A SURGERY FORMULA FOR THE $\bar{\mu}$–INVARIANT

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Abstract. We introduce a relative version, $\bar{\mu}'$, of the $\bar{\mu}$–invariant of Neumann and Siebenmann for graph 3–manifolds, and use it to prove a surgery formula for the $\bar{\mu}$–invariant. We further identify $\bar{\mu}'$ with the linking number of certain Montesinos links, and relate it to the Floer homology in the special case of Seifert manifolds.

The $\bar{\mu}$–invariant is an integral valued invariant defined by W. Neumann [8] and L. Siebenmann [12] in the 1970's for graph homology spheres. Its reduction modulo 2 equals the Rohlin invariant. We introduce in this paper a relative version of the $\bar{\mu}$–invariant, denoted by $\bar{\mu}'$ and defined for graph 3–manifolds having the integral homology of $S^1 \times S^2$.

Theorem 1. *Let $k \subset \Sigma$ be a graph knot and $\Sigma' = \Sigma \pm k$ the graph homology sphere resulting from a (± 1)–surgery of Σ along k. Let $\overline{K}$ be the graph manifold obtained by 0–surgery along k. Then the following surgery formula holds, $\bar{\mu}(\Sigma') = \bar{\mu}(\Sigma) \pm \bar{\mu}'(\overline{K})$. The modulo 2 reduction of $\bar{\mu}'(\overline{K})$ equals the Arf-invariant of the knot $k \subset \Sigma$.*

Another integral lift of the Arf-invariant, λ', has appeared in Casson's work, see e.g. [1], in connection with the surgery formula for his invariant. It equals one half the first derivative of the Alexander polynomial of k evaluated at 1. We should mention that the integers λ' and $\bar{\mu}'$ are in general different though they coincide modulo 2. In some special cases, the $\bar{\mu}'$–invariant has a gauge-theoretical interpretation in terms of the instanton Floer homology of manifolds which are homology $S^1 \times S^2$, see [4] for the definition.

Theorem 2. *Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere, k a singular fiber, and $\overline{K} = \Sigma + 0 \cdot k$ as in Theorem 1. Then $2\,\bar{\mu}'(\overline{K}) = \pm\,(-b_0 - b_1 + b_2 + b_3)$ where b_n is the n-th Betti number of the Floer homology $I_n(\overline{K})$, $n = 0, 1, 2, 3$.*

Note that $2\lambda'(\overline{K}) = \pm\,(b_0 - b_1 + b_2 - b_3)$ depending on the conventions, and that either all even or all odd Betti numbers b_n vanish, see [9]. The results of Section 6 on an interpretation of the $\bar{\mu}'$–invariant as a certain linking number imply the following curious relation.

Corollary 3. *In the setting of Theorem 2, represent $\overline{K}$ as a double branched cover of S^3 with the branch set a Montesinos link $\ell_1 \cup \ell_2$. Then $\mathrm{lk}(\ell_1, \ell_2) = \pm 4\,(-b_0 - b_1 + b_2 + b_3)$.*

1991 *Mathematics Subject Classification.* 57R57, 57M25, 57M12.

Key words and phrases. Floer homology, Seifert manifolds, Montesinos links.

Research was supported in part by NSF Grant DMS 9704204 and MPI Bonn.

This, together with the remarks after Theorems 1 and 2, allows one to express the Floer homology $I_*(\overline{K})$ in terms of such classical link invariants as the Alexander polynomial and the linking number.

The proofs are based on splitting the $\bar{\mu}'$–invariant along the spin-structures on $\overline{K}$, using Turaev's identification of spin structures on a double branched cover with quasiorientations of the branch set, and, in the part related to the Floer homology, on our paper [11]. I am thankful to Boris Apanasov, Leonid Parnovski, and Sergei Matveev for useful discussions.

1. The definition of $\bar{\mu}'$

Let Γ be a plumbing graph, that is a graph with no cycles, each of whose vertices x_i carries an integer weight e_i, $i = 1, \dots, s$. Associated to each vertex x_i is the D^2–bundle $Y(e_i)$ over S^2 with the Euler number e_i. If the vertex x_i has d_i edges connected to it in the graph Γ, we choose d_i disjoint discs in the base of $Y(e_i)$ and call the disc bundle over the j–th disc B_{ij}, so that $B_{ij} = D^2 \times D^2$. When two vertices, x_i and x_k, are connected by an edge, we identify B_{ij} with $B_{k\ell}$ by exchanging the base and fiber coordinates. This pasting operation is called plumbing, and the resulting smooth 4–manifold $P(\Gamma)$ is said to be obtained by plumbing according to Γ. Its boundary $M(\Gamma) = \partial P(\Gamma)$ is usually referred to as a graph manifold.

