew-field and (A_{ij}) a block decomposition of A (into submatrices of A). Consider the A_{ij}'s as elements of a matrix X. Then the quasideterminant of the matrix X will be a matrix B, and (under natural assumptions) the quasideterminant of B is equal to a suitable quasideterminant of A. Since determinants of block-matrices are not defined, there is no analog of this principle for ordinary (commutative) determinants.

The quasideterminant is not an analog of the commutative determinant but rather of a ratio of the determinant of $n \times n$-matrices to the determinant of an $(n-1) \times (n-1)$-submatrix.

Many areas of noncommutative mathematics (Ore rings, rings of differential operators, theory of factors, "quantum mathematics", Clifford algebras, etc) were developed separately from each other. Our approach shows an advantage of working with totally noncommutative variables (over free rings and skew-fields). It leads us to a large variety of results, and their specialization to different noncommutative areas implies known theorems with additional information.

The price one pays for this is a huge number of inversions in rational noncommutative expressions. The minimal number of successive inversions required to express an element is called the height of this element. This invariant (inversion height) reflects the "degree of noncommutativity" and it is of a great interest by itself.

Our experience shows that in dealing with noncommutative objects one should not imitate the classical commutative mathematics, but follow "the way it is" starting with basics. In this paper we concentrate on two problems: noncommutative Plücker coordinates (as a background of a noncommutative geometry) and the noncommutative Bezout and Vieta theorems (as a background of noncommutative algebra). We apply the obtained results to the theory of noncommutative symmetric functions started in [GKLLRT].

We already said that the universal notion of a determinant has a long history, both visible and invisible. The visible history of determinants comes from the fact that they are constructed from another universal objects: matrices.

The invisible history of determinants is related with the Heredity principle for matrices: matrices can be viewed as matrices with matrix entries (block matrices) and some matrix properties come from the corresponding properties of block matrices. In some cases, when the matrix entries of the block-matrix commute, the determinant of a matrix can be computed in terms of the determinants of its blocks, but in general it is not possible: the determinant of a matrix with matrix entries is not defined because the entries do not commute. In other words, the determinant does not satisfy the Heredity principle.

Quasideterminants are defined for matrices over division algebras and satisfy the Heredity Principle. Their definition can be specialized for matrices over a ring (including noncommutative rings) and can be connected with different "famous" determinants. This reflects another general principle: in many cases noncommutative algebra can be made simpler and more natural than commutative algebra.

I. A general theory and main identities

1.1. The skew-field of rational functions of free variables. The free skew-field is a natural framework for working with noncommutative variables. The theory was developed by Amitsur, Bergman and P. M. Cohn (see, for example [Co]). We consider here skew-fields as division algebras over fields of characteristic 0, but everything can be generalized to fields of characteristics p. Here we will recall Bergman's presentation of Amitsur's approach. Given a set $X = \{x_1, \dots, x_n\}$ we write $\mathcal{F}(X)$ for the algebra freely generated by the elements of X over rationals $\mathbb{Q}$ using the operations of addition, subtraction, multiplication and taking the inverse.

No relations are imposed so far, so that, e.g., $(x-x)^{-1}$ exists. The algebra $\mathcal{F}(X)$ is called the algebra of rational formulas on X. Let R be a ring with unit. Any map $\alpha : X \to R$ defines a partial map $\overline{\alpha}$ of $\mathcal{F}(X)$ to R by the following rules:

i) $\overline{\alpha}(m) = m,\ m \in \mathbb{Z}$,
ii) $\overline{\alpha}(x_i) = \alpha(x_i), \quad i = 1, \dots, n$,
iii) if $a = -b$, or $b + c$, or bc and $\overline{\alpha}(b), \overline{\alpha}(c)$ are defined, then $\overline{\alpha}(a) = -\overline{\alpha}(b)$, or $\overline{\alpha}(b + c)$, or $\overline{\alpha}(b) \cdot \overline{\alpha}(c)$,
iv) if $a = b^{-1}$ and $\overline{\alpha}(b)$ is defined and invertible in R, then $\overline{\alpha}(a) = (\overline{\alpha}(b))^{-1}$.

If $\bar{\alpha}(a)$ is defined for $a \in \mathcal{F}(X)$ we say that α can be evaluated on a.

Now let $\alpha : X \to D$ be a map to a skew-field D. Then $\bar{\alpha}(a)$ is not defined if and only if a has a subformula b^{-1} such that $\bar{\alpha}(b) = 0$. For each $\alpha : X \to D$ one can consider a subset $E(\alpha)$ of $\mathcal{F}(X)$, the domain of $\overline{\alpha}$, consisting of the formulas on which α can be evaluated. Similarly with each $a \in \mathcal{F}(X)$ one can associate its domain dom a, a subset of D^n consisting of the points $(\alpha(x_i), i = 1, \dots, n)$ such that $a \in E(\alpha)$ is defined; a is called *nondegenerate* if dom $a \neq \emptyset$.

One can show [Co] that if D is a skew-field which is an algebra over an infinite field k and a and b are nondegenerate, then d*om* $a \cap$ d*om* $b \neq \emptyset$.

Given $a, b \in \mathcal{F}(X)$, we write $a \sim b$ if a, b are nondegenerate and a, b have the same value at each point of dom $a \cap$ dom b. This is an equivalence relation. The equivalence classes are called the rational functions in $x_1, \dots, x_n$.

Theorem 1.1.1. [Co]
i) Let D be a skew-field with the center k of characteristic 0. *Then the equivalence classes of rational formulas of variables $x_1, \dots, x_n$ form a skew-field $F_D(X)$.*
ii) If D is infinite dimensional over k then $F_D(X)$ does not depend on the skew-field D.

Under the assumptions of i) and ii) we will identify all $F_D(X)$ and use the notation $F(X)$. For example, if $X = \{x\}$ then $F(X) = \mathbb{Q}(x)$. The skew-field $F(X)$ is called *the free skew-field* generated by X.

Remark. P.M. Cohn has shown ([Co] that the free skew-field is universal in the following sense. Consider the algebra $k\langle x_1, \dots, x_n\rangle$ of noncommutative polynomials over the commutative field k. The algebra $k\langle x_1, \dots, x_n\rangle$ is naturally embedded into the algebra $F(X)$. The universality means that if D is any division ring and

$$\alpha : k\langle x_1, \dots, x_n\rangle \to D$$

is a homomorphism then there is a subring R of $F(X)$ containing $k\langle x_1, \dots, x_n\rangle$ and an extension of α to a homomorphism

$$\beta : R \to D$$

such that if $a \in R$ and $\beta(a) \neq 0$ then $a^{-1} \in R$.

We recall the following definition (see, for example, [Re]).

Definition 1.1.2. The inversion height of a rational expression is the maximal number of nested inversions in it. The inversion height of an element in the free skew-field $F(X)$ is the smallest inversion height of its rational expressions.

Example. The inversion height of a polynomial in $x_1, \dots, x_n$ equals zero. The inversion height of the ratio of two polynomials PQ^{-1} equals 0 if P is divisible by Q and 1 otherwise. The inversion height of the rational *expressions* $r_1 = (1-x)^{-1} + (1-x^{-1})^{-1}$ and $r_2 = x^{-1} + x^{-1}(z^{-1}y^{-1} - x^{-1})^{-1}x^{-1}$ equals 2. However, $r_1 = 1$ and $r = (x - yz)^{-1}$ in the free skew-field generated by x, y and z. So, the height of r_1, r_2 as rational *functions* equals to 0 and 1 correspondigly. The rational function given by the expression $(x - yw^{-1}z)^{-1}$ has height 2.

1.2. Definition of quasideterminants. Let $A = (a_{ij}), i \in I, j \in J$ be a matrix over a ring R. Construct the following submatrices of A:

submatrix A^{ij}, $i \in I$, $j \in J$ obtained from A by deleting its i-th row and j-th column,

row submatrix r_i^j obtained from i-th row of A by deleting the element a_{ij},

column submatrix c_j^i obtained from j-th column of A by deleting the element a_{ij}.

Definition 1.2.1. *Let I, J be finite sets with the same number of elements. We say that the quasideterminant $|A|_{ij}$ is defined if the submatrix A^{ij} is invertible over the ring R. In this case*

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i. \tag{1.2.1}$$

It is sometimes convenient to adopt a more explicit notation boxing the element a_{ij}. For $A = (a_{pq})$, $p, q = 1, \dots, n$

$$|A|_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ & \dots & & \dots & \\ a_{i1} & \dots & \boxed{a_{ij}} & \dots & a_{in} \\ & \dots & & \dots & \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

We set $|A|_{ij} = a_{ij}$ if $I = \{i\}$, $J = \{j\}$.

Remark. The definition does not require any orderings of I and J or a bijective correspondence between I and J.

If all quasideterminants $|A^{ij}|_{pq}$ for $p \in I \setminus \{i\}, q \in J \setminus \{j\}$. are defined and invertible, then

$$|A|_{ij} = a_{ij} - \sum a_{iq} |A^{ij}|_{pq}^{-1} a_{pj}. \tag{1.2.2}$$

Here the sum is taken over all $p \in I \setminus \{i\}, q \in J \setminus \{j\}$.

If A is a generic $n \times n$-matrix (in the sense that all square submatrices of A are invertible), then there exist n^2 quasideterminants of A. However, a non-generic matrix may have k quaisdeterminants where $0 \leq k \leq n^2$. A square matrix $A = (a_{ij})$, $i \in I, j \in J$ is a generic matrix over the free skew-field generated by its entries.

Examples.

1) For a 2×2-matrix $A=(a_{ij}), i,j=1,2$ there are up to four quasideterminants

$$|A|_{11}=a_{11}-a_{12}\cdot a_{22}^{-1}\cdot a_{21},\qquad |A|_{12}=a_{12}-a_{11}\cdot a_{21}^{-1}\cdot a_{22},$$
$$|A|_{21}=a_{21}-a_{22}\cdot a_{12}^{-1}\cdot a_{11},\qquad |A|_{22}=a_{22}-a_{21}\cdot a_{11}^{-1}\cdot a_{12}.$$

Each is defined if the corresponding entry is invertible in R.

2) For a matrix $A=(a_{ij}), i,j=1,2,3$ there are 9 quasideterminants. We will write here only one

$$|A|_{11}=a_{11}-a_{12}(a_{22}-a_{23}a_{33}^{-1}a_{32})^{-1}a_{21}-a_{12}(a_{32}-a_{33}\cdot a_{23}^{-1}a_{22})^{-1}a_{31}-$$

$$-a_{13}(a_{23}-a_{22}a_{32}^{-1}a_{33})^{-1}a_{21}-a_{13}(a_{33}-a_{32}\cdot a_{22}^{-1}a_{23})^{-1}a_{31}$$

provided all inverses exist in R.

Remark. The examples show that if each a_{ij} is an invertible morphism $V_j\to V_i$ in an additive category then the quasideterminant $|A|_{pq}$ can be also viewed as a morphism from the object V_q to the object V_p if the matrix of morphisms A^{pq} is invertible.

Quasideterminant is not a generalization of a determinant over a commutative ring but a generalization of a ratio of two determinants.

Example. If in formulas (1.2.1), or (1.2.2) the variables a_{ij} commute then

$$|A|_{pq}=(-1)^{p+q}\frac{\det A}{\det A^{pq}}.$$

We will show in section III that similar expressions for quasideterminants can be given for quantum matrices, quaternionic matrices, Capelli matrices and other cases listed in the Introduction.

In general quasideterminants are not polynomials in their entries, but rational functions. The following theorem was conjectured by I. Gelfand and V. Retakh, and proved by Reutenauer [Re] in slightly different form.

Theorem 1.2.2. *Quasideterminants of the $n\times n$-matrix with formal entries have the inversion height $n-1$ over the free skew-field generated by its entries.*

In the commutative case determinants are finite sums of monomials with appropriate coefficients. In noncommutative case quasideterminants are formal power series with matrix entries or their inverses as indeterminants. A simple example of this type is described below.

Let $A=(a_{ij})$, $i,j=1,\dots,n$ be a matrix with formal entries. Denote by I_n the unit $n\times n$-matrix and by Γ_n the complete oriented graph with vertices $\{1,2,\dots,n\}$, the arrow from i to j is labelled by a_{ij}. A path $p: i\to k_1\to k_2\to\dots\to k_t\to j$ is labelled by the word $w=a_{ik_1}a_{k_1k_2}a_{k_2k_3}\dots a_{k_tj}$.

Denote by P_{ij} the set of words labelling paths going from i to j, i.e. the set of words of the form $w=a_{ik_1}a_{k_1k_2}a_{k_2k_3}\dots a_{k_tj}$. A simple path is a path p such that $k_s\neq i,j$ for every s. Denote by P'_{ij} the set of words labelling simple paths from i to j.

Proposition 1.2.3. *Let i,j be two distinct integers between 1 and n. In the ring of formal series in variables a_{ij}'s*

$$|I_n-A|_{ii}=1-\sum_{w\in P'_{ii}} w,\quad |I_n-A|_{ij}^{-1}=\sum_{w\in P_{ii}} w.$$

Example. For $n=2$,

$$|I_2-A|_{11}=1-a_{11}-\sum_{p\geq 0} a_{12}a_{22}^p a_{21}.$$

However, in some important special cases, quasideterminants are polynomials in their entries.

Proposition 1.2.4. *The following quasideterminant is a polynomial in its entries:*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ -1 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & -1 & a_{33} & \dots & a_{3n} \\ & & \dots & & \\ 0 & & \dots & -1 & a_{nn} \end{vmatrix}_{1n} =$$

$$=a_{1n}+\sum_{1\leq j_1<j_2<\dots<j_k<n} a_{1j_1}a_{j_1+1j_2}a_{j_2+1j_3}\dots a_{j_k+1n}.$$

1.3. Transformation properties of quasideterminants.

Let $A=(a_{ij})$ be a square matrix.

i) The quasideterminant $|A|_{pq}$ does not depend on the permutation of rows and columns in the matrix A if the *p-th* row and the *q-th* column are not changed;

ii) *The multiplication of rows and columns.* Let the matrix $B = (b_{ij}$ be constructed from the matrix A by multiplication of its i-th row by a scalar λ from the left, i.e. $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for $k \neq i$. Then

$$|B|_{kj} = \begin{cases} \lambda|A|_{ij} & \text{if } k = i \\ |A|_{kj} & \text{if } k \neq i \text{ and } \lambda \text{ is invertible.} \end{cases}$$

Let the matrix $C = c_{ij}$ be constructed from the matrix A by multiplication of its j-th column by a scalar μ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all i and $l \neq i$. Then

$$|C|_{i\ell} = \begin{cases} |A|_{ij}\mu & \text{if } \ell = j \\ |A|_{i\ell} & \text{if } \ell \neq j \text{ and } \mu \text{ is invertible.} \end{cases}$$

iii) *The addition of rows and columns.* Let the matrix B be constructed by adding to some row of the matrix A its k-th row multiplied by a scalar λ from the left. Then

$$|A|_{ij} = |B|_{ij}, \quad i = 1, \dots k-1, k+1, \dots n, j = 1, \dots, n.$$

Let the matrix C be constructed by addition to some column of the matrix A its ℓ-th column multiplied by a scalar λ from the right. Then

$$|A|_{ij} = |C|_{ij},\ i = 1, \dots, n, j = 1, \dots, \ell - 1, \ell + 1, \dots n.$$

1.4 General properties of quasideterminants

Quasideterminants and inverse matrices. [GR, GR1, GR2]

Theorem 1.4.1. *If $A = (a_{ij})$ is a generic matrix and $B = (b_{ij})$ equals to A^{-1} then $b_{ji} = |A|_{ij}^{-1}$ for all i, j.*

Two involutions. [GR4] For a square matrix $A = (a_{ij})$ denote by $IA = A^{-1}$ inverse matrix, and by $HA = (a_{ji}^{-1})$ the Hadamard inverse matrix. It is evident that if IA is defined, then $I^2A = A$, and if HA is defined, then $H^2A = A$.

Let $A^{-1} = (b_{ij})$. According to Theorem 1.4.1 $b_{ij} = |A|_{ji}^{-1}$. This formula can be written in the following form.

Theorem 1.4.2. *For a square matrix A with formal entries*

$$HI(A) = (|A|_{ij}) \tag{1.4.1}$$

Homological relations. [GR, GR1, GR2] The ratio of two quasideterminants of the same square matrix is the ratio of two rational functions of a smaller height. It follows from the following *homological* relations:

Theorem 1.4.3.
a) Row homological relations:

$$-|A|_{ij} \cdot |A^{i\ell}|_{sj}^{-1} = |A|_{i\ell} \cdot |A^{ij}|_{s\ell}^{-1} \qquad \forall s \neq i$$

b) Column homological relations:

$$-|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj} \qquad \forall r \neq j$$

Heredity (special case). Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a decomposition of $A = (a_{ij}), i,j = 1,\dots,n$, into a block-matrix. Let A_{11} be a $k \times k$-matrix and let the matrix A_{22} be invertible.

Theorem 1.4.4. $|A|_{ij} = |A_{11} - A_{12}A_{22}^{-1} \cdot A_{21}|_{ij}$ *for* $i,j = 1.\dots,k$. *In other words, the quasideterminant* $|A|_{ij}$ *of an* $n \times n$ *matrix can be computed in two steps: first, consider the quasideterminant* $|\tilde{A}|_{11} = A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21}$ *of a* 2×2*-matrix* $\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$; *and second, consider the corresponding quasideterminant of the* $k \times k$*-matrix* $|\tilde{A}|_{11}$.

Heredity (general case). Let $A = (a_{ij}), i,j = 1,\dots,n$, be a matrix and

$$A = \begin{pmatrix} A_{11} & \dots & A_{1s} \\ & & \\ A_{s1} & \dots & A_{ss} \end{pmatrix}$$

be its block decomposition. Denote by $\tilde{A} = (A_{ij}), i,j = 1,\dots,s$, the matrix with A_{ij}'s as entries. Suppose that

$$A_{pq} = \begin{pmatrix} a_{k\ell} & \dots & a_{k,\ell+m} \\ & \dots & \\ a_{k+m,\ell} & \dots & a_{k+m,\ell+m} \end{pmatrix}.$$

is a square matrix and that $|\tilde{A}|_{pq}$ is defined.

Theorem 1.4.5.

$$|A|_{k'\ell'} = \left||\tilde{A}|_{pq}\right|_{k'\ell'} \text{ for } k' = k,\dots,k+m,\ \ell' = \ell,\dots,\ell+m.$$

Example.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Set $A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}$, $A_{21} = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$, $A_{22} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$.

Set $\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Then

$$|\tilde{A}|_{12} = A_{12} - A_{11}A_{21}^{-1}A_{22} = \begin{pmatrix} a_{13} - \dots & a_{14} - \dots \\ a_{23} - \dots & a_{24} - \dots \end{pmatrix}$$

and

$$|A|_{13} = \begin{vmatrix} a_{13} - \dots & a_{14} - \dots \\ a_{23} - \dots & a_{24} - \dots \end{vmatrix}_{13}.$$

Quasideterminants and Kronecker tensor products.

Let $A = (a_{ij})$, $B = (b_{\alpha\beta})$ be matrices over a ring. Denote by $A \otimes B$ the Kronecker tensor product. The entries of matrix $A \otimes B$ are indexed by pairs $(ij, \alpha\beta)$.

