# On the cohomology of moduli spaces of rank two vector bundles over curves

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MPI/95-36

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## §1. INTRODUCTION AND MAIN RESULTS

Let $C$ be a Riemann surface, $L$ a line bundle over $C$, and $n$ a natural number. Then there is a moduli space of stable $n$-dimensional vector bundles $E$ over $C$ with determinant bundle $\Lambda^n(E) \equiv L$; this moduli space is smooth but in general non-compact and can be compactified by the suitable addition of semi-stable bundles to a projective, but in general singular, variety $\mathcal{N}_{C,n,L}$. The topology of this variety depends only on the genus $g$ of $C$ and the degree $d$ of $L$ (in fact, only on $d$ modulo $n$, since tensoring $E$ with a fixed line bundle $L_1$ replaces $L$ by $L \bigotimes L_1^n$), so we will also use the notation $\mathcal{N}_{g,n,d}$. We will be studying only the case $n = 2$, and hence will drop the $n$ and replace $d$ by $\varepsilon = (-1)^d$ in the notation. Thus for each $g$ we have two moduli spaces of stable 2-dimensional bundles $\mathcal{N}_g^-$ and $\mathcal{N}_g^+$, both projective varieties of complex dimension $3g - 3$. We will be looking mostly at the smooth space $\mathcal{N}_g^-$ and will often denote it simply $\mathcal{N}_g$.

The additive cohomology of $\mathcal{N}_g$ has been known for many years ([N1]; in fact, the Betti numbers of $\mathcal{N}_{g,n,d}$ for all $n$ and $d$ were found in beautiful and famous papers of Atiyah-Bott [AB] and Harder-Narasimhan [HN]), but the multiplicative structure was not: it was known that that the Künneth components of the Chern classes of a certain universal bundle over $\mathcal{N}_g$ generate the cohomology ring, but not what the relations were. Mumford (cf. [AB], p. 324) gave certain relations coming from the vanishing of the Chern classes beyond the dimension of another bundle and conjectured that these generate the ideal of all relations. The main objects of this paper are

(a) to prove Mumford's conjecture and give a complete additive and multiplicative description of the cohomology ring of $\mathcal{N}_g$ (over $\mathbb{Q}$) and of its intersection pairing,

(b) as an application, to prove the Verlinde formulas for the dimensions of spaces of sections of certain line bundles over $\mathcal{N}_g^{\pm}$, and

(c) as a further application, to give direct proofs of three conjectures of Newstead [N2] (or more properly, of Newstead and Ramanan) concerning the characteristic classes of $\mathcal{N}_g$.

Each of the main results on this list had or has been proved by other authors: Mumford's conjecture by Kirwan [K], the Verlinde formulas by Szenes and Bertram [S, BS] and others [DW, Do, NR, T2], and Newstead's conjectures by Gieseker [G], Thaddeus [T2], and Kirwan

[K], but since our proof (found in 1991) also yields detailed information about the structure of the cohomology which may be of some interest, it seemed worthwhile to publish it anyway. While this paper was in preparation, I learned of very recent work by several other authors [B, KN, ST]) which contains much of the same information on the cohomology ring.

Before describing the results on $H^*(\mathcal{N}_g)$, we recall briefly the statement of the Verlinde formulas. It is known that the canonical bundle of $\mathcal{N}_g^\varepsilon$ has the form $\mathcal{L}_\varepsilon^{-2}$ where $\mathcal{L}_\varepsilon$ is an ample line bundle which is, moreover, a square if $\varepsilon = +1$; in fact, both $\mathrm{Pic}(\mathcal{N}_g^+)$ and $\mathrm{Pic}(\mathcal{N}_g^-)$ are isomorphic to $\mathbb{Z}$, with generators $\mathcal{L}_+^{1/2}$ and $\mathcal{L}_-$, respectively, so that every line bundle over $\mathcal{N}_g^\varepsilon$ is equivalent to $\mathcal{L}_\varepsilon^{k/2-1}$ with $k \in \mathbb{Z}$, $\varepsilon^k = +1$. We set

$$D_\varepsilon(g,k) \;:=\; \chi(\mathcal{N}_g^\varepsilon\,,\; \mathcal{L}_\varepsilon^{k/2-1}) \qquad\qquad (g>0,\; k\in\mathbb{Z}\,,\; \varepsilon=\pm1,\; \varepsilon^k=1)\,. \tag{1}$$

The Kodaira vanishing and Serre duality theorems imply that

$$D_\varepsilon(g,k) = \begin{cases} \dim H^0(\mathcal{N}_g^\varepsilon\,,\;\; \mathcal{L}_\varepsilon^{k/2-1}) & \text{if } k \geqslant 2 \\ 0 & \text{if } |k| \leqslant 1 \\ (-1)^{g-1}\dim H^{3g-3}(\mathcal{N}_g^\varepsilon\,,\; \mathcal{L}_\varepsilon^{k/2-1}) & \text{if } k \leqslant -2 \end{cases}$$

and that $D_\varepsilon(g,k)$ is $(-1)^{g-1}$-symmetric under $k \mapsto -k$. (It is to achieve this simple symmetry that we made the shift $k \mapsto k-2$ in the definition of $D_\varepsilon$.) The formula which was conjectured by Verlinde is:

**Theorem 1.** $$D_\varepsilon(g,k) = \left(\frac{k}{2}\right)^{g-1} \sum_{\substack{j \pmod k \\ j\not\equiv 0 \pmod k}} \frac{\varepsilon^{j-1}}{\sin^{2g-2}\frac{\pi j}{k}}\,. \tag{2}$$

We remark that the numbers defined by the right-hand side of (2) can be given in many other ways. About a dozen formulas for $D_\pm(g,k)$ were collected in §1 of [Z]. For instance, $D_+(g,k)$ can be given by the generating function

$$\sum_{g=1}^{\infty} D_+(g,k)\,\bigl(\frac{2}{k}\sin^2 x\bigr)^{g-1} = \frac{k\sin(k-1)x}{\sin kx\,\cos x} = k - \frac{k\,\tan x}{\tan kx}$$

and for a fixed integer $g \geqslant 2$ is a polynomial in $k$ of degree $3g-3$ of the form

$$\begin{aligned} D_+(g,k) = \beta_{g-1}\,k^{3g-3} + \frac{g-1}{6}\,\beta_{g-2}\,k^{3g-5} + \frac{(g-1)(5g-4)}{360}\beta_{g-3}\,k^{3g-7} \;+\;\cdots\;+ \\ +\;\frac{2^{g-3}(1+\frac14+\cdots+\frac{1}{(g-2)^2})}{15(g-1)\binom{2g-2}{g-1}}\,k^{g+3} + \frac{2^{g-2}}{3(g-1)\binom{2g-2}{g-1}}\,k^{g+1} + C_g k^{g-1} \end{aligned}$$

where $\beta_r$ denotes $2^r|B_{2r}|/(2r)!$ ($B_n$ = $n$th Bernoulli number) and $C_g$, the only negative coefficient in the polynomial, is fixed by $D_+(g,1)=0$. There are similar results for $D_-$.

As mentioned above, independent (and earlier) proofs of Verlinde's formulas were given by Szenes [S] in the case of $\mathcal{N}_g^-$ and by Bertram-Szenes [BS] in the case of $\mathcal{N}_g^+$. The basic idea in the case of $\mathcal{N}_g^-$ is to suppose that the original curve $C$ is hyperelliptic (this is all right since the numbers $D_\epsilon(g,k)$ are independent of the complex structure of $C$), and then use an explicit description of Desale-Ramanan of $\mathcal{N}_g$ as a Grassmannian of linear spaces contained in the intersection of two quadrics. The proof in [BS] is based on a beautiful duality result which lets one compute the invariants $D_+(g,k)$ of the singular variety $\mathcal{N}_g^+$ as a *different* invariant of the smooth manifold $\mathcal{N}_g^-$, after which the proof is again completed by taking $C$ hyperelliptic. We will also use this duality result of Bertram-Szenes to reduce the proof of Theorem 1 to a Riemann-Roch type calculation on $\mathcal{N}_g^-$, but will then be able to give a much more direct computation by making use of our knowledge of the ring structure and intersection pairing in the cohomology of the latter space.

We now describe the results on the cohomology of $\mathcal{N}_g$ in more detail; the full statements are contained in §3. Over $\mathcal{N}_g \times C$ there is a universal 2-dimensional bundle $V$ whose fibre over $\{x\} \times C$ is the bundle over $C$ parametrized by $x$. This bundle is well-defined only up to tensoring with a line bundle over $\mathcal{N}_g$, but the combination of Chern classes $c_1^2(V) - 4c_2(V) \in H^4(\mathcal{N}_g \times C)$ is well-defined. Its Künneth components give cohomology classes

$$\alpha \in H^2(\mathcal{N}_g), \quad \psi_i \in H^3(\mathcal{N}_g) \quad (i = 1, \dots, 2g), \quad \beta \in H^4(\mathcal{N}_g)$$

which, as already mentioned, are known to generate the ring $H^*(\mathcal{N}_g)$. The relations given by Mumford are defined as follows. Let $\tilde{\mathcal{N}}_g$ denote the moduli space of all stable rank 2 bundles of degree $4g-3$ over the curve $C$. It is fibered over the Jacobian $J_g$ of $C$ and its rational cohomology splits as the tensor product of those of $\mathcal{N}_g$ and $J_g$. There is a 2-dimensional bundle $\tilde{V}$ over $\tilde{\mathcal{N}}_g \times C$ defined analogously to $V$, and its push-forward $f_!\tilde{V}$, where $f : \tilde{\mathcal{N}}_g \times C \to \tilde{\mathcal{N}}_g$ is the projection, has dimension $2g-1$, so the Chern class $c_i(f_!(\tilde{V}))$ vanishes for $i \geqslant 2g$. On the other hand, the Künneth components of these Chern classes can be computed as polynomials in the generators $\alpha$, $\beta$, and $\psi_i$ (this will be carried out in §6), so their vanishing gives relations. Mumford's conjecture is

**Theorem 2.** *The kernel of the map* $\mathbb{Q}[\alpha,\beta] \bigotimes \Lambda(\psi_1, \dots, \psi_{2g}) \twoheadrightarrow H^*(\mathcal{N}_g, \mathbb{Q})$ *is the ideal generated by the Künneth components of* $c_i(f_!(\tilde{V}))$, $i \geqslant 2g$.

The action of the genus $g$ mapping class group on $\mathcal{N}_g$ (via the interpretation of $\mathcal{N}_g$ as a space of unitary representations of $\pi_1(C)$ in the sense of Narasimhan-Seshadri) induces an action of the group $Sp(2g,\mathbb{Z})$ on $H^*(\mathcal{N}_g,\mathbb{Q})$ (leaving $\alpha$ and $\beta$ invariant and acting in the obvious way on the $\psi_i$). The ring $H_I^*(\mathcal{N}_g)$ of cohomology classes invariant under this action is of especial interest. Clearly it is generated by the classes $\alpha$, $\beta$, and $\gamma = -2\sum_{i=1}^{g} \psi_i\psi_{i+g} \in H^6(\mathcal{N}_g)$. We get relations in $H_I^*(\mathcal{N}_g)$ by taking the various Künneth components of $c_i(f_!\tilde{V})$ for $i \geqslant 2g$, in particular, the top-dimensional Künneth components $c_{r+g}(f_!\tilde{V}) \smallsetminus [J_g] \in H^{2r}(\mathcal{N}_g)$ $(r \geqslant g)$. These are essentially the classes $\xi_r$ defined in the following theorem, which completely describes the multiplicative and intersection structure of $H_I^*(\mathcal{N}_g)$.

**Theorem 3.** *Define elements* $\xi_r = \xi_r(\alpha,\beta,\gamma) \in \mathbb{Q}[\alpha,\beta,\gamma]$ *recursively by*

$$(r+1)\,\xi_{r+1} = \alpha\,\xi_r + r\beta\,\xi_{r-1} + 2\gamma\,\xi_{r-2} \qquad (r \in \mathbb{Z}) \tag{3}$$

*with initial conditions* $\xi_0 = 1$, $\xi_r = 0$ *for* $r < 0$. *Then*

*i) The kernel of the map* $\mathbb{Q}[\alpha,\beta,\gamma] \twoheadrightarrow H_I^*(\mathcal{N}_g)$ *is the ideal generated by* $\xi_g$, $\xi_{g+1}$ *and* $\xi_{g+2}$.

*ii) Define classes* $\xi_{r,s,t} \in \mathbb{Q}[\alpha,\beta,\gamma]$ *by*

$$\xi_{r,s,t} = \sum_{l=0}^{\min(r,s)} \binom{r+s-l}{r} \beta^{s-l} \frac{(2\gamma)^{l+t}}{l!\,t!}\, \xi_{r-l}\,, \qquad (r,\,s,\,t \geqslant 0). \tag{4}$$

*Then the* $\xi_{r,s,t}$ *with* $r+s+t \geqslant g$ *are a basis for the kernel of* $\mathbb{Q}[\alpha,\beta,\gamma] \to H_I^*(\mathcal{N}_g)$ *and the (images of) the* $\xi_{r,s,t}$ *with* $r+s+t \leqslant g-1$ *are a basis for the image of this map.*

*iii) The intersection pairing with respect to this basis is the product of a permutation and a diagonal matrix:* $\langle \xi_{r,s,t}\xi_{r',s',t'}, [\mathcal{N}_g]\rangle$ *is non-zero if and only if* $r' = s$, $s' = r$, $t' = g-1-r-s-t$.