The manifold $P(\Gamma)$ is simply connected. Its intersection form in the natural basis of $H_2(P(\Gamma))$ represented by the zero-sections of the plumbed bundles is given by the matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j; \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $\overline{K} = \partial P(\overline{\Gamma})$ be a 3-dimensional graph manifold obtained by plumbing according to a graph $\overline{\Gamma}$ such that $H_*(\overline{K}) = H_*(S^1 \times S^2)$. Then the matrix $A(\overline{\Gamma})$ has corank 1, and the equation

$$v \cdot x = x \cdot x \mod 2, \quad \text{for all } x \in H_2(P(\Gamma); \mathbb{Z}), \tag{2}$$

has exactly two solutions in $H_2(P(\overline{\Gamma}); \mathbb{Z}/2)$. They are called **the homology Wu classes**. Each homology Wu class v admits a unique integral lift $w \in H_2(P(\overline{\Gamma}); \mathbb{Z})$ such that the coordinates of w are either 0 or 1 in the natural basis. We call w an **integral homology Wu class**. Thus associated with $\overline{K}$ are two integral homology Wu classes, which we denote by $w_0 = (w_0^1, \dots, w_0^s)$ and $w_1 = (w_1^0, \dots, w_0^s)$. These two classes correspond to two different spin-structures on $\overline{K}$, see Section 2. We define the $\bar{\mu}'$–**invariant** up to a sign by the formula

$$\bar{\mu}'(\overline{K}) = \frac{1}{8}\,(w_0 \cdot w_0 - w_1 \cdot w_1). \tag{3}$$

This is an integer that, up to a sign, only depends on the manifold $\overline{K}$ and not on the particular choice of the plumbing graph $\overline{\Gamma}$, see Section 2. In order to fix the sign, we choose a plumbing graph $\overline{\Gamma}$ and assume that the first $(s-1)$ rows of the $s \times s$–matrix $A(\overline{\Gamma})$ are linearly independent over $\mathbb{Z}/2$. Then $w_0^s + w_1^s = 1 \mod 2$, and we choose w_0 and w_1 so that $w_0^s = 0$ and $w_1^s = 1$. This fixes the sign in the equation (2) but makes the resulting invariant dependent upon the choice of $\overline{\Gamma}$.

2. The $\bar{\mu}$–invariant of a spin-structure

Let Γ be a plumbing graph as in Section 1 but $M(\Gamma)$ is not necessarily a homology $S^1 \times S^2$. The various spin-structures on the spin-manifold $M(\Gamma)$ are classified by $H^1(M(\Gamma); \mathbb{Z}/2)$. To any spin-structure γ on $M(\Gamma)$ one has an obstruction γ in $H^2(P(\Gamma), M(\Gamma); \mathbb{Z}/2)$ to extending it over $P(\Gamma)$. The Poincaré dual of this obstruction, $v_\gamma \in H_2(P(\Gamma); \mathbb{Z}/2)$, is a homology Wu class. It lifts uniquely to an integral homology Wu class $w_\gamma \in H_2(P(\Gamma); \mathbb{Z})$ in the manner described in Section 1. For a given spin-structure γ, one can follow W. Neumann [8] and define the $\bar{\mu}$–invariant by the formula

$$\bar{\mu}(M(\Gamma), \gamma) = \frac{1}{8}\,(\operatorname{sign} P(\Gamma) - w_\gamma \cdot w_\gamma). \tag{4}$$

It is shown in [8], Section 4, that $\bar{\mu}(M(\Gamma), \gamma)$ only depends on the oriented spin diffeomorphism type of $M(\Gamma)$ and not on the choice of Γ.

One can rewrite the above definitions in the matrix language. Let us fix a spin-structure γ on $M(\Gamma)$. The property (2) of the homology Wu class $v_\gamma = (v_1, \dots, v_s)$ translates into the system

$$\sum_{i=1}^{s} a_{ij} v_j = a_{ii} \mod 2, \quad i = 1, \dots, s, \tag{5}$$

or shortly, into the matrix equation $A(\Gamma) v_\gamma = \operatorname{diag} A(\Gamma) \mod 2$ where $\operatorname{diag} A(\Gamma)$ is the column consisting of the diagonal elements of $A(\Gamma)$, i.e. $(\operatorname{diag} A(\Gamma))_i = a_{ii}$. One can show that if any two vertices of the graph Γ are connected by an edge, their corresponding coordinates in v cannot be both equal to 1. Therefore,

$$\bar{\mu}(M(\Gamma), \gamma) = \frac{1}{8}\,(\operatorname{sign} A(\Gamma) - \sum_{v_i = 1} a_{ii}).$$

An integral homology sphere $\Sigma = M(\Gamma)$ admits a unique spin-structure, so the $\bar{\mu}$–invariant is defined without referring to the spin-structures, and is called the **Neumann-Siebenmann invariant** $\bar{\mu}(\Sigma)$ of Σ. Its reduction modulo 2 equals the Rohlin invariant of Σ. In the case of $\overline{K} = M(\overline{\Gamma})$ a homology $S^1 \times S^2$, there exist exactly two different spin-structures, γ_0 and γ_1, and their integral homology Wu classes w_{γ_0} and w_{γ_1} are precisely the classes w_0 and w_1 used in the definition (3) of the $\bar{\mu}'$–invariant. In particular,

$$\bar{\mu}'(\overline{K}) = \bar{\mu}(\overline{K}, \gamma_1) - \bar{\mu}(\overline{K}, \gamma_0), \tag{6}$$

so the $\bar{\mu}'$–invariant is well-defined by the formula (3) up to the choice of spin-structures to call γ_0 and γ_1.

