On Hodge Structures and Non-Representability of Chow Groups

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0. Introduction. Let X be a smooth, projective variety of dimension n over an algebraically closed field. The Chow group, $CH_m(X)_{alg}$, constructed from m-dimensional cycles which are algebraically equivalent to zero by modding out by rational equivalence, is an important and tractible invariant when $m = n - 1$. In this case $CH_m(X)_{alg}$ is isomorphic in a natural way to the points of an Abelian variety. When $m < n-1$ there may or may not exist such an isomorphism. In the latter case we say that $CH_m(X)_{alg}$ is not weakly representable (see (1.6) for the precise definition). In this paper we take the complex numbers as the base field and ask

Question 0.1. *To what extent does the* $\mathbb{Q}$*-Hodge structure,* $H.(X_{\mathbb{C}})$*, determine whether or not* $CH_m(X_{\mathbb{C}})_{alg}$ *is weakly representable?*

The first result in this direction is Roitman's extension of Mumford's non-representablity theorem for surfaces with $p_g > 0$. To state this result, recall that the width, w, of a weight j Hodge structure, H, with $H_{\mathbb{C}} \simeq \oplus_{p+q=j} H^{p,q}$, is $max_{H^{p,q} \neq 0}\{|p-q|\}$. Note that $m = (|j| - w)/2$ is always an integer. With this terminology a version of Roitman's Theorem is

Theorem 0.2. *If* $H_j(X_{\mathbb{C}})$ *has width* j *for some* $j \geq 2$*, then* $CH_0(X_{\mathbb{C}})_{alg}$ *is not weakly representable.*

The first theorem we shall prove is the following conditional extension of (0.2) to higher dimensional algebraic cycles.

Theorem 0.3. *Assume that Grothendieck's generalized Hodge conjecture* [Gro1] *is true. Suppose that* $H_h(X_{\mathbb{C}})$ *has a* $\mathbb{Q}$*-Hodge substructure* V *of width* $w \geq 2$*. Set* $m = (h - w)/2$*. Then* $CH_m(X_{\mathbb{C}})_{alg}$ *is not weakly representable.*

This result was first obtained by James Lewis [Le2] using different methods.

The generalized Hodge conjecture is needed only to supply a smooth projective variety S of dimension w together with a family of m-cycles parametrized by S, $\Gamma \in Z_{w+m}(S \times X)$, such that the image of

$$\Gamma_* : H_w(S_{\mathbb{C}})(m) \to H_{w+2m}(X_{\mathbb{C}})$$

contains V.

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Theorem (0.3) can be applied to certain hypersurfaces in projective space. Let $X_{\mathbf{C}} \subset \mathbb{P}^{n+1}_{\mathbf{C}}$ be a non-singular hypersurface of degree d. Write w for the Hodge width of $H_n(X_{\mathbf{C}})$. We have

$$w = n - 2m > 0 \qquad \Leftrightarrow \qquad \frac{n}{2} > m \qquad \text{and} \qquad \frac{n+2}{m+1} \leq d < \frac{n+2}{m}.$$

The following result, first mentioned in [Cl1] and treated thoroughly in [Le3], is a special case of Grothendieck's generalized Hodge conjecture:

Theorem 0.4. *Suppose that $w = n - 2$ or equivalently $\frac{n}{2} > 1$ and $\frac{n}{2} + 1 \leq d < n + 2$. For any smooth degree d hypersurface, X, outside a proper, closed subset of moduli, the Hilbert scheme of lines on X, S, is smooth of dimension $2n - d - 1$. In this case the universal family of lines, $\mathcal{L} \subset S \times X$, induces a surjection, $H_{n-2}(S_{\mathbf{C}})(1) \to H_n(X_{\mathbf{C}})$.*

This is an important step in proving (compare [Le2,§3, Ex.1])

Theorem 0.5. *Let $X_{\mathbf{C}} \subset \mathbb{P}^{n+1}_{\mathbf{C}}$ $(n/2 > 1)$ be a smooth hypersurface of degree d, $\frac{n}{2} + 1 \leq d < n + 2$. Then $CH_1(X_{\mathbf{C}})_{alg}$ is not zero for $n = 3$ and is not weakly representable for $n \geq 4$.*

When $n = 3$ one is dealing with 1-cycles on the cubic and quartic threefolds. In this case the intermediate Jacobian has played a signifigant role in the study of $CH_1(X_{\mathbf{C}})_{alg}$ and much more is known. When $n \geq 4$ the intermediate Jacobian for one cycles is zero.

In order to more easily visualize the Hodge substructures which play a role in (0.3) we introduce the notion of an m-spanning Hodge substructure, $V \subset H_{2m+j}(X_{\mathbf{C}})$. V will be called m-spanning if $j \geq 0$ and

$$V_{\mathbf{C}} \simeq V^{-m-j,-m} \oplus \ldots \oplus V^{-m,-m-j}, \qquad V^{-m-j,-m} \neq 0.$$

In the Hodge diamond, V spans the cone with vertex $H^{-m,-m}_{2m}$ and sides extending out to $H^{-n,-m}_{n+m}$ and $H^{-m,-n}_{n+m}$.

$$\begin{array}{ccccccc}
 & & & H^{-m,-m}_{2m} & & & \\
 & & & / \quad \backslash & & & \\
 & & V^{-m-j,-m} \oplus & \ldots & \oplus V^{-m,-m-j} & & \\
 \ddots & & / & & \backslash & & \\
 H^{-n,-m}_{n+m} \oplus & & & \ldots & & & \oplus H^{-m,-n}_{n+m} \\
 & & \ddots & & & & \\
 & & & H^{-n,-n}_{2n} & & &
\end{array}$$

Now Theorem (0.3) says that if for some $j \geq 2$, $H_{2m+j}(X_{\mathbf{C}})$ has an m-spanning Hodge substructure, then the generalized Hodge conjecture implies that $CH_m(X_{\mathbf{C}})_{alg}$ is not weakly representable.

It is interesting to try to use this cone to further illuminate possible relationships between Hodge substructures and $CH_m(X_{\mathbf{C}})_{alg}$. We call a Hodge substructure, $V \subset H_{2m+j}(X_{\mathbf{C}})$ with $j > 0$, m-excessive if it extends beyond the boundaries of the cone (ie.

has $V^{p,q} \neq 0$ for some $p > -m$). A Hodge substructure $V \subset H_{2m+j}(X_{\mathbf{C}}), j > 0$ will be called m-deficient if $V^{p,q} = 0$ for all $p \geq -m$. In other words, an m-deficient Hodge substructure lies in the interior of the cone.

Suppose that $X_{\mathbf{C}}$ is a smooth projective variety for which no $H_j(X_{\mathbf{C}})$ with $j \geq 2$ has the maximal width, j. When $X_{\mathbf{C}}$ is a surface it has been conjectured that $CH_0(X_{\mathbf{C}})_{alg}$ is weakly representable [Bl2]. This has been verified in a substantial number of particular cases [BLK], [Bl1,Ex.1.5], [V]. The same conclusion holds for non-singular complete intersections of any dimension in projective space [R3] and in fact quite generally for non-singular Fano varieties [Ca]. One would like to know if these observations are specific examples of a general principle which pertains to higher dimensional cycles as well. We formulate a candidate for such a principle in the

Naive Question 0.6. *Let $X_{\mathbf{C}}$ be a smooth, projective variety. Suppose that there is a non-negative integer, m, with the property that for each $j \geq 2$ $H_{2m+j}(X_{\mathbf{C}})$ is m-deficient. Is $CH_m(X_{\mathbf{C}})_{alg}$ weakly representable?*

In §5 we show that the answer is yes for 1-cycles on smooth cubic hypersurfaces of dimension at least 6.

So far the discussion of $CH_m(X_{\mathbf{C}})_{alg}$ has ignored the case of a smooth, projective variety, $X_{\mathbf{C}}$, for which $H_{2m+j}(X_{\mathbf{C}})$ is m-excessive for some $j \geq 2$, but $H_{2m+j}(X_{\mathbf{C}})$ has no m-spanning Hodge substructure for any $j \geq 2$. Indeed in this case $CH_m(X_{\mathbf{C}})_{alg}$ seems even more mysterious, than in the cases discussed in (0.3) and (0.6). The following result will shed a small amount of light on an interesting example:

Theorem 0.7. *Let $X_{\mathbf{C}} \subset \mathbb{P}^{n+1}_{\mathbf{C}}$ $(n/2 > 1)$ be a geometric, generic hypersurface of degree $d, n+2 \leq d \leq 2n-1$. There exist two lines on $X_{\mathbf{C}}$ such that no positive multiple of their difference is rationally equivalent to zero.*

When $n = 3$, (0.7) together with [H] allows one to recover the fact that no positive multiple of the difference of two lines on a geometric generic, quintic threefold is rationally equivalent to zero. Of course, the theory of the intermediate Jacobian has been used to show the stronger result that no positive multiple is algebraicly equivalent to zero [Gri, 14.2]. The advantage of (0.7) is that it continues to give information when $n \geq 4$ in which case the intermediate Jacobian for one cycles is zero.

When $d \leq 2n-3$ all lines are known to be algebraicly equivalent [B-V], so we find $rk(CH_1(X_{\mathbf{C}})_{alg}) > 0$. Thus (0.7) gives an example of a smooth projective variety with $CH_m(X_{\mathbf{C}})_{alg} \neq 0$ and no m-spanning Hodge substructure of positive width. This statement can be amplified by the following general result

Theorem 0.8. *For a quasi-projective variety, $X'_{\mathbf{C}}$, $CH_m(X'_{\mathbf{C}})_{alg}$ is zero or has uncountable rank.*

We deduce (0.7) from (0.5) by a degeneration argument. Observe that (0.5) deals with Fano varieties while (0.7) deals with varieties of general type or having trivial canonical bundle.

We have chosen to take the complex numbers as the base field in order to formulate the results in the familiar language of Hodge structures. This choice is primarily a matter of convenience. Many arguments may be carried through with little change if $\mathbf{C}$ is replaced by an arbitrary algebraically closed field of infinite transendence degree over the prime field. We have included two remarks, (1.13) and (2.13), for those who find the category of varieties defined over $\mathbf{C}$ too restrictive. It is important to note that it would not be possible to extend our arguments to varieties defined over the algebraic closure of the prime field. This is in the spirit of the following conjecture (of which we state only a very special case).