As a corollary of part ii), we find that the Hilbert polynomial of $H_I^*(\mathcal{N}_g)$ is

$$\sum_{j=0}^{6g-6} \dim H_I^j(\mathcal{N}_g)\, t^j = \frac{(1-t^{2g})(1-t^{2g+2})(1-t^{2g+4})}{(1-t^2)(1-t^4)(1-t^6)}\,. \tag{5}$$

This is the same as the Hilbert polynomial of the cohomology of the Grassmannian of 3-planes in $\mathbb{C}^{g+2}$, but the ring structures are quite different and there seems to be no direct connection.

I would like to say a few words about the origin of the proof given here, which is a case history in backwards reasoning. I first learned of the Verlinde formulas from a beautiful lecture by Bott in Geneva in 1990, and discussed with him some of the elementary reformulations mentioned after Theorem 1. One of these formulas (the generating function) has a strong Riemann-Roch flavor, and this was the starting point for Michael Thaddeus to try to find what formulas for intersections numbers in the cohomology of $\mathcal{N}_g$ would give the desired answer if substituted into the Hirzebruch-Riemann-Roch theorem [T1]. Surprisingly, the answer turned out to be unique. In particular, one had to have

$$\begin{aligned} \langle \alpha^m\beta^n\gamma^p,\ [\mathcal{N}_g]\rangle &= \frac{(-1)^n 2^{2g-2-p} g!\,m!}{(g-p)!}\, b_{g-1-n-p} \qquad (m+2n+3p = 3g-3) \\ b_k &:= \text{ coefficient of } x^{2k} \text{ in } \frac{x}{\sin x} \qquad (\,=0 \text{ if } k<0) \end{aligned} \tag{6}$$

in order for the Verlinde formula to be correct. Thaddeus pointed out that knowing the intersection numbers in principle determines all the relations in $H^*(\mathcal{N}_g)$ (since by Poincaré duality $x = 0$ in $H^*$ if and only if $\langle xy, [\mathcal{N}_g]\rangle = 0$ for all $y \in H^*$), but added that actually finding these relations, and checking that the Betti numbers obtained agreed with the known Betti numbers of $\mathcal{N}_g$, would be a "hard exercise in number theory." To solve the exercise, the first step was to find the relations. Unfortunately, I had not read [AB] and did not know of the Mumford relations, whose fairly direct calculation by the Grothendieck-Hirzebruch-Riemann-Roch theorem (given in §6) would have simplified things considerably. Instead, following Thaddeus's hint, I took the conjectural intersection formula (6) as the starting point. Since the first relation among the generators $\alpha$, $\beta$, $\psi_i$ is known to be in degree $2g$ and to be unique, there had to be a unique (up to a constant) class $\xi_g$ of this degree whose intersection numbers with all generators vanished. A computation up to $g = 15$

and inspection of the coefficients of the classes obtained made it clear that the form of $\xi_g$ (normalized to begin $\alpha^g/g!$) was

$$\xi_g = \frac{\alpha^g}{g!} + \frac{2g-1}{6}\frac{\alpha^{g-2}\beta}{(g-2)!} + \frac{2}{3}\frac{\alpha^{g-3}\gamma}{(g-3)!} + \frac{20g^2-48g+7}{360}\frac{\alpha^{g-4}\beta^2}{(g-4)!} + \frac{10g-17}{45}\frac{\alpha^{g-5}\beta\gamma}{(g-5)!}$$
$$+ \frac{280g^3-1596g^2+1874g-93}{45360}\frac{\alpha^{g-6}\beta^3}{(g-6)!} + \frac{2}{9}\frac{\alpha^{g-6}\gamma^2}{(g-6)!} + \cdots$$

and in general that the coefficient of $\alpha^{g-2n-3p}\beta^n\gamma^p$ in $\xi_g$ had the form $\dfrac{2^p\,\phi_n(g,p)}{3^{n+p}\,(g-2n-3p)!p!}$ for some polynomials $\phi_n(g,p)$ of degree $n$, the first few of these being

$$\phi_0(g,p) = 1, \quad \phi_1(g,p) = g - \frac{6p}{5} - \frac{1}{2}, \quad \phi_2(g,p) = \frac{g^2}{2} - \frac{6(p+1)g}{5} + \frac{1008p^2+1872p+245}{1400}.$$

Inspection of these and further values led after some effort to the (then still conjectural) formula

$$\phi_n(g,p) = \text{Coefficient of } X^n \text{ in } \frac{\sqrt{3X}}{\sinh\sqrt{3X}}\left(\frac{1}{X} - \frac{\tanh\sqrt{3X}}{X\sqrt{3X}}\right)^p\left(\frac{\sqrt{3X}}{\tanh\sqrt{3X}}\right)^g,$$

which in turn is equivalent to a generating function (Proposition 4 below) and to the recursion (3). (The proof of the equivalence of these formulas, which is not difficult, will be omitted since we have given the "closed formula" only as a curiosity and will make no further use of it.) The next step was to show that the ideal defined as the radical of the intersection pairing (6) coincided with the ideal generated by the $\xi_r$ with $r \geqslant g$ (this proof will be given in §§2–3 below) and that the dimensions of the graded components of the quotient ring by this ideal coincided with the known Betti numbers of $\mathcal{N}_g$ (proof in §4). At this point (Fall 1991) I learned of the Mumford relations. A direct computation (§6) showed that they coincide essentially with the $\xi$'s, completing the proof. In the meantime different (and very pretty) proofs of the Verlinde formulas had been given by Szenes [S] for $\varepsilon = -1$ and by Bertram-Szenes [BS] for $\varepsilon = 1$, and since then several more proofs, both for the rank 2 and general rank cases, have been given by various authors and the Mumford conjecture had been proved by Frances Kirwan. These proofs used other methods, not relying on the explicit determination of $H^*(\mathcal{N}_g)$ and its intersection structure. Recently, as mentioned at the beginning of this introduction, several authors have also studied this cohomology ring and found many of the results given in this paper, including in particular the recursive formula (3) for the basic relations $\xi_r \in H^{2r}(\mathcal{N}_g)$. However, the approach and techniques here are a bit different and may be of independent interest, and some of the information obtained seems to be a little more detailed, so after some hesitation (and with the encouragement of several people) I decided to publish my original proof. I apologize to all concerned for the delay and for the duplication of results.

## §2. Calculus of the Thaddeus evaluation map

We will be concerned mostly with the classes $\alpha$, $\beta$, $\gamma$ and the subring $H_I^*(\mathcal{N}_g)$ which they generate, since the full structure of the cohomology ring can be deduced relatively easily once this part is known. (This will be carried out in §4.) We write $R$ for the graded ring $\mathbb{Q}[\alpha,\beta,\gamma]$, $I_g$ for the "$\xi$-ideal" $\xi_g R + \xi_{g+1}R + \xi_{g+2}R$, and $I_g^E$ for the "evaluation ideal" $\{x \in R \mid E_g[xy] = 0 \quad \forall y \in R\}$, where $E_g : R \to \mathbb{Q}$ is the "evaluation map" defined by

$$E_g[\alpha^m\beta^n\gamma^p] \;=\; \text{right-hand side of (6)} \qquad (\text{and } 0 \text{ if } m+2n+3p \neq 3g-3)\,.$$

Our goal in the next two sections is to show that $I_g^E = I_g$ and to compute the additive and multiplicative structure of the quotient ring $R/I_g$; we will show later that the $\xi_r$ ($r \geqslant g$) are the Mumford classes and then deduce by a comparison of dimensions that $I_g = I_g^E$ really coincides with the kernel of $R \to H^*(\mathcal{N}_g)$ and hence also that formula (6) is correct.

The key idea is to combine all the mappings $E_g$ into a single invariant

$$\mathcal{E}_T : R \to \mathbb{Q}[T]\,, \qquad \mathcal{E}_T[x] := \sum_{g=1}^{\infty} E_g[x]\,(-\tfrac14 T)^{g-1} \qquad (x \in R)$$

(the factor $-\frac14$ is included only for convenience). Notice that the sum is actually finite because $\mathcal{E}_T$ sends any monomial in $\alpha$, $\beta$, $\gamma$ to a monomial in $T$, namely

$$\mathcal{E}_T\big[\alpha^m\beta^n\gamma^p\big] = \begin{cases} (-\frac12)^p\, m!\,p!\,\binom{k+n+p+1}{p}\,a_k\,T^{n+p+k} & \text{if } m = n+3k,\ k \geqslant 0, \\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

where $a_k$ ($= (-1)^k b_k$ in the previous notation) is given by

$$a_k := \text{ coefficient of } x^{2k} \text{ in } \quad A(x) := \frac{x}{\sinh x}\,.$$

We will also consider the obvious extension of $\mathcal{E}_T$ to $R[[x_1,x_2,\ldots]] \to \mathbb{Q}[T][[x_1,x_2,\ldots]]$, where $x_1, x_2, \ldots$ are formal variables. The result of this extended map will then be a power series in $T$ but as a power series in the $x_i$ will have coefficients which are polynomials in $T$. In fact, the elements of $R[[x_1,x_2,\ldots]]$ and $\mathbb{Q}[T][[x_1,x_2,\ldots]]$ which are considered will always be homogeneous of degree 0 if the $x_i$ are assigned appropriate (negative) degrees, so the coefficient of any monomial in the $x_i$ will be a homogeneous polynomial in $\alpha$, $\beta$, $\gamma$ or a monomial in $T$. As a simple example, if $x$ is a formal variable then the value of $\mathcal{E}_T$ on the element $e^{\alpha x}$ of $\mathbb{Q}[\alpha][[x]]$ is

$$\mathcal{E}_T\big[e^{\alpha x}\big] = \sum_{m=0}^{\infty}\frac{x^m}{m!}\,\mathcal{E}_T[\alpha^m] = \sum_{k=0}^{\infty} a_k\,x^{3k}T^k = A(x^3T) \in \mathbb{Q}[T][[x]]$$

and all the functions occurring are homogeneous of degree 0 if $x$ and $T$ are given degrees $-2$ and 6, respectively. Similarly, if $x$ and $z$ are formal variables (of weights $-2$ and $-6$) and $n \geqslant 0$ an integer, then

$$\begin{aligned}\mathcal{E}_T\big[\beta^n\, e^{\alpha x + 2\gamma z}\big] &= \sum_{k,p\geqslant 0} x^{n+3k}(-z)^p\,a_k\,\binom{n+p+k+1}{p}\,T^{n+p+k} \\ &= \sum_{k\geqslant 0} a_k\,\frac{x^{n+3k}\,T^{n+k}}{(1+zT)^{n+k+2}} \;=\; \frac{x^nT^n}{(1+zT)^{n+2}}\,A\Big(\frac{x^3T}{1+zT}\Big)\end{aligned}$$

and hence

$$\mathcal{E}_T\big[f(\beta)\,e^{\alpha x+2\gamma z}\big] \;=\; \frac{1}{(1+zT)^2}\,f\Big(\frac{xT}{1+zT}\Big)\,A\Big(\frac{x^3T}{1+zT}\Big) \tag{8}$$

for any polynomial (or—with the proviso above—power series) $f(\beta)$.

It turns out that a more convenient basis for $R$ is given by the monomials in $\alpha$, $b$ and $\gamma^* = 2\gamma + \alpha\beta$ rather than $a$, $\beta$, $\gamma$. The analogue of (7) for this basis is the identity

$$\mathcal{E}_T\big[\alpha^m\beta^n\gamma^{*p}\big] = \begin{cases} m!\,p!\,\binom{2k-1}{p}\,a_k\,T^{n+p+k} & \text{if } m = n+3k,\ k \geqslant 0, \\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

To prove it, we note that

$$\mathcal{E}_T[\alpha^m\beta^n\gamma^{*p}] = \sum_{q=0}^{p} 2^{p-q}\binom{p}{q}\,\mathcal{E}_T[\alpha^{m+q}\beta^{n+q}\gamma^{p-q}]\,,$$

which vanishes by (7) unless $m$ has the form $n+3k$ for some $k \geqslant 0$ and in that case equals

$$\mathcal{E}_T[\alpha^{n+3k}\beta^n\gamma^{*p}] = (n+3k)!\,p!\,a_k\,T^{n+k+p}\sum_{q=0}^{p}(-1)^{p-q}\binom{n+3k+q}{q}\binom{n+k+p+1}{p-q}\,.$$

The assertion follows because the sum equals $\binom{2k-1}{p}$ by a standard binomial coefficient identity. (Recall that $\binom{x}{p}$ is defined for any integer $p \geqslant 0$ and any number $x$ as the polynomial $x(x-1)\cdots(x-p+1)/p!$. In particular, $\binom{2k-1}{p}$ is $(-1)^p$ if $k=0$ and for $k \geqslant 1$ is the usual binomial coefficient if $0 \leqslant p \leqslant 2k-1$ and $0$ if $p \geqslant 2k$.)