3. The surgery formula

In this section we prove the surgery formula in the form it is stated in Theorem 1. Let $\Sigma = M(\Gamma)$ be an integral homology sphere, and Γ the plumbing graph with s vertices shown in Figure 1. The arrow in the picture indicates a graph knot $k \subset \Sigma$. The reader is referred to [2] for basics on the graph knots.

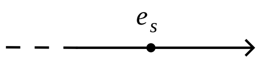

FIGURE 1. The graph Γ

Since Σ is an integral homology sphere, $\det A(\Gamma) = \pm 1$. The graph manifolds $\Sigma' = \Sigma + k$ and $\overline{K} = \Sigma + 0 \cdot k$ obtained from Σ by respectively $(+1)$– and 0–surgery along the knot k correspond to the plumbing graphs Γ' and $\overline{\Gamma}$ shown respectively in Figures 2 and 3.

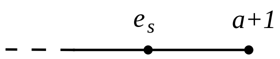

FIGURE 2. The graph Γ'

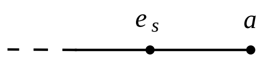

FIGURE 3. The graph $\overline{\Gamma}$

Here, $a = \det A(\Gamma_0)/\det A(\Gamma)$ where Γ_0 is the portion of the graph Γ obtained from it by deleting the vertex weighted by e_s and all the edges incident to it.

We are going to prove that $\bar{\mu}(\Sigma) = \bar{\mu}(\overline{K}, \gamma_0)$ and $\bar{\mu}(\Sigma') = \bar{\mu}(\overline{K}, \gamma_1)$ where γ_0 and γ_1 are the spin-structures on $\overline{K}$ such that, for the integral homology Wu classes $w_0 = w_{\gamma_0}$ and $w_1 = w_{\gamma_1}$, their last coordinates are equal respectively to 0 and 1, i.e. $w_0^{s+1} = 0$ and $w_1^{s+1} = 1$. This together with (6) will produce the surgery formula with a definite sign, $\bar{\mu}(\Sigma') = \bar{\mu}(\Sigma) + \bar{\mu}'(\overline{K})$.

Before we go on to prove the theorem, we prove the following useful technical result.

Lemma 4. *Let $A = (a_{ij})_{i,j=1,\dots,s}$ be a symmetric matrix over $\mathbb{Z}/2$ with det $A = 1$ mod 2, and $A^{-1} = (A_{ij})_{i,j=1,\dots,s}$ be the inverse matrix. If $(v_1, \dots, v_s)$ is a solution of the linear system (5) then $v_i = A_{ii}$ mod 2 for all $i = 1, \dots, s$.*

Proof. We work over $\mathbb{Z}/2$. Let us denote by diag (A) and diag (A^{-1}) the columns consisting of the diagonal elements of the matrices A and A^{-1}, respectively. Since the equation (5) has a unique solution, we only need to prove that

$$\operatorname{diag}(A) = A \operatorname{diag}(A^{-1}) \mod 2. \tag{7}$$

Let us call a matrix P elementary if its all diagonal elements and exactly one non-diagonal element are not zero. It is easy to verify that for any symmetric matrix X the following holds: $\operatorname{diag}(PXP^t) = P \operatorname{diag} X \mod 2$. Applying this identity to both sides of (7) we get $\operatorname{diag}(PAP^t) = P \operatorname{diag}(A) \mod 2$ and $PAP^t \operatorname{diag}((PAP^t)^{-1}) = PA \operatorname{diag}(A^{-1}) \mod 2$. Therefore, (7) is equivalent to

$$\operatorname{diag}(PAP^t) = PAP^t \operatorname{diag}((PAP^t)^{-1}) \mod 2.$$

The conclusion follows now from the fact that any matrix A as in the lemma can be represented in the form CEC^t where E is the identity matrix and C is a product of elementary matrices. $\square$

Let $v = (v_1, \dots, v_s)$ be the homology Wu class of $A(\Gamma)$. Then $v_0 = (v_1, \dots, v_s, 0)$ is a homology Wu class for $A(\overline{\Gamma})$: the first s equations of the system $A(\overline{\Gamma})v_0 = \operatorname{diag} A(\overline{\Gamma}) \mod 2$ are obviously satisfied while the $(s+1)$-st equation is of the form $v_s = a \mod 2$. It holds because both v_s and a are congruent to $\det A(\Gamma_0)/\det A(\Gamma)$ modulo 2, see Lemma 4. Since $\operatorname{sign} A(\Gamma)$ obviously coincides with the signature of the degenerate form $A(\overline{\Gamma})$ we conclude that $\bar\mu(\Sigma) = \bar\mu(\overline{K}, \gamma_0)$.