Conjecture 0.9. *(Beilinson and Bloch,* [Be,5.6]*) Suppose $X_{\bar{\mathbf{Q}}} \subset \mathbf{P}^{n+1}_{\bar{\mathbf{Q}}}$ is a smooth hypersurface defined over $\bar{\mathbf{Q}}$. If $n > 3$, then $CH_1(X_{\bar{\mathbf{Q}}})_{hom} \otimes \mathbf{Q} \simeq 0$.*

Note the striking contrast between (0.9) and (0.5) or (0.7).

Only after writing most of this paper did we become aware of work of James Lewis [Le2] and [Le3]. There is considerable overlap between his results and the first two sections of this paper. In particular Lewis proved in [Le2] a version of (0.3) which shows that $CH_m(X_{\mathbf{C}})_{alg}$ is "infinite dimensional" in the sense of Mumford and Roitman. He also treats (0.5) (first in a special case [Le1,§9] and then more generally [Le2,§3, Ex.1] and [Le3,15.44]) and (1.16) (cf. [Le2,§3, Ex. 4]). In spite of this overlap, we have not significantly changed the presentation in the first two sections. The viewpoint and techniques adopted here are to a large extent complementary to those of Lewis, and are of independent interest. Also these same techniques play a role in the proof of (0.7).

In a few words the two proofs of (0.3) may be compared as follows: Lewis uses the cycle, $\Gamma \in Z_{w+m}(S \times X)$, supplied by the generalized Hodge conjecture, to construct a mapping $CH_0(S_{\mathbf{C}})_{alg} \to CH_0(S_{\mathbf{C}})_{alg}$ which factors through $CH_m(X_{\mathbf{C}})_{alg}$. The Mumford-Roitman theory is used to show that the image of this map is infinite dimensional. The proof of (0.3) in this paper is based on Bloch's proof of non-representability for $CH_0(X_{\mathbf{C}})_{alg}$ when $X_{\mathbf{C}}$ is a surface with $h^{2,0} \neq 0$ [Bl,§1 Appendix]. When the cycle supplied by the generalized Hodge conjecture is substituted for the diagonal cycle in Bloch's argument, (0.3) falls out after a few modifications. Uwe Jannsen [Ja], working completely indepedently, has used a similar argument to establish

Theorem. *Let $X_{\mathbf{C}}$ be a smooth projective variety. If $CH_m(X_{\mathbf{C}})_{hom} \otimes \mathbf{Q} \simeq 0$ for all m, then the $\mathbf{Q}$-Hodge structure $\oplus_{j \geq 0} H_j(X_{\mathbf{C}})([-j/2])$ has pure type $(0,0)$ and is generated by the fundamental classes of algebraic cycles.*

Recently Madhav Nori has constructed smooth projective varieties $X_{\mathbf{C}}$ for which the Abel-Jacobi map on $CH_m(X_{\mathbf{C}})_{hom}/CH_m(X_{\mathbf{C}})_{alg}$ is not injective [No]. The search for complementary results provided the stimulus for the present work. It turns out that the techniques of [No] can be used to create further examples of smooth projective varieties with $CH_m(X_{\mathbf{C}})_{alg} \neq 0$ and no m-spanning Hodge substructures of positive width. These examples are quite different than those which arise from (0.7).

I wish to thank Madhav Nori for his inspiration, Bert van Geemen for suggesting the picture of a cone in the Hodge diamond as an efficient means of formulating the results, James Lewis for communicating insights arrising from his somewhat different viewpoint on many of the topics discussed here, and Sheldon Katz for a number of helpful discussions.

Notations.

$H_{\cdot}(X_{\mathbf{C}})$ = the singular homology with coefficients in $\mathbf{Q}$ of the analytic space associated to the complex variety $X_{\mathbf{C}}$.

$|X|_m = |X|^{dim(X)-m}$ the set m-dimensional points of a pure dimensional scheme X.

$Z_m(X)$ = the free abelian group on points of dimension m on a scheme X which is of finite type and separated over a field.

$Z_m(X)_{rat} \subset Z_m(X)_{alg} \subset Z_m(X)$ denote the subgroups of cycles rationally (repsectively algebraicly) equivalent to zero [F].

$CH_m(X) = Z_m(X)/Z_m(X)_{rat}$.
$CH_m(X)_{alg} = Z_m(X)_{alg}/Z_m(X)_{rat}$.
$cl(\Gamma)$ = the singular cohomology class of a cycle Γ.
$CH_m(X_{\mathbf{C}})_{hom} = \mathrm{Ker} : CH_m(X_{\mathbf{C}}) \to H_{2m}(X_{\mathbf{C}}, \mathbf{Z})$, where $X_{\mathbf{C}}$ is a projective variety.

Contents.

1. A conditional Mumford-Roitman Theorem for higher dimensional cycles.

Let X be a smooth, projective variety of dimension n over an algebraicly closed field $\bar{k}$.

Definition 1.1. Let m and r be non-negative integers. We say that a subgroup $M \subset CH_m(X)_{alg}$ is supported in dimension $m + r$ if there is a closed subscheme, $Z \subset X$ of dimension $m + r$ such that the image of M under the restriction map,

$$j_Z^* : CH_m(X)_{alg} \to CH_m((X - Z)), \tag{1.2}$$

is zero.

Remark 1.3. Suppose that $\bar{k} = \mathbf{C}$. In this case, if the image of (1.2) is torsion, then it is in fact zero. This follows from (0.8) and (4.3) below. We will not use this fact.

Remark 1.4. $CH_m(X)_{alg}$ is always a divisible group, since it is generated by the $\bar{k}$-points of Jacobians. Thus if $CH_m(X)_{alg}$ is supported in dimension m, it is finitely generated and hence 0. If $CH_m(X)_{alg}$ is supported in dimension $m + 1$ and if we assume resolution of singularities, then $CH_m(X)_{alg}$ may be identified with the quotient of the group of $\bar{k}$-points of an Abelian variety. To see this notice that $CH_m(X)_{alg}$ is contained in $i_*(CH^1(Z))$ where i is the inclusion of a closed subscheme $i : Z \to X$ of dimension $m + 1$. Let $\sigma : \tilde{Z} \to Z$ be a desingularization. Since the Neron-Severi group is finitely generated, the maximal divisible subgroup of $i_*\sigma_*(CH^1(\tilde{Z}))$ coincides with $i_*\sigma_*(CH^1(\tilde{Z})_{alg})$. This is contained in $CH_m(X)_{alg}$ and, since σ_* is surjective, it also contains $CH_m(X)_{alg}$. But $CH^1(\tilde{Z})_{alg}$ is well known to be the $\bar{k}$-points of the Abelian variety $Pic^0(\tilde{Z})$.

We now recall the concept of weak representability, whose precise formulation is based on the notion of a regular map to an Abelian variety. If S is a smooth variety of dimension p and $\Gamma \in Z_{p+m}(S \times X)$, then the moving lemma [Rob] or the Fulton-MacPherson intersection theory [F] gives a well defined map $\Gamma_* : S(\bar{k}) \to CH_m(X)$:

$$\Gamma_*(s) = pr_{X*}(\Gamma \cdot (s \times X)).$$

The dot here denotes intersection product in $CH(S \times X)$. Given a base point $s_0 \in S(\bar{k})$ we define $\gamma(s) = \Gamma_*(s) - \Gamma_*(s_0)$ and thus get a map $\gamma : S(\bar{k}) \to CH_m(X)_{alg}$. The maps Γ_* and γ depend only on the rational equivalence class of Γ.

Definition 1.5. Let $A/\bar{k}$ be an Abelian variety and let $\rho : CH_m(X)_{alg} \to A(\bar{k})$ be a group homomorphism. If for every (S, s_0, Γ) as above, the composition,$\rho \circ \gamma$, is a morphism of algebraic varieties, then ρ is said to be a regular map.

Definition 1.6. $CH_m(X)_{alg}$ is said to be weakly representable if there is an Abelian variety, $A/\bar{k}$, and a regular map, $\rho : CH_m(X)_{alg} \to A(\bar{k})$, which is a group isomorphism.

Remark 1.7. It is well known that $CH^1(X)_{alg}$ is representable by an Abelain variety [Gro2]. In particular it is weakly representable.

Lemma 1.8. *If $CH_m(X)_{alg}$ is not supported in dimension $m+1$, then $CH_m(X)_{alg}$ is not weakly representable.*

Proof. Suppose that $CH_m(X)_{alg}$ is weakly representable. One shows easily that there is a smooth, projective variety, S, whose dimension will be denoted by p, a base point, $s_0 \in S(\bar{k})$, and a cycle, $\Gamma \in Z_{p+m}(S \times X)$, such that $\rho \circ \gamma$ is surjective. We may assume that all components of $|\Gamma|$ map surjectively to S. There is a smooth, connected, pointed curve, $C, s_0 \subset S, s_0$, such that $\rho \circ \gamma(C(\bar{k}))$ generates the group $A(\bar{k})$. Write $\Gamma_C \in Z_{m+1}(C \times X)$ for a cycle which represents the pullback of Γ and set $Z = pr_{X*}(\Gamma_C)$. Then the restriction map,

$$j_Z^* : CH_m(X)_{alg} \to CH_m((X - Z)),$$

is clearly zero.

We shall use the following fact repeatedly:

Lemma 1.9. *If $k \subset K$ is an extension of fields and X is a variety defined over k, then the pullback map $CH_m(X_k) \to CH_m(X_K)$ has torsion kernel.*

Proof. [Bl1, p.1.21]

The following proposition is a slight variant of Bloch's zero cycle argument [Bl1, p.1.19].

Proposition 1.10. *Let $S_{\mathbf{C}}$ be a smooth, complex projective variety of dimension p. Let $\Gamma \in Z_{p+m}(S_{\mathbf{C}} \times X_{\mathbf{C}})$. Suppose given an integer w satisfying,*

(1) *$r < w$ and*
(2) *The image of the map of Hodge structures $\Gamma_* : H_w(S_{\mathbf{C}}) \to H_{w+2m}(X_{\mathbf{C}})(-m)$ has Hodge width w.*

Then $CH_m(X_{\mathbf{C}})_{alg}$ is not supported in dimension $m+r$.

Proof. The hypotheses imply that $p \geq w$. Let T be a smooth linear space section of S of dimension w. The Lefschetz hyperplane theorem implies that the natural map, $H_w(T_{\mathbf{C}}) \to H_w(S_{\mathbf{C}})$, is surjective. By replacing S by T and Γ by its restriction to $T \times X$ (which is well defined if T is chosen generally) we are reduced to proving the proposition in the case $p = w$.