Now, repeating the calculation leading to (8) with $\gamma^*$ instead of $\gamma$, we find

$$\begin{aligned}\mathcal{E}_T\big[\beta^n\,e^{\alpha x+\gamma^* z}\big] &= \sum_{k,p\geqslant 0}\binom{2k-1}{p}\,a_k\,x^{n+3k}\,z^p\,T^{n+p+k} \\ &= \sum_{k\geqslant 0} a_k\,x^{n+3k}\,T^{n+k}\,(1+zT)^{2k-1} \;=\; \frac{x^nT^n}{1+zT}\,A(x^3T(1+zT)^2)\end{aligned}$$

and hence

**Proposition 1.** *Let $x$ and $z$ be formal variables and $f$ a power series in one variable. Then*

$$\mathcal{E}_T\big[e^{\alpha x+\gamma^* z}\,f(\beta)\big] = \frac{x^{3/2}T^{1/2}\,f(xT)}{\sinh\big[x^{3/2}T^{1/2}(1+zT)\big]}\,. \tag{10}$$

This identity, like formula (8), is completely equivalent to the formulas (7) or (9) and hence determines the evaluation map $\mathcal{E}_T$ completely. However, for many purposes we will need a more general result in which $x$ and $z$ are replaced by functions of $\beta$. The following generalization of (10) is the basic identity of our "calculus of evaluation maps":

**Proposition 2.** *Let $f$, $u$ and $w$ be power series in one variable with $u(0) \neq 0$. Then*

$$\mathcal{E}_T\big[f(\beta)\, e^{u(\beta)\alpha+w(\beta)\gamma^*}\big] = \frac{\beta^{1/2} f(\beta)\, Q'(T)}{\sinh\big[\beta^{1/2}(u(\beta)+\beta w(\beta))\big]}\Bigg|_{\beta=Q(T)} \tag{11}$$

*where $Q(T)$ is the power series defined by $Q^{-1}(\beta) = \beta/u(\beta)$.*

*Proof.* We have

$$\begin{aligned}
\mathcal{E}_T\big[f(\beta)\, e^{u(\beta)\alpha+w(\beta)\gamma^*}\big] &= \sum_{m,n,p\geqslant 0} \frac{1}{m!p!}\, \mathbf{C}_{\beta^n}\big[f(\beta)u(\beta)^m w(\beta)^p\big]\, \mathcal{E}_T\big[\alpha^m \beta^n \gamma^{*p}\big] \\
&= \sum_{n,k,p\geqslant 0} \binom{2k-1}{p} a_k\, \mathbf{C}_{\beta^n}\big[f(\beta)u(\beta)^{n+3k} w(\beta)^p\big]\, T^{n+p+k} \\
&= \sum_{n,k\geqslant 0} a_k\, \mathbf{C}_{\beta^n}\big[f(\beta)u(\beta)^{n+3k}(1+Tw(\beta))^{2k-1}\big]\, T^{n+k} \\
&= \sum_{n\geqslant 0} \mathbf{C}_{\beta^n}\left[\frac{f(\beta)u(\beta)^{n+3/2}T^{1/2}}{\sinh\big[u(\beta)^{3/2}T^{1/2}(1+Tw(\beta))\big]}\right] T^n\,,
\end{aligned}$$

where $\mathbf{C}_{\beta^n}[F(\beta)]$ denotes the coefficient of $\beta^n$ in a power series $F(\beta)$. But by the (formal) residue calculus we have

$$\begin{aligned}
\mathbf{C}_{\beta^n}\big[F(\beta)u(\beta)^{n+1}\big] &= \mathrm{Res}_{\beta=0}\Big[\frac{F(\beta)\,d\beta}{(\beta/u(\beta))^{n+1}}\Big] \\
&= \mathrm{Res}_{t=0}\Big[\frac{F(Q(t))\,Q'(t)\,dt}{t^{n+1}}\Big] = \mathbf{C}_{t^n}\big[F(Q(t))\,Q'(t)\big]
\end{aligned}$$

for any power series $F(\beta)$. The proposition follows. ■

A similar proof gives the following even more general result, which will be used repeatedly in §§5–7.

**Proposition 3.** *Let $f$, $h$, $u$ and $w$ be power series in one variable with $h(0)u(0) \neq 0$. Then*

$$\sum_{g=1}^{\infty} E_g\big[f(\beta)\, h(\beta)^g\, e^{u(\beta)\alpha+w(\beta)\gamma^*}\big]\,\big(-\tfrac{1}{4}T\big)^{g-1} = \frac{\beta^{1/2} f(\beta)\, Q'(T)}{\sinh\big[\beta^{1/2}(u(\beta)+\beta w(\beta))\big]}\Bigg|_{\beta=Q(T)} \tag{12}$$

*where $Q(T)$ is the power series defined by $Q^{-1}(\beta) = \beta/u(\beta)h(\beta)$.*

## §3. The classes $\xi_r$ and $\xi_{r,s,t}$

In this section we will investigate in detail the properties of the elements $\xi_r$ and $\xi_{r,s,t}$ of $R$ defined in Theorem 3. Our object is to prove that the ideal $I_g$ of $R$ generated by $\xi_g$, $\xi_{g+1}$, and $\xi_{g+2}$ coincides with the "evaluation ideal" $I_g^E$ defined in the last section. Note that, by virtue of the recursion (3), all $\xi_r$ with $r \geqslant g$ belong to $I_g$ and $\gamma I_g \subset I_{g+1}$.

An explanation of the origin of the classes $\xi$, and a closed formula for them as polynomials in $\alpha$, $\beta$ and $\gamma$, was given in the introduction. A more useful description of these classes is in terms of a generating function.

**Proposition 4.** *Define* $F_0(x) = \sum_{r=0}^{\infty} \xi_r x^r \in R[[x]]$. *Then*

$$F_0(x) = \frac{e^{-2\gamma x/\beta}}{\sqrt{1-\beta x^2}} \left(\frac{1+x\sqrt{\beta}}{1-x\sqrt{\beta}}\right)^{\gamma^*/2\beta\sqrt{\beta}}, \tag{13}$$

*where* $\gamma^* = \alpha\beta + 2\gamma$ *as in §2.*

*Remark.* The quantities $\gamma/\beta$, $\sqrt{\beta}$, and $\gamma^*/\beta\sqrt{\beta}$ occurring in (13) are not in $R$, but one can rewrite (13) as

$$F_0(x) = \frac{e^{\alpha x + \gamma^* x^3 C(\beta x^2)}}{\sqrt{1-\beta x^2}}, \qquad C(x) = \frac{\tanh^{-1}\sqrt{x}}{x\sqrt{x}} - \frac{1}{x} = \frac{1}{3} + \frac{x}{5} + \frac{x^2}{7} + \cdots \in \mathbb{Q}[[x]],$$

which is clearly a power series with coefficients which are polynomials in $\alpha$, $\beta$, and $\gamma$.

*Proof.* The recursion and initial value given in (3) translate into the first order linear differential equation $(1-\beta x^2)\,F_0'(x) = (\alpha + \beta x + 2\gamma x^2)\,F_0(x)$ with initial condition $F_0(0) = 1$, and this can be solved by standard methods to give (13)—or, of course, one can simply verify that the right-hand side of (13) satisfies the differential equation. ∎

We next turn to the quantities $\xi_{r,s} \in R^{2r+4s}$ defined by (4) with $t = 0$. The general element $\xi_{r,s,t}$ is simply $\xi_{r,s}(2\gamma)^t/t!$, so it suffices to study these two-index classes. The generating function just given extends easily to them:

**Proposition 5.** *Define* $F(x,y) = \sum_{r,s\geqslant 0} \xi_{r,s} x^r y^s \in R[[x,y]]$. *Then*

$$F(x,y) = \frac{e^{-2\gamma x/\beta}}{\sqrt{(1-\beta y)^2 - \beta x^2}} \left(\frac{1+x\sqrt{\beta}-\beta y}{1-x\sqrt{\beta}-\beta y}\right)^{\gamma^*/2\beta\sqrt{\beta}}. \tag{14}$$

*Proof.* For fixed $r$ we have

$$\begin{aligned}\sum_{s\geqslant 0} \xi_{r,s}\, y^s &= \sum_{l=0}^{r} \frac{(2\gamma y)^l}{l!}\, \xi_{r-l} \sum_{s=l}^{\infty} \binom{r+s-l}{r} (\beta y)^{s-l} \\ &= \frac{1}{(1-\beta y)^{r+1}} \sum_{l=0}^{r} \frac{(2\gamma y)^l}{l!}\, \xi_{r-l}\,.\end{aligned}$$

Multiplying by $x^r$ and summing over $r \geqslant 0$ we obtain

$$F(x,y) = \frac{e^{2\gamma xy/(1-\beta y)}}{1-\beta y}\, F_0\Big(\frac{x}{1-\beta y}\Big),$$

and the desired result follows by substituting the formula given in Proposition 4. ∎

The following result shows that $\xi_{r,s}$ belongs to the ideal $I_{r+s}$ and hence (by virtue of the inclusion $\gamma I_g \subset I_{g+1}$ noted above) that $\xi_{r,s,t} \in I_{r+s+t}$ for all $r, s, t \geqslant 0$.

**Theorem 4.** *For $r,\ s \geqslant 0$ we have*

$$\xi_{r,s} = \sum_{l=0}^{s} (-1)^{s-l} \left[\binom{r+l}{r} + \binom{r+l-1}{r}\right] \xi_{s-l}\,\xi_{r+s+l}\,, \tag{15}$$

*where the binomial coefficient $\binom{r+l-1}{r}$ is to be taken as 0 if $r = l = 0$.*

*Proof 1.* Denote the right-hand side of (15) by $\xi^*_{r,s}$. We will prove the equality $\xi^*_{r,s} = \xi_{r,s}$ by induction on $s$, the case $s = 0$ being trivial. A trivial binomial coefficient identity shows—independently of the definition of the classes $\xi_i$—that $\xi_{r,s}$ satisfies the recursion relation

$$s\,\xi_{r,s} = (r+s)\,\beta\,\xi_{r,s-1} + 2\gamma\,\xi_{r-1,s-1}\,.$$

On the other hand, using the recursion for the $\xi$'s, and taking appropriate care with the $l = 0$ terms if $r = 0$ or 1, we find:

$$\begin{aligned}
s\,\xi^*_{r,s} &- (r+s)\,\beta\,\xi^*_{r,s-1} - 2\gamma\,\xi^*_{r-1,s-1} \\
&= \sum_{l=0}^{s}(-1)^{s-l}\left[(s-l)\binom{r+l}{r} + (r+s+l)\binom{r+l-1}{r}\right]\xi_{s-l}\,\xi_{r+s+l} \\
&\qquad - \beta\sum_{l=0}^{s-1}(-1)^{s-1-l}\left[(s-l)\binom{r+l-1}{r} + (r+s+l)\binom{r+l}{r}\right]\xi_{s-1-l}\,\xi_{r+s-1+l} \\
&\qquad - 2\gamma\sum_{l=0}^{s-1}(-1)^{s-1-l}\left[\binom{r+l}{r} - \binom{r+l-2}{r}\right]\xi_{s-1-l}\,\xi_{r+s-2+l} \\
&= \sum_{l=0}^{s-1}(-1)^{s-l}\binom{r+l}{l}\Big\{(s-l)\,\xi_{s-l}\,\xi_{r+s+l} - (r+s+l+1)\,\xi_{s-l-1}\,\xi_{r+s+l+1} \\
&\qquad\quad - \beta\left[(s-l-1)\,\xi_{s-l-2}\,\xi_{r+s+l} - (r+s+l)\,\xi_{s-l-1}\,\xi_{r+s+l-1}\right] \\
&\qquad\quad - 2\gamma\left[\xi_{s-l-3}\,\xi_{r+s+l} - \xi_{s-l-1}\,\xi_{r+s+l-2}\right]\Big\} \\
&= \sum_{l=0}^{s-1}(-1)^{s-l}\binom{r+l}{l}\left\{(\alpha\,\xi_{s-l-1})\,\xi_{r+s+l} - \xi_{s-l-1}\,(\alpha\,\xi_{r+s+l})\right\} = 0\,. \qquad \blacksquare
\end{aligned}$$

*Proof 2.* We can write the definition of $\xi^*_{r,s}$ in the alternative form ',

$$\xi^*_{r,s} = \sum_{i+j=r+2s} (-1)^i \binom{r+s-i}{r} \xi_i\,\xi_j$$

(the binomial coefficient vanishes unless $0 \leqslant i \leqslant s$ or $r+s < i \leqslant r+2s$; set $l = s - i$ in the first case and $l = i - r - s$ in the second). The corresponding generating function is then given by

$$F^*(x,y) = \sum_{r,s\geqslant 0} \xi^*_{r,s}\,x^r\,y^s = \sum_{i,j\geqslant 0} (-1)^i\,\xi_i\,\xi_j \left[\sum_{\substack{r,s\geqslant 0\\ r+2s=i+j}} \binom{r+s-i}{r} x^r\,y^s\right].$$