On the other hand, we have the following formula for $\bar\mu(\Sigma')$,

$$\bar\mu(\Sigma') = \frac{1}{8}\,(\operatorname{sign} A(\Gamma') - \sum_{v'_i=1} a'_{ii}) \tag{8}$$

where a'_{ii} are the diagonal elements of $A(\Gamma')$ and v'_i are to be found from the system of $(s+1)$ equations, $A(\Gamma')v' = \operatorname{diag} A(\Gamma') \mod 2$. If $v' = (v'_1, \dots, v'_{s+1})$ solves this system then it also solves the system $A(\overline{\Gamma})v' = \operatorname{diag} A(\overline{\Gamma}) \mod 2$. This can be seen as follows: again, we only need to check that the last equation, $v'_s + a \cdot v'_{s+1} = a \mod 2$ holds. As we know from (8), $v'_{s+1} + (a+1)\cdot v'_{s+1} = (a+1) \mod 2$. At the same time, Lemma 4 implies that

$$v'_{s+1} = \det A(\Gamma) = 1 \mod 2. \tag{9}$$

Therefore, the equation $v'_s + a \cdot v'_{s+1} = a \mod 2$ is satisfied. Now, $\operatorname{sign} A(\Gamma') = \operatorname{sign} A(\Gamma) + 1 = \operatorname{sign} A(\overline{\Gamma}) + 1$, and thus

$$\begin{aligned}\bar\mu(\Sigma') &= \frac{1}{8}\,(\operatorname{sign} A(\Gamma') - \sum_{v'_i=1} a'_{ii}),\\ &= \frac{1}{8}\,(\operatorname{sign} A(\overline{\Gamma}) + 1 - \sum_{v_i=1} a_{ii} - 1),\\ &= \bar\mu(\overline{K}, \gamma_1).\end{aligned}$$

To complete the proof of Theorem 1 we need to show that $\bar\mu'(\overline{K}) = \operatorname{Arf}(k) \mod 2$. This is seen from the following congruences:

$$\begin{aligned}\bar\mu'(\overline{K}) &= \bar\mu(\Sigma') - \bar\mu(\Sigma) \mod 2,\\ &= \mu(\Sigma') - \mu(\Sigma) \mod 2, \quad \text{where } \mu \text{ is the Rohlin invariant},\\ &= \operatorname{Arf}(k) \mod 2, \qquad \text{by the surgery formula for } \mu.\end{aligned}$$

Remark. The surgery formula will always have the form $\bar\mu(\Sigma') = \bar\mu(\Sigma) + \bar\mu'(\overline{K})$ with the plus sign if, in order to compute $\bar\mu'$, we use the particular plumbing graph $\overline{\Gamma}$ obtained from the graph Γ by adding one vertex as shown in Figure 3. A different choice of $\overline{\Gamma}$ may result in the formula $\bar\mu(\Sigma') = \bar\mu(\Sigma) - \bar\mu'(\overline{K})$. By keeping track of the signs, one can prove a more general surgery formula. Namely, if $\Sigma_n = \Sigma + 1/n \cdot k$ is the graph homology sphere obtained by $1/n$–surgery on a graph knot k, $n = 1, 2, \dots$, then $\bar\mu(\Sigma_n) = \bar\mu(\Sigma)$ if n is even, and $\bar\mu(\Sigma_n) = \bar\mu(\Sigma) + \bar\mu'(\overline{K})$ if n is odd.

4. Preferred involutions on graph manifolds

Following J. Montesinos [7], we construct in this section orientation preserving involutions on graph manifolds with the orbit space S^3 and the fixed point set a link in S^3, which will be called a link of plumbing type or (generalized) Montesinos link. The construction is similar to the one for graph homology spheres, see e.g. [11], Section 7.2.

Let D^2 be the unit disc in $\mathbb{C}$, $D^2 = \{ z \in \mathbb{C} \,|\, |z| \le 1 \}$, and $\sigma : D^2 \times D^2 \to D^2 \times D^2$ the complex conjugation involution, $(z, w) \mapsto (\bar{z}, \bar{w})$. The boundary S^3 of $D^2 \times D^2$ can be represented by stereographic projection onto $\mathbb{R}^3 \cup \infty$ in such a way that σ induces on it the $180°$-rotation about the y–axis. In this representation $\partial D^2 \times D^2$ is a regular neighborhood X of the unit circle in the xy–plane, the belt–sphere is the z–axis, and the belt–tube is $Y = S^3 \setminus \operatorname{int} X$.