We shall assume that an $m+r$-dimensional subscheme $Z \subset X$ exists such that the image of $j_{\mathbf{C}*} : CH_m(X_{\mathbf{C}})_{alg} \to CH_m((X - Z)_{\mathbf{C}})$ is torsion and derive a contradiction. Choose an algebraicly closed subfield $k \subset \mathbf{C}$ of finite transcendence degree over $\mathbf{Q}$ such that X, Z, S, and Γ can all be defined over k. Write η for the generic point of S_k and $\Gamma_\eta \in Z_m(X_\eta)$ for the restriction of Γ. Choose a point $s_0 \in S(k)$ such that the intersection $\gamma_0 = \Gamma \cdot (s_0 \times X)$ is defined. We may view γ_0 as an m-cycle on X_k. If Z does not already contain the support of γ_0 we enlarge Z so that it does. Write $\Gamma_0 \in Z_m(X_\eta)$ for the restriction of $pr_X^* \gamma_0 \in Z_{p+m}(S \times X)$ to the generic fiber.

Lemma 1.11. Γ_0 *and* $\Gamma_\eta \in Z_m(X_\eta)$ *are algebraicly equivalent.*

Proof. Write $p_{23} : S \times S \times X \to S \times X$ for the projection on the last two factors. Define $\delta, j_0 : S \to S \times S$ by $\delta(s) = (s,s)$ respectively $j_0(s) = (s, s_0)$. Then

$$(\Delta_S \times X) \cdot pr_{23}^*(\Gamma) \simeq (\delta \times Id_X)_* \Gamma\ , \qquad (S \times s_0 \times X) \cdot pr_{23}^*(\Gamma) = (j_0 \times Id_X)_* \circ pr_X^*(\gamma_0).$$

Now δ and j_0 map η to rational points of $S_\eta = \eta \times_k S$, and the fibers of the family $pr_{23}^*(\Gamma)|_{\eta \times_k S \times_k X}$ over these rational points have been identified with Γ_η and Γ_0. The lemma follows.

Choose an embedding of k-algebras, $k(S) \subset \mathbb{C}$. Since the kernel of the pullback map, $CH_m((X-Z)_\eta) \to CH_m((X-Z)_{\mathbb{C}})$ is torsion (1.9), there is a positive integer N such that $Nj_{Z_\eta}^*(\Gamma_\eta - \Gamma_0) = Nj_{Z_\eta}^* \Gamma_\eta \in CH_m((X-Z)_\eta)$ is zero. It follows from the localization sequence,

$$lim_{\mathcal{D} \in |S|^1} CH_{w+m}(\mathcal{D} \times X) \oplus CH_{w+m}(S \times Z) \to CH_{w+m}(S \times X) \to CH_m((X-Z)_\eta) \to 0,$$

that there is a divisor $D \subset S_k$ and cycles, Γ_1 and Γ_2 of dimension $w+m$, supported on $D \times X_k$, respectively $S_k \times Z_k$, such that $N\Gamma \sim_{rat} \Gamma_1 + \Gamma_2$.

To prove the proposition we need only show $\Gamma_{i*} : F^0 H_w(S_{\mathbb{C}}) \to F^0(H_{w+2m}(X_{\mathbb{C}})(-m))$ is zero for $i = 1, 2$. Although the computations are essentially the same as in [Bl,p.1.23] we repeat them here as we shall need a slight variant later. Begin with the case $i = 1$. Fix $\beta \in H_w^{0,-w}(S_{\mathbb{C}})$, write $\alpha \in H^{w,0}(S_{\mathbb{C}})$ for the Poincaré dual, and consider the commutative diagram,

$$\begin{tikzcd} \tilde{D} \times X \arrow[r, "h"] \arrow[d, "p_{\tilde{D}}"'] & S \times X \arrow[d, "pr_S"] \\ \tilde{D} \arrow[r, "i"] & S, \end{tikzcd}$$

where $\tilde{D}$ is a desingularization of D. The projections of $S \times X$ (respectively $\tilde{D} \times X$) on the individual factors are denoted pr_S and pr_X (respectively $p_{\tilde{D}}$ and p_X). There is $\gamma_1 \in Z_{w+m}(\tilde{D} \times X) \otimes \mathbb{Q}$ such that $h_* \gamma_1 = \Gamma_1$. Define

$$\gamma_{1*} : H^w(\tilde{D}_{\mathbb{C}}) \to H_{w+2m}(X_{\mathbb{C}})(-m), \qquad \gamma_{1*}(\tau) = p_{X*}([\gamma_1] \cap p_{\tilde{D}}^* \tau).$$

Compute

$$\Gamma_{1*}(\beta) = pr_{X*}([\Gamma_1] \cap pr_S^* \alpha) = pr_{X*} h_*([\gamma_1] \cap h^* pr_S^* \alpha) = \gamma_{1*} i^*(\alpha).$$

Since dim. $\tilde{D} < w$ the Hodge type of $i^*\alpha$ forces this expression to vanish.

To verify that Γ_{2*} is also zero write $\tilde{Z}$ for a desingularization of Z so that there is a commutative diagram,

$$\begin{tikzcd} S \times \tilde{Z} \arrow[r, "g"'] \arrow[d, "p_{\tilde{Z}}"'] & S \times X \arrow[d, "pr_X"] \\ \tilde{Z} \arrow[r, "\iota"] & X. \end{tikzcd}$$

There is $\gamma_2 \in Z_{w+m}(S \times \tilde{Z}) \otimes \mathbb{Q}$ such that $g_*(\gamma_2) = \Gamma_2$. For β and α as above compute

$$(1.12) \qquad \Gamma_{2*}\beta = pr_{X*}([\Gamma_2] \cap pr_S^* \alpha) = pr_{X*} g_*(\gamma_2 \cap g^* pr_S^* \alpha) = \iota_* \gamma_{2*}(\alpha),$$

Where the definition of γ_{2*} is analogous to the definition of γ_{1*} above. Since dim. $\tilde{Z} = m + r < w + m$, $H_{w+2m}^{-m,-w-m}(\tilde{Z})(-m) = 0$, whence (1.12) must vanish. This completes the proof of (1.10).

Remark 1.13. The proof of (1.10) makes essential use of two properties of the complex numbers. First, $\mathbb{C}$ is large enough to contain the function field of the parameter space S. Secondly, the cohomology has a natural filtration, the Hodge filtration, which contains the coniveau filtration. Resolution of singularities is used only for convenience. Thus the proof can be generalized to work over an arbitrary algebraicly closed field of infinite transendence degree if the Hodge filtration is replaced by any filtration which contains the coniveau filtration (cf. [Bl1, Appendix to §1]). The proof does not work if the base field is $\bar{\mathbb{Q}}$, which is consistent with the conjecture of Beilinson and Bloch [Be, 5.0, 5.2, 5.6].

For completeness we mention

Corollary 1.14. *([R1], [Bl1, Appendix to §1]) Suppose $w \geq 2$ and $H^{w,0}(X_{\mathbb{C}}) \neq 0$. Then $CH_0(X_{\mathbb{C}})_{alg}$ is not weakly representable.*

Proof. Take $m = 0, S = X, \Gamma = \Delta, r = 1$ in (1.10). Then $CH_0(X_{\mathbb{C}})_{alg}$ is not supported in dimension 1. The result follows from (1.8).

Now we prove (0.3) of the introduction.

Proposition 1.15. *Let $V \subset H_h(X_{\mathbb{C}})$ be a Hodge substructure of width w. There is a non-negative integer m such that $h = w + 2m$. Suppose that the generalized Hodge conjecture of Grothendieck holds. Then $CH_m(X_{\mathbb{C}})_{alg}$ is not supported in dimension $m + w - 1$. If $w \geq 2$, $CH_m(X_{\mathbb{C}})_{alg}$ is not weakly representable.*

Proof. By the generalized Hodge conjecture and resolution of singularities there is a smooth, projective $\mathbb{C}$-scheme, Z, of pure dimension $m + w$ and a morphism, $f : Z \to X$, with $V \subset f_* H_{w+2m}(Z)$ [St,§1]. Let S be a smooth linear space section of Z of dimension w. By Poincaré duality and the Lefschetz hyperplane theorem, $H_{w+2m}(Z) \simeq H_w(Z)(m) \simeq H_w(S)(m)$. By the Hodge conjecture applied to $S \times Z$ there is $\Gamma' \in Z_{w+m}(S \times Z)$ such that $\Gamma'_*(H_w(S)(m)) = H_{w+2m}(Z)$. Composing Γ' with f gives $\Gamma \in Z_{w+m}(S \times X)$ with $V \subset \Gamma_*(H_w(S)(m))$. Now (1.10) and (1.8) apply.

Corollary 1.16. *Let $X_{\mathbb{C}}$ be a smooth projective variety of dimension n. Suppose $H_w^{-w,0}(X_{\mathbb{C}}) \neq 0$ for some $w \geq 2$. Then $CH_0(X_{\mathbb{C}})_{alg}$ is not weakly representable, and if the ordinary Hodge conjecture for $X \times X$ is true, neither is $CH_m(X_{\mathbb{C}})_{alg}$ for $0 \leq m \leq n - w$.*

Proof. The first assertion is (1.14). The Hodge conjecture asserts the existence of $\Gamma' \in Z^{n-m}(X \times X)$ such that

$$H_w(X_{\mathbb{C}}) \to H_{w+2m}(X_{\mathbb{C}})(-m), \qquad \beta \to pr_{2*}((\beta \times [X]) \cap cl(\Gamma'))$$

is an isomorphism. Let $S_{\mathbb{C}} \subset X_{\mathbb{C}}$ be a general linear space section of dimension w with respect to a projective embedding of $X_{\mathbb{C}}$. Apply (1.10) with Γ the restriction of Γ' to $S \times X$.

Corollary 1.17. *Let $X_{\mathbb{C}}$ be an Abelian variety of dimension n. Then $CH_m(X_{\mathbb{C}})_{alg}$ is not weakly representable for $0 \leq m \leq n - 2$.*

Proof. Let $S, L \subset X$ denote linear space sections of dimensions 2 and m with respect to some embedding of X in projective space. Assume that S is non-singular. Write $\Gamma \subset S \times X$ for the subvariety obtained by translating L by the points of S. The map $\Gamma_* : H_2(S) \to H_{2+2m}(X)(-m)$ may be written in terms of Pontrjagin product: $\Gamma(\beta) = i_*(\beta) * [L]$, where $i_* : H_2(S) \to H_2(X)$ is the standard inclusion. This map is injective. The corollary now follows from (1.10).