Proposition 1.4.6. *If quasideterminants $|A|_{ij}$ and $|B|_{\alpha\beta}$ are defined then the quasideterminant $|A \otimes B|_{ij,\alpha\beta}$ is defined and*

$$|A \otimes B|_{ij,\alpha\beta} = |A|_{ij}|B|_{\alpha\beta}.$$

Note that in commutative case the corresponding identity looks differently: if A is a $m \times m$-matrix and B is a $n \times n$-matrix over a commutative ring then $\det(A \otimes B) = (\det A)^n (\det B)^m$.

Quasideterminants and matrix rank. Let $A = (a_{ij})$ be a matrix over a division algebra.

Proposition 1.4.7. *If the quasideterminant $|A|_{ij}$ be defined then the following statements are equivalent.*

i) $|A|_{ij} = 0$,
ii) *the i-th row of the matrix A is a left linear combination of A;*
iii) *the j-th column of the matrix A is a right linear combination of the other columns of A.*

Example. Let $i, j = 1, 2$ and $|A|_{11} = 0$. It means that $a_{11} - a_{12}a_{22}^{-1}a_{21} = 0$. It follows that $a_{11} = \lambda a_{21}$, where $\lambda = a_{12}a_{22}^{-1}$. Since $a_{12} = (a_{12}a_{22}^{-1})a_{22}$, the first row of A is proportional to the second row.

There exist notion of linear dependence of rows and of columns for matrices over a skew-field if rows are multiplied by the elements of this skew-field from the left and columns from the right. So there exist notions of the row rank and the column rank and these ranks are equal [Ja, Co]. It follows also from Proposition 1.4.7.

By definition, an r-quasiminor of a square matrix A is a quasideterminant of an $r \times r$-submatrix of A.

Proposition 1.4.8. *The rank of the matrix A over a skew field is greater or equal to r if and only if at least one of the r-quasiminors of the matrix A is defined and is not equal to zero.*

1.5 Basic identities

Row and columns decomposition. As it was pointed out in [KL], homological relations lead immediately to the following analogue of the classical expansion of a determinant by a row or a column.

Proposition 1.5.1. *For any $k \neq p$ and any $\ell \neq q$*

$$|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj}(|A^{pq}|_{kj})^{-1}|A^{pj}|_{kq}$$

$$|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pi}(|A^{pq}|_{i\ell})^{-1}a_{iq}$$

if all terms in these expressions are defined.

Sylvester identity. The following noncommutative analogue of Sylvester's identity stated in [GR, GR1, GR2] is closely related with the heredity property. Let $A = (A_{ij}), i, j = 1, \ldots, n$ be a matrix over a noncommutative skew-field. Suppose that the submatrix $A_0 = (a_{ij}), i, j = 1, \ldots, k$, is invertible. For $p, q = k+1, \ldots, n$ set

$$b_{pq} = \begin{vmatrix} & A_0 & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ a_{p1} & \ldots & a_{pk} & a_{pq} \end{vmatrix}_{pq};$$

$$B = (b_{pq}), p, q = k+1, \ldots, n.$$

We call the submatrix A_0 a *pivot* for matrix B.

Theorem 1.5.2. *For $i, j = k+1, \dots, n$,*

$$|A|_{ij} = |B|_{ij}$$

Example. A quasideterminant of an $n \times n$-matrix A is equal to the corresponding quasideterminant of a 2×2-matrix consisting of $(n-1) \times (n-1)$- quasiminors of the matrix A, or to the quasideterminant of an $(n-1) \times (n-1)$-matrix consisting of 2×2-quasiminors of the matrix A. One can use any of these procedures for an inductive definition of quasideterminants. In fact, A. Heyting [H] essentially defined inductively quasideterminants $|A|_{nn}$ for matrices $A = (a)_{ij}$, $i, j = 1, \dots, n$ starting with $n = 2$.

Theorem 1.5.3. (Sylvester)*: Let $A = (a_{ij}), i, j = 1, \dots, n$, be a matrix over a commutative ring. Suppose that its submatrix $A_0 = (a_{ij}), i, j = 1, \dots, k$, is invertible. For $p, q = k+1, \dots, n$, set*

$$\tilde{b}_{pq} = \det \begin{pmatrix} & A_0 & a_{1q} \\ & & \vdots \\ & & a_{kq} \\ & a_{p1} \dots a_{pk} & a_{pq} \end{pmatrix},$$

$$\tilde{B} = (\tilde{b}_{pq}), \ \ p, q = k+1, \dots, n.$$

Then

$$\det A = \frac{\det \tilde{B}}{(\det A_0)^{n-k-1}}.$$

Remark. In the special case when $A_0 = (a_{ij}), i, j = 2, \dots, n-1$, Theorem 1.3.1 is an analogue of a well-known commutative identity which is called the "Lewis Carroll identity" (see, for example, [Ho]).

Inversion for quasiminors. The following theorem was stated in [GR, GR1, GR2]. For a matrix $A = (a_{ij}), i \in I, j \in J, P \subset I, Q \subset J$, denote by A_{PQ} the submatrix

$$A_{PQ} = (a_{\alpha\beta}), \alpha \in P, \beta \in Q.$$

Let $|I| = |J|$ and $B = A^{-1} = (b_{rs})$. Suppose that $|P| = |Q|$.

Theorem 1.5.4. *Let $k \notin P, \ell \notin Q$. Then*

$$|A_{P \cup \{k\}, Q \cup \{\ell\}}|_{k\ell} \cdot |B_{I \setminus P, J \setminus Q}|_{\ell k} = 1.$$

Set $P = I \setminus \{k\}, Q = J \setminus \{\ell\}$. Then this theorem leads to the already mentioned identity

$$|A|_{k\ell} \cdot b_{\ell k} = 1.$$

Example. Theorem 1.5.4 implies the following identity for main quasiminors. Let $A = (a_{ij})$, $i, j = 1, \dots, n$ be an ivertible matrix. Set $A_{(k)} = (a_{ij})$, $i, j = 1, \dots, k$. Set $B = (b_{ij}) = A^{-1}$, $B^{(k)} = (b_{ij})$, $i, j = k, \dots, n$. Then

$$|A_{(k)}|_{kk} \cdot |B^{(k)}|_{kk} = 1.$$

A generalization of the homological relations. For a matrix $A = (a_{ij}), i \in I, j \in J$, and two subsets $L \subset I, M \subset J$ denote by $A^{L,M}$ the submatrix of the matrix A obtained from A by deleting the rows with the indexes $\ell \in L$ and the columns with the indexes $m \in M$.

Let A be a square matrix, $L = (\ell_1, \dots, \ell_k), M = (m_0, \dots, m_k)$. Set $M_i = M \setminus \{m_i\}, i = 0, \dots, k$.

Theorem 1.5.5. [GR1, GR2] *For* $p \notin L$

$$\sum_{i=0}^{k} |A^{L,M_i}|_{pm_i} \cdot |A|^{-1}_{\ell m_i} = \delta_{p\ell},$$

$$\sum_{i=0}^{k} |A|^{-1}_{m_i \ell} \cdot |A^{M_i,L}|_{m_i p} = \delta_{\ell p}.$$

Multiplicative properties of quasideterminants. Let $X = (x_{pq}), Y = (y_{rs})$ be $n \times n$-matrices.

Theorem 1.5.6.

$$|XY|^{-1}_{ij} = \sum_{p=1}^{n} |Y|^{-1}_{pj} |X|^{-1}_{pi}.$$

1.6 Noncommutative linear algebra

In this section we use quasideterminants to formulate basic theorems in noncommutative linear algebra [GR, GR1, GR2].

Solutions of systems of linear equations.

Theorem 1.6.1. *If the system*

$$\begin{cases} a_{11}x_1 + \dots \quad + a_{1n}x_n = \xi_1 \\ \dots \\ a_{n1}x_1 + \dots \quad + a_{nn}x_n = \xi_n \end{cases}$$

is given and for the matrix $A = (a_{ij})$ *the quasideterminants of* A *are defined and invertible, then*

$$x_i = \sum_{j=1}^{n} |A|^{-1}_{ji} \xi_j.$$

for $i = 1, \dots, n$.

Cramer's rule. Let $A_\ell(\xi)$ be an $n\times n$-matrix constructed by replacing ℓ-th column of the matrix A by the column $(\xi_1,\dots,\xi_n)$.

Theorem 1.6.2. *In notations of Theorem 1.4.1, if the quasideterminants $|A|_{ij}$ and $|A_j(\xi)|_{ij}$ are defined, then*

$$|A|_{ij}x_j = |A_j(\xi)|_{ij}.$$

Cayley-Hamilton theorem. Let t be a formal variable and $A = (a_{ij}), i,j = 1,\dots,n$ a matrix over a skew field. We do not assume that t commutes with variables $a_{ij}, i,j = 1,\dots,n$. Let I_n be $n\times n$-identity matrix, $\tilde{A}_{ij} = a_{ij}I_n$, and $\tilde{A} = (\tilde{A}_{ij})$ the block matrix. Let $T = (\delta_{ij}t)$ also be the block matrix, and $f_{ij}(t) = |T - \tilde{A}|_{ij}$. Consider the expression $\tilde{f}_{ij}(t)$ as a matrix function of a matrix variable changing in $f_{ij}(t)$ elements a_{ij} by matrices $\tilde{a}_{ij} = a_{ij}\cdot I_n$. For example, if $n = 2$, then $\tilde{f}_{11}(t) = t - \tilde{A}_{11} - \tilde{A}_{12}(t-\tilde{A}_{22})^{-1}\tilde{A}_{21}$.

The following theorem was stated in [GR1], [GR2]. An independent proof (similar to the original one) was given by C. Reutenauer in his Montreal lectures.

Theorem 1.6.3. *For all $i,j = 1,\dots,n$*

$$f_{ij}(A) = 0.$$

II. Important example: Quaternionic quasideterminants

As an example we compute here quasideterminants of quaternionic matrices.

2.1 Norms of quaternionic matrices.

Let $\mathbb{H}$ be the algebra of quaternions. For $a\in\mathbb{H}$, $a = x+y\mathbf{i}+z\mathbf{j}+t\mathbf{k}$, $x,y,z,t\in\mathbb{R}$, we denote by $\bar{a} = x-y\mathbf{i}-z\mathbf{j}-t\mathbf{k}$, the conjugate of a, and by $\nu(a)$ the norm, $\nu(a) = a\bar{a}$.

Let $M(n,\mathbb{H})$ be the $\mathbb{R}$-algebra of quaternionic matrices of order n. There exists a multiplicative functional $\nu : M(n,\mathbb{H}) \to \mathbb{R}_{\geq 0}$ such that

i) $\nu(A) = \nu(a)$ if $A\in M(1,\mathbb{H})$, $A = (a)$,

ii) $\nu(A) = 0$ if and only if the matrix A is non-invertible,

iii) If A' is obtained from A by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $\nu(A') = \nu(A)$.

We will show later that for a generic quaternionic matrix $A = (a_{ij})$, $i,j = 1,\dots,n$, a norm can be defined by induction on n by formula $\nu(A) = \nu(|A|_{11})\nu(A^{11})$ and that $\nu(A)$ is a polynomial in a_{ij}'s, $\bar{a}_{ij}$'s.

2.2. Quasideterminants of quaternionic matrices.

This section contains results from [GRW1].

Let $A = (a_{ij})$, $i,j = 1,\dots,n$, be a quaternionic matrix. Let I and J be two ordered sets of natural numbers $1\leq i_1,i_2,\dots,i_k\leq n$ and $1\leq j_1,j_2,\dots,j_k\leq n$ such that the i_p's are distinct and the j_p's are distinct. For $k = 1$ set $m_{I,J}(A) = a_{i_1 j_1}$. For $k \geq 2$ set

$$m_{I,J}(A) = a_{i_1 j_2} \bar{a}_{i_2 j_2} a_{i_2 j_3} \bar{a}_{i_3 j_3} a_{i_3 j_4} \dots \bar{a}_{i_k j_k} a_{i_k j_1}.$$

If the matrix A is Hermitian, i.e. $a_{ji} = \bar{a}_{ij}$ for all i, j then

$$m_{I,J}(A) = a_{i_1 j_2} a_{j_2 i_2} a_{i_2 j_3} a_{j_3 i_3} a_{i_3 j_4} \dots a_{j_k i_k} a_{i_k j_1}.$$

Remark. The monomials $m_{I,J}(A)$ are conviniently described in terms of paths in a graph. Let Γ_n be the complete oriented n-graph with vertices $1, \dots, n$, the arrows from p to q being labelled by a_{pq} and $\bar{a}_{qp}$. The monomial $m_{I,J}(A)$ corresponds then to a path from i_1 to j_1.

To a quaternionic matrix $A = (a_{pq})$, $p, q = 1, \dots, n$, and to a fixed row index i and a column index j we associate a polynomial in a_{pq}'s, $\bar{a}_{pq}$'s which we call the (i, j)-th double permanent of A.

Definition 2.2.1. *The (i, j)-th double permanent of A is the sum*

$$\pi_{ij}(A) = \sum m_{I,J}(A),$$

taken over all orderings $I = \{i_1, \dots, i_n\}$, $J = \{j_1, \dots, j_n\}$ of $\{1, \dots, n\}$ such that $i_1 = i$ and $j_1 = j$.

Example. For $n = 2$

$$\pi_{11}(A) = a_{12} \bar{a}_{22} a_{21}.$$

For $n = 3$

$$\pi_{11}(A) = a_{12}\bar{a}_{32}a_{33}\bar{a}_{23}a_{21} + a_{12}\bar{a}_{22}a_{23}\bar{a}_{33}a_{31} + a_{13}\bar{a}_{33}a_{32}\bar{a}_{22}a_{21} + a_{13}\bar{a}_{23}a_{22}\bar{a}_{32}a_{31}.$$

For any submatrix B of A denote by B^c its complementary submatrix. The matrix B^c is obtained from A by omitting all rows and columns containing elements from B. If B is a $k \times k$ matrix then B^c is a $(n-k) \times (n-k)$ matrix.

Theorem 2.2.2. *If the quasideterminant $|A|_{ij}$ of a quaternionic matrix is defined then*

$$\nu(A^{ij})|A|_{ij} = \sum (-1)^{k(B)-1} \nu(B^c) \pi_{ij}(B) \tag{2.2.1}$$

where the sum is taken of all square submatrices B of A containing a_{ij} and $k(B)$ is the order of B.

Recall that the quasideterminant $|A|_{ij}$ is defined if the matrix A^{ij} is invertible. In this case $\nu(A^{ij})$ is invertible and formula (2.2.1) gives an expression for $|A|_{ij}$.

Note that both sides in formula (2.2.1) are polynomials in the a_{ij}'s, the $\bar{a}_{ij}$'s with real coefficients. The right-hand side in (2.2.1) is a linear combination of monomials of lengths $1, 3, \dots, 2n-1$. Let $\mu(n)$ be the number of such monomials for a matrix of order n. Then $\mu(n) = 1 + (n-1)^2 \mu(n-1)$.

Example. For $n = 2$

$$\nu(a_{22})|A|_{11} = \nu(a_{22})a_{11} - a_{12}\bar{a}_{22}a_{21}.$$

For $n = 3$

$$\begin{aligned}\nu(A^{11})|A|_{11} = \nu(A^{11})a_{11} - \nu(a_{33})a_{12}\bar{a}_{22}a_{21} - \nu(a_{23})a_{12}\bar{a}_{32}a_{31}-\\ -\nu(a_{32})a_{13}\bar{a}_{23}a_{21} - \nu(a_{22})a_{13}\bar{a}_{33}a_{31} + a_{12}\bar{a}_{32}a_{33}\bar{a}_{23}a_{21}+\\ +a_{12}\bar{a}_{22}a_{23}\bar{a}_{33}a_{31} + a_{13}\bar{a}_{33}a_{32}\bar{a}_{22}a_{21} + a_{13}\bar{a}_{23}a_{22}\bar{a}_{32}a_{31}.\end{aligned}$$

The example shows how to simplify the general formula for quasideterminants of matrix of order 3 (see subsection 1.2) for quaternionic matrices.

Formula (2.2.1) gives a recurrence relation for norms of quaternionic matrices. From this relation one can obtain an expression of $\nu(A)$ as a polynomial in a_{ij}'s and their conjugates (see also subsection 3.4 below).

Corollary 2.2.3.

$$\nu(A)\nu(A^{ij}) = \sum(-1)^{K(B)-1}\nu(B^c)\pi_{ij}(B))(\sum(-1)^{K(B)-1}\nu(B^c)\overline{\pi_{ij}(B)}).$$

Example. For $n = 2$

$$\nu(A) = \nu(a_{11})\nu(a_{22}) + \nu(a_{12})\nu(a_{21}) - a_{12}\bar{a}_{22}a_{21}\bar{a}_{11} - a_{11}\bar{a}_{21}a_{22}\bar{a}_{12}.$$

For a square quaternionic matrix $A = (a_{ij})$ set $Q_{ij}(A) = |A|_{ij}\nu(A^{ij})$. Then one can rewrite formula (2.2.1) as

$$Q_{ij}(A) = \nu(A^{ij})a_{ij} - \sum_{p\neq i, q\neq j} a_{iq}\overline{Q_{pq}(A^{ij})}a_{pj}.$$

If A is a Hermitian matrix then $\bar{Q}_{pq}(A^{ij}) = Q_{qp}(A^{ij})$ and

$$Q_{ij}(A) = \nu(A^{ij})a_{ij} - \sum_{p\neq i, q\neq j} a_{iq}Q_{qp}(A^{ij})a_{pj}.$$

When A is a complex matrix, formula (2.2.1) can be written as

$$\det A \det \bar{A}^{ij} = (-1)^{i+j}(a_{ij}\det \bar{A}^{ij} - \sum_{p\neq i, q\neq j} a_{iq}\det \bar{A}^{ij}\det A^{ip,jq}a_{pj}).$$

This implies the standard row/column expansion of the determinant.

III. Noncommutative determinants

In this section we describe some results from [GR, GR1, GR2, GRW1].

3.1. Noncommutative determinants as products of quasiminors.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix over a division algebra R such that all square submatrices of A are invertible. For any orderings $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ define $A^{i_1 \dots i_k, j_1 \dots j_k}$ to be submatrices of A obtained by deleting rows with indices $i_1, \dots, i_k$ and columns with indices $j_1, \dots, j_k$. Then set

$$D_{I,J}(A) = |A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2 j_2} |A^{i_1 i_2, j_1 j_2}|_{i_3 j_3} \dots a_{i_n j_n}.$$

In the commutative case all $D_{I,J}(A)$ are equal up to a sign to the determinant of A. When A is a quantum matrix all $D_{I,J}(A)$ are equal up to a scalar to the quantum determinant of A [GR, GR1, KL]. The same is true for some other well-known noncommutative algebras. This gives us a reason to call the $D_{I,J}(A)$ the (I, J)-predeterminants of A. From a "categorical point of view" the expressions $D_{I,I'}$ when $I' = (i_2, i_3, \dots, i_n, i_1)$ are particularly important. Set $D_I = D_{I,I'}$. There are $n!$ expressions D_I. It is convenient to have a basic predeterminant $\Delta = D_{\{12\dots n\},\{12\dots n\}}$.