The fixed point set of the involution σ on $D^2 \times D^2$ is the unit disc $B = D^1 \times D^1 \subset D^2 \times D^2$. The orbit space of σ is a 4–dimensional ball D^4, so that the projection $p : D^2 \times D^2 \to D^4$ is a 2–fold branched cover of D^4 with the branch set the 2–disc $B' = p(B)$. The restriction

$$p \,\big|_{\partial(D^2 \times D^2)} \colon\ \partial(D^2 \times D^2) \to S^3$$

is the standard covering over the trivial knot $\partial B' \subset S^3$.

The branch set $B' \subset D^4$ can be described as follows. Let $S^3 = X \cup Y$ as above. The restriction of p onto X turns X into a double branched cover of the 3–ball $X' = X/\sigma$ with the branch set consisting of the two arcs shown on the left in Figure 4. These two arcs represent the intersection $B' \cap X' \subset S^3$.

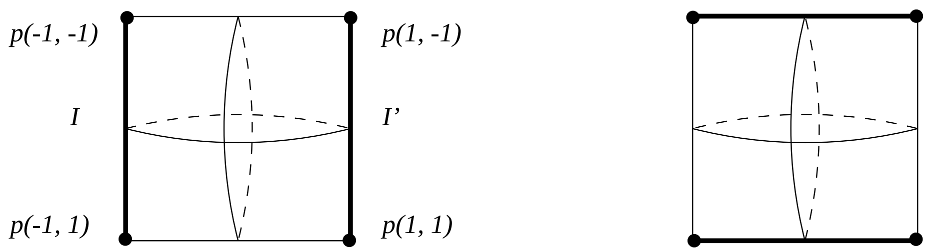

Figure 4. The branch sets

Similarly, $Y' = Y/\sigma$ is a 3–ball with the branch set the arcs shown on the right in Figure 4. The 3–balls X' and Y' are glued together to form S^3, the arcs in X' and Y' being glued along their endpoints into the trivial knot $\partial B' \subset S^3$. Now, let B'' be a 2–disc in S^3 with $\partial B'' = \partial B'$. Then the branch set $B' \subset D^4$ can be viewed as a 2–disc obtained by pushing the interior of B'' radially into D^4 so that $B' \cap S^3 = \partial B' = \partial B''$.

The complex conjugation involution on $D^2 \times D^2$ can be extended over the 2–disc bundles $Y(e)$ over S^2, which are the building blocks for the plumbed manifolds. The 4–manifold $Y(e)$ can be obtained by pasting two copies of $D^2 \times D^2$,

$$Y(e) = D^2 \times D^2 \cup_{h_e} D^2 \times D^2,$$

along $\partial D^2 \times D^2$ where $h_e : \partial D^2 \times D^2 \to \partial D^2 \times D^2$ is given by the formula

$$(e^{i\phi}, \lambda e^{i\psi}) \mapsto (e^{-i\phi}, \lambda e^{i(\psi - e\phi)}), \quad 0 \le \phi, \psi < 2\pi, \ 0 \le \lambda \le 1.$$

The complex conjugation involution σ on $D^2 \times D^2$ obviously commutes with the map h_e, so σ can be extended to an involution (also denoted by σ) on $Y(e)$. The quotient $Y(e)/\sigma$ is glued from two 4–balls along a 3–ball, so that $Y(e)/\sigma = D^4$. The branch set in D^4 can be described as follows.

Let us represent $Y(e)/\sigma = D^4$ as

$$D^4 = D^4 \cup_{D^3} \mathrm{Cyl}\,(\bar{h}_e) \cup_{D^3} D^4 \tag{10}$$

where $\mathrm{Cyl}\,(\bar{h}_e)$ is the mapping cylinder of the map $\bar{h}_e : D^3 \to D^3$ induced by h_e on the quotient of $\partial D^2 \times D^2$. The branch set in $Y(e)/\sigma$ then consists of four bands glued together, see Figure 5, two untwisted bands in the two copies of D^4, and two more bands in $\mathrm{Cyl}\,(\bar{h}_e)$ generated by the arcs I' and I, see Figure 4. One can easily see that the former is untwisted while the latter one is twisted e half–twists in the right–hand direction (in case e is positive).

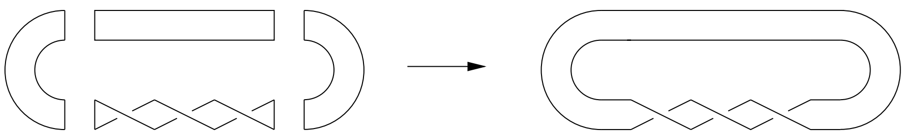

FIGURE 5. A twisted band with $e = 3$

Let $P(\Gamma)$ be a plumbed manifold obtained by plumbing D^2–bundles $Y(e_i)$ according to a connected plumbing tree Γ. The involutions σ on the manifolds $Y(e_i)$ fit together to form an involution on $P(\Gamma)$ whose orbit space is D^4, and the branch set $B(\Gamma)$ is obtained by plumbing twisted bands (like one in Figure 5) according to the tree Γ, see Figure 6 and [12].