2. Lines on hypersurfaces with 1-spanning Hodge structures.

Definition 2.1. Let $n/2 > 1$. A hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is said to be ordinary for lines, if the Hilbert scheme of lines on X, denoted S, is smooth of pure dimension $2n - d - 1$.

Write $\dot{P}_d \subset \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$ for the parameter space of smooth hypersurfaces of degree d.

Proposition 2.2. ([B-V]) *The smooth hypersurfaces which are ordinary for lines form a non-empty Zariski open subset $U_d \subset \dot{P}_d$.*

Remark 2.3. $U_3 = \dot{P}_3$. This follows from the determination of the possible normal bundles for lines. The proof is by induction on the dimension of the hypersurface [A-K,1.10].

Remark 2.4. The Fermat hypersurface of dimension n and degree $n+1$ is not ordinary for lines. In fact there are $n+1$ hyperplane sections which are cones with a common vertex over Fermat varieties of dimension $n-2$. Thus $U_d = \dot{P}_d$ does not hold when $d = n+1 > 3$.

Theorem 2.5. *Write $\mathcal{L} \subset S \times X$ for the universal family of lines. If X is ordinary for lines, then the map,*

$$H^{n-1,1}(X)(1) \to H^{n-2,0}(S), \qquad \beta \to pr_{S*}(cl(\mathcal{L}) \cdot pr_X^*\beta),$$

is not zero.

Proof. The reader is referred to [Cl1] and [Shi] for sketches of an argument. A different approach is treated in detail in [Le3, §13]

Remark 2.6. (2.5) is equivalent to the dual map on homology $\mathcal{L}_* : H^{2-n,0}_{n-2}(S)(1) \to H^{1-n,-1}_n(X)$ not being zero.

Remark 2.7. Recall that the monodromy representation on the primitive cohomology is irreducible. Thus, if X is chosen to be sufficiently general, the Hodge structure $H^n_{prim}(X)$ is irreducible. In this case (2.5) implies that $\mathcal{L}^* : H^n_{prim}(X) \to H^{n-2}(S)$ is injective.

The main purpose of this section is to prove the following result of which (0.5) is an obvious corollary (cf. [Le3, 15.44]).

Theorem 2.8. *Let $\frac{n}{2} > 1$ and $\frac{n}{2}+1 \leq d < n+2$. If $X_{\mathbb{C}} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth hypersurface of degree d, then $CH_1(X_{\mathbb{C}})_{alg}$ is not supported in dimension $n-2$.*

Proof. Suppose first of all that the Hilbert scheme of lines on $X_{\mathbb{C}}$ is smooth of dimension $2n - d - 1$. By (2.2) this is the case on a non-empty, Zariski open subset in the moduli of degree d hypersurfaces. Apply (1.10) and (2.6) with S the Hilbert scheme of lines on X, $\Gamma \subset S \times X$ the universal family of lines, and $w = n-2$. This gives the desired result. In fact it shows

Corollary 2.9. *Suppose $X_{\mathbb{C}}$ in the statement of (2.8) is regular for lines. Then the subgroup of $CH_1(X_{\mathbb{C}})_{alg}$ which is generated by differences of two lines is not supported in dimension $n-2$.*

The case where X is not ordinary for lines is dealt with be means of a broadly applicable lemma. This says that, if certain natural conditions are imposed, then the generalized Hodge conjecture is true for a special fiber in a family, if it is true for the general fiber. This result is best stated in the following context: Let $U \subset C$ be a non-empty, Zariski open subset of a smooth, connected curve over the field of complex numbers. Let $\pi_X : \mathcal{X} \to C$ (respectively $\pi_S : \mathcal{S} \to U$) be a smooth, projective morphism with connected fibers of relative dimension n (respectively p). Let $\Gamma \in Z_{p+m+1}(\mathcal{S} \times_C \mathcal{X})$ be a linear combination of subvarieties each of which is flat over C. Let $\mathcal{V} \subset R^{w+2m}\pi_{X*}\mathbb{Q}$ be a subvariation of Hodge structure of width w. Now Γ gives rise to $\Gamma^* \in$ Hom $(\mathcal{V}|_U, R^w\pi_{S*}\mathbb{Q}(-m))$ as follows: By the Leray spectral sequence for the map $p : \mathcal{S} \times_C \mathcal{X} \to C$ and the Künneth decomposition, the cohomology class of Γ gives rise to a class

$$\{\Gamma\} \in H^0(U, R^{2n-w-2m}\pi_{X*}\mathbb{Q}(n) \otimes R^w\pi_{S*}\mathbb{Q}(-m)).$$

Write $tr : R^{2n}\pi_{X*}\mathbb{Q}(n) \simeq \mathbb{Q}$ for the orientation isomorphism and define

$$\text{(2.10)} \qquad \Gamma^* \,:\, \mathcal{V}|_U \to R^w\pi_{S*}\mathbb{Q}(-m) \qquad \text{by} \qquad \Gamma^*(\beta) = (tr \otimes 1)(\{\Gamma\} \cdot (\beta \otimes 1)).$$

Finally let $c \in C - U$. Write $X = \pi_X^{-1}(c)$ and $V = \mathcal{V}_c$. Now the lemma we need is

Lemma 2.11. *Suppose that Γ^* is injective. Then there is a smooth projective scheme, S, of dimension p, and a cycle $\gamma \in Z_{p+m}(S \times X)$ such that $\gamma^* : V \to H^w(S)(-m)$ is injective.*

Prior to proving (2.11) we take a moment to discuss its significance and its application to (2.8). The hypothesis that Γ^* is injective is a strong version of the generalized Hodge conjecture for the stalks $\mathcal{V}_u, u \in U$. Indeed, $\mathcal{V}_u$ has width w and, exactly as the generalized Hodge conjecture predicts, there is an algebraic correspondence Γ_u^* which maps $\mathcal{V}_u$ injectively to the degree w cohomology of a smooth projective variety. We have made the minor additional assumption that all of these corrsepondences fit together in a family over U. With this hypothesis the lemma says that even for points $c \in C - U$ the generalized Hodge conjecture is true for $\mathcal{V}_c$. Thus (2.11) is a device for establishing the generalized Hodge conjecture at a special point in a family, if it is known to hold at the general point.

In order to apply this to (2.8), fix $c \in \dot{P}_d - U_d$. Take for C a general curve in $\dot{P}_d$ through c. Let $U = C \cap U_d$ and let $\mathcal{X}$ be the pullback of the universal family of degree d hypersurfaces to C. $\mathcal{S}$ is the relative Hilbert scheme for lines on $\mathcal{X}|_U$, $\mathcal{V} = (R^n\pi_{X*}\mathbb{Q})_{prim}$, and Γ is the pullback of the universal family of lines. Since C is general, there is a point $u \in U$ where the map on stalks

$$\Gamma_u^* \,:\, \mathcal{V}_u \to R^{n-2}\pi_{S*}\mathbb{Q}(-m)_u$$

is injective (2.7). Since $\mathcal{V}$ is locally constant and U is connected, Γ^* is injective. The lemma now gives us a smooth projective scheme S of dimension $n-2$ and an algebraic cycle γ with the property that $\gamma_* : H_{n-2}(S) \to H_n(X)_{prim}(-1)$ is surjective. This is precisely what we need to apply (1.10). Now (2.8) follows even when X is not ordinary for lines.

Proof of 2.11. The semi-stable reduction theorem says that by replacing C by a finite branched cover, we may assume that π_S extends to a projective morphism, $\bar{\pi}_S : \bar{\mathcal{S}} \to C$, where $\bar{\mathcal{S}}$ is a non-singular variety, $\bar{\mathcal{S}}|_U \simeq \mathcal{S}$, and $\bar{\pi}_S^{-1}(c)$ is a reduced normal crossing divisor [Ke,§II]. Taking the closures of the components of Γ leads to a cycle $\bar{\Gamma} \in$

$Z_{p+m+1}(\bar{\mathcal{S}} \times_C \mathcal{X})$ whose restriction to $\mathcal{S} \times_C \mathcal{X}$ is Γ. The class of $\bar{\Gamma}$ in the cohomology of the non-singular variety $\bar{\mathcal{S}} \times_C \mathcal{X}$ gives rise to

$$\{\bar{\Gamma}\} \in H^0(C, R^{2n-w-2m}\pi_{X*}\mathbb{Q}(n) \otimes R^w \bar{\pi}_{S*}\mathbb{Q}(-m))$$

via the Leray spectral sequence and the Kuenneth decomposition. As in (2.9) $\{\bar{\Gamma}\}$ defines a homomorphism

$$\bar{\Gamma}^* \ : \ \mathcal{V} \to R^w \bar{\pi}_{S*}\mathbb{Q}(-m).$$

This map is injective since $\mathcal{V}$ is a locally constant sheaf and $\bar{\Gamma}^*|_U = \Gamma^*$ is injective. Write $\nu : S \to \bar{\pi}_S^{-1}(c)$ for the normalization and $i : \bar{\pi}_S^{-1}(c) \times X \to \mathcal{S} \times_C \mathcal{X}$ for the inclusion. Then $i^* : H^{2n-2m}(\mathcal{S} \times_C \mathcal{X})(n-m) \to H^{2n-2m}(\bar{\pi}_S^{-1}(c) \times X)(n-m)$ is a morphism of mixed Hodge structures. Thus $\xi := i^*(cl(\bar{\Gamma}))$ has Hodge type $(0,0)$ and gives a morphism of mixed Hodge structures

$$\xi^* : H^{w+2m}(X) \to H^w(\bar{\pi}_S^{-1}(c))(-m), \qquad \xi^*(\beta) = (tr \otimes 1)(\xi \cdot (\beta \otimes 1)).$$

Now ξ^* is injective since it is the restriction of $\bar{\Gamma}^*$ to the stalk at c. The composition with the normalization,

$$\nu^* \circ \xi^* \in \text{ Hom } (H^{w+2m}(X), H^w(S)(-m)),$$

is also injective by a standard weight argument [De,8.2.7]. Let $\gamma = (i \circ (\nu \times 1))^* \bar{\Gamma} \in CH_{p+m}(S \times X)$ denote the pullback of $\bar{\Gamma}$, in the sense of intersection theory. Then

$$\begin{aligned} \nu^* \circ \xi^*(\beta) = \nu^*(tr \otimes 1)(\xi \cdot (\beta \otimes 1)) &= (tr \otimes 1)(\nu^*(\xi) \cdot (\beta \otimes 1)) \\ &= (tr \otimes 1)(\gamma \cdot (\beta \otimes 1)) = \gamma^*(\beta). \end{aligned}$$

The lemma follows.