We use the homological relations for quasideterminants to compare different $D_{I,J}$'s. For example, if $I = (i_1, i_2, \dots, i_n)$, $J = (j_1, j_2, \dots, j_n)$ and $I' = (i_2, i_1, \dots, i_n)$, $J' = (j_2, j_1, \dots, j_n)$ it is enough to compare $|A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2 j_2}$ and $|A|_{i_2 j_2} |A^{i_2 j_2}|_{i_1 j_2}$. The homological relations show that these expressions are in a sense "conjugate":

$$|A|_{i_2 j_2} |A^{i_2 j_2}|_{i_1 j_1} = |A^{i_1 j_1}|_{i_2 j_2} |A^{i_2 j_1}|^{-1}_{i_1 j_2} |A|_{i_1 j_1} |A^{i_1 j_1}|^{-1}_{i_2 j_2} |A^{i_2 j_1}|_{i_1 j_2} |A^{i_2 j_2}|_{i_1 j_1}.$$

Let R be a division algebra, R^* be the monoid of invertible elements in R and $\pi : R^* \to R^*/[R^*, R^*]$ be the canonical homomorphism. The group $R^*/[R^*, R^*]$ is abelian. To this group, we adjoin a zero element 0 with obvious multiplication, and denote the semi-group thus obtained by $\tilde{R}$. For $\mu \in R$ set $\tilde{\mu} = \pi(\mu)$ if $\mu \neq 0$ and $\tilde{\mu} = 0$ otherwise

We recall here a classical notion of the Dieudonne determinant [D, A]. There exist a unique homomorphism

$$\det : M_n(R) \to \tilde{R}$$

such that

i) $\det A' = \tilde{\mu} \det A$ for any matrix A' obtained from $A \in M_n(R)$ by multiplying one row of A from the left by μ,

ii) $\det A'' = \det A$ for any matrix A'' obtained from A by adding one row to another,

iii) the identity matrix has determinant 1.

The homomorphism det is called the Dieudonne determinant.

Proposition 3.1.1. *$\pi(\Delta(A))$ is the Dieudonne determinant of A and $\pi(D_{I,J}) = p(I)p(J)\Delta$ where $p(I)$ is the parity of ordering I.*

Note that in [Dr] P. Draxl introduced a Dieudonne predeterminant, denoted $\delta\epsilon\tau$. For a generic matrix A over a division algebra there exists the Gauss decomposition $A = UDL$ where U, D, L are upper-triangular, diagonal, and low-triangular matrices. Draxl denoted the product of diagonal elements in D from top to the bottom by $\delta\epsilon\tau(A)$ and called it the Dieudonne predeterminant of A. For non-generic matrices Draxl used the Bruhat decomposition instead of Gauss decomposition.

Proposition 3.1.2. $\delta\epsilon\tau(A) = \Delta(A)$.

Proof. Let $y_1, \dots, y_n$ be the diagonal elements in D from top to the bottom. As shown in [GR1, GR2] (see also Section 4.9 of the review), $y_k = |A^{12\dots k-1,12\dots k-1}|_{kk}$. Then $\delta\epsilon\tau(A) = y_1 y_2 \dots y_n = \Delta(A)$. $\square$

Below we consider below special examples of noncommutative determinants.

3.2. Dieudonne determinant for quaternions.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$ be a generic quaternionic matrix. Recall that for any orderings $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ we defined expressions

$$D_{I,J}(A) = |A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2, j_2} |A^{i_1 i_2, j_1 j_2}|_{i_3 j_3} \dots a_{i_n j_n}.$$

In the quaternionic case the Dieudonne determinant D maps

$$D : M_n(\mathbb{H}) \to \mathbb{R}_{\geq 0}$$

(see [As]).

The following proposition generalizes a result from [VP].

Proposition 3.2.1. *In the quaternionic case $|D_{I,J}(A)|$, the absolute value of $D_{I,J}(A)$, is equal to $D(A)$ for any I, J.*

The proof of Proposition 3.2.1 follows from the homological relations for quasideterminants.

3.3. Moore determinants of Herimitian quaternionic matrices.

A quaternionic matrix $A = (a_{ij})$, $i, j = 1, \dots, n$ is Hermitian if $a_{ji} = \bar{a}_{ji}$ for all i, j. It follows that all diagonal elements of A are real numbers and that the submatrices A^{11}, $A^{12,12}$, $\dots$ are Hermitian.

A determinant of Hermitian matrices was introduced by E.M. Moore in 1922 [M, MB]. Here we recall the original definition.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix over a ring. Let σ be a permutation of $\{1, \dots, n\}$. Write σ as a product of disjoint cycles such that each cycle starts with a smallest number. Since disjoint cycles commute, we may write

$$\sigma = (k_{11} \dots k_{1j_1})(k_{21} \dots k_{2j_2}) \dots (k_{m1} \dots k_{mj_m})$$

where for each i, we have $k_{i1} < k_{ij}$ for all $j > 1$, and $k_{11} > k_{21} \dots > k_{m1}$. This expression is unique. Let $p(\sigma)$ be a parity of σ. Define Moore determinant $M(A)$ by

$$M(A) = \sum_{\sigma \in S_n} p(\sigma) a_{k_{11},k_{12}} \dots a_{k_{1j_1},k_{11}} a_{k_{21},k_{22}} \dots a_{k_{mj_m},k_{m1}}. \tag{3.3.1}$$

One can write cycles starting with the largest element. This will lead to the same result.

If A is Hermitian quaternionic matrix then $M(A)$ is a real number. Moore determinants have nice features and are widely used (see, for example, [Al, As, Dy1]).

The following Theorem 3.3.2 shows that determinants of Hermitian quaternionic matrices can be obtained via our general approach. First we prove that for a quaternionic Hermitian matrix A determinants $D_{I,I}(A)$ coincide up to a sign.

Recall that $\Delta(A) = D_{I,I}(A)$ for $I = \{1, \dots, n\}$ and that $\Delta(A)$ is a pre-Dieudonne determinant in the sense of [Dr]. If A is Hermitian then $\Delta(A)$ is a product of real numbers and, therefore, $\Delta(A)$ is real.

Proposition 3.3.1. *Let $p(I)$ be the parity of the ordering I. Then $\Delta(A) = p(I)p(J)D_{I,J}(A)$.*

The proof follows from homological relations for quasideterminants.

Theorem 3.3.2. *Let A be a Hermitian quaternionic matrix. Then $\Delta(A) = M(A)$.*

Proof. We use noncommutative Sylvester formula for quasideterminants (Theorem 1.5.2)

For $i, j = 2, \dots, n$ set

$$B_{ij} = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}.$$

B_{ij} is a Hermitian matrix. Let $b_{ij} = M(B_{ij})$ and $c_{ij} = |B_{ij}|_{11}$.

Note that $B = (b_{ij})$ and $C = (c_{ij})$ also are Hermitian matrices. It follows from (3.3.1) that $M(A) = a_{nn}^{2-n} M(B)$. Note, that $M(B) = a_{nn}^{n-1} M(C)$, therefore, $M(A) = a_{nn} M(C)$.

By Theorem 1.5.2 , $|A|_{11} = |C|_{11}$, $|A^{11}|_{22} = |C^{11}|_{22}$, ... So,

$$|A^{11}|_{22} |A^{11}|_{22} \dots |A^{12\dots n-1,12\dots n-1}|_{n-1,n-1} = |C^{11}|_{22} |C^{11}|_{22} \dots |C^{12\dots n-1,12\dots n-1}|_{n-1,n-1}.$$

The left-hand product equals $\Delta(A) a_{nn}^{-1}$ and the right-hand product equals $\Delta(C)$, so $\Delta(A) = \Delta(C) a_{nn} = M(A)$. □

3.4. Moore determinants and norms of quaternionic matrices.

For a generic quaternionic matrix $A = (a_{ij})$, $i, j = 1, \dots, n$, define quaternionic norm $\nu(A)$ by induction:

i) $\nu(A) = \nu(a)$ if A is the 1×1-matrix (a),

ii) $\nu(A) = \nu(|A|_{11})\nu(A^{11})$.

It follows that $\nu(A) \in \mathbb{R}_{\geq 0}$.

Proposition 3.4.1. *For generic matrices A, B*

i) $\nu(AB) = \nu(A)\nu(B)$,

ii) $\nu(A) = \Delta(A)\Delta(A^*) = \Delta(AA^*)$.

Since AA^* is a Hermitian matrix one has the following

Corollary 3.4.2. $\nu(A) = M(AA^*)$.

It follows that $\nu(A)$ is a polynomial in the a_{ij}'s and the $\bar{a}_{ij}$'s and so it can be defined for all matrices A by continuity. The continuation posseses the multiplicative property.

3.5. Study determinants.

An embedding of the field of complex numbers $\mathbb{C}$ into $\mathbb{H}$ is defined by an image of $\mathbf{i} \in \mathbb{C}$. Chose the embedding given by $x + y\mathbf{i} \mapsto x + y\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, where $x, y \in \mathbb{R}$ and identify $\mathbb{C}$ with its image in $\mathbb{H}$. Then any quaternion a can be uniquely written as $a = \alpha + \mathbf{i}\beta$ where $\alpha, \beta \in \mathbb{C}$.

Let $M(n, F)$ be the algebra of matrices of order n over a field F. Define a homomorphism $\theta : \mathbb{H} \to M(2, \mathbb{C})$ by setting

$$\theta(a) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$

The homomorpism can be extended to homomorhism of matrix algebras

$$\theta_n : M(n, \mathbb{H}) \to M(2n, \mathbb{C}).$$

Let $A = (a_{ij}) \in M(n, \mathbb{H})$, set $\theta_n(A) = (\theta(a_{ij}))$.

In 1920, E. Study [S] defined a determinant $S(A)$ of a quaternionic matrix A of order n by setting $S(A) = \det \theta_n(A)$. Here det is the standard determinant of a complex matrix. The following proposition is well known (see [As]).

Proposition 3.5.1. *For any quaternionic matrix A*

$$S(A) = M(AA^*).$$

The proof in [As] was based on properties of eigenvalues of quasideterminant matrices. Our proof based on Sylvester identity and homological relations actually shows that $S(A) = \nu(A)$ for a generic matrix A.

3.6. Quantum determinants.

Recall that $A = (a_{ij}), i, j = 1, \dots, n$ is a *quantum* matrix over a field F if $q \in F$ is given such that

$$a_{ik}a_{il} = q^{-1}a_{il}a_{ik} \quad \text{for} \quad k < l, \quad a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik} \quad \text{for} \quad i < j,$$

$$a_{il}a_{jk} = a_{jk}a_{il} \quad \text{for} \quad i < j, \quad k < l,$$

$$a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)a_{il}a_{jk} \quad \text{for} i < j, \quad k < l.$$

The quantum determinant $\det_q A$ by definition equals to

$$\sum_{\sigma \in S_n} (-q)^{-l(\sigma)} a_{1\sigma(1)}a_{1\sigma(2)} \dots a_{1\sigma(n)},$$

where $l(\sigma)$ is the number of inversions in σ.

If A is a quantum matrix then any square submatrix of A also is a quantum matrix with the same scalar q.

Theorem 3.6.1. [GR, GR1, GR2, KL]

$$det_q\ A = (-q)^{i-j}|A|_{ij} \cdot det_q\ A^{ij} = (-q)^{i-j} det_q\ A^{ij} \cdot |A|_{ij}.$$

Corollary 3.6.2. [GR, GR1, GR2, KL]

$$det_q\ A = |A|_{11}|A^{11}|_{22} \dots a_{nn}$$

and all factors in the right-hand side commute.

An essential generalizations of this result for matrices satisfying Faddeev-Reshetikhin-Takhtadjan relations was given in [ER].

3.7. Capelli determinants.

Let $X = (x_{ij})$, $i, j = 1, \dots, n$ be a matrix of formal commutative variables and X^T the transposed matrix. Let $D = (\partial_{ij})$, $\partial_{ij} = \partial/\partial_{ij}$, be a matrix of differential operators. Then $X^T D = (f_{ij})$, where $f_{ij} = \sum_k x_{ki}\partial/\partial_{kj}$.

Let W be a diagonal matrix, $W = \mathrm{diag}(0, 1, 2, \dots, n)$. By definition, the Capelli determinant $\det_{Cap}$ of $X^T D - W$ equals to the sum

$$\sum_{\sigma \in S_n} (-1)^{l(\sigma)} f_{\sigma(1)1}(f_{\sigma(2)2} - \delta_{\sigma(2)2} \dots (f_{\sigma(n)n} - (n-1)\delta_{\sigma(n)n}).$$

Set $Z = X^T D - I_n$. It was shown in [GR1, GR2] that the Capelli determinant can be expressed as a product of quasideterminants. More precisely, one has

Theorem 3.7.1.

$$|Z|_{11}|Z^{11}|_{22}\dots z_{nn} = \det X \det D$$

and all the factors in the left hand side commute.

It is known [We] that the right hand side in the theorem is the Capelli determinant. The theorem can also be interpreted in a different way.

Let $A = (e_{ij})$, $i, j = 1, \dots n$ be the matrix of the standard generators of the universal enveloping algebra $U(gl_n)$. Recall that these generators satisfy the relations

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

Let I_n be an identity $n \times n$-matrix. It is well-known (see, for example, [Ho]) that coefficients of the polynomial in a central variable t

$$\det(I_n + tA) := \sum_{\sigma \in S_n} (-1)^{l(\sigma)}(\delta_{\sigma(1)1} + te_{\sigma(1)1})\dots(\delta_{\sigma(n)n} + t(e_{\sigma(n)n} - (n-1)\delta_{\sigma(n)n}))$$

generate the center of $U(gl_n)$.

Theorem 3.7.1 can be stated in the following way [GKLLRT].

Theorem 3.7.2. $\det(I_n + tA)$ *can be factored in the algebra of formal power series in t with coefficients in $U(gl_n)$*

$$\det(I_n + tA) = (1 + te_{11}) \begin{vmatrix} 1 + t(e_{11} - 1) & te_{12} \\ te_{21} & \boxed{1 + t(e_{22} - 1)} \end{vmatrix} \cdot \dots$$

$$\dots \cdot \dots \begin{vmatrix} 1 + t(e_{11} - n + 1) & \dots & te_{1n} \\ \dots & \dots & \dots \\ te_{n1} & \dots & \boxed{1 + t(e_{nn} - n + 1)} \end{vmatrix}$$

and the factors in the right-hand side commute with each other.

The above version is obtained by using the classical embedding of $U(gl_n)$ into the Weyl algebra generated by $(x_{ij}, \partial/\partial x_{ij})$, $i, j = 1, \dots, n$ where e_{ij} corresponds to

$$f_{ij} = \sum_{k=1}^{n} x_{ki}\partial/\partial x_{kj}$$

.

3.8 Berezinians.

Let $p(k)$ be a parity of an integer k, i.e. $p(k)=0$ if k is even and $p(k)=1$ if k is odd. A (commutative) super-ring over R^0 is a ring $R=R^0\oplus R^1$, such that

i) $a_ia_j\in R^{p(i+j)}$ for any $a_m\in R^m$, $m=0,1$,

ii) $ab=ba$ for any $a\in R^0$, $b\in R$, and $cd=-dc$ for any $c,d\in R^1$.

Let $A=\begin{pmatrix} X & Y\\ Z & T\end{pmatrix}$ be a $(m+n)\times(m+n)$-block-matrix over a super-ring $R=R^0\oplus R^1$, X be a $m\times m$-matrix over R^0, T be a $n\times n$-matrix over R^0, and Y,Z be matrices over R^1. If T is an invertible matrix, then $X-YT^{-1}Z$ is an invertible matrix over commutative ring R^0. Super-determinant, or Berezinian of A is defined by the following formula

$$\mathrm{Ber}A=\det(X-YT^{-1}Z)\det T^{-1}.$$

Note that this expression belongs to R^0.

Theorem 3.8.1. *Let R^0 be a field. Set $J_k=\{1,2,\dots,k\}$, $k\le m+n$ and $A^k=A^{J_k,J_k}$. Then $BerA$ is a product of elements from R^0:*

$$BerA=|A|_{11}|A^{(1)}|_{22}\dots|A^{(m-1)}|_{mm}|A^{(m)}|^{-1}_{m+1,m+1}\cdots|A^{(m+n-1)}|^{-1}_{m+n,m+n}.$$

3.9 Cartier-Foata determinants.

Let $A=(a_{ij})$, $i,j=1,\dots,n$ be a matrix such that the entries a_{ij} and a_{kl} commute when $i\ne k$. In this case Cartier and Foata [CF, F] defined a determinant of A as

$$\det{}_{CF}\ A=\sum_{\sigma\in S_n}(-1)^{l(\sigma}a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)}.$$

The order of factors in monomials $a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)}$ is insignificant.

Theorem 3.9.1. *Let $A=(a_{ij})$, $i,j=1,\dots,n$ be a matrix such that the entries a_{ij} and a_{kl} commute when $i\ne k$.*

$$|A|_{pq}=(-1)^{p+q}det_{CF}\ (A^{pq})^{-1}det_{CF}\ A$$

if all expressions in the formula are defined.

IV. Noncommutative Plücker and Flag Coordiniates

In this section we described results obtained in [GR4].

4.1. Commutative Plücker coordinates. Let $k \leq n$ and A be a $k \times n$-matrix over a commutative ring R. Denote by $A(i_1, \dots, i_k)$ the submatrix of A consisting of columns labelled by indices $i_1, \dots, i_k$. Define $p_{i_1 \dots i_k}(A) := \det A(i_1, \dots, i_k)$. Expressions $p_{i_1 \dots i_k}(A)$ are called Plücker coordinates of matrix A.

The Plücker coordinates $p_{i_1 \dots i_k}(A)$ satisfy the following properties:

i) (invariance) $p_{i_1 \dots i_k}(XA) = \det X p_{i_1 \dots i_k}(A)$ for any $k \times k$-matrix X over R,

ii) (skew-symmetry) $p_{i_1 \dots i_k}(A)$ are skew-symmetric in indices; in particular, $p_{i_1 \dots i_k}(A) = 0$ if a pair of indices coincides,

iii) (Plücker relations) Let $i_1, \dots, i_{k-1}$ be $k-1$ distinct numbers which are chosen from the set $1, \dots, n$, and $j_1, \dots, j_{k+1}$ be $k+1$ distinct numbers chosen from the same set. Then

$$\sum_{t=1}^{k} (-1)^t p_{i_1 \dots i_{k-1} j_t}(A) p_{j_1 \dots j_{t-1} j_{t+1} \dots j_{k+1}}(A) = 0.$$

Example. For $k = 2$ and $n = 4$ the Plücker relations in iii) imply the famous identity

$$p_{12}(A)p_{34}(A) - p_{13}(A)p_{24}(A) + p_{23}(A)p_{14}(A) = 0. \tag{4.1.1}$$

A $k \times n$-matrix over a field F is non-degenerate if at least one of its Plücker coordinates is invertible. Let M be a set of all non-degenerate $k \times n$-matrices over the field. There is a natural action of the group $GL(k, F)$ of invertible $k \times k$-matrices on M defined by the multiplication from the right. The invariance of Plücker coordinates implies that they depend of $GL(k, F)$-orbits only.