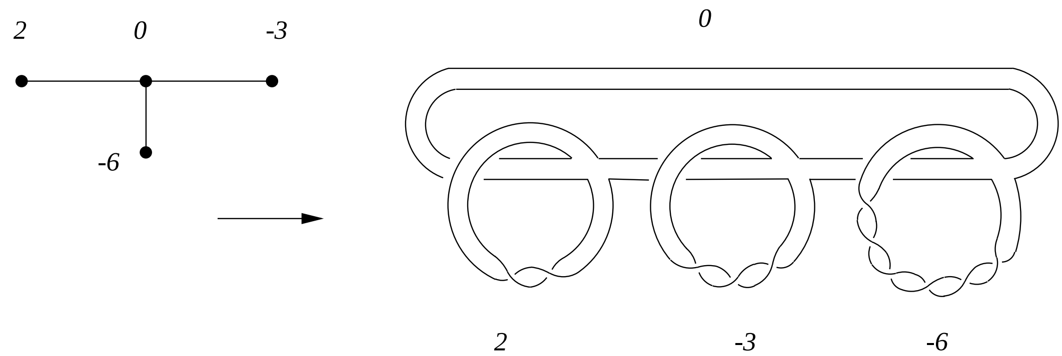

FIGURE 6. A Montesinos link

In particular, this construction represents the boundary $M(\Gamma)$ of a plumbed manifold $P(\Gamma)$ as a double branched cover of S^3 branched over the boundary of the surface $B(\Gamma)$.

5. Spin structures and quasiorientations

Let $L \subset S^3$ be a link. By orientation of L we mean an orientation of the manifold L. Following [13], we call every unordered pair { an orientation of L, the opposite orientation of L } a **quasiorientation** of L. Each knot has a unique quasiorientation; a link of n components has 2^{n-1} quasiorientations.

Let M be a double branched cover of S^3 branched along L. In [13], V. Turaev established a one-to-one correspondence between quasiorientations of L and spin-structures on M. In their turn, the spin structures on $M(\Gamma)$ are classified by the group $H^1(M(\Gamma), \mathbb{Z}/2)$. In particular, if $M(\Gamma)$ is an integral homology sphere (or more generally a $\mathbb{Z}/2$–homology sphere), it admits a unique spin structure, so the branch set is a knot. If $M(\Gamma)$ is a homology $S^1 \times S^2$ then the branch set is a link of two components.

Theorem 5. *Let $M(\Gamma)$ be a graph manifold with a fixed spin–structure γ, and suppose that $M(\Gamma)$ is represented, with respect to an involution σ described in Section 4, as a double branched cover of S^3 with the branch set a link L_σ of plumbing type with the quasiorientation determined by γ. Then*

$$\bar\mu(M(\Gamma), \gamma) = \frac{1}{8}\, \mathrm{sign}(L_\sigma).$$

By $\mathrm{sign}\, L_\sigma$ we mean the classical link signature of the quasioriented link L_σ. More precisely, we define $\mathrm{sign}\, L_\sigma$ to be the signature of $S_F + S_F^t$ where S_F is the Seifert matrix for any orientable spanning surface F of L_σ which has a quasiorientation inducing the given quasiorientation of L_σ.

Proof. Let $M(\Gamma) = \partial P(\Gamma)$. The 4–manifold $P(\Gamma)$ is a double branched cover of D^4 whose branched set is the surface $F = B(\Gamma) \subset D^4$ such that $\partial F \cap S^3 = L_\sigma$. The surface F is orientable if and only if all the weights in the plumbing graph Γ are even.

Suppose that Γ has only even weights. Then, since the intersection form of $P(\Gamma)$ is even, the integral homology Wu class w_γ vanishes, and

$$\begin{aligned} \mathrm{sign}\, L_\sigma &= \mathrm{sign}\, P(\Gamma), \qquad \text{see e.g. [5], Corollary 5'}, \\ &= 8\, \bar\mu(M(\Gamma), \gamma). \end{aligned}$$

Now suppose that Γ has at least one odd weight. According to C. Gordon and R. Litherland [5], Theorem 3 and Corollary 5",

$$\mathrm{sign}\, L_\sigma = \mathrm{sign}\, P(\Gamma) + \frac{1}{2}\, \bar e(F). \tag{11}$$

The number $\bar e(F)$ is defined as follows. Let L_σ^F be a parallel copy of the link L_σ missing the surface F, and let $L_\sigma = \ell_1 \cup \ldots \cup \ell_m$ and $L_\sigma^F = \ell_1^F \cup \ldots \cup \ell_m^F$. Then

$$\bar e(F) = -\sum_{i,j=1}^{m} \mathrm{lk}(\ell_i, \ell_j^F) = -\sum_i \mathrm{lk}(\ell_i, \ell_i^F) - 2 \sum_{i<j} \mathrm{lk}(\ell_i, \ell_j). \tag{12}$$

Note that $\bar{e}(F) = 0$ is F is orientable. To complete the proof we only need to show that

$$-\frac{1}{2}\,\bar{e}(F) = w_\gamma \cdot w_\gamma$$

where w_γ is the Wu class of the manifold $P(\Gamma)$.