Remark 2.12. ([Le3,15.??]) At the referee's request we mention an alternative proof of (2.8). Suppose that $X_{\mathbb{C}}$ is a smooth, projective variety of dimension $n \geq 2$ for which $CH_0(X_{\mathbb{C}})_{alg}$ is supported in dimension $n-1$. Then Lewis shows that there exists a smooth projective variety S of dimension p and a cycle $\Gamma \in Z_{p+1}(S \times X)$ such that $\Gamma_* : H_{n-2}(S_{\mathbb{C}}) \to H_n(X_{\mathbb{C}})$ is surjective. Now the hypotheses are satisfied when $X_{\mathbb{C}}$ is as in (2.8). Indeed $CH_0(X_{\mathbb{C}})_{alg} = 0$ [R3,§4]. Thus (1.10) implies (2.8). Unfortunately this approach does not yield the information about lines which is needed in the proofs of (2.9), (3.1), and (3.18).

Remark 2.13. (Positive characteristic.) Theorem (2.8) is not quite true if $\mathbb{C}$ is replaced by an algebraically closed field $\bar{k}$ of infinite transcendence degree over the prime field, $\mathbb{F}_p, p > 0$. The point is that in positive characteristic it can occassionally happen that there is a subscheme $Z \subset X$ of dimension less than $n-1$ such that the induced map $H_n(Z, \mathbb{Q}_l) \to H_n(X, \mathbb{Q}_l)$ is surjective. When this occurs one might hope that 1-cycles are supported in dimesnion $n-2$. However this is frequently difficult to verify in practice. In the following example we can overcome these difficulties. Presumably the result illustrates what to expect in general in positive characterisitic.

Proposition 2.14. *Let $\bar{k}$ be an algebraically closed field of infinite transcendence degree over the prime field in characteristic $p > 3$. Suppose given for $i \in \{1,2\}$ two smooth plane cubics,*

$$E_i \subset \mathbb{P}^2_{\bar{k}} \ : \ f_i(x_0, x_1, x_2) = 0.$$

Define a (non-singular) cubic hypersurface $X \subset \mathbb{P}^5_{\bar{k}}$ by

$$f_1(x_0, x_1, x_2) + f_2(x_3, x_4, x_5) = 0.$$

Then $CH_1(X_{\bar{k}})_{alg}$ is not supported in dimension 2 unless both E_1 and E_2 are supersingular. If this is the case, then $CH_1(X_{\bar{k}})_{alg} = 0$.

Before proving the proposition, we recall that for each prime p, the set of isomorphism classes of supersingular elliptic curves defined over $\bar{k}$ is non-empty and finite. In fact it contains approximately $\frac{p}{12}$ elements [Ha,IV.4.23].

Proof. The geometric set up is taken from [Sh-K,§1] (especially Remark 1.10) to which we refer for details. Let $Y_i \subset \mathbb{P}^3_{\bar{k}}$ denote the smooth cubic surface defined by

$$f_i(x_0, x_1, x_2) + x_3^3 = 0.$$

Consider the inclusions

$$E_i \to Y_i, \qquad (x_0, x_1, x_2) \to (x_0, x_1, x_2, 0)$$
$$E_1 \to X, \ (x_0, x_1, x_2) \to (x_0, x_1, x_2, 0, 0, 0) \qquad E_2 \to X, \ (x_0, x_1, x_2) \to (0, 0, 0, x_0, x_1, x_2).$$

Write $\tilde{Y}$ (respectively $\tilde{X}$) for the blow up of $Y_1 \times Y_2$ (respectively X) along $E_1 \times E_2$ (respectively $E_1 \amalg E_2$). Multiplying the coordinate x_3 by roots of unity, gives an operation of μ_3 on Y_i. The fixed locus of the corresponding diagonal action on $Y_1 \times Y_2$ is $E_1 \times E_2$. There is an induced action on $\tilde{Y}$ and the quotient is isomorphic to $\tilde{X}$. The exceptional fiber $E_1 \times E_2 \times \mathbb{P}^1 \subset \tilde{Y}$ maps to both $E_1 \times E_2$ and X. The resulting correspondence $\Gamma \in Z_3(E_1 \times E_2 \times X)$ gives a map $\Gamma_* : H_2(E_1 \times E_2, \mathbb{Q}_l(-1)) \to H_4(X, \mathbb{Q}_l(-2))$. Use the subscript t to denote that part of the homology which is othogonal under the intersection pairing to the classes of algebraic cycles. If at least one of the E_i's is not supersingular, then $H_2(E_1 \times E_2, \mathbb{Q}_l(-1))_t \neq 0$. Furthermore, $\Gamma_* : H_2(E_1 \times E_2, \mathbb{Q}_l(-1))_t \to H_4(X, \mathbb{Q}_l(-2))_t$ is well defined and injective [Sh-K, Proposition 2.4]. Thus there is no surface $Z \subset X$ such that the image of $H_4(Z, \mathbb{Q}_l(-2))$ contains $\Gamma_* H_2(E_1 \times E_2, \mathbb{Q}_l(-1))$. Now the argument used to prove (1.10) shows that $CH_1(X_{\bar{k}})_{alg}$ is not supported in dimension two.

Suppose now that both E_1 and E_2 are supersingular. Then the regular map $CH_0(E_1 \times E_2)_{alg} \to Alb_{E_1 \times E_2}(\bar{k})$ is an isomorphism [Bl3, A.10,A.11(i)] and [Shio, Theorem 1.1]. Consider the diagram

$$E_1 \times E_2 \xleftarrow{p} E_1 \times E_2 \times \mathbb{P}^1 \xrightarrow{i} \tilde{Y} \xrightarrow{\mu} \tilde{X} \xleftarrow{r} (E_1 \amalg E_2) \times \mathbb{P}^2,$$

where the last map is the inclusion of the exceptional divisor in $\tilde{X}$. We claim that $CH_1(\tilde{Y})$ is the direct sum of the divisible group $\mu^* \circ r_*(CH_1((E_1 \amalg E_2) \times \mathbb{P}^2)_{alg})$ with a finitely generated group. This would certainly suffice to prove the propostion since $r_*(CH_1((E_1 \amalg E_2) \times \mathbb{P}^2)_{alg})$ maps to zero in $CH_1(X)$ while $CH_1(\tilde{Y})$ maps surjectively to $CH_1(X)$. Thus $CH_1(X)$ would be finitely generated, which implies that the divisible group $CH_1(X)_{alg}$ is zero.

To check the claim we use the exact sequence for a blow up [Fu, 6.7e]

$$0 \to CH_1(E_1 \times E_2) \to CH_1(E_1 \times E_2 \times \mathbb{P}^1) \oplus CH_1(Y_1 \times Y_2) \to CH_1(\tilde{Y}) \to 0.$$

First note that $CH_1(Y_1 \times Y_2)$ is finitely generated. In fact, Y_i is the blow up of $\mathbb{P}^2$ at six points. So subvarieties isomorphic to $\mathbb{P}^1 \times Y_2$ and $Y_1 \times \mathbb{P}^1$ can be removed from $Y_1 \times Y_2$ in such a way that one is left with an open subset of $\mathbb{P}^2 \times \mathbb{P}^2$. The exact sequence allows us to identify the maximal divisible subgroup of $CH_1(\tilde{Y})$ with the image of $i_* \circ p^* : CH_0(E_1 \times E_2)_{alg} \to CH_1(\tilde{Y})$. This map is injective, since $p_* \circ i^*$ is a left inverse. Using elementary facts about the cohomology of blow-ups we deduce easily that

$$p_* i^* \mu^* r_* : H^3((E_1 \amalg E_2) \times \mathbb{P}^2, \mathbb{Q}_l) \to H^1(E_1 \times E_2, \mathbb{Q}_l)$$

is an isomorphism. Thus the morphism of Abelian varieties

$$p_* i^* \mu^* r_* : CH_1((E_1 \amalg E_2) \times \mathbb{P}^2)_{alg} \to CH_0(E_1 \times E_2)_{alg}$$

is an isogeny. We may thus identify $\mu^* \circ r_*(CH_1((E_1 \amalg E_2) \times \mathbb{P}^2)_{alg}$ with the maximal divisible subgroup of $CH_1(\tilde{Y})$. The claim follows.

3. Lines on hypersurfaces with 1-excessive Hodge structures.

For any positive integer d define $P_d = \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbf{Q}}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$. Let $\mathcal{F} \subset P_d \times \mathbb{P}^{n+1}_{\mathbf{Q}}$ be the universal families of hypersurfaces of degree d. Let $\mathcal{F}|_{\mathbb{Q}(P_d)}$ denote the generic fiber of $\mathcal{F}/P_d$ and let $\mathcal{F}|_{\mathbb{C}}$ denote the complex variety obtained by base changing with respect to an embedding $\mathbb{Q}(P_d) \subset \mathbb{C}$. The purpose of this section is to prove

Theorem 3.1. *Suppose $n/2 > 1$ and $n+2 \leq d \leq 2n-1$. Then there exist two lines on $\mathcal{F}|_{\mathbb{C}}$ whose difference has infinite order in $CH_1(\mathcal{F}|_{\mathbb{C}})_{hom}$.*

The idea of the proof of (3.1), and hence of (0.7), can be described very crudely as follows: Take a family of hypersurfaces of degree d, which is parametrized by a smooth, but not necessarily complete curve. We specify that d is in the range $n+2 \leq d \leq 2n-1$. (Indeed if $d \geq 2n$, the general hypersurface contains no lines.) Now suppose that a special fiber is the union of two smooth hypersurfaces, one of which we call G. The degree of G will be assumed to lie in the range of applicability of Theorem 2.8; that is $\frac{n}{2} + 1 \leq d_G < n+2$. We wish to find two lines on the general fiber which specialize to two lines on G. Having done this, we would next like to use (2.9) to show that the two lines on G are not rationally equivalent. The final step would be to deduce from this, that the original two lines on the general fiber are not rationally equivalent.