The set of orbits of this action is canonically identified with the set of k-dimensional subspaces in n-dimensional vector space over F. The latter is called Grassmanian $G_{k,n}$. The Plücker coordinates are projective coordinates on $G_{k,n}$.

4.2. Quasi-Plücker coordinates for $n \times (n+1)$- and $(n+1) \times n$-matrices. Let $A = (a_{ij})$, $i = 1, \dots, n$, $j = 0, 1, \dots, n$ be a matrix over a division algebra R. Denote by $A^{(k)}$ the submatrix of A obtained from A by removing its k-th column. Suppose that all $A^{(k)}$ are invertible, choose any $s \in \{1, \dots, n\}$ and set $p_{ij}(A) = |A^{(j)}|_{si}^{-1} |A^{(i)}|_{sj}$.

Proposition 4.2.1. *The elements $p_{ij}(A)$ do not depend on s. If g is an invertible $n \times n$-matrix over R then $p_{ij}(gA) = p_{ij}(A)$.*

Proof. One can see that there exist nonzero elements $x_0, \dots, x_n \in R$ such that

$$A \begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix} = 0.$$

It follows that

$$A^{(j)}\begin{pmatrix} x_0 \\ \dots \\ \hat{x}_j \\ \dots \\ x_n \end{pmatrix} = -\begin{pmatrix} a_{1j} \\ \\ \dots \\ \\ a_{nj} \end{pmatrix} x_j.$$

Cramer rules and transformations properties for quasideterminanats imply that $|A^{(j)}|_{si}x_i = -|A^{(i)}|_{sj}x_j$. Therefore,

$$p_{ij}(A) = |A^{(j)}|_{si}^{-1}|A^{(i)}|_{sj} = -x_i x_j^{-1} \tag{4.2.1}$$

does depend on s.

Let g be an invertible $n \times n$-matrix. Note that $gA\begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix} = 0$. It follows that $p_{ij}(gA) = -x_i x_j^{-1} = p_{ij}(A)$. □

In the commutative case $p_{ij}(A)$ is a ratio of two Plücker coordinates: $p_{ij}(A) = \frac{\det A^{(j)}}{\det A^{(i)}}$. We call $p_{ij}(A)$'s the left quasi-Plücker coordinates of $n \times (n+1)$-matrix A.

Remark. The identity (4.2.1) reflects an important relation between left quasi-Plücker coordinates of the Grassmanian $G_{n,n+1}$ of n-dimensional subspaces in R^{n+1} considered as a left R-module and the right projective coordinates on a line in R^{n+1} viewed as the right R-module. Note that in (4.2.1) the left fraction equals to the right fraction. We discuss the connection below.

Define also the right quasi-Plücker coordinates $r_{ij}(B)$ for $(n+1) \times n$-matrix $B = (b_{ji})$. Denote by $B^{(k)}$ the submatrix of B obtained from B by removing its k-th row. Suppose that all $B^{(k)}$ are invertible, choose any $s \in \{1, \dots, n\}$ and set $r_{ij}(B) = |B^{(j)}|_{is}|B^{(i)}|_{js}^{-1}$.

Proposition 4.2.2. *The elements $r_{ij}(B)$ do not depend of s. If g is an invertible $n \times n$-matrix over R then $r_{ij}(Bg) = r_{ij}(B)$.*

Proof. The proof is dual the proof of Proposition 4.2.1. It is enough to find non-zero elements $y_0, \dots, y_n \in R$ such that the product $(y_0, \dots, y_n)B = 0$ and show that $-y_i^{-1}y_j = r_{ij}(B)$. □

In the commutative case $r_{ij}(A) = \frac{\det B^{(j)}}{\det A^{(i)}}$.

4.3. Definition of left quasi-Plücker coordinates. Let $A = (a_{pq})$, $p = 1, \dots, k$, $q = 1, \dots, n, k < n$ be a matrix over a division algebra R. Fix $1 \leq i, j, i_1, \dots, i_{k-1} \leq n$ such that $i \notin I = \{i_1, \dots, i_{k-1}\}$. Let $A(i, j, i_1, \dots, i_{k-1})$ be a $(k+1) \times k$-submatrix of A with columns labelled by $i, j, i_1, \dots, i_{k-1}$.

For $1 \leq s \leq k$ set

$$p_{ij}^I(A) = \begin{vmatrix} a_{1i}a_{1i_1} & \dots & a_{1i_{k-1}} \\ & \dots & \\ a_{ki}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{si}^{-1} \cdot \begin{vmatrix} a_{1j}a_{1i_1} & \dots & a_{1i_{k-1}} \\ & \dots & \\ a_{kj}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{sj}.$$

Note that $p_{ij}^I(A) = p_{ij}(A(i,j,i_1,\dots,i_{k-1}))$.

Proposition 4.3.1.
i) $p_{ij}^I(A)$ *does not depend on* s*;*
ii) $p_{ij}^I(gA) = p_{ij}^I(A)$ *for any invertible* $k \times k$ *matrix* g*.*

Proof. Use Proposition 4.2.1 or the homological relations.

Definition 4.3.2. We call $p_{ij}^I(A)$ left quasi-Plücker coordinates of the matrix A.

One may consider p_{ij}^I as quasi-Plücker coordinates of a k-dimensional left subspace generated in the left F-linear space F^n by rows of the matrix A. In the commutative case $p_{ij}^I = \frac{p_{jI}}{p_{iI}}$, where $p_{\alpha_1\dots\alpha_k}$ is the standard Plücker coordinate.

4.4. Identities for the left Quasi-Plücker coordinates.
The following properties of p_{ij}^I's trivially follow from their definition:

i) p_{ij}^I does not depend on reordering on elements in I;
ii) $p_{ij}^I = 0$ if $j \in I$;
iii) $p_{ii}^I = 1$ and $p_{ij}^I \cdot p_{jk}^I = p_{ik}^I$.

Theorem 4.4.1. (Skew-Symmetry) *Let* N *be a set of indices,* $|N| = k+1, i, j, m \in N$*. Then*

$$p_{ij}^{N\smallsetminus\{i,j\}} \cdot p_{jm}^{N\smallsetminus\{j,m\}} \cdot p_{mi}^{N\setminus\{m,i\}} = -1.$$

Theorem 4.4.2. (Plücker relations) *Fix* $M = (m_1, \dots, m_{k-1})$*,* $L = (\ell_1, \dots, \ell_k)$*. Let* $i \notin M$*. Then*

$$\sum_{j \in L} p_{ij}^M \cdot p_{ji}^{L\setminus\{j\}} = 1.$$

Examples. Suppose that $k = 2$.
1) From Theorem 4.4.1 it follows that

$$p_{ij}^{\{\ell\}} \cdot p_{j\ell}^{\{i\}} \cdot p_{\ell i}^{\{j\}} = -1.$$

In the commutative case one has $p_{ij}^{\{\ell\}} = \frac{p_{j\ell}}{p_{i\ell}}$ and $p_{ij} = -p_{ji}$, so this identity is valid.

2) From Theorem 4.4.2 it follows that for any i, j, ℓ, m

$$p_{ij}^{\{\ell\}} \cdot p_{ji}^{\{m\}} + p_{im}^{\{\ell\}} \cdot p_{mi}^{\{j\}} = 1.$$

In the commutative case this identity implies the standard identity (cf. (4.1.1))

$$p_{ij} \cdot p_{\ell m} - p_{i\ell} \cdot p_{jm} + p_{im} \cdot p_{\ell j} = 0.$$

Remark. In fact the products $p_{ij}^{\{\ell\}} p_{ji}^{\{m\}}$ (which are equal to $\frac{p_{j\ell}}{p_{i\ell}} \cdot \frac{p_{im}}{p_{jm}}$ in the commutative case) are noncommutative cross-ratios.

We will consider different specializations of the relations of quasi-Plücker coordinates in a separate paper.

To prove Theorems 4.4.1 and 4..4.2 we need the following lemma. Let $A = (a_{ij}), i = 1, \dots, k, j = 1, \dots, n,\ k < n$, be a matrix over a skew-field. Denote by $A_{j_1,\dots,j_\ell}, \ell \leq n$, its submatrix $(a_{ij}), i = 1, \dots, k,\ j = j_1, \dots, j_\ell$. Consider $n \times n$-matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & I_{n-k} \end{pmatrix},$$

where I_m is a $m \times m$-identity matrix.

Lemma 4.4.3. *Let $j < k < i$. If $p_{ij}^{1\dots\hat{j}\dots k}(A)$ is defined then $|X|_{ij}$ is defined and*

$$|X|_{ij} = -p_{ij}^{1\dots\hat{j}\dots k}(A) \tag{4.4.1}$$

Proof. We have to prove

$$|X|_{ij} = -|A_{1\dots\hat{j}\dots ki}|_{si}^{-1} \cdot |A_{1\dots k}|_{sj} \tag{4.4.2}$$

if the right-hand side is defined. We will prove this by induction on $\ell = n - k$. We suppose that formula (2.2) is valid for $l = m$ and prove it for $\ell = m + 1$. Without loss of generality we may set $j = 1, i = k + 1$. By homological relations (Theorem 1.4.3)

$$|X|_{k+1,1} = -|X^{k+1,1}|_{s,k+1}^{-1} \cdot |X^{k+1,k+1}|_{s1}$$

for an appropriate $1 \leq s \leq k$. Here

$$X^{k+1,1} = \begin{pmatrix} A_{2\dots k+1} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix},$$

$$X^{k+1,k+1} = \begin{pmatrix} A_{1\dots k} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix}.$$

By the induction assumption

$$|X^{k+1,1}|_{s,k+1} = -|A_{23\dots kk+2}|^{-1}_{s,k+2} \cdot |A_{23\dots k+1}|_{s,k+1},$$

$$|X^{k+1,k+1}|_{s1} = -|A_{23\dots kk+2}|^{-1}_{s,k+2} \cdot |A_{1\dots k}|_{s1}$$

and $|X|_{k+1,1} = -p^{23\dots k}_{k+1,1}$. □

To prove Theorem 4.4.2 we apply the second formula in Theorem 1.5.5 to the matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & E_{n-k} \end{pmatrix}$$

for $M = (k+1, \dots, n)$ and any L such that $|L| = n-k-1$.

By Lemma 4.4.3 $|X|_{m_i\ell} = -p^{1\dots\hat{\ell}\dots k}(A)$, $|X^{M_i,L}|_{m_iq} = -p^{1\dots n\setminus L}_{m_iq}(A)$, and Theorem 4.4.2 follows from Theorem 1.5.5. □

To prove Theorem 4.4.1 consider matrix X for $n = k+1$ and use homological relations.

Theorem 4.4.4. *Let $A = (a_{ij}), i = 1, \dots, k, j = 1, \dots, n$, be a matrix with formal entries, and $f(a_{ij})$ a rational function over a free skew-field F generated by a_{ij}. Let f be invariant under all transformations*

$$A \to gA$$

where g is an invertible $k \times k$-matrix over f. Then f is a rational function of the quasi-Plücker coordinates.

Proof. Let $b_{ij} = a_{ij}$ for $i, j = 1, \dots k$. Consider matrix $B = (b_{ij})$. Then $B^{-1} = (|B|^{-1}_{ji})$. Set $C = (c_{ij}) = B^{-1}A$. Then

$$c_{ij} = \begin{cases} \delta_{ij} & j \le k, \\ p^{1\dots\hat{i}\dots k}_{ij}(A) & j > k. \end{cases}$$

By invariance we have $f(A) = f(1, C)$. □

4.5. Right quasi-Plücker coordinates. Consider a matrix $B = (b_{ij}), i = 1, \dots, n; j = 1, \dots, k, k < n$ over a skew-field F. Fix $1 \leq i, j, i_1, \dots, i_{k-1} \leq n$ such that $j \notin I = (i_1, \dots, i_{k-1})$. Also fix $1 \leq t \leq k$ and set

$$r^I_{ij}(B) = \begin{vmatrix} b_{i1} & \dots & b_{ik} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1}1} & \dots & b_{i_{k-1}k} \end{vmatrix}_{it} \cdot \begin{vmatrix} b_{j1} & \dots & b_{jk} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1}1} & \dots & b_{i_{k-1}k} \end{vmatrix}^{-1}_{jt} .$$

Proposition 4.5.1.
i) $r^I_{ij}(B)$ does not depend of t;
ii) $r^I_{ij}(Bg) = r^I_{ij}(B)$ for any invertible $k \times k$-matrix g over F

Definition 4.5.2. *We call $r^I_{ij}(B)$ right quasi-Plücker coordinates of the matrix B.*

Proposition 4.5.1 ii) shows that $r^I_{ij}(B)$ are quasi-Plücker coordinates of the right k-dimensional subspace generated by columns of B in the right vector space F^n.

4.6. Identities for the right quasi-Plücker coordinates.
Generating identities for the r^I_{ij}'s are dual to generating identities for the left quasi-Plücker coordinates p^I_{ij}.

i) r^I_{ij} does not depend on reordering on elements in I;
ii) $r^I_{ij} = 0$ if $i \in I$;
iii) $r^I_{ii} = 1$ and $r^I_{ij} \cdot r^I_{jk} = r^I_{ik}$.

Theorem 4.6.1. (Skew-Symmetry) *Let N be a set of indices, $|N| = k+1, i, j, m \in N$. Then*

$$r^{N \setminus \{i,j\}}_{ij} \cdot r^{N \setminus \{j,m\}}_{jm} \cdot r^{N \setminus \{m,i\}}_{mi} = -1.$$

Theorem 4.6.2. (Plücker relations) *Fix $M = (m_1, \dots, m_{k-1})$, $L = (\ell_1, \dots, \ell_k)$. Let $i \notin M$. Then*

$$\sum_{j \in L} r^{L \setminus \{j\}}_{ij} r^M_{ij} = 1.$$

4.7. The duality between quasi-Plücker coordinates. Let $A = (a_{ij}), i = 1, \dots, k; j = 1, \dots, n$; and $B = (b_{rs}), r = 1, \dots, n; s = 1, \dots, n-k$. Suppose that $AB = 0$. This is equivalent to the statement that the subspace generated by rows of A in the left linear space is orthogonal to the subspace generated by columns of B in the right linear space. Fix $1 \leq i, j \leq n$, $I \subset [1, n], i \notin I, |I| = k-1$. Set $J = ([1, n] \setminus I) \setminus \{i, j\}$.

Theorem 4.7.1.

$$p^I_{ij}(A) + r^J_{ij}(B) = 0.$$

4.8. Quasi-Plücker coordinates for $k \times n$-matrices for different k. Let $A = (a_{\alpha\beta}), \alpha = 1, \dots, k, \ \beta = 1, \dots, n$, be any $k \times n$ - matrix over a noncommutative skew-field and A' a $(k-1) \times n$-submatrix of A. Fix $1 \le i, j, m, j_1, \dots, j_{k-2} \le n$ such that $i, m \notin J = \{j_1, \dots, j_{k-2}\}$ and $i \ne m$.

Proposition 4.8.1.

$$p_{ij}^J(A') = p_{ij}^{J \cup \{m\}}(A) + p_{im}^J(A') \cdot p_{mj}^{J \cup \{i\}}(A).$$

4.9. Applications of quasi-Plücker coordinates

Row and column expansion of a quasideterminant.

Some of the results obtained in [GR], [GR1], [GR2] and partially listed in section I can be rewritten in terms of quasi-Plücker coordinates. Let $A = (a_{ij}), i, j = 1, \dots, n$, be a matrix over a skew-field.

Fix $1 \le \alpha, \beta \le n$. Using the notations from section I let $B = A^{\{\alpha\}, \emptyset}, C = A^{\emptyset, \{\beta\}}$ be the $(n-1) \times n$ and $n \times (n-1)$ submatrices of A obtained by deleting the α-th row and the β-th column. For any $j \ne \beta$ and $i \ne \alpha$ set

$$p_{j\beta} = p_{j\beta}^{1 \dots \hat{j} \dots \hat{\beta} \dots n}(B),$$
$$r_{\alpha i} = r_{\alpha i}^{1 \dots \hat{\alpha} \dots \hat{i} \dots n}(C).$$

Proposition 4.9.1.

i) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{j \ne \beta} a_{\alpha j} p_{j\beta}$,
ii) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{i \ne \alpha} r_{\alpha i} a_{i\beta}$
if terms in the right-hand sides are defined.

Homological relations.

Proposition 4.9.2. *In the previous notations*
i) $|A|_{ij}^{-1} \cdot |A|_{i\ell} = -p_{j\ell}$ *(row relations)*
ii) $|A|_{ij} \cdot |A|_{kj}^{-1} = -r_{ik}$ *(column relations).*

Corollary 4.9.3. *In the previous notations, let* $(i_1, \dots, i_s), (j_1, \dots, j_t)$ *be sequences of indices such that* $i \ne i_1, i_1 \ne i_2, \dots, i_{s-1} \ne i_s; j \ne j_1, j_1 \ne j_2, \dots, j_{t-1} \ne j_t$.
Then

$$|A|_{i_s j_t} = p_{i_s i_{s-1}} \dots p_{i_2 i_1} p_{i_1 i} \cdot |A|_{ij} \cdot r_{j j_1} r_{j_1 j_2} \dots r_{j_{t-1} j_t}.$$

Example.

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have

$$a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{22}^{-1} \cdot a_{22} = |A|_{22},$$
$$a_{12} a_{22}^{-1} a_{21} a_{11}^{-1} |A|_{11} a_{21}^{-1} a_{22} a_{12}^{-1} a_{11} = |A|_{11}$$

Matrix multiplication. The following formula has been already used in the proof of Theorem 4.4.4. Let $A = (a_{ij}), i = 1, \dots, n; j = 1, \dots, m, n < m; B = (a_{ij}), i = 1, \dots, n, j = 1, \dots, n; C = (a_{ik}), i = 1, \dots, n; k = n+1, \dots, m$.

Proposition 4.9.4. *Let the matrix B be invertible. Then the $p_{ik}^{1\dots\hat{i}\dots n}(A)$ are defined for $i = 1, \dots n, k = n+1, \dots m$, and*

$$B^{-1}C = (p_{ik}^{1\dots\hat{i}\dots n}(A)), i = 1, \dots, n; k = n+1, \dots, m.$$

Quasideterminant of a product. Let $A = (a_{ij}), B = (b_{ij}), i, j = 1, \dots n$ be matrices over a skew-field. Fix $1 \le k \le n$. Consider the $(n-1) \times n$-matrix $A' = (a_{ij})$, $i \neq k$, and the $n \times (n-1)$-matrix $B'' = (b_{ij})$, $j \neq k$.

Proposition 4.9.5.

$$|B|_{kk} \cdot |AB|_{kk}^{-1} |A|_{kk} = 1 + \sum_{\alpha \neq k} r_{k\alpha} \cdot p_{\alpha k},$$

where

$$r_{k\alpha} = r_{k\alpha}^{1\dots\hat{\alpha}\dots n}(B'') - \text{ right quasi-Plücker coordinates, and}$$

$$p_{\alpha k} = p_{\alpha k}^{1\dots\hat{\alpha}\dots n}(A') - \text{ left quasi-Plücker coordinates}$$

if all the expressions are defined.