Using the residue theorem we find that the coefficient in square brackets equals

$$\mathbf{C}_{t^{i+j}}\left[\sum_{r,s\geqslant 0}\binom{r+s-i}{r}x^r\,y^s\,t^{r+2s}\right]=\mathbf{C}_{t^{i+j}}\left[\sum_{s\geqslant 0}\frac{y^s\,t^{2s}}{(1-xt)^{s-i+1}}\right]$$
$$=\mathbf{C}_{t^{i+j}}\left[\frac{(1-xt)^i}{1-xt-yt^2}\right]=\mathrm{Res}_{t=0}\left[\frac{(1-xt)^i\,dt}{t^{i+j+1}(1-xt-yt^2)}\right]$$
$$=-\Big(\mathrm{Res}_{t=1/A}+\mathrm{Res}_{t=1/B}\Big)\left[\frac{(1-xt)^i\,dt}{t^{i+j+1}(1-xt-yt^2)}\right]=(-1)^i\,\frac{A^{j+1}\,B^i-B^{j+1}\,A^i}{A-B}\,,$$

where $A$ and $B$ are the roots of $A+B=x$, $AB=-y$, so

$$F^*(x,y)=F_0(A)\,F_0(B)=\frac{e^{-2\gamma(A+B)/\beta}}{\sqrt{(1-A^2\beta)(1-B^2\beta)}}\left(\frac{1+A\sqrt{\beta}}{1-A\sqrt{\beta}}\,\frac{1+B\sqrt{\beta}}{1-B\sqrt{\beta}}\right)^{\gamma^*/2\beta\sqrt{\beta}}\,,$$

and this equals $F(x,y)$ by (14) and the definitions of $A$ and $B$. ■

By inverting (15), or by expanding in powers of $A$ and $B$ the identity $F_0(A)F_0(B)=F(A+B,-AB)$ just proved, we get the following formula, which will be generalized in §7:

**Corollary.** $$\xi_{r_1}\,\xi_{r_2}=\sum_{j=0}^{\min(r_1,r_2)}(-1)^j\binom{r_1+r_2-2j}{r_1-j}\,\xi_{r_1+r_2-2j,j}\,.$$

We now give the main result relating the classes $\xi_{r,s}$ and the Thaddeus evaluation map.

**Theorem 5.** *The Thaddeus intersection numbers of the classes $\xi_{r,s,t}$ are given by*

$$E_g\big[\xi_{r,s,t}\,\xi_{r',s',t'}\big]=(-1)^{r+s}4^{g-1}\,\frac{g!}{(r+s+1)\,r!\,s!\,t!\,t'!}\,\delta_{r,s'}\,\delta_{r',s}\,\delta_{r+s+t+t',g-1}\,.$$

*Proof.* We have $\xi_{r,s,t}=\xi_{r,s}(2\gamma)^t/t!$ and (from the definition) $E_g[x\gamma^h]=2^h\dfrac{g!}{(g-h)!}E_{g-h}[x]$ for all $x\in R$, so it suffices to prove the theorem for $t=t'=0$. We can rewrite the generating function identity (14) in the form

$$F(x,y)=\frac{\sinh\theta}{x\sqrt{\beta}}\,e^{\alpha x+(\theta/\beta\sqrt{\beta}-x/\beta)\gamma^*}\,,\qquad\theta=\theta(\beta)=\tanh^{-1}\left(\frac{x\sqrt{\beta}}{1-\beta y}\right),$$

so, denoting by $\theta'=\theta'(\beta)$ the corresponding power series with $x'$, $y'$ in place of $x$, $y$,

$$F(x,y)\,F(x',y')=\frac{\sinh\theta}{x\sqrt{\beta}}\cdot\frac{\sinh\theta'}{x'\sqrt{\beta}}\cdot e^{(x+x')\alpha+w(\beta)\gamma^*}\,,\qquad w(\beta)=\frac{\theta+\theta'}{\beta\sqrt{\beta}}-\frac{x+x'}{\beta}\,.$$

Applying Proposition 2 (in the easy case when $u(\beta)$ is constant, here $x+x'$), we obtain

$$\mathcal{E}_T\big[F(x,y)F(x',y')\big]=\frac{\sinh\theta}{x\sqrt{\beta}}\cdot\frac{\sinh\theta'}{x'\sqrt{\beta}}\cdot\frac{(x+x')\sqrt{\beta}}{\sinh(\theta+\theta')}\bigg|_{\beta=(x+x')T}$$
$$=\frac{x+x'}{xx'\sqrt{\beta}\,(1/\tanh\theta+1/\tanh\theta')}\bigg|_{\beta=(x+x')T}$$
$$=\frac{x+x'}{x'\,(1-\beta y)+x\,(1-\beta y')}\bigg|_{\beta=(x+x')T}=\frac{1}{1-(xy'+x'y)\,T}\,.$$

Comparing coefficients of $T^{g-1}$, we find

$$E_g\big[F(x,y)\,F(x',y')\big] = (-4)^{g-1}\,(xy'+x'y)^{g-1}$$

and the desired identity follows by equating the coefficients of $x^r y^s x'^{r'} y'^{s'}$ on both sides. ∎

**Corollary.** *$I_g^E = I_g$ for all $g \geqslant 1$.*

*Proof of the corollary.* From the theorem we know that $E_g\left[\xi_{r,s,t}\xi_{r',s',t'}\right]$ is non-zero if $r'=s$, $s'=r$, $r+s+t+t'=g-1$ and zero in all other cases (as asserted in part (iii) of Theorem 3). On the other hand, from $\xi_{r,s,t} = (\neq 0)\alpha^r\beta^s\gamma^t +$ lower order terms (in a lexicographic ordering of the exponents), we see that the elements $\xi_{r,s,t}$ $(r,s,t\geqslant 0)$ form an additive basis for $R$. Together, these facts imply that the elements $\xi_{r,s,t}$ with $r+s+t\geqslant g$ belong to the ideal $I_g^E$ (because their intersection numbers with all basis elements vanish) and that they in fact form a basis of it. The first of these two statements shows that $I_g \subseteq I_g^E$ (because $I_g$ is the ideal generated by $\xi_g$, $\xi_{g+1}$, and $\xi_{g+2}$ and each of these belongs to the ideal $I_g^E$), and the second that $I_g^E \subseteq I_g$ (because $\xi_{r,s,t}\in I_g$ for $r+s+t\geqslant g$ by virtue of Theorem 4). ∎

Another consequence of Theorem 5 is the formula (5) for the Poincaré polynomial of the graded ring $R/I_g^E$ (which we have not yet proved coincides with $H_I^*(\mathcal{N}_g)$), since

$$\sum_{\substack{r,s,t\geqslant 0\\ r+s+t\leqslant g-1}} T^{r+2s+3t} = \mathbf{C}_{u^{g-1}}\left[\frac{1}{(1-u)(1-uT)(1-uT^2)(1-uT^3)}\right]$$
$$= \frac{(1-T^g)(1-T^{g+1})(1-T^{g+2})}{(1-T)(1-T^2)(1-T^3)}$$

(use a partial fraction decomposition). This formula for the dimensions of the graded components of $R/I_g^E$ is a reflection of the free resolution (syzygy)

$$0 \to R \xrightarrow{\begin{pmatrix}\xi_g & \xi_{g+1} & \xi_{g+2}\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}0 & -\xi_{g+2} & \xi_{g+1}\\ \xi_{g+2} & 0 & -\xi_g\\ -\xi_{g+1} & \xi_g & 0\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}\xi_g\\ \xi_{g+1}\\ \xi_{g+2}\end{pmatrix}} R \to R/I_g^E \to 0\,,$$

## §4. Computation of the Betti numbers

In this section we complete the solution of Thaddeus's "number-theoretic exercise" by computing the Poincaré series of the quotient ring of $R_g = \mathbb{Q}[\alpha,\beta]\otimes\Lambda(\psi_1,\dots,\psi_{2g})$ by the ideal (which we again denote by $I_g^E$, although it is of course bigger than the corresponding ideal in $R$) of elements $x\in R_g$ with $E_g(xR_g)=0$.

In Sections 2 and 3 we considered only the ring $R=\mathbb{Q}[\alpha,\beta,\gamma]$ and its image in $H^*(\mathcal{N}_g)$, under the assumption of the intersection formula (6). Here we extend this analysis to the study of the full ring

$$R_g = \mathbb{Q}_0[\alpha,\beta]\,\otimes\,\Lambda(\psi_1,\dots,\psi_{2g})$$

With the natural extension of the intersection functional $E_g$ from $R$ to $R_g$, to be explained in a moment, we get an ideal $I_g^E = \{x\in R_g \mid E_g[xy]=0\ \forall y\in R_g\}$ (this ideal is the natural extension to $R_g$ of the ideal in $R$ denoted $I_g^E$ up to now) and we will show that the Poincaré polynomial

$$\mathcal{P}_t(R_g/I_g^E) = \sum_{i\geqslant 0} \dim((R_g/I_g^E)_i)\, t^i$$

agrees with the known Poincaré polynomial of $H^*(\mathcal{N}_g)$. On the other hand, from the equality $I_g^E = I_g$ proved in the last section together with the fact (to be proved in §6) that $\xi_r$ coincides with the Mumford class and hence maps to zero in $H^*(\mathcal{N}_g)$, we deduce that there is a surjection $R_g/I_g^E \twoheadrightarrow H^*(\mathcal{N}_g)$, and the equality of the dimensions then proves at the same time the isomorphism of these two rings and the correctness of the intersection formula (6).

We first must introduce some notations. It is convenient to use both the notations $\psi_1, \dots, \psi_{2g}$ and $\psi_1, \dots, \psi_g, \psi_1^*, \cdots, \psi_g^*$ with $\psi_i^* = \psi_{i+g}$. We write $\gamma_i$ $(1 \leqslant i \leqslant g)$ for the class $\psi_i\psi_{i+g} = \psi_i\psi_i^* \in H^6(\mathcal{N}_g)$, so $\gamma = \sum_{i=1}^g \gamma_i$. Clearly $R_g$ has a basis consisting of the elements

$$\alpha^r\,\beta^s\,\psi_A\,\psi_B^*\,\gamma_C \qquad (r,\, s \geqslant 0, \quad A,\, B,\, C \subseteq [g], \quad A,\, B,\, C \text{ pairwise disjoint}) \tag{16}$$

where $[g] = \{1, \cdots, g\}$ and $\psi_A$, $\psi_B^*$, $\gamma_C$ are defined as $\prod_{i\in A} \psi_i$, $\prod_{i\in B} \psi_i^*$, $\prod_{i\in C} \gamma_i$, the products in the first two cases being taken in ascending order of the indices. In [T1], Thaddeus showed that for $A$ and $B$ disjoint one has

$$\langle \alpha^r\,\beta^s\,\psi_A\,\psi_B^*\,\gamma_C,\, [\mathcal{N}_g]\rangle = \begin{cases} \langle \alpha^r\,\beta^s,\, [\mathcal{N}_{g-|C|}]\rangle & \text{if } A = B = \emptyset\,, \\ 0 & \text{otherwise.} \end{cases}$$

This formula, which includes as a special case the identity $\langle x\gamma^h, [\mathcal{N}_g]\rangle = \frac{2^h g!}{(g-h)!}\langle x, [\mathcal{N}_{g-h}]\rangle$ used in the proof of the Corollary to Theorem 5, reduces the calculation of all $\langle x, [\mathcal{N}_g]\rangle$ to the case when $x \in \mathbb{Q}[\alpha,\beta]$. Because of it, if we extend the map $E_g : R \to \mathbb{Q}$ to $R_g$ by setting

$$E_g[\alpha^r\,\beta^s\,\psi_A\,\psi_B^*\,\gamma_C] = \begin{cases} E_{g-|C|}[\alpha^r\,\beta^s] & \text{if } A = B = \emptyset\,, \\ 0 & \text{otherwise } (A,\, B,\, C \text{ pairwise disjoint}), \end{cases}$$

then the desired equality $E_g[x] = \langle x, [\mathcal{N}_g]\rangle$ will hold for all $x \in R_g$ if it is true for $x \in R$.

With this extension of $E_g$ to $R_g$, we can state the result we want to prove as

**Theorem 6.** $$\mathcal{P}_t(R_g/I_g^E) = \frac{(1+t^3)^{2g} - (t+t^2)^{2g}}{(1+t^3)^2 - (t+t^2)^2}\,.$$

(Note that the expression on the right has the form $\frac{A^g - B^g}{A-B}$, so is a polynomial in $t$.)