To transform this rough idea into a rigorous argument, we will produce a finitely generated field K, a smooth, geometrically irreducible curve C/K, and a map τ : $\operatorname{Spec} K(C) \to P_{d\,\mathbf{Q}}$ satisfying the following

List of Properties 3.2.

(1) *The image of τ is $\operatorname{Spec} \mathbb{Q}(P_d)$.*
(2) *The pullback of the universal family, $\tau^*\mathcal{F}$, can be spread out to a regular model $p : \mathbf{F} \to C_K$, where p is projective and flat.*
(3) *There is a K-rational point $c \in C(K)$ such that $p^{-1}(c) = G \cup \tilde{H}$ is a normal crossing divisor with G non-singular.*

(4) *There are ruled surfaces $\rho_i : \mathcal{L}_i \to C_K$ and embeddings of C_K-schemes $\varphi_i : \mathcal{L}_i \to$ $\mathbf{F}$.*
(5) *The intersection, $\varphi_i(\mathcal{L}_i) \cdot G = L_i$, is a line for $i \in \{1,2\}$.*
(6) *Write $G' = G - G \cap \tilde{H}$. Then $(L_1 - L_2)|_{G'} \in CH_1(G')$ has infinite order.*

Assuming the set up (3.2) we now prove (3.1). This is not difficult. Let $\mathbf{F}' = \mathbf{F} - G \cap \tilde{H}$. Consider the exact sequence

$$\begin{array}{ccccc}
\oplus_{t \in |C|_0} CH_2(p^{-1}(t) \cap \mathbf{F}') & \xrightarrow{\oplus i_{t*}} & CH_2(\mathbf{F}') & \longrightarrow & CH_1(\tau^*(\mathcal{F})) \longrightarrow 0. \\
& & \Big\downarrow{\scriptstyle i_{G'}^*} & & \\
& & CH_1(G') & &
\end{array}$$

Observe that $p^{-1}(t) \cap G' = \emptyset$ for $t \neq c$ and that the normal bundle, $\mathcal{N}_{G'/\mathbf{F}'}$ is trivial. Thus

$$i_{G'}^* \circ i_{t*} = 0 \qquad (c \neq t) \qquad \text{and} \qquad i_{G'}^* \circ i_{c*}(Z) = i_{G'}^* \circ i_{G'*}(Z|_{G'}) = c_1(\mathcal{N}_{G'/\mathbf{F}'}) \cdot Z|_{G'} = 0$$

[F, Proposition 2.6(c)]. It follows that $i_{G'}^*$ induces a specialization homomorphism

$$sp : CH_1(\tau^*\mathcal{F}) \to CH_1(G').$$

By (6) $sp((\mathcal{L}_1 - \mathcal{L}_2)|_{K(C)}) = L_1 - L_2 \in CH_1(G')$ has infinite order. Given an embedding $\mathbb{Q}(P_d) \to \mathbb{C}$, there is a factorization $\mathbb{Q}(P_d) \to K(C) \to \mathbb{C}$. By (1.9) $(\mathcal{L}_1 - \mathcal{L}_2)|_{\mathbf{C}} \in CH_1(\tau^*(\mathcal{F})|_{\mathbf{C}})$ has infinite order as desired.

The remainder of this section is devoted to the explicit construction of the field K and the varieties $C, G, \tilde{H}, \mathbf{F}, \ldots$ of (3.2). This requires considerable care. We proceed in several rather lengthy steps.

Step 1. For K we take the function field, $\mathbb{Q}(Q)$, of the variety, Q, which parametrizes 5-tuples $(H, G, \ell_1, \ell_2, F)$ where: H, G, F are hypersurfaces of degrees d_H, d_G, and d; ℓ_1 and ℓ_2 are disjoint lines on G meeting H transversely, and $H \cap \ell_i \subset F \cap \ell_i$ for $i \in \{1,2\}$. Here the degrees satisfy $n+2 \leq d \leq 2n-1$, $d = d_G + d_H$, and $n \geq d_G > d_H$. We also introduce the notations $\mathcal{G} \subset P_{d_G} \times \mathbb{P}_{\mathbb{Q}}^{n+1}$ and $\mathcal{H} \subset P_{d_H} \times \mathbb{P}_{\mathbb{Q}}^{n+1}$ for the universal families of hypersurfaces of degrees d_G and d_H. For the definition of K to make sense we must of course check

Lemma 3.3. *Q is irreducible.*

Proof. (cf. [Ka, §3 Lemma]) We apply the familiar irreducibility criterion that a finite type scheme over a field, V, is irreducible if the there is a morphism, $f : V \to W$, with irreducible image and all fibers irreducible of the same dimension [Shaf, I.6 Thm 8]. Write $\mathfrak{g} \subset Gr(\mathbb{P}^1, \mathbb{P}^{n+1})^2$ for the open subset parametrizing pairs of non-incident lines. Now $Q \subset P_{d_H} \times P_{d_G} \times \mathfrak{g} \times P_d$ projects surjectively to the factor $\mathfrak{g}$. This one sees by considering the natural action of $Aut(\mathbb{P}^{n+1})$ on Q and the corresponding action on $\mathfrak{g}$ which is transitive. It also follows that the fiber over a pair of lines (ℓ_1, ℓ_2), call it $Q_{(\ell_1,\ell_2)}$, has dimension independent of the choice of pair. Again by the transitivity of the $Aut(\mathbb{P}^{n+1})$ action, the linear spaces in P_{d_G} which are the images of the various $Q_{(\ell_1,\ell_2)}$ under the projection have the same dimension. Each fiber of this projection, $Q_{(\ell_1,\ell_2,G)}$, dominates P_{d_H}. A fiber of this last map, $Q_{(\ell_1,\ell_2,G,H)}$, is a linear subspace of P_d of codimension $2d_H$. Indeed we are dealing with the space of degree d hypersurfaces which contains a set $\mathfrak{z}$ of $2d_H < d$ distinct points. Such points always impose independent conditions on degree d hypersurfaces, since the evaluation map, $H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \to H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathfrak{z}}(d))$, is clearly surjective.

Lemma 3.4. *The projection of Q to P_d is dominant.*

Proof. To prove (3.4) we may assume that the base field is algebraicly closed. Fix a reduced hypersurface F_0 of degree d and two disjoint lines ℓ_1 and ℓ_2 which meet F_0 transversely. There is a degree d_G hypersurface, G_0, containing ℓ_1 and ℓ_2. Now choose a hypersurface of degree d_H, H_0, which meets both ℓ_1 and ℓ_2 transversely at points contained in $F_0 \cap (\ell_1 \cup \ell_2)$. For instance, take for H_0 a union of d_H hyperplanes. Now $(H_0, G_0, \ell_1, \ell_2, F_0)$ is a point of Q which maps to $F_0 \in P_d$.

Step 2. We turn now towards the construction of $p : \mathbf{F} \to C_K$. Write l_1, l_2 for the pair of universal lines on $\mathfrak{g}$. The generic fibers of the pullbacks of $\mathcal{H}, \mathcal{G}, \mathcal{F}, l_1, l_2$ with respect to the projections of Q to $P_{d_H}, P_{d_G}, P_d, \mathfrak{g}$ are denoted

$$H_K, G_K, F_K, L_1, L_2. \tag{3.5}$$

The starting point in the construction of $p : \mathbf{F} \to C_K$ is the pencil of degree d hypersurfaces in $\mathbb{P}^{n+1}_K$

$$tF + GH = 0. \tag{3.6}$$

Here and subsequently we use the same letter to denote a hypersurface in (3.5) and a homogeneous polynomial which defines it.

Lemma 3.7.

(1) *H_K and G_K meet transversely.*
(2) *F_K meets H_K, G_K, and $H_K \cap G_K$ transversely.*

Proof. Describe a point in $Q(\bar{\mathbb{Q}})$ by fixing two disjoint lines, ℓ_1 and ℓ_2, a smooth hypersurface G_0 containing them, and hypersurfaces H_0 and F_0 to be described presently. By Bertini, we may choose a non-singular H_0 to meet G_0 and the two lines transversely. It is possible to choose hyperplanes $T_1, ..., T_{2d_H}$ such that the intersection of $T_1 + ... + T_{d_H}$ with ℓ_1 coincides with $H_0 \cap \ell_1$. Similarly $T_{d_H+1} + ... + T_{2d_H} \cap \ell_2$ coincides with $H_0 \cap \ell_2$. We may arrange that each T_i meets H_0, G_0, and $H_0 \cap G_0$ transversely. The base locus of the linear system $H^0((H_0 \cap G_0), \mathcal{I}_{(\ell_1 \cup \ell_2) \cap H_0}(d))$ is exactly $(\ell_1 \cup \ell_2) \cap H_0$, since adding an arbitrary hypersurface section of degree $d - 2d_H > 0$ to $T_1 + ... + T_{2d_H}$ gives an element of this linear system. As an element, which is non-singular on the base locus has been exhibited, the general member is non-singular everywhere by the characteristic 0 Bertini Theorem [Ha, 10.9.2]. We apply this argument also to the corresponding linear systems on H_0 and G_0. This allows us to select a non-singular, degree d hypersurface F_0 with the desired transversality properties. Now that we know that there are closed points on the irreducible variety Q for which the corresponding varieties, H_0, G_0, F_0 etc., meet transversely the corresponding statement at the generic point follows.