The proof follows from the multiplicative property of quasideterminants and Proposition 4.9.2.

Gauss decomposition. Consider matrix $A = (a_{ij}), i, j = 1, \dots, n$, over a skew-field. Let $A_k = (a_{ij}), i, j = k, \dots n$, $B_k = (a_{ij}), i = 1, \dots n,\ j = k, \dots n$, and $C_k = (a_{ij}), i = k, \dots n, j = 1, \dots n$. These are submatrices of sizes $(n-k+1) \times (n-k+1)$, $n \times (n-k+1)$, and $(n-k+1) \times n$ respectively.

Suppose that the quasideterminants

$$y_k = |A_k|_{kk},\ k = 1, \dots, n$$

are defined and invertible. From [GR1], [GR2] it follows

Theorem 4.9.6.

$$A = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix},$$

where

$$x_{\alpha\beta} = r_{\alpha\beta}^{\beta+1\dots n}(B_\beta),\ 1 \le \alpha < \beta \le n$$

$$z_{\beta\alpha} = p_{\beta\alpha}^{\beta+1\dots n}(C_\beta).\ 1 \le \alpha < \beta \le n$$

An important example of Quasi-Plücker coordinates of Vandermonde matrix will be considered later.

4.10. Flag coordinates. Noncommutative flag coordinates were introduced in [GR1, GR2].

Let $A = (a_{ij}), i = 1, \dots, k,\ j = 1, \dots, n$, be a matrix over a division algebra R. Let F_p be the subspace of the left vector space R^n generated by the first p rows of A. Then $\mathcal{F} = (F_1 \subset F_2 \subset \cdots \subset F_k)$ is a flag in R^n. Put

$$f_{j_1 \dots j_k}(\mathcal{F}) = \begin{vmatrix} a_{1j_1} & \dots & a_{1j_k} \\ & \dots & \\ a_{kj_1} & \dots & a_{kj_k} \end{vmatrix}_{kj_1}.$$

In [GR1, GR2] functions $f_{j_1 \dots j_k}(\mathcal{F})$ were called the *flag coordinates* of $\mathcal{F}$. Transformations properties of quasideterminants imply that $f_{j_1 \dots j_k}(\mathcal{F})$ does not depend on indices $j_2, \dots, j_k$.

Proposition 4.10.1. [GR1], [GR2] *The functions $f_{j_1 \dots j_m}(\mathcal{F})$ do not change under left multiplication of A by an upper triangular matrix with units as its diagonal entries.*

The functions $f_{j_1 \dots j_k}(\mathcal{F})$'s satisfy the following relations [GR1, GR2].

Theorem 4.10.2.

$$f_{j_1 j_2 j_3 \dots j_k}(\mathcal{F}) f_{j_1 j_3 \dots j_k}(\mathcal{F})^{-1} = -f_{j_2 j_1 \dots j_k}(\mathcal{F}) f_{j_2 j_3 \dots j_k}(\mathcal{F})^{-1},$$

$$f_{j_1 \dots j_k}(\mathcal{F}) f_{j_1 \dots j_{k-1}}(\mathcal{F})^{-1} + f_{j_2 \dots j_k j_1}(\mathcal{F}) f_{j_2 \dots j_k}(\mathcal{F})^{-1} + \dots + f_{j_k \dots j_{k-1}}(\mathcal{F}) f_{j_k \dots j_{k-2}}(\mathcal{F})^{-1} = 0$$

Example. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. Then $f_{12}(\mathcal{F}) a_{11}^{-1} = -f_{21}(\mathcal{F}) a_{12}^{-1}$ and $f_{12}(\mathcal{F}) a_{11}^{-1} + f_{23}(\mathcal{F}) a_{12}^{-1} + f_{31}(\mathcal{F}) a_{13}^{-1} = 0$.

It is easy to see that

$$p_{ij}^{i_1 \dots i_{k-1}}(A) = \left(f_{i i_1 \dots i_{k-1}}(\mathcal{F})\right)^{-1} \cdot f_{j i_1 \dots i_{k-1}}(\mathcal{F}).$$

Theorems 4.4.1 and 4.4.2 may be deduced from Theorem 4.10.2.

V. Bezout and Vieta Theorems

In this section we will discuss analogues of the Bezout and Vieta Theorems for polynomials over a noncommutative division algebra. We will prove the Vieta Theorem which was formulated in [GR3, GR4] using our noncommutative form of Sylvester's identity. The proof of this theorem, using differential operators, appeared in [EGR]. Another noncommutative version of the Vieta Theorem, based on notions of traces and determinants, was given by A. Connes and A. Schwarz in [CS].

5.1. Vandermonde quasideterminants.

Let $x_1, x_2, \ldots, x_k$ be a set of elements of a skew-field F. For $k > 1$ define a *Vandermonde* quasideterminant

$$V(x_1, \ldots, x_k) = \begin{vmatrix} x_1^{k-1} & \ldots & x_k^{k-1} \\ & \ldots & \\ x_1 & \ldots & x_k \\ 1 & \ldots & 1 \end{vmatrix}_{1k}.$$

We say that a sequence $x_1, \ldots, x_n \in F$ is *independent* if the quasideterminants $V(x_1, \ldots, x_k)$, $k = 2, \ldots, n$, are all defined and invertible. For independent sequences $x_1, \ldots, x_n$ and $x_1, \ldots, x_{n-1}, z$, where $x_1, \ldots, x_n, z \in F$ set

$$y_1 = x_1,\ z_1 = z$$
$$y_k = V(x_1, \ldots, x_k) x_k V(x_1, \ldots, x_k)^{-1},\ k \geq 2$$
$$z_k = V(x_1, \ldots, x_{k-1}, z) z V(x_1, \ldots, x_{k-1}, z)^{-1},\ k \geq 2.$$

In the commutative case $y_k = x_k$, and $z_k = z$ for $k = 1, \ldots, n$.

5.2. Bezout and Vieta decompositions of Vandermonde quasideterminants. We will prove here:

Theorem 5.2.1. *(Bezout decomposition of a Vandermonde quasideterminant) Suppose that sequences $x_1, \ldots, x_n$ and $x_1, \ldots, x_{n-1}, z$ are independent. Then*

$$V(x_1, \ldots, x_n, z) = (z_n - y_n)(z_{n-1} - y_{n-1}) \ldots (z_1 - y_1). \tag{5.2.1}$$

Note that if z commutes with $x_i, i = 1, \ldots n$ then

$$V(x_1, \ldots, x_n, z) = (z - y_n)(z - y_{n-1}) \ldots (z - y_1).$$

Theorem 5.2.2. *(Vieta decomposition of a Vandermonde quasideterminant) For an independent sequence $x_1, \ldots, x_n, z$*

$$V(x_1, \ldots, x_n, z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \tag{5.2.2}$$

where

$$a_k = (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} y_{i_k} \cdot y_{i_{k-1}} \ldots y_1. \tag{5.2.3}$$

In particular

$$a_1 = -(y_1 + \cdots + y_n),$$
$$a_2 = \sum_{1 \leq i < j \leq n} y_j y_i,$$
$$\ldots$$
$$a_n = (-1)^n y_n \ldots y_1.$$

5.3. Proof of Theorem 5.2.2. We will show, by induction on n, that Theorem 5.2.2 follows from Theorem 5.2.1. For $n = 1$ one has $V(x_1, z) = z - x_1$ and formulas (5.2.1) and (5.2.2) are valid. Suppose that these formulas are valid for $m = n - 1$. By Theorem 5.2.1

$$V(x_1, \dots, x_n, z) = (z_n - y_n)V(x_1, \dots, x_{n-1}, z) =$$

$$= (V(x_1, \dots, x_{n-1}, z) \cdot z) - (y_n \cdot V(x_1, \dots, x_{n-1}, z)).$$

By induction,

$$V(x_1, \dots, x_{n-1}, z) = z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1},$$

where

$$b_1 = -(y_1 + \dots + y_{n-1}),$$
$$\dots$$
$$b_{n-1} = (-1)^n y_{n-1} \cdot \dots \cdot y_1.$$

So,

$$V(x_1, \dots, x_n, z) = z^n + (b_1 - y_n)z^{n-1} + (b_2 - y_n b_1)z^{n-2} + \dots - y_n b_n =$$

$$= z^n + a_1 z^{n-1} + \dots + a_n,$$

where $a_1, \dots, a_n$ are given by formulas (3.3). □

5.4. A division lemma. To prove Theorem 5.2.1 we need the following

Lemma 5.4.1.

$$V(x_1, \dots, x_n, z) = V(\hat{x}_2, \dots, \hat{x}_n, \hat{z})(z - x_1),$$

where

$$\hat{x}_k = (x_k - x_1)x_k(x_k - x_1)^{-1}, k = 2, \dots, n$$
$$\hat{z} = (z - x_1)z(z - x_1)^{-1}.$$

Proof. By definition

$$V(x_1, \dots, x_n, z) = \begin{vmatrix} x_1^n & x_2^n & \dots & z^n \\ x_1^{n-1} & x_2^{n-1} & \dots & z^{n-1} \\ & & \dots & \\ x_1 & x_2 & \dots & z \\ 1 & 1 & \dots & 1 \end{vmatrix}_{1n+1}.$$

Multiply the k-th row by x_1 from the left and subtract it from the $(k-1)$-th row for $k = 2, \ldots, n$. The quasideterminant will not change, and so

$$V(x_1, \ldots, x_n, z) = \left| \begin{array}{cccc} 0 & x_2^n - x_1 x_2^{n-1} & \ldots & z^n - x_z^{n-1} \\ 0 & x_2^{n-1} - x_1 x_2^{n-2} & & z^{n-1} - x_1 z^{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & x_2 - x_1 & & z - x_1 \\ 1 & 1 & & 1 \end{array} \right|_{1n+1} =$$

$$= \left| \begin{array}{cccc} 0 & (x_2 - x_1) x_2^{n-1} & \ldots & (z - x_1) z^{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & x_2 - x_1 & & z - x_1 \\ 1 & 1 & & 1 \end{array} \right|_{1n+1} .$$

Applying Sylvester's theorem to the last quasideterminant with the element of index $(n1)$ as a pivot one has

$$V(x_1, \ldots, x_n, z) = \left| \begin{array}{ccc} (x_2 - x_1) x_2^{n-1} & \ldots & (z - x_1) z^{n-1} \\ \ldots & & \ldots \\ x_2 - x_1 & & z - x_1 \end{array} \right|_{1n} .$$

Multiply the k-th column on the right by $(x_{k+1} - x_1)^{-1}$ for $k = 1, \ldots, n-1$ and the last column by $(z - x_1)^{-1}$. According to elementary properties of quasideterminants,

$$V(x_1, \ldots, x_n, z) =$$

$$= \left| \begin{array}{ccc} (x_2 - x_1) x_2^{n-1} (x_2 - x_1)^{-1} & \ldots & (z - x_1) z^{n-1} (z - x_1)^{-1} \\ \vdots & & \vdots \\ 1 & & 1 \end{array} \right|_{1n} (z - x_1) =$$

$$= \left| \begin{array}{ccc} \hat{x}_2^{n-1} & \ldots & \hat{z}^{n-1} \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{array} \right|_{1n} \cdot (z - x_1) = V(\hat{x}_2, \ldots, \hat{x}_n, \hat{z}) \cdot (z - x_1).$$

5.5. Proof of Theorem 5.2.1. We proceed by induction on n. By Lemma 5.4.1, Theorem 5.2.1 is valid for $n = 2$. Also by Lemma 5.4.1

$$V(x_1, \ldots, x_n, z) = V(\hat{x}_2, \ldots, \hat{x}_n, \hat{z})(z - x_1). \tag{5.5.1}$$

Suppose that our theorem is valid for $m = n - 1$. Then

$$V(\hat{x}_2, \ldots, \hat{x}_n, \hat{z}) = (z'_n - y'_n) \ldots (z'_2 - y'_2),$$

where

$$\begin{aligned}
z'_2 &= \hat{z},\\
y'_2 &= \hat{x}_2,\\
z'_k &= V(\hat{x}_2,\dots,\hat{x}_{k-1},\hat{z})\hat{z}V^{-1}(\hat{x}_2,\dots,\hat{x}_{k-1},\hat{z}),\\
y'_k &= V(\hat{x}_2,\dots,\hat{x}_k)\hat{x}_kV^{-1}(\hat{x}_2,\dots,\hat{x}_k) \text{ for } k=3,\dots,n.
\end{aligned}$$

It is enough to show that

$$z'_k = z_k, \text{ and } y'_k = y_k \text{ for } k=2,\dots,n.$$

For $k=2$, this is obvious. By Lemma 5.4.1

$$V(\hat{x}_2,\dots,\hat{x}_{k-1},\hat{z}) = V(x_1,\dots,x_{k-1},z)(z-x_1)^{-1},$$

and by definition $\hat{z}=(z-x_1)z(z-x_1)^{-1}$. So,

$$\begin{aligned}
z'_k = \{V(x_1,\dots,x_{k-1},z)(z-x_1)^{-1}\}(z-x_1)z(z-x_1)^{-1}\cdot\\
\cdot\{(z-x_1)V^{-1}(x_1,\dots,x_{k-1},z)\} = z_k \text{ for } k=3,\dots,n.
\end{aligned}$$

Similarly, $y'_k=y_k$ for $k=3,\dots,n$ and from (3.4) we have

$$V(x_1,\dots,x_n,z) = (z_n-y_n)\dots(z_2-y_2)(z_1-y_1).$$

□

5.6. Another expression for the coefficients in Vieta decomposition. Another expression for the coefficients $a_1,\dots a_n$ in Vieta decomposition of $V(x_1,\dots,x_n,z)$ may be obtained from Proposition 1.5.1.

Theorem 5.6.1. [GKLLRT]*:*

$$V(x_1,\dots,x_n,z) = z^n + a_1z^{n-1}+\dots+a_n,$$

where for $k=1,\dots,n$

$$a_k = -\begin{vmatrix} x_1^n & \dots & x_n^n\\ & \dots & \\ x_1^{n-k+1} & \dots & x_n^{n-k+1}\\ x_1^{n-k-1} & \dots & x_n^{n-k-1}\\ & \dots & \\ 1 & \dots & 1\end{vmatrix}_{1n} \cdot \begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1}\\ & \dots & \\ x_1^{n-k} & \dots & x_n^{n-k}\\ & \dots & \\ 1 & \dots & 1\end{vmatrix}_{kn}^{-1}. \tag{5.6.1}$$

From Theorem 5.6.1 we will get the Bezout and Vieta formulas expressing the coefficients of the equation

$$z^n + a_1z^{n-1}+\dots+a_n = 0 \tag{5.6.2}$$

5.7. The Bezout and Vieta Theorems.

Lemma 5.7.1. *Suppose that $x_1, \dots, x_n$ is an independent set of roots of the equation (5.6.2). Then the coefficients $a_1, \dots, a_n$ may be written in the form (5.6.1).*

Proof. Consider a system of right linear equations

$$x_i^n + a_1 x_i^{n-1} + \dots + a_{n-1} x_i + a_n = 0, i = 1, \dots, n$$

with unknowns $a_1, \dots, a_n$ and use Cramer's rule. □

Theorem 5.7.2. (Noncommutative Bezout Theorem). *Let $x_1, \dots, x_n$ be an independent set of roots of equation (5.6.2). In notations of Theorem 5.2.1*

$$z^n + a_1 z^{n-1} + \dots + a_n = (z_n - y_n) \dots (z_1 - y_1).$$

Proof. Use Lemma 5.7.1, Theorem 5.6.1 and Theorem 5.2.1. □

Theorem 5.7.2. *(Noncommutative Vieta Theorem)* [GR3] *Let $x_1, \dots, x_n$ be an independent set of roots of equation (3.6). Then coefficients $a_1, \dots, a_n$ of the equation are given by formulas (5.2.3).*

Proof Use Lemma 3.2.1, Theorem 3.1.4 and Theorem 3.1.2 □

VI. Noncommutative symmetric functions

A general theory of noncommutative symmetric functions was developed in [GKLLRT]. In fact, [GKLLRT] was devoted to the study of different systems of multiplicative and linear generators in a free algebra **Sym** generated by any system of noncommuting variables $\Lambda_i, i = 1, 2, \dots$. In [GKLLRT] these variables were called elementary symmetric functions but the theory was developed independently of the origin of Λ_i's.

In this section we apply the general theory to noncommutative symmetric functions generated by specific Λ_i's. As in the commutative case, they depend of a set of roots of a polynomial equation.

6.1. Formal noncommutative symmetric functions.

This theory was started in [GKLLRT] and developed in several papers (see, for example, [KLT LST]). An extensive review was given in [Thi]. Here we just remind some basic constructions.

The algebra **Sym** is a free graded associative algebra over a field F generated by an infinite sequence of indeterminants (Λ_k), deg $\Lambda_k = k$, $k \geq 1$. The homogeneous component of deg n is denoted by $\mathbf{Sym}_n$.The direct sum $\oplus_{n\geq 1}\mathbf{Sym}_n$ is denoted by $\mathbf{Sym}_+$. Initially Λ_k's were regarded as as the elementary symmetric functions of some virtual set of arguments. A natural set of arguments was found later [GR3, GR4].

Recall that the algebra of commutative symmetric functions Sym in variables $t_1, t_2, t_3, \dots$ is defined as the inductive limit of algebras of symmetric polynomails Sym_N in commuting variables $t_1, \dots, t_N$.

Let $\mathcal{I}_N$ be an ideal **Sym** generated by all Λ_k, $k > N$. The canonical map π_N of **Sym** onto the algebra Sym_N of symmetric polynomials in commuting variables $t_1, \dots, t_N$ is given by the formulas

(6.1.1a) $$\pi_N(\Lambda_k) = \sum_{i_1 < \dots < i_k} t_{i_1} \dots t_{i_k} \quad \text{for} \quad k \leq N,$$

(6.1.1b) $$\pi_N(\Lambda_k) = 0 \quad \text{for} \quad k > N.$$

Obviously, $\pi_N(\Lambda k)$ is the k-th elementary symmetric function in N variables and $\pi_N(\mathcal{I}_N) = 0$. The inductive limit of π_N gives the canonical homomorphism $\pi :$ **Sym** $\rightarrow Sym$.

Let t be a formal variable commutating with all Λ_k's. Set $\Lambda_0 = 1$. Define the generating series

$$\lambda(t) := \sum_{k \geq 0} \Lambda_k t^k.$$

Definition 6.1.1. *The complete homogeneous symmetric functions are defined by*

$$\sigma(t) := \sum_{k \geq 0} S_k t^k = \lambda(-t)^{-1}.$$

The power sums symmetric functions of the first kind Ψ_k are defined by

$$\sum_{k \geq 1} \Psi_k t^{k-1} := \sigma(t)^{-1} \frac{d}{dt} \sigma(t).$$

The power sums symmetric functions of the second kind Φ_k are defined by

$$\sum_{k \geq 1} \Phi_k t^{k-1} := \frac{d}{dt} \log \ \sigma(t).$$

By using formula (6.1.1) one can prove that $\pi_N(S_k)$ is the k-th complete symmetric function and $\pi_N(\Psi_k) = \pi_N(\Phi_k)$ is the k-th power symmetric function in N commuting variables.