Let us first color in black the surface F, and orient its boundary $\partial F = L_\sigma$. In a knot projection there are two types of double points, see Figure 7.

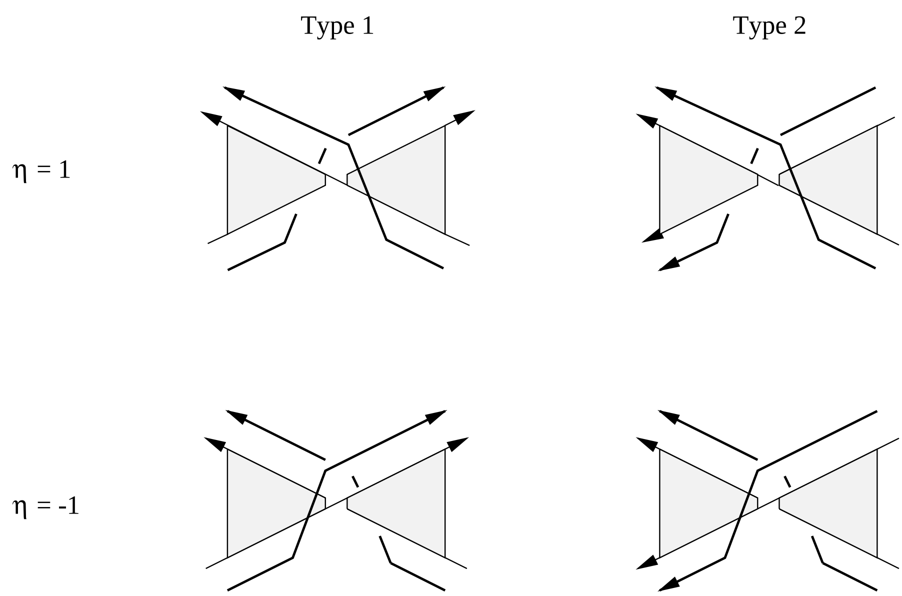

FIGURE 7. Types of double points

One can see that $-1/2 \cdot \bar{e}(F)$ is the sum of the incidence numbers η of double points of type 2, see [5], Lemma 7". In a regular projection of L_σ, like one seen in Figure 6, the double points only occur because of twisting the bands, and all the double points on the same band have the same type and the same incidence number.

Let x_i be a vertex in Γ weighted by a_{ii} (remember that the a_{ij}'s are the entries of the matrix $A(\Gamma)$). Let $w_i = 1$ if the double points on the band corresponding to the vertex x_i are of type 2, and $w_i = 0$ otherwise. Then

$$-\frac{1}{2}\,\bar{e}(F) = \sum_{i=1}^{s} a_{ii} w_i.$$

On the other hand, by comparing the orientations on the boundary of the i-th band with the orientations on the boundaries of bands adjacent to it one can easily check that

$$\sum_{j=1}^{s} a_{ij} w_j = a_{ii} \mod 2, \quad i = 1, \dots, s.$$

Therefore, the vector $w = (w_1, \dots, w_s)$ is the Wu vector w_γ of the matrix $A(\Gamma)$, and $w.w = \sum w_i a_{ii}$. $\square$

Remark. In our proof of the surgery formula we have shown that, for a graph homology sphere Σ and the manifold $\overline{K} = \Sigma + 0 \cdot k$ obtained by 0–surgery along a graph knot $k \subset \Sigma$, we have that $\bar\mu(\Sigma) = \bar\mu(\overline{K}, \gamma)$ for a proper spin–structure γ on $\overline{K}$. Theorem 5 allows one to rewrite this identity in terms of the link signatures, $\operatorname{sign}(k_\sigma) = \operatorname{sign}(L_\sigma)$, where k_σ and L_σ are the fixed point sets of the involutions on Σ and $\overline{K}$. The equality of these signatures can be proved more directly by using a link cobordism argument: the manifolds Σ and $\overline{K}$ are connected by a 4-dimensional cobordism X, the trace of the 0–surgery along k. The involutions on the two boundary components of X extend to an involution $\sigma : X \to X$ so that $X/\sigma = S^3 \times [0,1]$. The fixed point set of this involution is a surface in $S^3 \times [0,1]$ which provides a cobordism between k_σ and L_σ. Since the signature is a cobordism invariant, we are done.