Step 3. The next step is to blow up the base locus in the pencil (3.6). The homogeneous ideal, $I = (F, GH)$, defines an $n-1$ dimensional scheme consisting of two smooth components which meet transversely along the variety, W, defined by the ideal (F, G, H). Blowing up $\mathbb{P}^{n+1}$ along (F, GH) gives a variety, $B_I\mathbb{P}^{n+1}$, which is non-singular outside a codimension two family of A_1 singularities parametrized by W. In fact locally at any point of W we may extend F, G, H to a system of local parameters, $F, G, H, x_4, ..., x_{n+1}$. Locally in the étale topology the blow up is obtained by gluing the spectra of the rings

$$K[F, G, H, U, x_4, ..., x_{n+1}]/GHU - F \qquad \text{and} \qquad K[F, G, H, V, x_4, ..., x_{n+1}]/GH - FV.$$

The strict transform of H (respectively G) is defined in the second chart by (H,V) (respectively (G,V)). This subvariety is isomorphic to H (respectively G) since the ideal sheaf associated to (F,GH) restricts to an invertible ideal sheaf on H (respectively G). Blowing up the strict transform of H in $B_I\mathbb{P}^{n+1}$ yields a non-singular variety, $\tilde{\mathbb{P}}^{n+1}_K$, with a natural morphism $\delta : \tilde{\mathbb{P}}^{n+1} \to \mathbb{P}^{n+1}$. The strict transform of G remains unchanged in this second blow up since (H,V) defines a principal ideal in $K[F,G,H,V,x_4,...,x_{n+1}]/(G,V)$. The function $-GH/F$ induces a morphism, $f : \tilde{\mathbb{P}}^{n+1}_K \to \mathbb{P}^1_K$. The fiber, $f^{-1}(0)$, consists of two components, denoted $\tilde{H}$ and G. The former is isomorphic to H blown up along W and the latter to the original hypersurface G. The intersection of these two components sits in G as $G \cap H$. Now the lines $L_1, L_2 \subset G_K$ defined in (3.5) may be viewed as living in $f^{-1}(0)$.

Step 4. We now apply the deformation theory of Katz [Ka] to deform the lines L_1 and L_2 off the fiber $f^{-1}(0)$. This is the first step in the construction of the ruled surfaces $\mathcal{L}_i$ of (3.2)(4).

Lemma 3.8. *Let L be a line on G. Suppose that*

(1) $H^1(L, \mathcal{N}_{L/G}) \simeq 0$,
(2) *L meets H transversely,*
(3) $L \cap H \subset L \cap F$,

then L deforms to first order in the pencil (3.6).

Proof. For the reader's convenience we recall briefly the argument from [Ka,§1] and [Cl2,1.24]. The line L is the image of a map $\vec{\alpha}^0 : \mathbb{P}^1 \to \mathbb{P}^{n+1}$ given by an $(n+2)$-tuple of linear forms in two variables, $\vec{\alpha}^0 = (\alpha^0_0 : ... : \alpha^0_{n+1})$. The problem is to solve

$$(tF + GH)(\vec{\alpha}^0 + \vec{\alpha}^1 t) \equiv 0 \tag{3.9}$$

for $\vec{\alpha}^1$ when $t^2 = 0$. Define a map

$$\Phi_G : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{n+2} \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_G)), \qquad \Phi_G(\vec{\sigma}) = \sum_{0 \le j \le n+1} \sigma_j \frac{\partial G}{\partial X_j}(\vec{\alpha}^0).$$

By the chain rule solving (3.9) reduces to solving

$$F \circ \vec{\alpha}^0 + H \circ \vec{\alpha}^0 \cdot \Phi_G(\vec{\alpha}^1) = 0. \tag{3.10}$$

Hypotheses 2 and 3 imply $-F \circ \vec{\alpha}^0 / H \circ \vec{\alpha}^0 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_G))$. To solve (3.10) and prove the lemma it remains only to check

Lemma 3.11. *Φ_G is surjective if and only if $H^1(L, \mathcal{N}_{L/G}) \simeq 0$.*

Proof. Consider the standard exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes (\vec{\alpha}^0)^* \mathcal{O}_{\mathbb{P}^{n+1}}(1) \xrightarrow{r_1} (\vec{\alpha}^0)^* \Theta_{\mathbb{P}^{n+1}} \to 0$$

and the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Theta_{\mathbf{P}^1} \arrow[r] \arrow[d, equal] & (\vec{\alpha}^0)^*\Theta_G \arrow[r] \arrow[d] & \mathcal{N}_{L/G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta_{\mathbf{P}^1} \arrow[r] & (\vec{\alpha}^0)^*\Theta_{\mathbf{P}^{n+1}} \arrow[r, "\tau_3"] \arrow[d, "\tau_2"'] & \mathcal{N}_{L/\mathbf{P}^{n+1}} \arrow[r] \arrow[d, "\tau_4"'] & 0 \\
 & & (\vec{\alpha}^0)^*\mathcal{N}_{G/\mathbb{P}^{n+1}} \arrow[r, equal] \arrow[d] & \mathcal{N}_{G/\mathbf{P}^{n+1}}|_L \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Now $\Phi_G = H^0(\tau_2) \circ H^0(\tau_1)$. Since $H^0(\tau_1)$ and $H^0(\tau_3)$ are surjective, Φ_G will be surjective if and only if $H^0(\tau_4)$ is. Since $H^1(\mathcal{N}_{L/\mathbb{P}^{n+1}}) \simeq 0$, this is true if and only if $H^1(\mathcal{N}_{L/G}) \simeq 0$.

Remark 3.12. The lines L_1 and L_2 of (3.5) satisfy the hypotheses of (3.8). Indeed $H^1(L, \mathcal{N}_{L/G}) \simeq 0 \Leftrightarrow h^0(L, \mathcal{N}_{L/G}) = 2n - d_G - 1 \Leftrightarrow$ the Hilbert scheme of G_K is smooth at $[L]$. By (2.2) the Hilbert scheme of lines of G_K is smooth.

Write $\xi : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) \to \mathcal{O}_{\mathbf{P}^1}(d) \otimes \mathcal{O}_{L\cap H}$ for the evaluation map.

Proposition 3.13. *Suppose $L \subset G$ satisfies the hypotheses of (3.8). Define Φ_F and Φ_H analogously to Φ_G. Suppose that*

$$\xi \circ (\Phi_F - \frac{F \circ \vec{\alpha}^0}{H \circ \vec{\alpha}^0}\Phi_H) : \ \mathrm{Ker}\,(\Phi_G) \to \mathcal{O}_{\mathbf{P}^1}(d) \otimes \mathcal{O}_{L\cap H}$$

is surjective. Then there exists a formal power series, $\vec{\alpha}(t) = \sum_{i\geq 0} \vec{\alpha}^i t^i$, whose coefficients are $n+2$-tuples of linear forms in two variables such that

$$(tF + GH)(\vec{\alpha}(t)) \equiv 0.$$

Proof. [Ka, §3].

Lemma 3.14. *The hypotheses of (3.13) are satisfied for the lines L_1 and L_2 of (3.5).*

Proof. Consider the special line, $L_0 : \ x_2 = \ldots = x_{n+1} = 0$, and the special hypersurfaces

$$G_0 = \sum_{1\leq j\leq d_G} x_{j+1}x_1^{j-1}x_0^{d_G-j} \qquad \text{and} \qquad F_0 = \sum_{1\leq j\leq n} x_{j+1}x_0^{j-1}x_1^{d-j}.$$

The definition of G_0 is legitimate since we continue to assume $n \geq d_G > d_H$. Both hypersurfaces contain L_0 and are smooth in a neighborhood of L_0. Now

$$\Phi_{G_0}(\sigma_0, \ldots, \sigma_{n+1}) = \sum_{1\leq j\leq d_G} \sigma_{j+1}x_1^{j-1}x_0^{d_G-j} \qquad \Phi_{F_0}(\sigma_0, \ldots, \sigma_{n+1}) = \sum_{1\leq j\leq n} \sigma_{j+1}x_0^{j-1}x_1^{d-j}.$$

Clearly the image of Φ_{G_0} is all of $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_G))$. Write e_i (respectively f_i) for the element of $\oplus_{0\leq i\leq n+1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ which is x_0 (respectively x_1) in the i^{th} place and zero elsewhere. For $2 \leq i \leq d_G$, $e_{i+1} - f_i \in$ Ker (Φ_{G_0}) and

$$\Phi_{F_0}(e_{i+1} - f_i) = -x_1^{d+2-i} x_0^{i-2} + x_1^{d-i} x_0^i \tag{3.15}$$

are $d_G - 1 \geq d_H$ linearly independent elements of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$. One can choose d_H points on L_0 such that the evaluation map, ξ, restricted to the span of (3.15) is surjective. Now choose a degree d_H hypersurface $H_0 \subset \mathbb{P}^{n+1}$ which cuts out this set of points on L_0. Since $L_0 \subset F_0$,

$$\Phi_{F_0} = \Phi_{F_0} - \frac{F_0 \circ \vec{\alpha}^0}{H_0 \circ \vec{\alpha}^0} \Phi_{H_0}.$$

This verifies that the hypothesis of (3.13) holds for a special choice of lines and hypersurfaces. Thus it certainly holds for the general lines L_1 and L_2 and the the general hypersurface G_K, H_K, F_K and the lemma follows.

Step 5. Having now shown that the lines deform formally in the pencil (3.6), we turn to constructing algebraic families of lines. Define open subschemes,

$$Hom(\mathbb{P}^1, \mathbb{P}^{n+1})_0 \subset Hom(\mathbf{P}^1, \mathbb{P}^{n+1}), \qquad Hom(\mathbb{P}^1, \tilde{\mathbb{P}}^{n+1})_0 \subset Hom(\mathbb{P}^1, \tilde{\mathbb{P}}^{n+1}),$$

by requiring that the image of $\mathbb{P}^1$ not be contained in $F \cap GH$ (respectively in $\delta^{-1}(F \cap GH)$). Then $Hom(\mathbb{P}^1, \tilde{\mathbb{P}}^{n+1})_0 \simeq Hom(\mathbb{P}^1, \mathbb{P}^{n+1})_0$. Define

$$\Xi = \{\vec{\alpha} \in Hom(\mathbb{P}^1, \tilde{\mathbb{P}}^{n+1})_0 : (\delta \circ \vec{\alpha})^* \mathcal{O}_{\mathbf{P}^{n+1}}(1) \simeq \mathcal{O}_{\mathbf{P}^1}(1), (f \circ \vec{\alpha})^* \mathcal{O}_{\mathbf{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1}\}.$$

Let ${}_i\vec{\alpha}^0 \in \Xi(K)$ correspond to the line $L_i \in f^{-1}(0)$ of (3.5). By (3.13) ${}_i\vec{\alpha}^0$ is the constant term in a power series ${}_i\vec{\alpha}(t)$ satisfying

$$t = -GH({}_i\vec{\alpha}(t))/F({}_i\vec{\alpha}(t)).$$

This power series may be viewed as a morphism over $\mathbf{P}^1$, $\hat{\kappa}_i$: Spec $\hat{\mathcal{O}}_{\mathbf{P}^1,0} \to \Xi$. By [Ar,2.5] there is an étale neighborhood (C_i, c_i) of $(\mathbb{P}^1, 0)$ and a morphism $\kappa_i : (C_i, c_i) \to (\Xi, {}_i\vec{\alpha}^0)$ of schemes over $\mathbb{P}^1$. Let $\bar{C}$ denote the connected component of a smooth projective model of the fiber product $C_1 \times_{\mathbf{P}^1} C_2$ with the property that there is $c \in \bar{C}$ which maps to (c_1, c_2). The map κ_i gives rise to a ruled surface $\mathcal{L}_i \subset \tilde{\mathbb{P}}^{n+1} \times_{\mathbf{P}^1} \bar{C}$. Since $\bar{C}$ is étale over $\mathbb{P}^1$ in a neighborhood of c, we may identify the fiber $\mathcal{L}_i|_c$ with L_i. Since $\bar{C}$ is smooth and has a K-rational point, it is geometrically irreducible. By removing the ramification locus of $\bar{C}/\mathbb{P}^1$ one obtains an open neighborhood, $C \subset \bar{C}$, of c such that $\mathbf{F} := \tilde{\mathbb{P}}^{n+1} \times_{\mathbf{P}^1} C$ is non-singular.