Definition 6.1.1 leads to the following quasideterminantal formulas.

Proposition 6.1.2. *For every $k \geq 1$, one has*

$$S_k = (-1)^{k-1} \begin{vmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_{k-1} & \boxed{\Lambda_k} \\ 1 & \Lambda_1 & \dots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \dots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Lambda_k = (-1)^{k-1} \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 & \\ S_2 & S_1 & 1 & \dots & 0 & \\ S_3 & S_2 & S_1 & 1 & \dots & 0 \\ & & \dots & & & \\ \boxed{S_k} & S_{k-1} & S_{k-2} & \dots & S_1 & \end{vmatrix},$$

$$kS_k = \begin{vmatrix} \Psi_1 & \Psi_2 & \dots & \Psi_{k-1} & \boxed{\Psi_k} \\ -1 & \Psi_1 & \dots & \Psi_{k-2} & \Psi_{k-1} \\ 0 & -2 & \dots & \Psi_{k-3} & \Psi_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & -n+1 & \Psi_1 \end{vmatrix},$$

$$k\Lambda_k = \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ \Psi_2 & \Psi_1 & 2 & \dots & 0 \\ \Psi_3 & \Psi_2 & \Psi_1 \dots & 0 & \\ & & \dots & & \\ \boxed{\Psi_k} & \Psi_{k-1} & \Psi_{k-2} & \dots & \Psi_1 \end{vmatrix},$$

$$\Psi_k = \begin{vmatrix} \Lambda_1 & 2\Lambda_2 & \dots & (k-1)\Lambda_{k-1} & \boxed{k\Lambda_k} \\ 1 & \Lambda_1 & \dots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \dots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \dots & & \\ 0 & 0 & \dots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Psi_k = \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 & \\ 2S_2 & S_1 & 1 & \dots & 0 & \\ 3S_3 & S_2 & S_1 & 1 & \dots & 0 \\ & & \dots & & & \\ \boxed{kS_k} & S_{k-1} & S_{k-2} & \dots & S_1 & \end{vmatrix}.$$

Each of the four sequences (Λ_k), (S_k), (Ψ_k), and (Φ_k) constitues a set of generators in **Sym**. Therefore, aach of the four sets of products $F_{i_1} \dots F_{i_N}$, $i_1, \dots, i_N \geq 1$, where F_k equals to Λ_k, S_k, Ψ_k, or Φ_k, $k = 1, 2, \dots$ constitues a linear basis in $\mathbf{Sym}_+$. Another important example of a linear basis in $\mathbf{Sym}_+$ is given by *ribbon Schur functions.*

6.2. Ribbon Schur Function.

Let $I = (i_1, \dots, i_k)$, $i_1, \dots, i_k \geq 1$ be an ordered set.

Definition 6.2.1. [GKLLRT] *The ribbon Schur function R_I is defined by*

$$R_I = (-1)^{k-1} \begin{vmatrix} S_{i_1} & S_{i_1+i_2} & S_{i_1+i_2+i_3} & \dots & \boxed{S_{i_1+\dots+i_k}} \\ 1 & S_{i_2} & S_{i_2+i_3} & \dots & S_{i_2+\dots+i_k} \\ 0 & 1 & S_{i_3} & \dots & S_{i_3+\dots+i_k} \\ & & & \dots & \\ 0 & 0 & 0 & \dots & S_{i_k} \end{vmatrix}.$$

In [GKLLRT] a similar description of R_I as quasideterminants of matrices with entries Λ_k's was given.

Proposition 6.1.2 implies that $R_I = S_k$ if $I = \{k\}$ and $R_I = \Lambda_k$ if $i_1 = \dots = i_k = 1$. The homomorphisms π_N map R_I's into ribbon Schur functions was introduced by MacMahon [M].

Theorem 6.2.2. [GKLLRT] *Ribbon Schur functions R_I's constitue a linear basis in* $\mathbf{S}ym$.

The homological relations for quasideterminants imply the following multiplication rule, whose commutative version is due to MacMahon [M]. Let $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$, $i_p \geq 1$, $j_q \geq 1$ for all p, q. Set $I+J = (i_1, \dots, i_{r-1}, i_r+j_1, j_2, \dots, j_s)$ and $I \cdot J = (i_1, \dots, i_r, j_1, \dots, j_s)$.

Theorem 6.2.3. [GKLLRT]

$$R_I R_J = R_{I+J} + R_{I.J}.$$

In commutative case ribbon Schur functions are not linearly independent. It reflects the fact that the kernel of $\pi : \mathbf{Sym} \to Sym$ is generated by linear combinations of R_I's. A description of Kerπ is given by the following theorem.

Let $I = (i_1, \dots, i_r$ and $J = (j_1, \dots, j_s)$. We say that $I \leq J$ if $i_1 = j_1+j_2+\dots+j_{t_1}$, $i_2 = j_{t_1+1} + \dots + j_{t_2}$, $\dots$, $i_s = j_{t_{s-1}+1} + \dots + j_s$. Recall that I is an ordered set. Denote by $u(I)$ the corresponding unordered set.

Theorem 6.2.4. *The kernel of π is linearly generated by differences*

$$\Delta_{J,J'} = \sum_{I \leq J} R_I - \sum_{I' \leq J} R_{I'},$$

where $u(J) = u(J')$.

Example. Let $J = (12)$, $J' = (21)$. Then $\Delta_{J,J'} = (R_{12} + R_3) - (R_{21} + R_3) = R_{12} - R_{21}$ and $\pi(R_{12} - R_{21}) = 0$.

A naturality of ribbon Schur functions R_I can be explained via the following construction.

6.3. Algebras with two multiplications.

The relations between functions Λ_k and functions S_k may be illuminated by defining two associative products on the ideal $\mathbf{Sym}_+$. Recall, that the monomials $(\Lambda_{i_1} \ldots \Lambda_{i_r})$ as well as monomials $(S_{i_1} \ldots S_{i_r})$ constitue linear basises for the ideal.

Definition 6.3.1. *Define a linear map* $*_1 : \mathbf{Sym}_+ \otimes \mathbf{Sym}_+ \to \mathbf{Sym}_+$ *by*

$$(\Lambda_{i_1} \ldots \Lambda_{i_r}) *_1 (\Lambda_{j_1} \ldots \Lambda_{j_s}) = \Lambda_{i_1} \ldots \Lambda_{i_{r-1}} \Lambda_{i_r + j_1} \Lambda_{j_2} \ldots \Lambda_{j_s}$$

and a linear map $*_2 : \mathbf{Sym}_+ \otimes \mathbf{Sym}_+ \to \mathbf{Sym}_+$ *by*

$$(S_{i_1} \ldots S_{i_r}) *_1 (S_{j_1} \ldots S_{j_s}) = S_{i_1} \ldots S_{i_{r-1}} S_{i_r + j_1} S_{j_2} \ldots S_{j_s}.$$

Write $ab = a * b$ for $a, b \in \mathbf{S}ym_+$. Then it is clear that $a * (b *_i c) = (a * b) *_i c$, $a *_i (b * c) = (a *_i b) * c$ and $a *_i (b *_i c) = (a *_i b) *_i c$ for all $a, b, c \in \mathbf{S}ym_+$, $i = 1, 2$. When it is convinient we will write $* = *_0$.

In fact, $*$, $*_1$ and $*_0$ are closely related.

Lemma 6.3.2. $* = *_1 + *_2$.

Proof.

$$\lambda(-t)^{-1} = (1 - \sum_{i>0} \Lambda_i t^i)^{-1} = 1 + \sum_{j>0} \sum_{i_1 + \ldots i_l = j} (-1)^{l+j} \Lambda_{i_1} \ldots \Lambda_{i_l} t^j.$$

Since $\Lambda_i = \Lambda_1 *_1 \Lambda_1 *_1 \cdots *_1 \Lambda_1$, where there are $i - 1$ occurances of $*_1$, we have that the coefficient of t^j in $\lambda(-t)^{-1}$ is

$$\sum_{u_1, \ldots u_{j-1} \in \{0,1\}} (-1)^{k+1} \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \cdots *_{u_{j-1}} \Lambda_1,$$

where k is the number of u_t equal to 0.

Since $1 + \sum S_i t^i = \lambda(-1)^{-1}$ we have that

$$S_i = \sum_{u_1, \ldots u_{j-1} \in \{0,1\}} (-1)^{k+1} \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \cdots *_{u_{j-1}} \Lambda_1.$$

It follows that $S_i *_1 S_j = S_i S_j - S_{i+j}$ and hence that $*_1 = * - *_2$, as required.

This gives an isomorphism of vector spaces

$$\theta : \mathbf{Sym}_+ = \sum_{l\geq 1}(\mathbf{Sym}_+)_l \to F\langle U\rangle = \sum_{l\geq 0} F\langle U\rangle_l$$

where U is the two dimensional vector space with basis $\{u_0, u_1\}$.

This isomorphism shifts degree, restricting to an isomorphism between $(\mathbf{Sym}_+)_{l+1}$ and $F\langle U\rangle_l$.

The monomial $u_{l_1} \dots u_{i_l}$ in $F\langle U\rangle_l$ is mapped to $\Lambda_1 *_{i_1} \Lambda_1 *_{i_2} \cdots *_{i_l} \Lambda_1 \in (\mathbf{Sym}_+)_{l+1}$.

If $u_2 = u_0 - u_1$, and ϕ is the automorphism of U defined by $\phi(u_0) = u_0$, $\phi(u_1) = u_2$, then ϕ extends to an endomorphism of $F\langle U\rangle$. Then we see that that

$$\theta^{-1}\phi\theta(\Lambda_{i_1} \dots \Lambda_{i_r}) = S_{i_1} \dots S_{i_r},$$

recovering Proposition 4.13 of [GKLLRT].

Similarly, the identity

$$a^{n-1} + (-1)^n b^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} a^{k-1}(a+b)b^{n-k-1} = 0,$$

valid in any associative algebra, becomes upon setting $a = u_2$, $b = u_1$ and applying θ^{n-1}, $0 = \sum_{k=0}^{n} (-1)^{n-k} \Lambda_k S_{n-k}$.

These identities may be written in terms of quasideterminants (Proposition 6.1.2).

Write each finite sequence ω of symbols 1 and 2 as $(2\dots2)(1)(2\dots2)(1)\dots$ bracketing maximal number of 2's and each 1. Labell the sequence as $I_\omega = (i_1, \dots, i_k)$ where i_m is the number of elements in the m-th bracket. For example, sequences 2212 and 12212 will be labelled as $2, 1, 1$ and and $1, 2, 1, 1$ correspondingly. The sequence 222 is labelled as $I = (3)$ and the sequence 111 is labelled as $I = (111)$.

The bijective correspondences between monomials in $F\langle U\rangle$ and finite sequences in $1, 2$'s and between the finite sequences and ordered sets $I = (i_1, \dots, i_k)$. Thus, to any monomial $u_{j_1} \dots u_{j_l} \in F\langle U\rangle$ corresponds a ribbon Schur function R_I establishing a bijective correspondence between two natural linear basises.

6.4. Quasi-Schur functions.

Quasi-Schur functions were defined in [GKLLRT]. They are not elements in **Sym** but in the free-skew field generated by $S_1, S_2 \dots$. Let $I = (i_1, \dots, i_k)$.

Definition 6.4.1. *Define $\check{S}_I$ by setting*

$$\check{S}_I = (-1)^{k-1} \begin{vmatrix} S_{i_1} & S_{i_2+1} & \dots & \boxed{S_{i_k+k-1}} \\ S_{i_1-1} & S_{i_2} & \dots & S_{i_k+k-2} \\ & & \dots & \\ S_{i_1-k+1} & S_{i_2-k+2} & \dots & S_{i_k} \end{vmatrix}. \tag{6.4.1}$$

The expression $\check{S}_I$ is called the generalized quasi-Schur function. A generalized quasi-Schur function $\check{S}_I$ is called the quasi-Schur function if $I = (i_1, \dots, i_k)$ is a partition, i.e. weakly increasing sequence of nonnegative sequences.

In particular, $\check{S}_k = S_k$ and $\check{S}_{1^k} = \Lambda_k$ where $1^k := (1 \dots 1)$ with k occurances of 1.

Definition 6.4.1 is a noncommutative analog of Jacobi-Trudi formula written in French notations. In the commutative case $\check{S}_I$ for a partition I is the *ratio* of two Schur functions S_I/S_J where $J = (i_1 - 1, \dots, i_{k-1} - 1)$. It shows that in general $\check{S}$ is not polynomial in S_k's.

Remark. The homological relations and trasformation properties of quasideterminants imply that any generalized quasi-Schur function $\check{S}_I$ can be expressed as a rational function in the quasi-Schur functions. For example

$$\check{S}_{42} = -\begin{vmatrix} S_4 & \boxed{S_3} \\ S_3 & S_2 \end{vmatrix} = -\begin{vmatrix} \boxed{S_3} & S_4 \\ S_2 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & \boxed{S_4} \\ S_2 & S_3 \end{vmatrix} S_3^{-1} S_2 = \check{S}_{33} S_3^{-1} S_2.$$

6.5. Symmetric functions in noncommutative variables. Fix n independent indeterminants $x_1, x_2, \dots, x_n$ and construct new variables $y_1, \dots, y_n$ which are *rational* functions in $x_1, \dots, x_n$. Recall that

$$V(x_1, \dots, x_k) = \begin{vmatrix} x_1^{k-1} & \dots & \boxed{x_n^{k-1}} \\ x_1^{k-2} & \dots & x_n^{k-2} \\ & \dots & \\ 1 & \dots & 1 \end{vmatrix}.$$

Set

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= V(x_1, x_2) x_2 V(x_1, x_2)^{-1} = (x_2 - x_1) x_2 (x_2 - x_1)^{-1}, \\ &\dots \\ y_n &= V(x_1, \dots, x_n) x_n V(x_1, \dots, x_n)^{-1} \end{aligned}$$

In the commutative case $x_i = y_i, i = 1, \dots n$. In the noncommutative case they are obviously different.

Remark. Consider the free skew-field R generated by $x_1, \dots, x_n$. Define on R differential operators ∂_i by formulas $\partial_i x_j = \delta_{ij}$ and satisfying Leibniz rule $\partial_i(fg) = \partial_i(f)g + f\partial_i(g)$, for $i = 1, \dots n$.

It easy to see that $\partial_i y_j \neq \delta_{ij}$. Consider, however, $\partial = \partial_1 + \dots + \partial_n$. Then

$$\partial(V(x_1, \dots, x_k)) = 0, \ k = 2, \dots n$$

and

$$\partial(y_i) = \partial(x_i) = 1, \ i = 1, \dots, n.$$

Elementary symmetric functions.

Defintion 6.5.1. *The functions*

$$\Lambda_1(x_1,\dots,x_n)=y_1+y_2\dots+y_n,$$
$$\Lambda_2(x_1,\dots,x_n)=\sum_{i<j}y_jy_i,$$
$$\dots$$
$$\Lambda_n(x_1,\dots,x_n)=y_n\dots y_1$$

are called elementary symmetric functions in $x_1,\dots,x_n$.

In the commutative case these functions are the standard elementary symmetric functions of $x_1,\dots,x_n$. By the Noncommutative Vieta Theorem (Theorem 5.7.1) $\Lambda_i(x_1,\dots,x_n)=(-1)^ia_i, i=1,\dots,n$ where $x_1,\dots,x_n$ are roots of the equation

$$x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n=0.$$

This implies

Proposition 6.5.2. *The functions* $\Lambda_i(x_1,\dots,x_n)$ *are symmetric in* $x_1,\dots,x_n$.

Denote by $\overline{\mathbf{Sym}}_n$ the subalgebra of of algebra of rational functions in $x_1,\dots,x_n$ generated by $\Lambda_k(x_1,\dots,x_n)$, $k=1,\dots,n$. Construct a surjective homomorphism

$$\phi:\mathbf{Sym}\to\overline{\mathbf{Sym}}_n \tag{6.5.1}$$

by setting $\phi(\Lambda_k)=\Lambda_k(x_1,\dots,x_n)$.

Theorem 6.5.3. *The kernel of* ϕ *is generated by* Λ_k *for* $k>n$.

Remark. The order of $y_1,\dots,y_n$ is essential in the definition of $\Lambda_i(x_1,\dots,x_n)$, $i=1,\dots,n$. For example, $\Lambda_2(x_1,x_2)=y_2y_1$ is symmetric in x_1,x_2 but the product y_1y_2 is not symmetric. To see this set $\Delta=x_2-x_1$. The product y_1y_2 is symmetric in x_1,x_2 if and only if $x_1\Delta^2=\Delta^2x_1$.

Complete symmetric functions.

Definition 6.5.4. *The functions*

$$S_k(x_1,\dots,x_n)=\sum_{i_1\le i_2\le\dots\le i_k}y_{i_1}\dots y_{i_k}, k=1,2,3,\dots$$

are called complete symmetric functions in $x_1,\dots,x_n$.

In the commutative case these functions are the standard complete symmetric functions in $x_1,\dots,x_n$.

Let t be a formal variable commuting with $x_i, i = 1, \dots n$. Define the generating functions

$$\lambda(t) = 1 + \Lambda(x_1, \dots, x_n)t + \dots + \Lambda_n(x_1, \dots, x_n)t^n$$

and

$$\sigma(y) = 1 + \sum_i S_i(x_1, \dots, x_n)t^i = \lambda(-t)^{-1}$$

.

Proposition 6.5.5.

$$\sigma(t)\lambda(-t) = 1.$$

In the commutative case the $S_k(x_1, \dots, x_n)$'s are the standard complete symmetric functions.

Proposition 6.5.6. *Functions $S_k(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$.*

Proof. Apply Proposition 6.5.5, Theorem 6.5.3 and Proposition 6.1.2.

Remark. The order of $y's$ in the definition of S_k's is essential: $S_2(x_1, x_2) = y_1^2 + y_1y_2 + y_2^2$ is symmetric in $x_1, \dots, x_n$ but $y_1^2 + y_2y_1 + y_2^2$ is not symmetric. Otherwise, the product y_1y_2 will be also symmetric. The remark after Theorem 6.5.3 shows that this is not true.

Ribbon Schur Functions.
We define "ribbon Schur functions with arguments" $R_I(x_1, \dots, x_n)$ similar to Definition 6.2.1. One have to replace R_I by $R_I(x_1, \dots, x_n)$ and S_k by $S_k(x_1, \dots, x_n)$. Evidently, ribbon Schur functions $R_I(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$ and form a linear basis in $\overline{\mathbf{Sym}}_n$.

Proposition 6.5.7. *$R_I(x_1, \dots, x_n) = \phi(R_I)$ where ϕ is defined by formula (6.5.1).*

To express $R_I(x_1, \dots, x_n)$ as a sum of monomials in $y_1, \dots, y_n$ we need a vocabulary. Let $w = a_{i_1} \dots a_{i_k}$ be a word in ordered letters $a_1 < \dots < a_n$. An integer m is called a descent of w if $1 \le m \le k-1$ and $i_m > i_{m+1}$.

Let $J = (j_i, \dots, j_k)$ be a set of positive integers.