6. The $\bar\mu'$–invariant as a linking number

In the special case of $M(\Gamma)$ a homology $S^1 \times S^2$, Theorem 5 together with an argument from [5] implies the following result.

Corollary 6. *Let $\overline{K}$ be a manifold obtained by 0–surgery along a graph knot $k \subset \Sigma$, and suppose that $\overline{K}$ is represented as a double branched cover over S^3 with the branch set a link $\ell_1 \cup \ell_2$ of plumbing type. Then*

$$\bar\mu'(\overline{K}) = \pm\frac{1}{4}\,\operatorname{lk}(\ell_1, \ell_2).$$

Proof. Let us fix a quasiorientation on $L_\sigma = \ell_1 \cup \ell_2$ then $8\,\bar\mu'(\overline{K}) = \pm(\operatorname{sign}(\ell_1 \cup \ell_2) - \operatorname{sign}(\ell_1 \cup -\ell_2)$ where $-\ell_2$ stands for ℓ_2 with reversed orientation. According to (11) and (12), the right hand side equals $\pm(\operatorname{lk}(\ell_1, \ell_2) - \operatorname{lk}(\ell_1, -\ell_2)) = \pm 2\ \operatorname{lk}(\ell_1, \ell_2)$, which completes the proof. $\square$

7. Relation to Floer homology

This section is devoted to a proof of Theorem 2 of the introduction relating the $\bar\mu'$–invariant to Floer homology.

Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere with Seifert invariants $a_1, \dots, a_n$, and $a = a_1 \cdots a_{n-1}$ the product of the first $n-1$ Seifert invariants. Let $k \subset \Sigma(a_1, \dots, a_n)$ be the n-th singular fiber in Σ. Then the result of (-1)–surgery of Σ along k is the Seifert fibered homology sphere $\Sigma' = \Sigma(a_1, \dots, a_{n-1}, a + a_n)$. The Floer homology groups of the two homology spheres are related by the Floer exact triangle, see [4],

$$\begin{tikzcd}
& I_*(\overline{K}) \arrow[dl] & \\
I_*(\Sigma') \arrow[rr] & & I_*(\Sigma) \arrow[ul]
\end{tikzcd}$$

Here $\overline{K}$ is as usual the manifold obtained by 0–surgery of Σ along k. The manifold $\overline{K}$ is Seifert fibered. The groups $I_*(\Sigma)$ are $\mathbb{Z}/8$–graded while there are only four groups $I_n(\overline{K})$, $n = 0, 1, 2, 3$, so $I_*(\overline{K})$ in the exact triangle stands for $I_*(\overline{K}) \oplus I_{*+4}(\overline{K})$.

The Floer homology groups of Seifert fibered homology spheres are very specific in that they vanish in odd dimensions, and are free abelian in even dimensions, see e.g. [3] and [6]. The same is true for the Floer homology of the manifold $\overline{K}$, see [9], with the only difference that the only grading intrinsically defined on $I_*(\overline{K})$ is a relative one. It can be made into an absolute grading by requiring that the right arrow in the Floer exact triangle above be of degree 0; then the arrow on the left will have degree 1. In any case, the Floer exact triangle splits into four short exact sequences, and in particular, we get the following identity

$$\nu(\Sigma') = \nu(\Sigma) + (-b_0 - b_1 + b_2 + b_3) \tag{13}$$

where b_n are the Betti numbers of $I_*(\overline{K})$ (such that in fact $b_0 = b_2 = 0$), and the ν–invariant is defined as in [11] by the formula

$$\nu(\Sigma) = \frac{1}{2} \sum_{j=0}^{3} (-1)^{j+1} \operatorname{rk} I_{2j}(\Sigma),$$

and similarly for Σ'. Now we are in position to prove Theorem 2 and Corollary 3.

Proof of Theorem 2. Suppose that $\overline{K}$ is represented as a double branched cover over S^3 with the branch set a Montesinos link $L = \ell_1 \cup \ell_2$. According to the main theorem of [11] we have $\nu(\Sigma) = \bar{\mu}(\Sigma)$ and $\nu(\Sigma') = \bar{\mu}(\Sigma')$, therefore,

$$\begin{aligned}
-b_0 - b_1 + b_2 + b_3 &= \nu(\Sigma') - \nu(\Sigma), && \text{according to (13)}, \\
&= \bar{\mu}(\Sigma') - \bar{\mu}(\Sigma), && \\
&= \pm\, \bar{\mu}'(\overline{K}), && \text{by the surgery formula}, \\
&= \pm \frac{1}{4}\ \operatorname{lk}(\ell_1, \ell_2), && \text{by Corollary 6.}
\end{aligned} \tag{14}$$

This completes the proof. □

Remark. Note that the sign ambiguity in the definition of the $\bar{\mu}'$–invariant reflects the ambiguity in assigning an absolute grading to the groups $I_*(\overline{K})$.

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