*Proof.* It is clear that instead of the basis (16) we can take the elements

$$\xi_{r,s}\,\psi_A\,\psi_B^*\,\gamma_C \qquad (r,\, s \geqslant 0, \quad A,\, B,\, C \subseteq [g], \quad A,\, B,\, C \text{ disjoint}) \tag{17}$$

since $\xi_{r,s}$ is a polynomial in $\alpha$ and $\beta$ with coefficients in $\mathbb{Q}[\gamma]$ and leading term (in a lexicographic ordering) a non-zero multiple of $\alpha^r\beta^s$. Theorem 5 and the formulas above imply that the intersection number of two such elements is given by

$$\begin{aligned} &E_g\left[\xi_{r,s}\,\psi_A\,\psi_B^*\,\gamma_C \cdot \xi_{r',s'}\,\psi_{A'}\,\psi_{B'}^*\,\gamma_{C'}\right] \\ &\qquad = (*)\,\delta_{r,s'}\,\delta_{r',s}\,\delta_{A,B'}\,\delta_{A',B}\,\delta_{C\cap C',\emptyset}\,\delta_{r+r'+|A|+|B|+|C|+|C'|,g-1} \end{aligned}$$

where $\delta$ is the Kronecker delta and $(*)$ is a non-zero factor depending only on $g$, $r$, $s$, $A$ and $B$. Therefore, if we decompose $R_g$ as a direct sum

$$R_g = \bigoplus_{\substack{r,s,c\geqslant 0\\ A,B\subseteq[g],\ A\cap B=\emptyset}} R_g^{(r,s,A,B,c)}\,, \qquad R_g^{(r,s,A,B,c)} = \xi_{r,s}\,\psi_A\,\psi_B^*\cdot \bigoplus_{\substack{C\subseteq[g]\smallsetminus A\cup B\\ |C|=c}} \mathbb{Q}\gamma_C\,,$$

then (i) each piece $R_g^{(r,s,A,B,c)}$ is orthogonal (with respect to the pairing $E_g[xx']$) to all of $R_g$ if $r+s+|A|+|B|+c \geqslant g$ and to all but a single block $R_g^{(s,r,B,A,g-1-r-s-|A|-|B|-c)}$ if $r+s+|A|+|B|+c \leqslant g-1$, and (ii) in the latter case the matrix of the pairing between the blocks has rank $\rho(n-|A|-|B|, c, g-1-r-s-|A|-|B|-c)$, where by definition

$$\rho(n;i,j) := \text{rank of the } \tbinom{n}{i}\times\tbinom{n}{j} \text{ matrix } \left(\delta_{I\cap J,\emptyset}\right)_{I,J\subseteq[n],\,|I|=i,\,|J|=j}\,. \tag{18}$$

It follows that the rank of the full pairing on $R_g$, which is the dimension of the quotient $R_g/I_g^E$, equals

$$\sum_{\substack{r,s,c,c'\geqslant 0\\ A,B\subseteq[g],\ A\cap B=\emptyset\\ r+s+|A|+|B|+c+c'=g-1}} \rho(g-|A|-|B|,c,c')\,,$$

and, more precisely, that the Poincaré polynomial of $R_g/I_g^E$ is given by

$$\mathcal{P}_t(R_g/I_g^E) = \sum_{\substack{r,s,c,c'\geqslant 0\\ A,B\subseteq[g],\ A\cap B=\emptyset\\ r+s+|A|+|B|+c+c'=g-1}} \rho(g-|A|-|B|,c,c')\,t^{2r+4s+3|A|+3|B|+6c}\,. \tag{19}$$

(Actually, it would suffice for us to compute the total dimension, since once we have established $\dim(R_g/I_g^E) = \dim H^*(\mathcal{N}_g)$ and $I_g^E \subseteq \operatorname{Ker}(R_g \to H^*(\mathcal{N}_g))$ it would follow that $R_g/I_g^E \cong H^*(\mathcal{N}_g)$ and hence that all individual Betti numbers agree. But it will be no harder to compute the Poincaré polynomial directly.) We thus have two problems: to calculate the ranks $\rho(n;i,j)$ and to calculate the sum (19).

**Lemma.** *Let $n$, $i$, $j$ be integers with $n \geqslant i, j \geqslant 0$. Then the number $\rho(n;i,j)$ defined by (18) is given by*

$$\rho(n;i,j) = \begin{cases} 0 & \text{if } i+j>n,\\ \dbinom{n}{\min(i,j)} & \text{if } i+j\leqslant n.\end{cases} \tag{20}$$

*Proof.* Clearly the rank is 0 if $i+j>n$, since then no subsets of $[n]$ of cardinalities $i$ and $j$ can be disjoint. We thus assume $i+j\leqslant n$ and, by symmetry, that $i\leqslant j$; then we have to prove that the $\binom{n}{i}\times\binom{n}{j}$ matrix $P$ with entries $P_{I,J}=\delta_{I\cap J,\emptyset}$ $(I,J\subseteq[n],\ |I|=i,\ |J|=j)$ has maximal rank. We will prove this by showing that the $\binom{n}{j}\times\binom{n}{i}$ matrix $Q$ with entries

$$Q_{J,I} = \frac{(-1)^k\binom{n-i-j+k-1}{k}}{\binom{j}{k}\binom{n-i}{j}} \qquad (J,\ I\subseteq[n],\ |J|=j,\ |I|=i,\ |J\cap I|=k)$$

is a right inverse of $P$. Let $I_1$, $I_2$ be two subsets of $[n]$ of cardinality $i$. Then

$$(P \cdot Q)_{I_1,I_2} = \sum_{J \subseteq [n]\, |J|=j} P_{I_1,J}\, Q_{J,I_2} = \sum_{J \subseteq I_1'\, |J|=j} Q_{J,I_2}\,.$$

where $I_1' = [n] \setminus I_1$ denotes the complement of $I_1$ in $[n]$. Write the cardinality of $I_1 \cap I_2$ as $i-l$. Then each $J$ in the last sum has an intersection with $I_1' \cap I_2$ of some cardinality $k \leqslant l$ and an intersection with $I_1' \setminus I_2$ of cardinality $j-k$, and conversely for each $k \leqslant l$ there are $\binom{l}{k}\binom{n-i-l}{j-k}$ ways of choosing these two intersections. Hence

$$\begin{aligned}(P \cdot Q)_{I_1,I_2} &= \sum_{k=0}^{l} \binom{l}{k}\binom{n-i-l}{j-k} \cdot \frac{(-1)^k\binom{n-i-j+k-1}{k}}{\binom{j}{k}\binom{n-i}{j}} \\ &= \binom{n-i}{l}^{-1} \sum_{k=0}^{l} (-1)^k \binom{n-i-j}{l-k}\binom{n-i-j+k-1}{k} \\ &= \delta_{l,0} \;=\; \delta_{I_1,I_2}\end{aligned}$$

as claimed. ■

Substituting the result of the lemma into (19) we find

$$\mathcal{P}_t(R_g/I_g^E) = \sum_{\substack{r,s,c,c' \geqslant 0 \\ A,B \subseteq [g],\ A \cap B = \emptyset \\ r+s+|A|+|B|+c+c'=g-1}} \binom{g-|A|-|B|}{\min(c,c')}\, t^{2r+4s+3|A|+3|B|+6c}\,.$$

(Notice that $c+c' \leqslant g-|A|-|B|$ is automatically satisfied, so only the second case of (20) applies.) For each integer $h \leqslant g$ there are $2^h\binom{g}{h}$ ways to choose disjoint subsets $A$ and $B$ of $[g]$ with $|A|+|B|=h$, so we can rewrite this

$$\mathcal{P}_t(R_g/I_g^E) = \sum_{\substack{r,s,h,c,c' \geqslant 0, \\ r+s+h+c+c'=g-1}} 2^h \binom{g}{h}\binom{g-h}{\min(c,c')}\, t^{2r+4s+3h+6c}\,.$$

There are now two ways of finishing the computation. The first, mechanical, way is to make a standard generating function construction, writing $\mathcal{P}_t$ as the coefficient of $X^{g-1}$ in the series obtained from the above expression by omitting the second summation condition and inserting a factor $X^{r+s+h+c+c'}$. This series can be summed in closed form and one can then compute the required coefficient by residue calculus using the substitution $u = t/(1+t^3X)^2$; the details are a bit tedious. A nicer method is as follows. Write $p = \min(c,c')$, $q = |c-c'|$. Then the summation over $c$ and $c'$ can be replaced by a summation over $p$ and $q$, but if $q \neq 0$ then there are two pairs $(c,c') = (p,p+q)$ and $(p+q,p)$ corresponding to $(p,q)$, so

$$\mathcal{P}_t(R_g/I_g^E) = \sum_{\substack{p,q,r,s,h \geqslant 0, \\ 2p+q+r+s+h=g-1}} 2^h \binom{g}{h}\binom{g-h}{p}\, t^{2r+4s+3h+6p}\left(1+t^{6q}-\delta_{0,q}\right).$$

Now using the identity

$$\sum_{\substack{q,r,s\geqslant 0\\ q+r+s=n-1}} t^{2r+4s}\,(1+t^{6q}-\delta_{0,q}) = \frac{(1-t^{2n})(1-t^{4n})}{(1-t^2)(1-t^4)} \qquad (n\geqslant 0),$$

whose proof is just an exercise in summing geometric series, we find

$$\begin{aligned}\mathcal{P}_t(R_g/I_g^E) &= \sum_{\substack{p,n,h\geqslant 0\\ 2p+n+h=g}} \binom{g}{h}\binom{g-h}{p}(2t^3)^h\,\frac{t^{6p}-t^{6p+2n}-t^{6p+4n}+t^{6p+6n}}{(1-t^2)(1-t^4)}\\ &= \sum_{\substack{h,i,j\geqslant 0\\ h+i+j=g}} \frac{g!}{h!\,i!\,j!}\,(2t^3)^h\,\frac{t^{0i+6j}-t^{2i+4j}}{(1-t^2)(1-t^4)},\end{aligned}$$

where in the second line we have repeated the initial trick $\{c,c'\}=\{p,p+q\}$ in reverse by setting $\{p,p+n\}=\{i,j\}$ with one pair $(i,j)$ corresponding to two pairs $(p,n)$. The trinomial theorem now gives the desired result $(1-t^2)(1-t^4)\mathcal{P}_t=(1+2t^3+t^6)^g-(t^2+2t^3+t^4)^g$. ∎

*Remark.* The proof of the theorem has given us an additive basis of $R_g/I_g^E$ ($\cong H^*(\mathcal{N}_g)$) with "anti-diagonal intersection pairing," i.e., such that each basis element has a non-zero intersection with exactly one basis element. This basis is the union of bases of the blocks $R_g^{(r,s,A,B,c)}$ for all $(r,s,A,B,c)$ with $r+s+|A|+|B|+c\leqslant g-1$, the latter being given as follows: if $c\leqslant c':=g-1-r-s-|A|-|B|-c$, then take all $\binom{g-|A|-|B|}{c}$ elements (17) with $|C|=c$, while if $c>c'$ then take the $\binom{g-|A|-|B|}{c'}$ elements $\xi_{r,s}\psi_A\psi_B^*\sum_C Q_{C,C'}\gamma_C$ where $C'$ runs over the subsets of $[g]\smallsetminus A\cup B$ of cardinality $c'$.

## §5. Characteristic class computations I: proof of the Newstead and Verlinde conjectures

To complete the proofs of Theorems 2 and 3 and the intersection formula (6), we still must show that $\xi_r$ coincides with the classes defined by Mumford and hence vanishes in $H^*(\mathcal{N}_g)$ for $r\geqslant g$. This will be done in the next section, whereas here we still assume formula (6) and use it to compute various numerical invariants of $\mathcal{N}_g$ by the Hirzebruch-Riemann-Roch formula. Since there are several calculations to be done, we have divided the section into three subsections: in **A** we calculate the Chern classes of the tangent bundle $T_g$ of $\mathcal{N}_g$, in **B** we prove the three formulas

$$\text{(a) } \beta^g=0 \text{ in } H^{2g}(\mathcal{N}_g);\quad \text{(b) } c_i(T_g)=0 \text{ for } i\geqslant 2g-1;\quad \text{(c) } \chi(\mathcal{N}_g,T_g)=3-3g \qquad (21)$$

conjectured in §5 of [N2], and in **C** we prove the two Verlinde formulae (2), all under the assumption of (6). Although it would be more logical to interchange this section and the next one to remove the (temporary) hypotheticality of the results proven here, we have preferred this order because the calculation of the Mumford classes uses the same techniques as the calculations in this section, but is more difficult, so that it is easier to "practise" by calculating the numerical invariants first.