Theorem 6.5.8.

$$R_J(x_1, \dots, x_n) = \sum y_{i_1} \dots y_{i_m}, \tag{6.5.2}$$

where the sum is running over all words $w = y_{i_1} \dots y_{i_m}$ whose descent set is exactly equal to $\{j_1, j_1 + j_2, \dots, j_1 + j_2 + \dots + j_{k-1}\}$.

The proof of the theorem was essentially given in [GKLLRT] in Section VII.

6.6. Main Theorem for noncommutative symmetric functions.

In the commutative case the main theorem of the theory of symmetric functions says that every symmetric polynomial of n variables is a polynomial of (elementary) symmetric functions of these variables. Its analogue for a noncommutative case is given by the following theorem. Denote $\Lambda_k(x_1,\dots,x_n$ as $\Lambda_k(X)$, $k=1,\dots,n$.

Theorem 6.6.1. [W]. *Let a polynomial $P(y_1,\dots,y_n)$ over $\mathbb{Q}$ be symmetric as a function in $x_1,\dots,x_n$. Then $P(y_1,\dots,y_n)=Q(\Lambda_1,\dots,\Lambda_n)$ where Q is a noncommutative polynomial over $\mathbb{Q}$.*

Remark. Recall that $P(y_1,\dots,y_n)$ is a polynomial in y's and not in x's. Roughly speaking, noncommutative symmetric polynomials are not polynomials in the natural variables.

Corollary 6.6.2. *Any polynomial with rational coefficient function $P(y_1,\dots,y_n)$ is symmetric in $x_1,\dots,x_n$ if and only iff $P(y_1,\dots,y_n)$ as a function of $x_1,\dots,x_n$ is a linear combination of $R_J(x_1,\dots,x_n)$.*

6.7. Quasi-Plücker coordinates of Vandermonde matrices and symmetric functions.

We study here right quasi-Plücker coordinates $r_{ij}^{i_1,\dots,i_{n-1}}(V_n)$ where $V_n=(x_j^i)$, $i\in\mathbb{Z}$, $j=1,\dots,n$ is Vandermonde matrix, x_j, $j=1,\dots,n$ are noncommuting variables. Matrices V_n have n columns and infinetly many rows. The following proposition shows an importance of such coordinates.

Proposition 6.7.1.

$$r_{n+k-1,n-1}^{0,1,\dots,n-2}(V_n)=S_k(x_1,\dots,x_n),\quad k=0,1,2,\dots,$$

$$r_{n,n-k}^{0,1,\dots,n-k-1,n-k+1,\dots,n-1}(V_n)=(-1)^{k-1}\Lambda_k(x_1,\dots,x_n),\quad k=0,1,\dots,n.$$

Examples.

$$S_k(x_1,x_2)=\begin{vmatrix} x_1^{k+1} & \boxed{x_2^{k+1}} \\ 1 & 1\end{vmatrix}\begin{vmatrix} x_1 & \boxed{x_2} \\ 1 & 1\end{vmatrix}^{-1}=(x_2^{k+1}-x_1^{k+1})(x_2-x_1)^{-1}=$$

$$=y_2^k+y_2^{k-1}y_1+\dots+y_2y_1^{k-1}+y_1^k,$$

where $y_1=x_1$, $y_2=(x_2-x_1)x_2(x_2-x_1)^{-1}$.

$$\Lambda_1(x_1,x_2)=\begin{vmatrix} x_1^2 & \boxed{x_2^2} \\ 1 & 1\end{vmatrix}\begin{vmatrix} x_1 & \boxed{x_2} \\ 1 & 1\end{vmatrix}^{-1}=(x_2^2-x_1^2)(x_2-x_1)^{-1}=y_1+y_2,$$

$$\Lambda_2(x_1,x_2) = -\begin{vmatrix} x_1^2 & \boxed{x_2^2} \\ x_1 & x_2 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ 1 & \boxed{1} \end{vmatrix}^{-1} = (x_2^2 - x_1x_2)(1 - x_1^{-1}x_2)^{-1} = y_2y_1.$$

Remark. Formulas for S_k in Proposition 1 are valid for all $k \in \mathbb{Z}$.

An important "periodic" property of quasi-Plücker coordinates of Vandermonde matrices is given by the following proposition.

Proposition 6.7.2.

$$r_{ij}^{i_1,\dots,i_{n-1}}(V_n) = r_{i+k,j+k}^{i_1+k,\dots,i_{n-1}+k}(V_n)$$

for any $k \in \mathbb{Z}$.

Proposition 6.7.1 can be generalized as follows. Recall that in section 6.4 we defined generalized quasi-Schur functions. Let $I = \{i_1, i_2, \dots, i_m\}$.

Proposition 6.7.3. *Let $I = \{i_1, i_2, \dots, i_m\}$ and $i_1 \geq i_2 \geq \dots \geq i_m$. Set $J = \{0, 1, \dots, n-m-1, n-m+i_1, \dots, n-2+i_{m-1}\}$. Then*

$$r_{n-1+i_1,n-m}^{J}(V_n) = \hat{S}_I.$$

Examples. In particular,

$$r_{n+k-1,n-1}^{01\dots n-2} = S_k$$

and

$$r_{n-1,n-k}^{01\dots n-k-1,n-k+1,\dots,n-2} = \Lambda_k.$$

VII. Universal quadratic algebras associated with pseudo-roots of noncommutative polynomials and noncommutative differential polynomials

7.1. Pseudo-roots of noncommutative polynomials.

During the last years the authors introduced and studied universal algebras associated with pseudo-roots of noncommutative polynomials. The results appeared in our papers [GRW, GGR, GGRWS].

Let C be an algebra with a unit and $P(t) \in C[t]$ be a polynomial (where t is a formal variable commuting with elements of C). We say that an element $c \in C$ is a *pseudo-root* of $P(t)$ if there exist polynomials $L_c(t), R_c(t) \in C[t]$ such that $P(t) = L_c(t)(t-c)R_c(t)$. If $P(t) = a_0t^n + a_1t^{n-1} + \dots + a_{n-1}t + a_n$ and c is a pseudo-root of $P(t)$ with $R_c(t) = 1$ then

$$a_0c^n + a_1c^{n-1} + \dots + a_{n-1}c + a_n = 0,$$

i.e. c is a root of the polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ (where x is a non-commuting indeterminant). Our theory shows that an investigation of noncommutative polynomials is impossible without studying their pseudo-roots.

Let $x_1, \ldots, x_n$ be roots of a *generic* monic polynomial $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ over an algebra C. There are two important classical problems: a) to express the coefficients $a_1, \ldots, a_n$ via the roots, and b) to determine all factorizations of $P(x)$, or $P(t)$.

When C is a division algebra, the first problem was solved in [GR3,GR4] using the theory of quasideterminants and presented here in Section V. Let $V(x_{i_1}, \ldots, x_{i_k})$ be the Vandermonde quasideterminant defined by the sequence $x_{i_1}, \ldots, x_{i_k}$ ([GR3, GR5].) For any ordering $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$, elements $x_{\emptyset, i_1} = x_{i_1}$,

$$x_{\{i_1,i_2,\ldots,i_{k-1}\},i_k} = V(x_{i_1}, \ldots, x_{i_k})x_{i_k}V(x_{i_1}, \ldots, x_{i_k})^{-1}, \quad k = 2, \ldots, n,$$

in C were constructed such that for every $m = 1, \ldots, n$,

$$(-1)^m a_m = \sum_{j_1 > j_2 > \ldots j_m} y_{j_1} y_{j_2} \ldots y_{j_m}, \tag{7.1.1}$$

where $y_1 = x_{i_1}$, $y_k = x_{\{i_1,\ldots,i_{k-1}\},i_k}$, $k = 2, \ldots, n$.

It is surprising that the left-hand side in formula (7.1.1) does not depend on the ordering of $\{1, \ldots, n\}$ but the right-hand side *a priori* depends on the ordering. The independence of the right-hand side in (7.1.1) on the ordering of $\{1, \ldots, n\}$ was a key point in the theory of noncommutative symmetric functions developed in [GR3, GR4] and presented here in Section VI.

The element $x_{\{i_1,\ldots,i_{k-1}\},i_k}$ has an interesting structure. It is symmetric in $x_{i_1}, \ldots, x_{i_{k-1}}$. It is a rational function in $x_{i_1}, \ldots, x_{i_k}$ containing $k-1$ inversions in the generic case. In other words, $x_{\{i_1,\ldots,i_{k-1}\},i_k}$ is a rational expression of height $k-1$.

Set $A_k = \{i_1, \ldots, i_{k-1}\}$ for $k = 2, \ldots, n$. Recall that formulas (7.1.1) are equivalent to the decomposition

$$P(t) = (t - x_{A_n,i_n})(t - x_{A_{n-1},i_{n-1}}) \ldots (t - x_{i_1}). \tag{7.1.2}$$

This formula shows that elements $x_{A,i}$ are pseudo-roots of $P(t)$, and in the generic case the polynomial $P(t)$ has at least $n!$ different pseudo-roots. We study all such pseudo-roots together by constructing a universal algebra of pseudo-roots Q_n.

7.2. Universal algebra of pseudo-roots.

It is easy to see that elements $x_{A,i}$, $i \notin A$ satisfy simple relations:

$$x_{A \cup \{i\},j} + x_{A,i} = x_{A \cup \{j\},i} + x_{A,j}, \tag{7.2.1a}$$

(7.2.1b) $$x_{A\cup\{i\},j}\cdot x_{A,i}=x_{A\cup\{j\},i}\cdot x_{A,j},$$

for all $A\subseteq\{1,\dots,n\}$, $i,j\notin A$.

In order to avoid inversions and to make our construction independent of the algebra C, we define the universal algebra of pseudo-roots Q_n over a field F to have generators $z_{A,i}$, $A\subseteq\{1,\dots,n\}$, $i\notin A$, and relations corresponding to (7.2.1) (with x replaced by z).

This is a natural algebra containing a free associative algebra generated by the $z_i=z_{\emptyset,i}$, $i=1,\dots,n$.

We studied the internal structure of algebra Q_n. It has a natural derivation and a natural anti-involution. For each pair $A,B\subseteq\{1,\dots,n\}$, with $A\cap B=\emptyset$ we define an element $z_{A,B}\in Q_n$ such that $z_{\emptyset,\emptyset}=0$ and $z_{A\cup\{i\},B}-z_{A,B\cup\{i\}}=z_{A,B}$ for $i\notin A,B$.

When B contains more than one element, the element $z_{A,B}$ is "invisible" in the commutative case, i.e., under the natural commutative specialization of Q_n, its image is zero.

In [GRW] it was proved that

$$z_{A,i}-z_i=\sum_{\emptyset\neq C\subseteq A} z_{\emptyset,C\cup\{i\}}.$$

Under the natural commutative specialization

(7.2.2) $$\pi_n: Q_n\to F[t_1,\dots,t_n],\quad \pi(z_{A,i})=t_i,\quad i=1,\dots,n,$$

the image of $z_{A,i}-z_i$ equals zero. This shows that the elements from the right hand side in (7.2.2) measure the "noncommutativity" of $z_{A,i}$. Moreover, the "degree of noncommutativity" carried by $z_{\emptyset,B}$ depends on the size of B.

Remark. One can try to construct a "noncommutative logic" using these elements; roughly speaking, the element $z_{A,B}$ is marked "yes" for indices from A, "no" for indices from B and "do not know" for indices from $\{1,\dots,n\}\setminus(A\cup B)$.

7.3. Basises in the universal algebra of pseudo-roots.

One can see that elements $z_{A,\emptyset}$, and similarly, the elements $z_{\emptyset,A}$, for all $A\subseteq\{1,\dots,n\}$, $A\neq\emptyset$, constitute a basis for the subspace of Q_n spanned by all generators. These elements satisfy simple quadratic relations.

The algebras Q_n seem to be very interesting. They may be viewed as a Galois type extension of the algebra of symmetric functions in n variables. Our study of Q_n is based on our construction of a basis for Q_n [GRW]. It is a hard combinatorial problem.

For $A\subseteq\{1,...,n\}$ let $min(A)$ denote the smallest element of A. Then set $A^{(0)}=A$, $A^{(1)}=A\backslash\{min(A)\}$, $A^{(i+1)}=(A^{(i)})^{(1)}$. Set $r(A)=z_{\emptyset,A}$.

Theorem 7.3.1. *The set of all monomials*

$$r_{A_1^{(0)}} r_{A_1^{(1)}} \dots r_{A_1^{(j_1)}} r_{A_2^{(0)}} r_{A_2^{(1)}} \dots r_{A_2^{(j_2)}} \dots r_{A_l^{(0)}} r_{A_l^{(1)}} \dots r_{A_l^{(j_l)}}$$

where $A_1, \dots, A_l \subseteq \{1, \dots, n\}$ *and, for* $1 \le i \le l-1$, *either* $A_{i+1} \not\subseteq A_i$ *or* $|A_{i+1}| \neq |A_i| - j_i - 1$, *is a basis in* Q_n.

Remark. There are natural problems related to this construction. It would be interesting to study in details combinatorial properties of the basis and to give constructions of similar bases in Q_n using different techniques including noncommutative Gröbner bases and Bergman's Diamond lemma.

7.4. Algebra of noncommutative symmetric polynomials as a subalgebra in Q_n.

For each ordering $I = (i_1, \dots, i_n)$ of $\{1, \dots, n\}$, there is a natural free subalgebra, denoted $Q_{n,I}$, of Q_n generated by $\{z_{\{i_1,\dots,i_k\},\emptyset} \mid 1 \le k \le n\}$. Using the basis theorem in [GRW], we described arbitrary intersections of the $Q_{n,I}$. In particular, we proved the following theorem. Let $\mathbf{S}_n$ be a subalgebra in Q_n generated by all coefficients a_m (see formula (7.1.1.)). One can identify $\mathbf{S}_n$ with algebra $\overline{\mathbf{Sym}}_n$ of noncommutative symmetric functions in $x_1, \dots, x_n$.

Theorem 7.4.1. *The intersection of all subalgebras* $Q_{n,I}$ *coincides with algebra* $\mathbf{S}_n$.

This is a pure noncommutative phenomenon: under the commutative specialization π_n all algebras $Q_{n,I}$ map to the algebra of all polynomials and algebra $\mathbf{S}_n$ maps to the algebra of symmetric funtions.

7.5. The dual algebra $Q_n^!$.

Since Q_n is a quadratic algebra, it is natural to ask if Q_n is a Koszul algebra. This was proved by S. Serconek and R. Wilson [SW].

The algebra Q_n has a natural graded structure $Q_n = \sum_{i \ge 0} Q_{n,i}$ where $Q_{n,i}$ is the span of all products of i generators. As usual, we denote the Hilbert series of Q_n by $H(Q_n, t) = \sum_{i \ge 0} dim(Q_{n,i}) t^i$.

In [GGRWS] we computed the Hilbert series of Q_n and of its dual quadratic algebra $Q_n^!$. In particular, we proved:

Theorem 7.5.1.

$$H(Q_n, t) = \frac{1-t}{1-t(2-t)^n},$$

$$H(Q_n^!, t) = \frac{1+t(2+t)^n}{1+t}.$$

It follows that the dual algebra $Q_n^!$ is finite dimensional. It also has a rich and interesting structure and I am going to investigate it. Clearly

$$H(Q_n, t) \cdot H(Q_n^!, -t) = 1.$$

This also follows from the Koszulity of Q_n.

7.6. Quotient algebras of Q_n.

There are at least two reasons to study quotient algebras of Q_n. The noncommutative nature of Q_n can be studied by looking at quotients of Q_n by ideals generated by some $z_{\emptyset,A}$. These quotients are "more commutative" than Q_n. For example, the quotient of Q_n by the ideal generated by all $z_{\emptyset,A}$ with $|A| \geq 2$ is isomorphic to the algebra of commutative polynomials in n variables.

To consider more refined cases, we need to turn to a "noncommutative combinatorial topology." In our approach the algebra Q_n corresponds to an n-simplex Δ_n, and we consider quotients of Q_n by ideals generated by some $z_{\emptyset,A}$ corresponding to subcomplexes of Δ_n. In [GGR] we described generators and relations for those quotients. Special attention was given to the quotients of Q_n corresponding to 1-dimensional subcomplexes of Δ_n (they are close to algebras of commutative polynomials). However, we need to study other quotient algebras of this type and to be able to "glue" together such quotient algebras. This will lead to a construction of a "noncommutative combinatorial topology."

It would also be interesting to study quotients of algebras Q_n associated with special types of polynomials (such as $x^n = 0$ or polynomials with multiple roots).

Example. Let F be a field. We consider here F-algebra Q_2 and its quotients. This algebra is generated by z_1, z_2, $z_{1,2}$, $z_{2,1}$. It corresponds to a polynomial $P(t) = t^2 - pt + q$ with $p = z_1 + z_{1,2} = z_2 + z_{2,1}$, $q = z_{1,2}z_1 = z_{2,1}z_2$.

There is an "invisible element" $z_{\emptyset,12} = z_{1,2} - z_2 = z_{2,1} - z_1$. This element satisfies the relation $z_{\emptyset,12}(z_1 - z_2) = z_1z_2 - z_2z_1$.

There are three quotient algebras corresponding to special cases of $P(t)$:

i) The algebra corresponding to the polynomial t^2 equals to $Q_2/(p, q)$. This algebra is isomorphic to $F < z_1, z_2 > /(z_1^2, z_2^2)$.

ii) The algebra corresponding to a polynomial with multiple roots equals $Q_2/(z_1 - z_2)$. It is isomorphic to a free algebra with generators z_1 and $z_{1,2}$.

iii) The algebra $Q_2/(z_{\emptyset,12})$ is isomorphic to an algebra of commutative polynomials with two variables.

We are going to investigate such algebras for noncommutative polynomials of any degree.

7.7. Quadratic algebras associated with differential noncommutative polynomials.

In [GRW] similarly to algebras Q_n, we constructed universal algebras of pseudo-roots of noncommutative differential polynomials.

Let F be a field and (C, D) a differential division algebra with a unit and a derivation D. Sometimes we use the notation $Da = a'$. Consider an operator $P(D) : C \to C$, $P = a_0 D^n + a_1 D^{n-1} + ... + a_n$, $a_i \in C$ for $i = 0, 1, \dots, n$. The action of P on C is given by the formula $P(\phi) = a_0 D^n \phi + a_1 D^{n-1}\phi + ... + a_n \phi$.

We say that an element $c \in A$ is a pseudo-root of $P(D)$ if there exist polynomials $L_c(D)$ and $R_c(D)$ with coefficients from A such that $P(D) = L_c(D)(D - c)R_c(D)$. We say that c is a root of $P(D)$ if $R_c(D) = 1$. We do not assume here that differential polynomials have any solutions.

Suppose that $a_0 = 1$ and the differential polynomial $P(D)$ has n different roots $f_1, \dots, f_n$. Following [EGR], for any ordering $(i_1, \dots, i_n)$ of $\{1, \dots, n\}$, in a generic case we constructed in [GRW] pseudo-roots f_{i_1,i_2}, ..., $f_{i_1,\dots i_{n-1},i_n}$ such that

$$P(D) = (D - f_{i_1,\dots,i_{n-1},i_n}) \dots (D - f_{i_1,i_2})(D - f_{i_1}).$$

The elements $f_{i_1,\dots,i_{k-1},i_k}$ for $k = 2, \dots, n$ do not depend on the ordering in $(i_1, \dots, i_{k-1})$.