We first fix some notations needed here and in §6. Let $\sigma \in H^2(C)$ be the cohomology fundamental class of our genus $g$ Riemann surface $C$ and $e_1, \ldots, e_{2g}$ a basis of $H^1(C)$ with $e_i e_{i+g} = \sigma$ $(1 \leqslant i \leqslant g)$ and all other intersection numbers equal to 0. As in the introduction, $J_g$ and $\tilde{\mathcal{N}}_g$ denote the moduli spaces of isomorphism classes of 1-dimensional and of stable 2-dimensional bundles, respectively, of degree $4g-3$ over $C$. (The reason for this choice of degree will be recalled below.) Then $\tilde{\mathcal{N}}_g$ is fibred over $J_g$ with typical fibre $\mathcal{N}_g$ by the determinant map $\pi : \tilde{\mathcal{N}}_g \to J_g$ and its rational cohomology ring is the tensor product of $H^*(\mathcal{N}_g)$ with $H^*(J_g)$, the latter being the free exterior algebra on the $2g$ classes $d_i \in H^1(J_g)$ defined by the formula

$$c_1(U) = (4g-3) \otimes \sigma + \sum_{i=1}^{2g} d_i \otimes e_i + x \otimes 1 \in H^2(J_g \times C) \cong \sum_{r=0}^{2} H^r(J_g) \otimes H^{2-r}(C).$$

Here $U$ is the "tautological" line bundle over $J_g \times C$ with fibre over $[L] \times C$ isomorphic to $L$ and the class $x \in H^2(J_g)$ is not uniquely defined and can be taken to be 0, since $U$ is unique only up to tensoring by the pull-back of a line bundle over $J_g$. There is also a tautological 2-dimensional bundle $\tilde{V}$ over $\tilde{\mathcal{N}}_g \times C$ whose restriction to $\{E\} \times C$ is isomorphic to $E$ and whose restrictions to the fibres of $\pi \circ f$, where $f : \tilde{\mathcal{N}}_g \times C \to \tilde{\mathcal{N}}_g$ is the projection map, are the tensor products of a fixed 2-dimensional bundle $V$ over $\mathcal{N}_g \times C$ with a variable line bundle. The bundles $\tilde{V}$ and $V$ are unique only up to tensoring by a line bundle over $\tilde{\mathcal{N}}_g$ and $\mathcal{N}_g$, respectively, but the 4-dimensional bundle $\mathrm{End}(\tilde{V}) = \tilde{V} \otimes \tilde{V}^*$ is uniquely defined and is the pull-back of the uniquely defined bundle $\mathrm{End}(V)$ over $\mathcal{N}_g \times C$, and similarly for the 3-dimensional subbundles $\tilde{W}$ and $W$ of endomorphisms of trace 0. Thus, while the Chern classes of $V$ are unique only up to a transformation $c_1(V) \mapsto c_1(V) + 2u$, $c_2(V) \mapsto c_2(V) + c_1(V)u + u^2$ with $u \in H^2(\mathcal{N}_g)$, the Chern classes $c_1(W) = 0$, $c_2(W) = 4c_2(V) - c_1(V)^2$ and $c_3(W) = 0$ of $W$ are well-defined. The classes $\alpha$, $\beta$, $\psi_i$ of Newstead's theorem are then defined by the formula

$$c_2(W) = 2\alpha \otimes \sigma + 4 \sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1 \in H^4(\mathcal{N}_g \times C) \cong \sum_{r=2}^{4} H^r(\mathcal{N}_g) \otimes H^{4-r}(C).$$

(The coefficients 2 and 4 arise because $c_2(W) = 4c_2(V) - c_1(V)^2$.) We also write

$$D = \sum_{i=1}^{2g} d_i \otimes e_i \in H^2(J_g \times C) \qquad \Psi = \sum_{i=1}^{2g} \psi_i \otimes e_i \in H^4(\mathcal{N}_g \times C). \tag{22}$$

Then the formulas for $c_1(U)$ and $c_2(W)$ become

$$c_1(U) = (4g-3) \otimes \sigma + D + x \otimes 1, \qquad c_2(W) = 2\alpha \otimes \sigma + 4\Psi - \beta \otimes 1 \tag{23}$$

and the intersection numbers for the $e_i$ imply the relations

$$D^2 = -2A \otimes \sigma, \quad D^3 = 0, \quad D\Psi = B \otimes \sigma, \quad \Psi^2 = \gamma \otimes \sigma, \quad \Psi^3 = 0 \tag{24}$$

with $A = \sum_{i=1}^{g} d_i d_{i+g} \in H^2(J_g)$, $B = \sum_{i=1}^{g} (-d_i \psi_{i+g} + d_{i+g} \psi_i) \in H^4(\tilde{\mathcal{N}}_g)$, and $\gamma = -2 \sum_{i=1}^{g} \psi_i \psi_{i+g} \in H^6(\mathcal{N}_g)$.

Finally, we recall the relationship between the Chern character and (total) Chern class of a vector bundle:

**Lemma 1.** *Let $\xi$ be a $d$-dimensional bundle over a base space $B_\xi$. Then the total Chern class and the Chern character of $\xi$ are related by*

$$\log c(\xi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} s_n \quad \leftrightarrow \quad ch(\xi) = d + \sum_{n=1}^{\infty} \frac{s_n}{n!} \qquad (s_n = s_n(\xi) \in H^{2n}(B_\xi)).$$

The proof is immediate from the definitions and the splitting principle. We will also use the notation $c(\xi)_t$ for the generating function $\sum_{i\geqslant 0} c_i(\xi)\,t^i = \exp\bigl(-\sum_{n\geqslant 1} s_n(\xi)\,(-t)^n/n\bigr)$ of the Chern classes of $\xi$.

**A. The tangent bundle of $\mathcal{N}_g$.** To prove parts (b) and (c) of (21), as well as for the Verlinde formulas, we will need the Chern class of $T_g$. Their calculation was performed in principle in §4 of [N2], but the final result was not given explicitly, so we repeat it here.

The choice of $4g-3$ for the degrees of the bundles parametrized by $J_g$ and $\tilde{\mathcal{N}}_g$ implies that $H^0(C, W_y) = \{0\}$, $H^1(C, W_y) \cong T(\tilde{\mathcal{N}}_g)_y$ for $y \in \tilde{\mathcal{N}}_g$ and hence that the (K-theoretical) push-forward $f_!(W)$ equals $-T_g$ ([N2], Lemma in §4, or [AB], p. 582). Hence by the Grothendieck-Hirzebruch-Riemann-Roch theorem

$$\mathrm{ch}(T_g) = -f^*\bigl(\mathrm{ch}(W)\cdot \mathrm{td}(C)\bigr) = -f_*\bigl(\mathrm{ch}(W)\cdot(1-(g-1)\sigma)\bigr).$$

Applying Lemma 1 to $\mathrm{ch}(W) = 1 + c_2(W)$ we find that $s_n(W) = 0$ for $n$ odd and

$$\begin{aligned} s_{2r}(W) &= 2\,(-c_2(W))^r = 2\,(\beta\otimes 1 - 4\Psi - 2\alpha\otimes\sigma)^r \\ &= 2\beta^r\otimes 1 - 8r\sum_{i=1}^{2g}\beta^{r-1}\psi_i\otimes e_i + \bigl(16r(r-1)\beta^{r-2}\gamma - 4r\alpha\beta^{r-1}\bigr)\otimes\sigma \end{aligned}$$

and therefore

$$s_n(T_g) = f_*\Bigl((g-1)\,s_n(W)\,\sigma - \frac{s_{n+1}(W)}{n+1}\Bigr) = \begin{cases} 2\,(g-1)\,\beta^r & \text{if } n = 2r > 0, \\ 2\alpha\beta^r - 8r\beta^{r-1}\gamma & \text{if } n = 2r+1. \end{cases}$$

This last equation, which is Theorem 2 of §4 of [N2], implies on the one hand the formulas

$$c_1(T_g) = 2\alpha \qquad p(T_g) = \exp\Bigl(-2(g-1)\sum_{r=1}^{\infty}\frac{(-\beta)^r}{r}\Bigr) = (1+\beta)^{2g-2}$$

(=Corollaries 1 and 2 of [N2], §4) and hence

$$\mathrm{td}(T_g) = e^{\alpha}\left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^{2g-2}, \tag{25}$$

and on the other hand gives the "closed" formulas

$$\mathrm{ch}(T_g) = (g-1)(1+2\cosh\sqrt{\beta}) + 2\alpha\cosh\sqrt{\beta} - 2\Bigl(\frac{\cosh\sqrt{\beta}}{\beta} - \frac{\sinh\sqrt{\beta}}{\beta\sqrt{\beta}}\Bigr)\gamma^* \tag{26}$$

for the Chern character (recall that $\gamma^* = 2\gamma + \alpha\beta$) and, by Lemma 1 again,

$$c(T_g) = \bigl(1-\beta\bigr)^{g-1}\exp\Bigl(\frac{2\alpha}{1-\beta} + 2\Bigl(\frac{\tanh^{-1}\sqrt{\beta}}{\beta\sqrt{\beta}} - \frac{1}{\beta(1-\beta)}\Bigr)\gamma^*\Bigr) \tag{27}$$

for the total Chern class of the bundle $T_g$.

**B. The Newstead conjectures.** In this subsection we prove the three assertions of (21) under the assumption of the intersection formula (6). Part (a) is then trivial, since, as discussed in §4, a class $x \in R = \mathbb{Q}[\alpha,\beta,\gamma]$ vanishes in $H^*(\mathcal{N}_g)$ if and only if $E_g[xy] = 0$ for all $y \in R$, and the truth of this for $x = \beta^g$ is a special case of (6). Part (c) of (21) also follows easily, since by (25), (26) and the Hirzebruch-Riemann-Roch theorem we have

$$\begin{aligned}
\chi(\mathcal{N}_g, T_g) &= E_g\big[\mathrm{td}(\mathcal{N}_g)\,\mathrm{ch}(T_g)\big] \\
&= (g-1)\,E_g\big[e^\alpha\,\big(\tfrac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\big)^{2g-2}\,(1+2\cosh\beta)\big] \\
&\quad + 2\,E_g\big[e^\alpha\,\big(\tfrac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\big)^{2g-2}\,\big(\alpha\,\cosh\sqrt{\beta} - \big(\tfrac{\cosh\sqrt{\beta}}{\beta} - \tfrac{\sinh\sqrt{\beta}}{\beta\sqrt{\beta}}\big)\,\gamma^*\big)\big] \\
&= (g-1)(1-2) + 2(1-g),
\end{aligned}$$

the two expressions $E_g[\ldots]$ being calculable by a direct, though tedious, application of Proposition 3 to the corresponding generating functions. To get part (b), we have to show that for all monomials $y = \alpha^i\beta^j\gamma^{*k}$ in $R$ the expression $x^{2g-2}E_g[y\,c(T_g)_{1/x}]$, which *a priori* is a Laurent polynomial in $x$, is actually a polynomial. Replacing $y$ by a generating function $\beta^r\,e^{u\alpha+w\gamma}$, we see that that we have to prove that for each $r \geqslant 0$ the power series

$$C_r = C_r(x,u,w,T) = \sum_{g=1}^{\infty} x^{2g-2}\,E_g\big[\beta^r\,e^{u\alpha+w\gamma^*}\,c(T_g)_{1/x}\big]\,(-T/4)^{g-1}$$

in $u$, $w$ and $T$ has coefficients in $\mathbb{Q}[x]$ rather than $\mathbb{Q}[x,1/x]$. Substituting for $c(T_g)$ from (27), we find that $C_r$ has the form of the left-hand side of (12) with

$$\begin{aligned}
&f(\beta) = \frac{\beta^r}{x^2-\beta}, \quad h(\beta) = x^2-\beta, \quad u(\beta) = u + \frac{2x}{x^2-\beta}, \\
&w(\beta) = w + 2\frac{\tanh^{-1}(\sqrt{\beta}/x)}{\beta\sqrt{\beta}} - \frac{2x}{\beta(x^2-\beta)}, \\
&T = Q^{-1}(\beta) = \frac{\beta}{u(x^2-\beta)+2x}, \quad \beta = Q(T) = \frac{(2+ux)xT}{1+uT}, \qquad Q'(T) = \frac{(2+ux)x}{(1+uT)^2}.
\end{aligned}$$

Now using (12) and the addition law for $\sinh$ we find after some computation

$$\begin{aligned}
C_r &= \frac{\dfrac{x\sqrt{\beta}}{x^2-\beta}\cdot\dfrac{2+ux}{(1+uT)^2}\cdot\beta^r}{\sinh[u\sqrt{\beta}+w\beta\sqrt{\beta}+2\tanh^{-1}(\sqrt{\beta}/x)}\Bigg|_{\beta=Q(T)} \\
&= \frac{\dfrac{2+ux}{(1+uT)^2}\,\beta^r}{\big(\dfrac{x}{\sqrt{\beta}}+\dfrac{\sqrt{\beta}}{x}\big)\sinh(u\sqrt{\beta}+w\beta\sqrt{\beta}) + 2\cosh(u\sqrt{\beta}+w\beta\sqrt{\beta})}\Bigg|_{\beta=Q(T)} \\
&= \frac{(2+ux)^r\,x^r\,T^r}{(1+uT)^{r+1}\,(1+2uT)\,(1+\mathrm{O}(xT))},
\end{aligned}$$

which indeed contains only nonnegative powers of $x$. Moreover, taking $x=0$ we see that

$$C_r(0,u,w,T) = \delta_{r,0}\cdot\frac{1}{(1+uT)(1+2uT)} = \delta_{r,0}\,\sum_{g=1}^{\infty}(-1)^{g-1}\,(2^g-1)\,u^{g-1}\,T^{g-1}$$

and hence that the top non-vanishing Chern class of $T_g$ satisfies

$$E_g\left[\alpha^i\beta^j\gamma^{*h}\,c_{2g-2}(T_g)\right] = \begin{cases} 4^{g-1}(2^g-1)(g-1)! & \text{if } i=g-1,\ j=0,\ h=0 \\ 0 & \text{otherwise.} \end{cases}$$

By (9), this is equivalent to the closed formula

$$c_{2g-2}(T_g) = (-1)^{g-1}\,(2^g-1)\,\beta^{g-1}\,, \tag{28}$$

a result which will be generalized in §7 (Theorem 7), where we will give another proof of (21b).

**C. The Verlinde formulas.** We have to prove the two formulas stated in Theorem 1. The case $\varepsilon=-1$ is straightforward: $\mathcal{N}_g^- = \mathcal{N}_g$ is smooth, so the Hirzebruch-Riemann-Roch theorem together with (25) and $c_1(\mathcal{L}_-)=\alpha$ gives

$$D_-(g,k) = \chi(\mathcal{N}_g, \mathcal{L}_-^{k/2-1}) = E_g\left[e^{k\alpha/2}\left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^{2g-2}\right]$$

and hence (by Proposition 3 with $f=\frac{4}{\beta}\sinh\frac{\sqrt{\beta}}{2}$, $h=\frac{1}{f}$, $u=\frac{k}{2}$, $w=0$ and $T=-\frac{8}{k}\sin^2 x$)

$$\sum_{g=1}^{\infty}\frac{D_-(g,k)}{(k/2)^{g-1}}\sin^{2g-2}x = \frac{k\,\tan x}{\sin kx} = 1+\frac{k^2+2}{6}\sin^2 x+\frac{7k^4+40k^2+88}{360}\sin^4 x+\cdots\,,$$

which is equivalent to formula (2) for $\varepsilon=-1$ by an elementary identity ([Z], eq. (5)). That this works so easily is of course not surprising, since Thaddeus wrote down the formula (6), which we are still assuming, precisely to produce the Verlinde formula in the case $\varepsilon=-1$.

For $\varepsilon=+1$ we can no longer apply the Riemann-Roch theorem directly, since $\mathcal{N}_g^+$ is singular. However, as mentioned in the discussion following Theorem 1, Bertram and Szenes [BS] gave a formula for $D_+(g,k)$ in terms of invariants of the *non-singular* variety $\mathcal{N}_g=\mathcal{N}_g^-$, namely

$$D_+(g,k) = \dim H^0(\mathcal{N}_g,\,\mathrm{Sym}^{k-2}V_0)\,,$$

where $V_0$ is the twist by a line bundle of the restriction of $V$ to $\mathcal{N}_g\times\{P\}$, $P\in C$, the line bundle being chosen so that $c_1(V_0)=\alpha$. The right-hand side of this formula equals $\chi(\mathcal{N}_g,\mathrm{Sym}^{k-2}V_0)$ and hence can be computed by the Riemann-Roch theorem as $E_g\left[\mathrm{ch}(\mathrm{Sym}^{k-2}V_0)\,\mathrm{td}(\mathcal{N}_g)\right]$. Now using (25) and computing $\mathrm{ch}(\mathrm{Sym}^{k-2}V_0)$ from

$$c(V_0) = (1+x_1)(1+x_2) \;\Rightarrow\; x_1+x_2=c_1=\alpha,\quad (x_1-x_2)^2=c_1^2-4c_2=-\beta,$$

$$\mathrm{ch}(\mathrm{Sym}^{k-2}V_0) = \frac{e^{(k-1)x_1}-e^{(k-1)x_2}}{e^{x_1}-e^{x_2}} = e^{(k-2)\alpha/2}\,\frac{\sinh{(k-1)\sqrt{\beta}/2}}{\sinh\sqrt{\beta}/2}\,,$$

and then calculating the generating series of Proposition 3 in the usual way, we obtain

$$\sum_{g=1}^{\infty}\frac{D_+(g,k)}{(k/2)^{g-1}}\sin^{2g-2}x = \frac{k\,\sin(k-1)x}{\cos x\,\sin kx}\,,$$

which, as remarked after Theorem 1, is equivalent to formula (12) in the case $\varepsilon=1$.

## §6. Characteristic class computations II: the Mumford classes

In this section we compute the Chern classes of the push-forward $f_!(\tilde{V})$. As mentioned in the introduction, this is a $(2g-1)$-dimensional bundle over $\tilde{\mathcal{N}}_g$ (by the Riemann-Roch theorem and the choice of $4g-3$ as the degree of the underlying 2-dimensional bundles), so $c_i(f_!(\tilde{V})) \in H^{2i}(\tilde{\mathcal{N}}_g)$ vanishes for $i \geqslant 2g$ and we wish to show that the Künneth component of this class in $H^{2i-2g}(\mathcal{N}_g) \otimes H^{2g}(J_g) \cong H^{2i-2g}(\mathcal{N}_g)$ is essentially the class $\xi_{i-g}$, completing the proof that the ideal $I_g = I_g^E$ is contained in, and hence coincides with, the kernel of the map $R_g \to H^*(\mathcal{N}_g)$.

Under the identification of $H^*(\tilde{\mathcal{N}}_g)$ with $H^*(\mathcal{N}_g) \otimes H^*(J_g)$ the determinant of $\tilde{V}$ is just the (pull-back of the) line bundle $U$, while the bundle $\tilde{W} \subset \mathrm{End}(\tilde{V})$ is the pull-back of $W$, so by (23) we have

$$c_1(\tilde{V}) = c_1(U) = (4g-3)\otimes\sigma + D + x\otimes 1\,, \quad 4c_2(\tilde{V}) - c_1(\tilde{V})^2 = c_2(W) = 2\alpha\otimes\sigma + 4\Psi - \beta\otimes 1$$

and hence formally

$$c(\tilde{V}) = 1 + c_1(\tilde{V}) + c_2(\tilde{V}) = \left(1 + \frac{c_1(\tilde{V}) + \sqrt{\beta}\otimes 1}{2}\right)\left(1 + \frac{c_1(\tilde{V}) - \sqrt{\beta}\otimes 1}{2}\right) - \Psi - \frac{\alpha}{2}\otimes\sigma\,.$$

Write $\delta = \Psi + \frac{\alpha}{2}\otimes\sigma$ so that $\delta^2 = \gamma\otimes\sigma$, $\delta^3 = 0$ by (24). We apply to this situation the following lemma.

**Lemma 2.** *Let $\xi$ be a 2-dimensional bundle over a base space $X$, and suppose that the Chern class of $\xi$ has a (formal) decomposition of the form $c(\xi) = (1+x_1)(1+x_2) - \delta$ with $x_1$, $x_2 \in H^2(X)$, $\delta \in H^4(X)$, $\delta^3 = 0$. Then*

$$\mathrm{ch}(\xi) = e^{x_1} + e^{x_2} + \frac{e^{x_1} - e^{x_2}}{x_1 - x_2}\,\delta - \left(\frac{e^{x_1} - e^{x_2}}{(x_1 - x_2)^3} - \frac{e^{x_1} + e^{x_2}}{2(x_1 - x_2)^2}\right)\delta^2\,.$$

*Proof.* Applying Lemma 1, we find

$$\begin{aligned}
\log c(\xi) &= \log(1+x_1) + \log(1+x_2) + \log\Big(1 + \frac{\delta}{(1+x_1)(1+x_2)}\Big) \\
&= \sum_{n=1}^{\infty} (-1)^{n-1}\,\frac{x_1^n + x_2^n}{n} + \frac{\delta}{(1+x_1)(1+x_2)} - \frac{\delta^2}{2(1+x_1)^2(1+x_2)^2}\,, \\
s_n(\xi) &= x_1^n + x_2^n - n\,\frac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2}\,\delta - \left(n\,\frac{x_1^{n-1} - x_2^{n-1}}{(x_1 - x_2)^3} - \frac{n(n-1)}{2}\,\frac{x_1^{n-2} + x_2^{n-2}}{(x_1 - x_2)^2}\right)\delta^2
\end{aligned}$$

for $n \geqslant 1$, and computing $\mathrm{ch}(\xi) = 2 + \sum_{n=1}^{\infty} s_n(\xi)/n!$ yields the formula given. Alternatively, we can write

$$\mathrm{ch}(\xi) = e^{\frac{1}{2}c_1(\xi)}\,\cosh\sqrt{\tfrac{1}{4}c_1(\xi)^2 - c_2(\xi)} = e^{\frac{1}{2}(x_1 + x_2)}\,\cosh\sqrt{\tfrac{1}{4}(x_1 - x_2)^2 - \delta}$$

and expand the right-hand side in a Taylor series in $\delta$ which breaks off at the $\delta^2$ term. Note, by the way, that the coefficients of $\delta$ and $\delta^2$ in the lemma are power series in $x_1$ and $x_2$, despite appearances. ∎

We now apply this to the bundle $\tilde{V}$. Note that for our purposes we can suppose that the class $x \in H^2(J_g)$ is 0, since this class will survive the pushing-down by $f$ and an easy lemma says that the effect of twisting by a line bundle $L$ on the total Chern class of a bundle $F$ is given by

$$c(F \otimes L)_t = (1+tx)^N\, c(F)_{t/(1+xt)} \qquad (N = \dim F, \quad x = c_1(L)),$$

which makes sense formally even for rational classes $x$ and preserves the property of $c(F)_t$ of being a polynomial of degree $\leqslant N$ in $t$. We find

$$\mathrm{ch}(\tilde{V}) = e^{\frac{1}{2}D+(2g-\frac{3}{2})\sigma}\left[2\cosh\frac{\sqrt{\beta}}{2} + \frac{2\sinh\sqrt{\beta}/2}{\sqrt{\beta}}\,\delta - \left(\frac{2\sinh\sqrt{\beta}/2}{\beta\sqrt{\beta}} - \frac{\cosh\sqrt{\beta}/2}{\beta}\right)\delta^2\right]$$

and multiplying this out we find that $\mathrm{ch}(\tilde{V})$ has the Künneth-decomposed form $X \otimes 1 + \sum_{i=1}^{2g} Y_i \otimes e_i + Z \otimes \sigma$ with $X = 2\cosh\sqrt{\beta}/2$ and

$$Z = 2\cosh\frac{\sqrt{\beta}}{2}\left(\frac{4g-3}{2} - \frac{A}{4}\right) - \frac{\sinh\sqrt{\beta}/2}{\sqrt{\beta}}\,B - \left(\frac{2\sinh\sqrt{\beta}/2}{\beta\sqrt{\beta}} - \frac{\cosh\sqrt{\beta}/2}{\beta}\right)\gamma\,.$$

But by the Grothendieck-Hirzebruch-Riemann-Roch theorem and the fact that $\mathrm{td}(C) = 1-(g-1)\sigma$ we have $\mathrm{ch}(f_!(\tilde{V})) = f_*(\mathrm{ch}(\tilde{V})\mathrm{td}(C)) = -(g-1)\,X + Z$, so this yields a formula for the Chern character of $f_!(\tilde{V})$. In terms of the notations of Lemma 1 this formula says that $\dim f_!(\tilde{V}) = 2g-1$ (which we knew) and

$$2^n\, s_n(f_!(\tilde{V})) = \begin{cases} (2g-1)\,\beta^{\frac{n}{2}} - 2n\,\beta^{\frac{n-2}{2}}\,B & \text{for } n \text{ even,}\\ -n\,\beta^{\frac{n-1}{2}}\,A + 2(n-1)\,\beta^{\frac{n-3}{2}}\,\gamma - \alpha\,\beta^{\frac{n-1}{2}} & \text{for } n \text{ odd.}\end{cases}$$

Lemma 1 then gives the formula

$$\begin{aligned}\log c(f_!(\tilde{V}))_{-2t} &= -\sum_{n=1}^{\infty} 2^n s_n(f_!(\tilde{V}))\,\frac{t^n}{n}\\ &= (g-\frac{1}{2})\,\log(1-\beta t^2) + \frac{At}{1-\beta t^2} + \frac{2Bt^2}{1-\beta t^2} - \frac{2\gamma t}{\beta(1-\beta t^2)} + \frac{\tanh^{-1}(t\sqrt{\beta})}{\beta\sqrt{\beta}}\,\gamma^*\end{aligned}$$

or

$$c(f_!(\tilde{V}))_{-2t} \;=\; \left(1-\beta\,t^2\right)^{g-\frac{1}{2}}\left(\frac{1+t\sqrt{\beta}}{1-t\sqrt{\beta}}\right)^{\gamma^*/2\beta\sqrt{\beta}} \exp\!\left(\frac{At+2Bt^2-2\gamma t/\beta}{1-\beta\,t^2}\right) \tag{29}$$

for the total Chern class of $f_!(\tilde{V})$.

The last step is to evaluate the slant product with $[J_g]$. This is performed with the help of the following lemma.