Set $f_{\emptyset,i} = f_i$. It was proved in [GRW] that for any $A \subset \{1, \dots, n\}$ such that $|A| < n - 1$, and for any $i, j \notin A$ we have

$$f_{A\cup i,j} + f_{A,i} = f_{A\cup j,i} + f_{A,j}, \tag{7.7.1a}$$

$$f_{A\cup i,j} f_{A,i} - f'_{A,i} = f_{A\cup j,i} f_{A,j} - f'_{A,j}. \tag{7.7.1b}$$

Based on these formulas one can define universal algebras DQ_n of pseudo-roots of noncommutative differential polynomials. They are defined by elements $f_{A,i}$ for $i \notin A$ satisfying relations (7.7.1).

We are planning to develop a theory of algebras DQ_n and to use it in noncommutative integrable systems. The theory must include a theory of symmetric functions related to algebras DQ_n, constructions of bases, studies of quotient algebras of

VIII. Noncommutative traces, determinants and eigenvalues

In this section we discuss noncommutative traces, determinants and eigenvalues. Classical (commutative) determinants play a key role in representation theory. Frobenius developed his theory of group characters by studying factorizations of group determinants (see [L]). Therefore, one cannot start a noncommutative representation theory without looking at possible definition of noncommutative determinants and traces. The definition of a noncommutative determinant given in this Section is different from the definition given in Section III.

8.1. Determinants and cyclic vectors.

Let A be an $m\times m$-matrix over an algebra R with a unit. A vector $v\in R^m$ is an A-*cyclic vector* if $v, Av, \dots, A^{m-1}v$ is a basis in R^m considered as a right R-module. In this case there exist $\Lambda_i(v,A)\in R$, $i=1,\dots,m$, such that

$$(-1)^m v\Lambda_m(v,A)+(-1)^{m-1}(Av)\Lambda_{m-1}(v,A)+\dots-(A^{m-1}v)\Lambda_1(v,A)+A^m v=0.$$

We call $\Lambda_m(v,A)$ the *determinant* of (v,A) and $\Lambda_1(v,A)$ the *trace* of (v,A). We may express $\Lambda_i(v,A)\in R$, $i=1,\dots,m$, as quasi-Plücker coordinates of the $m\times(m+1)$ matrix with columns $v, Av, \dots, A^n v$ (following [GR4]).

When R is commutative $\Lambda_m(v,A)$ is the determinant of A and $\Lambda_1(v,A)$ is the trace of A.

This definition of noncommutative determinants and traces was essentially used in [GKLLRT] for a unit vector v. In this case $\Lambda_i(v,A)$ are quasi-Plücker coordinates of the corresponding Krylov matrix.

Example. Let $A=(a_{ij})$ be an $m\times m$-matrix and $v=(1,0,\dots,0)^T$. Denote by $a_{ij}^{(k)}$ the corresponding entries of of A^k. Then

$$\Lambda_m(v,A)=\begin{vmatrix} \boxed{a_{11}^{(m)}} & a_{12}^{(m)} & \dots & a_{1m}^{(m)} \\ a_{11}^{(m-1)} & a_{12}^{(m-1)} & \dots & a_{1m}^{(m-1)} \\ & \dots & \dots & \\ a_{11} & a_{12} & \dots & a_{1m} \end{vmatrix}.$$

For $m=2$ the "noncommutative trace" Λ_1 equals $a_{11}+a_{12}+a_{12}a_{22}^{-1}a_{21}$ and the "noncommutative determinant" Λ_2 equals $a_{12}a_{22}^{-1}a_{21}a_{11}-a_{12}a_{21}$.

It was shown in [GKLLRT] that when A is a quantum matrix then Λ_m equals $\det_q A$ and when A is a Capelli matrix, Λ_m equals Capelli determinant.

A construction of a noncommutative determinant and a noncommutative trace via cyclic vectors in a special case was later used in [Ki].

When R is noncommutative, the expressions $\Lambda_i(v,A)\in R$, $i=1,\dots,m$, depend on vector v. However, they provide some information about A. For example, the following statement is true.

Proposition 8.1.1. *If the determinant $\Lambda_m(v,A)$ equals zero, then matrix A is not invertible.*

We will study properties of $\Lambda_i(v,A)$ for $i=1,\dots,m$ in a separate paper.

8.2. Noncommutative determinants and noncommutative eigenvalues.

One can also express $\Lambda_i(v,A)\in R$ via left eigenvalues of A.

Definition 8.2.1. *A row-vector $u = (u_1, \dots, u_m)$ is a left eigenvector of A if there exists $\lambda \in R$ such that $uA = \lambda u$.*

We call λ a *left eigenvalue* of A corresponding to vector u. Note, that the eigenvalue λ is defined up to a conjugation: if $\alpha \in R$, then $(\alpha u)A = \alpha\lambda\alpha^{-1}(\alpha u)$.

Proposition 8.2.2. *Suppose that $u = (u_1, \dots, u_m)$ is a left eigenvector of A with eigenvalue λ, $v = (v_1, \dots, v_m)^T$ is a cyclic vector of A, and the scalar product $u \cdot v = u_1v_1 + \dots u_mv_m = 1$.*

The eigenvalue λ satisfies the equation

$$(8.2.1) \quad (-1)^m\Lambda_m(v, A) + (-1)^{m-1}\lambda\Lambda_{m-1}(v, A) + \cdots - \lambda^{m-1}\Lambda_1(v, A) + \lambda^m = 0.$$

Equation (8.2.1) and the corresponding Vieta theorem (see Section V) show that if the matrix A has m distinct left eigenvectors u^i such that $u^i \cdot v = 1$ for $i = 1, \dots, m$ and m distinct corresponding eigenvalues $\lambda_1, \dots, \lambda_m$, then $\lambda_1, \dots, \lambda_m$ completely determine $\Lambda_i(v, A) \in R$, $i = 1, \dots, m$, as "noncommutative elementary symmetric functions" by formulas similar to formulas from Definition (6.5.1).

We are planning to investigate relations between noncommutative traces, determinants and eigenvalues in greater detail and to construct a corresponding "universal algebra" similar to Section VII.

8.3. Multiplicativity of determinants.

In the commutative case the multiplicativity of determinants and the additivity of traces are related to computations of determinants and traces with diagonal block-matrices. In the noncommutative case we suggest to consider the following construction.

Let A be an $m \times m$-matrix and D an $n \times n$-matrix over an algebra R with unit. Let $v \in R^m$ be a cyclic vector for A and $w \in R^n$ be a cyclic vector for D. Suppose that vector $z = (v, w)^T \in R^{m+n}$ is a cyclic vector for matrix $C = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$.

Then one can express functions $\Lambda_k(z, C)$, $k = 1, \dots, m+n$ via functions $\Lambda_i(v, A)$, $i = 1, \dots, m$ and $\Lambda_j(w, D)$, $j = 1, \dots, n$. In the commutative case these formulas imply the multiplicativity of determinants and the additivity of traces.

We are going to study relations between symmetric functions associated with matrices A and D and with the block-matrix C.

IX. Some applications

In this Section we present some results from [GR1, GR2, GR4].

9.1. Continued fractions and almost triangular matrices.

Consider an infinite matrix A over a skew-field:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n}\dots \\ -1 & a_{22} & a_{23} & \dots & a_{2n}\dots \\ 0 & -1 & a_{33} & \dots & a_{3n}\dots \\ 0 & 0 & -1 & \dots & \dots \end{pmatrix}$$

It was pointed out in [GR1], [GR2] that the quasideterminant $|A|_{11}$ can be written as a generalized continued fraction

$$|A|_{11} = a_{11} + \sum_{j_1 \neq 1} a_{1j_1} \frac{1}{a_{2j_1} + \sum_{j_2 \neq 1, j_1} a_{2j_2} \frac{1}{a_{3j_2} + \dots}}.$$

Let

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ -1 & a_{22} & \dots & a_{2n} \\ 0 & -1 & \dots & a_{3n} \\ & & \dots & \\ \dots & 0 & -1 & a_{nn} \end{pmatrix}.$$

The following proposition was formulated in [GR1], [GR2].

Proposition 9.1.1. $|A_n|_{11} = P_n Q_n^{-1}$, *where*

$$P_n = \sum_{1 \leq j_1 < \dots < j_k < n} a_{1j_1} a_{j_1+1 j_2} a_{j_2+1 j_3} \dots a_{j_k+1 n} \tag{9.1.1}$$

and

$$Q_n = \sum_{2 \leq j_1 < \dots < j_k < n} a_{2j_1} a_{j_1+1 j_2} a_{j_2+1 j_3} \dots a_{j_k+1 n}. \tag{9.1.2}$$

Proof. From the homological relations one has

$$|A_n|_{11} |A_n^{1n}|_{21}^{-1} = -|A_n|_{1n} |A_n^{11}|_{2n}^{-1}.$$

We will apply formula (1.2.2) to compute $|A_n|_{1n}$, $|A_n^{11}|_{2n}$, and $|A_n^{1n}|_{21}$. It is easy to see that $|A_n^{1n}|_{21} = -1$. To compute the two other quasideterminants, we have to invert triangular matrices. Setting $P_n = |A_n|_{1n}$ and $Q_n = |A_n^{11}|_{2n}$ we arrive at formulas (9.1.1), (9.1.2). $\square$

Remark. In the commutative case Proposition 9.1.1 is well-known. In this case $P_n = |A_n|_{1n} = (-1)^n \det A_n$ and $Q_n = (-1)^{n-1} \det A_n^{11}$.

Formulas (9.1.1), (9.1.2) imply the following result of [GR1], [GR2]:

Corollary 9.1.2. *The polynomials P_k for $k \geq 0$ and Q_k for $k \geq 1$ are related via the formulas:*

$$P_k = \sum_{s=0}^{k-1} P_s a_{s+1,k}, \qquad P_0 = 1, \tag{9.1.3}$$

and

$$Q_k = \sum_{s=1}^{k-1} Q_s a_{s+1,k}, \qquad Q_1 = 1. \tag{9.1.4}$$

Corollary 9.1.3. *Consider the matrix A_n. Suppose that for any $i \neq j$ and any p, q we have*

$$a_{ij} a_{pq} = a_{pq} a_{ij}$$

and

$$a_{jj} a_{ii} - a_{ii} a_{jj} = a_{ij}, \ 1 \leq i < j \leq n.$$

Then

$$P_n = |A_n|_{1n} = a_{nn} a_{n-1n-1} \dots a_{11}. \tag{9.1.5}$$

The proof follows from formula (9.1.3).

Corollary 9.1.4. [GR1], [GR2] *For a Jacobian matrix*

$$A = \begin{pmatrix} a_1 & 1 & 0 & \dots \\ -1 & a_2 & 1 & \\ 0 & -1 & a_3 & \dots \end{pmatrix}$$

we have:

$$|A|_{11} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}},$$

and

$$P_k = P_{k-1} a_k + P_{k-2}, \ k \geq 2; \ P_0 = 1, \ P_1 = a_1,$$

$$Q_k = Q_{k+1} a_k + Q_{k-2}, \ k \geq 3; Q_1 = 1, \ Q_2 = a_2.$$

In this case P_k is a polynomial in $a_1, \dots, a_k$ and Q_k is a polynomial in $a_2, \dots, a_k$.

9.2 Continued fractions and formal series. In the notation of the previous subsection the infinite continued fraction $|A|_{11}$ may be written as a ratio of formal series in the letters a_{ij} and a_{ii}^{-1}. Namely, set

$$P_\infty = \sum_{\substack{1\le j_1<j_2\dots<j_k<r-1\\ r=1,2,3,\dots}} a_{1j_1}a_{j_1+1j_2}\dots a_{j_k+1r}a_{rr}^{-1}\cdot\dots\cdot a_{11}^{-1} =$$

$$= 1 + a_{12}a_{22}^{-1}a_{11}^{-1} + a_{13}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{11}a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \dots,$$

and

$$Q_\infty = a_{11}^{-1} + \sum_{\substack{2\le j_1<j_2\dots<j_k<r-1\\ r=2,3\dots}} a_{2j_1}a_{j_1+1j_2}\dots a_{j_k+1r}a_{rr}^{-1}\cdot\dots\cdot a_{11}^{-1} =$$

$$= a_{11}^{-1} + a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{24}a_{44}^{-1}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \dots.$$

As no monomial appears more than once in these sums, these are rigorously defined formal series.

The following theorem was proved in [PPR]. Another proof was given in [GR4].

Theorem 9.2.1.

$$|A|_{11} = P_\infty \cdot Q_\infty^{-1}.$$

Proof. Set $b_{ij} = a_{ij}a_{jj}^{-1}$ and consider matrix $B = (b_{ij})$, $i, j = 1, 2, 3, \dots$. According to a property of quasideterminants $|A|_{11} = |B|_{11}a_{11}$.

Applying the noncommutative Sylvester Theorem to B with matrix (b_{ij}), $i, j \ge 3$, as the pivot, one has

$$|B|_{11} = 1 + |B^{21}|_{12}|B^{11}|_{22}^{-1}a_{11}^{-1}.$$

It follows

$$|A|_{11} = (a_{11}|B^{11}|_{22}a_{11}^{-1} + |B^{21}|_{12}a_{11}^{-1})(|B^{11}|_{22}a_{11}^{-1})^{-1}. \tag{9.2.1}$$

From [GKLLRT, Proposition 2.4] it follows that the first factor in (9.2.1) equals P_∞, and the second one equals Q_∞^{-1}. $\square$

9.3. Noncommutative Rogers-Ramanujan continued fraction. The following application of Theorem 9.2.1 to Rogers-Ramanujan continued fraction was given in [PPR]. Consider a continued fraction with two formal variables x and y:

$$A(x,y) = \frac{1}{1 + x\frac{1}{1+x\frac{1}{1+\dots}y}y}.$$

It is easy to see that

$$A(x,y) = \begin{vmatrix} 1 & x & & \cdot & & \\ -y & 1 & x & & \cdot & 0 \\ & -y & 1 & x & & \cdot \\ & & & 1 & & \cdot \\ & 0 & & \ddots & \ddots & \cdot \end{vmatrix}_{11}^{-1} =$$

$$= \begin{vmatrix} 1 & x & 0 & & \\ -1 & y^{-1} & xy^{-1} & & \\ 0 & -1 & y^{-1} & xy^{-1} & \\ & & -1 & y^{-1} & \ddots \end{vmatrix}_{11}$$

Theorem 9.2.1 implies:

Corollary 9.3.1. *$A(x,y) = P \cdot Q^{-1}$, where $Q = yPy^{-1}$ and*

$$P = 1 + \sum_{\substack{k\geq 1 \\ n_1,\dots,n_k \geq 1}} y^{-n_1}xy^{-n_2}x \dots y^{-n_k}xy^{k+n_1+n_2+\dots+n_k}.$$

Following [PPR] suppose that $xy = qyx$, where q commutes with x and y. Set $z = yx$.

Then Corollary 9.3.1 implies Rogers-Ramanujan continued fraction identity

$$A(x,y) = \frac{1}{1 + \frac{qz}{1+\frac{q^2z}{1+\dots}}} =$$

$$= \frac{1 + \sum_{k\geq 1} \frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)} z^k}{1 + \sum_{k\geq 1} \frac{q^{k^2}}{(1-q)\dots(1-q^k)} z^k}.$$

9.4. Determinants of almost triangular matrices.

Consider now a general almost triangualr matrix

$$B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ b_{21} & a_{22} & \dots & a_{2n} \\ 0 & b_{32} & \dots & a_{3n} \\ & & \dots & \\ \dots & 0 & b_{nn-1} & a_{nn} \end{pmatrix}.$$

In this subsection we suppose that all b_{i+1i}'s are invertible.

Denote by B_k, $k = 1, 2, \dots, n-1$, submatrices of B obtained by deleting the first k rows and columns from matrix B.

There is one more definition of a determinant-like expression for an almost triangular matrix given by the formula

$$D(1, 2, \dots, n) = |B|_{11} b_{21}^{-1} |B_1|_{22} b_{32}^{-1} |B_2|_{33} b_{43}^{-1} \dots b_{nn-1}^{-1} a_{nn}.$$

The homological relations imply the following proposition [GR1], [GR2].

Proposition 9.4.1.

$$(-1)^{n+1} D(1, 2, \dots n) = |B|_{1n} =$$

$$= \sum_{1 \leq j_1 < \dots < j_k < n} (-1)^{k+1} a_{1j_1} b_{j_1+1j_1}^{-1} a_{j_1+1j_2} b_{j_2+1j_2}^{-1} a_{j_2+1j_3} \dots b_{j_k+1j_k}^{-1} a_{j_k+1n}.$$

Remark. When $b_{i+1i} = 1$, $i = 1, \dots, n-1$ and all a_{ij}'s commute with each other, then the expression $D(1, 2, \dots, n)$ is the determinant of B.

It is interesting to determine other quasideterminants of an almost triangular matrix.

Proposition 9.4.2. [GR1], [GR2] *Set $D(\emptyset) = 1$. For $i \leq j$ we have*

$$|B|_{ij} = D(1, \dots, i-1)^{-1} D(1, \dots, n) D(j+1, \dots, n)^{-1}.$$

9.5. Quasideterminants and characterisric functions of graphs.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, and let a_{ij}'s be formal noncommuting variables. Fix $p, q \in \{1, \dots, n\}$ and a set $J \subset \{1, \dots, \hat{p}, \dots, n\} \times \{1, \dots, \hat{q}, \dots, n\}$ such that $|J| = n-1$ and both the projections of J onto $\{1, \dots, \hat{p}, \dots, n\}$ and $\{1, \dots, \hat{q}, \dots, n\}$ are surjective. Introduce new variables b_{kl}, $k, l = 1, \dots, n$, such that $b_{kl} = a_{kl}$ for $(l, k) \notin J$, $b_{kl} = a_{lk}^{-1}$ for $(l, k) \in J$. Let F_J be a ring of formal series in variables b_{kl}.

Proposition 9.5.1. *The quasideterminant $|A|_{ij}$ is defined in the ring F_J and is given by the formula*

$$|A|_{ij} = b_{ij} - \sum (-1)^s b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}. \tag{9.5.1}$$

The sum is taken over all sequences $i_1, \dots, i_s$ such that $i_k \neq i, j$, $k = 1, \dots, s$.

Proposition 9.5.2. *The inverse to $|A|_{ij}$ is also defined in the ring F_J and is given by the following formula*

$$|A|_{ij} = b_i j - \sum (-1)^s b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}. \tag{9.5.2}$$

The sum is taken over all sequences $i_1, \dots, i_s$.

All relations between quasideterminants including Sylvester identity can be deduced from formulas (9.5.1) and (9.5.2).

Formulas (9.5.1) and (9.5.2) can be interpreted via graph theory. Let Γ_n be a complete oriented graph with vertices $1, \dots, n$ and edges e_{kl}, where $k, l = 1, \dots, n$. Introduce a bijective correspondence beween edges of the graph and elements b_{kl}: $e_{kl} \mapsto b_{kl}$. Then there exist a bijective correspondence between monomials $b_{ii_1} b_{i_1 i_2} \dots b_{i_s j}$ and paths from vertex i to vertex j.

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