**Lemma 3.** *Let $A \in H^2(J_g)$, $B \in H^4(\tilde{N}_g)$ be the classes appearing in* (24). *Then*

$$\frac{A^r}{r!}\,\frac{B^s}{s!} \diagdown [J_g] = \begin{cases} \dfrac{(\gamma/2)^p}{p!} & \text{if } r = g-p,\ s = 2p \quad (0 \leqslant p \leqslant g),\\ 0 & \text{if } 2r+s \neq 2g.\end{cases}$$

**Corollary.** *Let $\kappa \in H^*(\mathcal{N}_g)$. Then $e^{(A+2Bt)\kappa} \setminus [J_g] = \kappa^g\, e^{2\kappa\gamma t^2}$.*

*Proof.* Applying the principle

$$x_i^2 = 0, \quad x_i x_j = x_j x_i \quad (i \in S) \quad \Rightarrow \quad \frac{1}{r!}\Bigl(\sum_{i\in S} x_i\Bigr)^r = \sum_{I\subseteq S,\,|I|=r} \prod_{i\in I} x_i \tag{30}$$

we find

$$\frac{A^r}{r!} = \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=r}} \prod_{i\in I} d_i d_{i+g}\,, \qquad \frac{B^s}{s!} = \sum_{\substack{J,K\subseteq\{1,\dots,g\}\\ |J|+|K|=s}} \prod_{j\in J} \psi_j d_{j+g} \prod_{k\in K} -\psi_{k+g} d_k\,.$$

Multiplying these expressions and taking the slant product with $[J_g]$, which means picking out the coefficient of $\prod_{i=1}^g d_i d_{i+g}$, we find that the only terms that contribute are those with $J = K = \{1,\dots,g\} \setminus I$ and hence that

$$\frac{A^r}{r!}\,\frac{B^s}{s!} \setminus [J_g] = \delta_{s,2(g-r)} \sum_{I\subseteq\{1,\dots,g\},\,|I|=r} \prod_{i\notin I} -\psi_i\psi_{i+g}\,,$$

and applying (30) again we see that this agrees with the formula given in the lemma. The corollary follows immediately by expanding the exponential. ∎

Applying the lemma with $\kappa = t/(1-\beta t^2)$ to formula (29) and using (13), we find

$$c(f_!(\tilde V))_{-2t} \setminus [J_g] = t^g\, F_0(t),$$

and this says precisely that $c_{g+r}(f_!(\tilde V)) \setminus [J_g]$ is a multiple of $\xi_r$, as desired.

## §7. Complements

In this final section we give refinements of two of the calculations of the paper and mention some related problems. The two calculations in question concern the Chern classes of $T_g = T(\mathcal{N}_g)$ and the ring structure of $R$, respectively.

At the end of subsection **A** of §5 it was shown that the $n$th Chern class of $T_g$ vanishes for $n > 2g-2$ (Newstead conjecture) and is a multiple of $\beta^{g-1}$ for $n = 2g-2$ (equation (28)). Continuing the calculations given there to other values $c_{2g-2-i}$ with $i$ small, we find empirically that $c_{2g-3}$ is a linear combination of $\alpha\beta^{g-2}$ and $\beta^{g-3}\gamma$ and more generally that $c_{2g-2-i}(T_g)$ belongs to the ideal $\beta^{g-1-2i}(\beta,\gamma)^i$ of $R$, or equivalently, that it is a linear combination of monomials $\alpha^r\beta^s\gamma^t$ with $r+t \leqslant i$. The following result gives a proof of this and at the same time an explicit way to compute these Chern classes in terms of our canonical basis $\{\xi_{r,s,t}\}_{r+s+t\leqslant g-1}$ of $H^*(\mathcal{N}_g)$, in principle generalizing formula (28).

**Theorem 7.** *Let $\tilde F(x,y,z) = \sum \xi_{r,s,t} x^r y^s z^t$ be the generating function for the basis $\xi_{r,s,t}$ of $R$. Then*

$$\sum_{g=1}^{\infty} E_g\bigl[c(T_g)\,\tilde F(x,y,z)\bigr]\,\bigl(-\frac{T}{4}\bigr)^{g-1} = \frac{1}{\sqrt{\Delta}}\cdot\frac{2}{1+(3x-4y+z)T+\sqrt{\Delta}} \tag{31}$$

*where $\Delta = (1+(x+z)T)^2 - 4z\,(2+x)\,T^2$.*

**Corollary.** *The total Chern class of $T_g$ belongs to the subspace of $H^*(\mathcal{N}_g)$ spanned by the classes $\xi_{r,s,t}$ with $r+s+2t \leqslant g-1$.*

*Proof.* Denote the left-hand side of (31), a power series in $x$, $y$, $z$ and $T$, by $\Phi(x,y,z,T)$. Since $\xi_{r,s,t} = \xi_{r,s}(2\gamma)^t/t!$, we have $\tilde{F}(x,y,z) = e^{2\gamma z} F(x,y)$ where $F(x,y) = \tilde{F}(x,y,0)$ is the generating function of the two-index $\xi$'s. Substituting for the latter from (14), we find that $\Phi(x,y,z,T)$ has the form of the left-hand side of (12) with

$$f(\beta) = \frac{\sinh\theta}{x\sqrt{\beta}} \cdot \frac{1}{1-\beta}\,, \qquad h(\beta) = 1-\beta\,, \qquad u(\beta) = \frac{2}{1-\beta} + x - z\beta\,,$$

$$w(\beta) = \frac{2}{\beta}\Big(\frac{\tanh^{-1}\sqrt{\beta}}{\sqrt{\beta}} - \frac{1}{1-\beta}\Big) + \frac{\theta}{\beta\sqrt{\beta}} - \frac{x}{\sqrt{\beta}} + z\,,$$

$$Q^{-1}(\beta) = T = \frac{\beta}{2+(1-\beta)(x-z\beta)}\,, \qquad Q'(T) = \frac{1}{dT/d\beta} = \frac{\beta^2/T^2}{2+x-z\beta^2}\,,$$

where $\theta = \theta(\beta) = \tanh^{-1}\big(\frac{x\sqrt{\beta}}{1-\beta y}\big)$ as in the proof of Theorem 5. Equation (12) then gives

$$\begin{aligned}\Phi(x,y,z,T) &= \frac{\beta^2/T^2}{2+x-z\beta^2} \cdot \frac{\sinh\theta}{x\,(1-\beta)} \cdot \frac{1}{\sinh(\theta+2\tanh^{-1}\beta)}\bigg|_{\beta=Q^{-1}(T)} \\ &= \frac{\beta^2/T^2}{2+x-z\beta^2}\,\frac{1}{x\,(1+\beta)+2\,(1-\beta y)}\bigg|_{\beta=Q^{-1}(T)}\,.\end{aligned}$$

We get (31) by substituting the solution of the quadratic equation $Q(\beta) = T$. The right-hand side of (31) is a power series in $xT$, $yT$, $zT$ and $zT^2$ and therefore contains only monomials $x^r y^s z^t T^{g-1}$ with $r+s+2t \geqslant g-1$, so the corollary follows from Theorem 5. ∎

Note that the Corollary gives another proof of (21b), since $r+2s+3t \leqslant 2(r+s+2t)$.

From equation (31) and Theorem 5 we can in principle calculate the total Chern class $c(T_g)$ as a combination of classes $\xi_{r,s,t}$ with $r+s+2t \leqslant g-1$, this representation being unique because there are no relations in $H^*(\mathcal{N}_g)$ among the classes $\xi_{r,s,t}$ with $r+s+t \leqslant g-1$. This would give for each $g \geqslant 1$ a specific element $c^{(g)} \in R$ and it would be of interest to compute the generating funcion $\sum_{g=1}^{\infty} c^{(g)}\,(-T/4)^{g-1}$. However, I have not done this computation and have no guess as to the form of the answer.

The second point concerns the ring structure in $R$. Of course, from one point of view $R$ is just the polynomial algebra $\mathbb{Q}[\alpha,\beta,\gamma]$ on three generators of degrees 2, 4 and 6, so its ring structure is known. However, we have now replaced the "obvious" basis of $R$ consisting of monomials $\alpha^r\beta^s\gamma^t$ by the basis consisting of the elements $\xi_{r,s,t}$, independent of $g$, which have a good intersection behavior in the quotient $R_g$ of $R$ for every $g$ (Theorem 5). The question then arises how to compute the structure constants of the multiplication of $R$ with respect to this basis, i.e., the rational numbers $C^{r,s,t}_{r_1,s_1,t_1;r_2,s_2,t_2}$ defined by the formula

$$\xi_{r_1,s_1,t_1}\,\xi_{r_2,s_2,t_2} = \sum_{\substack{r,\,s,\,t\geqslant 0\\ r+2s+3t=r_1+r_2+2s_1+2s_2+3t_1+3t_2}} C^{r,s,t}_{r_1,s_1,t_1;r_2,s_2,t_2}\,\xi_{r,s,t}\,.$$

Because of the usual relation $\xi_{r,s,t} = \xi_{r,s}\,(2\gamma)^t/t!$ we have

$$C^{r,s,t}_{r_1,s_1,t_1;r_2,s_2,t_2} = \frac{t!}{t_1!\,t_2!\,(t-t_1-t_2)!}\,C^{r,s,t-t_1-t_2}_{r_1,s_1,0;r_2,s_2,0}$$

($= 0$ unless $t \geqslant t_1 + t_2$), so we can suppose that $t_1 = t_2 = 0$. If also $s_1 = s_2 = 0$, then the corollary to Theorem 4 gives the answer. A more general result, whose proof (similar to those we have given) we will omit, is that in general

$$\tilde{\xi}_{r_1,s_1}\,\tilde{\xi}_{r_2,s_2} = \sum_{j=0}^{\min(r_1,r_2)} \frac{(-1)^j\,(r_1+r_2-j)!}{j!\,(r_1-j)!\,(r_2-j)!}\,\tilde{\xi}_{r_1+r_2-2j,s_1+s_2+j} + \sum_{\substack{r,s\geqslant 0\\ 1\leqslant t\leqslant s_1+s_2}} (*)\,\xi_{r,s,t}\,, \tag{32}$$

where $\tilde{\xi}_{r,s} := \binom{r+s}{r}^{-1}\xi_{r,s}$. This formula describes the multiplication in the ring $R/\gamma R$ with respect to the basis $\xi_{r,s}$. To describe the multiplication in the full ring $R$, i.e., to determine all the $C$'s, is equivalent to computing the numbers $E_g[\xi_{r_1,s_1}\xi_{r_2,s_2}\xi_{r_3,s_3}]$ for all $r_i$, $s_i$ and $g$ (subject to $3g-3 = \sum(r_i+2s_i)$), because by Theorem 5 we have

$$C^{r,s,t}_{r_1,s_1,0;r_2,s_2,0} = \frac{(-1)^{r+s}\,(r+s+1)\,r!\,s!\,t!}{4^{r+s+t}\,(r+s+t+1)!}\,E_{r+s+t+1}[\xi_{r_1,s_1}\xi_{r_2,s_2}\xi_{s,r}]\,. \tag{33}$$

The new problem has the extra attraction that $E_g[\xi_{r_1,s_1}\xi_{r_2,s_2}\xi_{r_3,s_3}]$ is symmetric in all three pairs of indices $(r_i,s_i)$, whereas the structure constants have only a 2-fold symmetry. As usual, the answer is expressed in terms of generating functions:

**Theorem 8.** *Let $F(x,y)$ be the generating function of the $\xi_{r,s}$ as in §3 and $\mathcal{E}_T : R \to \mathbb{Q}[[T]]$ the intersection number invariant defined in §2. Then*

$$\mathcal{E}_T\big[F(x_1,y_1)F(x_2,y_2)F(x_3,y_3)\big] = \frac{1}{1-AT+BT^2} \tag{34}$$

*with* $A = \sum_{i\neq j} x_iy_j - x_1x_2x_3$, $B = (\sum_i x_i)(\sum_i x_iy_{i+1}y_{i+2})$ *(indices modulo 3).*

*Proof.* Just like that of Theorem 5, except that now we use the trigonometric identity

$$\frac{\sinh(\theta_1+\theta_2+\theta_3)}{\sinh\theta_1\,\sinh\theta_2\,\sinh\theta_3} = 1 + \frac{1}{\tanh\theta_1\,\tanh\theta_2} + \frac{1}{\tanh\theta_1\,\tanh\theta_3} + \frac{1}{\tanh\theta_2\,\tanh\theta_3}$$

instead of the corresponding simpler identity with only two $\theta$'s. ∎

**Corollary.** *The number $E_g[\xi_{r_1,s_1}\xi_{r_2,s_2}\xi_{r_3,s_3}]$ vanishes unless $r_1+r_2+r_3 = g-1+2j$, $s_1+s_2+s_3 = g-1-j$ with $0 \leqslant j \leqslant \min(r_1,r_2,r_3)$.*

*Proof.* Replace each $x_i$ by $ux_i$, each $y_i$ by $uy_i$, and $T$ by $u^{-2}T$ in formula (34). Then the right-hand side has the form $(X + ux_1x_2x_3T)^{-1}$ with $X$ independent of $u$, so its Taylor series contains no negative powers of $u$ and any term divisible by $u^j$ ($j \geqslant 0$ is also divisible by $x_1^jx_2^jx_3^j$. This is easily seen to be equivalent to the statement of the corollary. ∎

From (33), it also follows that the triple intersection number $E_g[\xi_{r_1,s_1}\xi_{r_2,s_2}\xi_{r_3,s_3}]$ is 0 unless $g-1 \geqslant \max(r_1+s_1, r_2+s_2, r_3+s_3)$, the cases with equality being described by equation (32), and one can use Theorem 8 to give explicit formulas in various other limiting cases. It would be nice to find a simple formula for the coefficient of arbitrary monomials $x_1^{r_1}y_1^{s_1}x_2^{r_2}y_2^{s_2}x_3^{r_3}y_3^{s_3}\,T^{g-1}$ in the Taylor series of the rational function in (34), thus determining the structure coefficients of $R$ completely, but I have not been able to do this.

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