Lemma 3.16. *$p : \mathbf{F} \to C_K$ satisfies (4.2)(1).*

Proof. The tautological composition $C_K \to \mathbb{P}^1_K \to Q \times \mathbf{P}^1_{\mathbf{Q}}$ is dominant. The pencil $tF + GH = 0$ in $\mathbb{P}^{n+1}_K$ corresponds to a flat family of degree d hypersurfaces in $U \times \mathbb{P}^{n+1}_{\mathbf{Q}}$, where $U \subset Q \times \mathbf{P}^1_{\mathbf{Q}}$ is a non-empty Zariski open subset which contains the generic point η_∞ of $Q \times \infty$. This flat family is obtained from the universal family over $P_{d\ \mathbf{Q}}$ by pulling back with respect to a morphism, $\tau' : U \to P_{d\ \mathbf{Q}}$. The restriction of τ' to η_∞ gives rise to the hypersurface F_K. By (3.4), $\tau'(\eta_\infty) = \operatorname{Spec} \mathbb{Q}(P_d)$. It follows that τ' sends the generic point of U to $\operatorname{Spec} \mathbb{Q}(P_d)$. Now (3.2)(1) is immediate.

Step 6. We have now arranged that conditions (1)-(5) of (3.2) are fulfilled. To show that (3.2)(6) also holds we use an argument similar to the proof of (1.10). Let $\mathcal{S}$ denote the relative Hilbert scheme for lines in the fibers of $\mathcal{G}/P_{d_G\mathbf{Q}}$. This is a projective bundle over $Gr(\mathbb{P}^1,\mathbb{P}^{n+1})$, which as a set is the incidence correspondence $\{(G,\ell) \in P_{d_G} \times Gr(\mathbb{P}^1,\mathbb{P}^{n+1}) : \ell \subset G\}$. Let $K_1 = \mathbb{Q}(P_{d_H} \times \mathcal{S})$ and write G_{K_1} for the generic fiber of $P_H \times \mathcal{S} \times_{P_{d_G}} \mathcal{G}$. The universal family of lines in $\mathcal{S} \times_{P_{d_G}} \mathcal{G}$ pulls back to give a line $L_{1K_1} \subset G_{K_1}$. Write Spec K_2 for the generic point of $\mathcal{S}|_{K_1} :=$ Spec $K_1 \times_{P_{d_G}} \mathcal{S}$. There are two natural lines L_1 and L_2 on G_{K_2}. The first comes from $L_{1K_1} \subset G_{K_1}$ by base change and the second by restricting the univeral family of lines $\mathrm{l}_1 \subset \mathcal{S}|_{K_1} \times_{P_{d_G}} \mathcal{G}$ to the generic fiber. By construction, $Q \subset P_{d_H} \times \mathcal{S} \times_{P_{d_G}} \mathcal{S} \times P_d$. The projection $Q \to P_{d_H} \times \mathcal{S} \times_{P_{d_G}} \mathcal{S}$ sends the generic point Spec K to Spec K_2. Base changing G_{K_2}, L_1, L_2 by this map gives the lines (3.5) on G_K. Since H and hence G' are defined over K_2, (3.2)(6) will follow from (1.9) and

Claim 3.17. $N(L_1 - L_2) \neq 0$ in $CH_1(G'_{K_2})$ for any positive integer N.

Fix an embedding of $\mathcal{S}|_{K_1}$ in a projective space over K_1. Let K_3 be the field of definition of a generic linear space section of $\mathcal{S}|_{K_1}$ of dimension $n-2$. Denote this linear space section by T_{K_3}. Let $\mathbf{L}_2 \subset T_{K_3} \times G_{K_3}$ be the restriction of the universal family of lines on $\mathcal{S}|_{K_3} \times G_{K_3}$. Let $K_4 = K_3(T)$. Since $K_2 \subset K_4$ (3.16) follows from (1.9) and

Lemma 3.18. *For any positive integer N, $N(L_1 - L_2) \neq 0$ in $CH_1(G'_{K_4})$.*

Proof. As in the proof of (1.10), if $N(L_1 - L_2) = 0$ in $CH_1(G'_{K_4})$, then $N\mathbf{L}_2 \sim_{rat} \Gamma_1 + \Gamma_2 + \Gamma_3$, where each $\Gamma_i \in Z_{n-1}(T_{K_3} \times G_{K_3})$ and Γ_1 is supported on $D_{K_3} \times G_{K_3}$ with $D \subset T$ a divisor, Γ_2 is supported on $T_{K_3} \times L_{1\ K_3}$, and Γ_3 is supported on $T_{K_3} \times (G \cap H)_{K_3}$. By (2.6) and the Lefschetz hyperplane theorem

$$(N\mathbf{L}_2)_* : F^0H_{n-2}(T_{\mathbf{C}}) \to F^0(H_n(G_{\mathbf{C}})(-1))$$

is not zero for any $N \neq 0$. The proof of (1.10) shows however that Γ_{1*} and Γ_{2*} annihilate $F^0H_{n-2}(T_{\mathbf{C}})$. To show that Γ_{3*} also annihilates $F^0H_{n-2}(T_{\mathbf{C}})$, consider the commutative diagram

$$\begin{tikzcd} T \times (G \cap H) \arrow[r, "g"'] \arrow[d, "pr_{G\cap H}"'] & T \times G \arrow[d, "pr_G"] \\ G \cap H \arrow[r, "\iota"] & G \end{tikzcd}$$

With $\alpha \in H^{n-2,0}(T_{\mathbf{C}})$, $\beta \in H^{0,n-2}_{n-2}(T_{\mathbf{C}})$, and $\gamma_3 \in Z_{n-1}(T \times (G \cap H)) \otimes \mathbf{Q}$ essentially as in (1.12),

$$\Gamma_{3*}\beta = pr_{G*}([\Gamma_3] \cap pr_T^*\alpha) = pr_{G*}g_*([\gamma_3] \cap g^*pr_T^*\alpha) = \iota_*pr_{G\cap H*}([\gamma_3] \cap g^*pr_T^*\alpha).$$

Since $G \cap H$ is a non-singular complete intersection of dimension $n-1 > 1$ in projective space, $F^0(H_n(G \cap H)(-1))$ is zero. But $pr_{G\cap H*}(\gamma_3 \cap g^*pr_T^*\alpha) \in F^0(H_n(G \cap H)(-1))$. Thus $\Gamma_{3*} = 0$.

This contradiction proves (3.18). It follows that (3.2)(6) holds and thus the proof of (3.1) is complete.

Remark 3.19. If $n \geq 3$ and $X_{\mathbf{C}} \subset \mathbb{P}^{n+1}_{\mathbf{C}}$ is a geometric generic hypersurface of very high degree, then $CH_1(X_{\mathbf{C}})_{alg}$ remains mysterious. See [G-H] for further discussion.

4. A general result about $CH_m(X_{\mathbb{C}})_{alg}$.

Theorem 4.1. *Let $X'_{\mathbb{C}} \subset X_{\mathbb{C}}$ be a non-empty open subset of a complex projective variety. The group $CH_m(X'_{\mathbb{C}})_{alg}$ is isomorphic to 0 or has uncountable rank.*

The first step in the proof is

Lemma 4.2. *$CH_m(X'_{\mathbb{C}})_{tors}$ is a countable group.*

Proof. The inclusion $X' \subset X$ is defined over a countable, algebraicly closed subfield $k \subset \mathbb{C}$. The Hilbert scheme of X_k has countably many components, each with countably many k-rational points. Thus the group of m-cycles, $Z_m(X_k)$, is countable. Certainly $Z_m(X'_k)$ and $CH_m(X'_k)_{tors}$ must also be countable. According to [L], base change, $X'_{\mathbb{C}} \to X'_k$ induces an isomorphism $CH_m(X'_k)_{tors} \to CH_m(X'_{\mathbb{C}})_{tors}$.

Lemma 4.3. *The restriction map $r : CH_m(X_{\mathbb{C}})_{alg} \to CH_m(X'_{\mathbb{C}})_{alg}$ is surjective.*

Before proving the lemma we introduce some notation. Let T be a variety and let C be a smooth projective variety. Let $W \subset C \times T$ be a closed subscheme, flat over C of relative dimension m. Write $p_C : W \to C$ and $p_T : W \to T$ for the projections restricted to W. For each closed point $c \in |C|_0$ denote by $[W_c] \in Z_m(T)$ the cycle $p_{T*}(p_C^*(c))$. The image of $[W_c]$ in $CH_m(T)$ will be denoted $\langle W_c \rangle$.

Proof of 4.3. $CH_m(X'_{\mathbb{C}})_{alg}$ is generated by classes $\langle W'_{c_1} \rangle - \langle W'_{c_2} \rangle$, where $W' \subset C \times X'$ is a subvariety of dimension $m+1$, flat over a smooth projective irreducible curve, C. The closure W of W' in $C \times X$, taken with its reduced scheme structure is flat over C. Now $r(\langle W_{c_1} \rangle - \langle W_{c_2} \rangle) = \langle W'_{c_1} \rangle - \langle W'_{c_2} \rangle$.

For $\mathcal{V} \in CH_m(X_{\mathbb{C}})_{alg}/\operatorname{Ker} r$ define

$$R_{\mathcal{V}} = \{(c_1, c_2) \in |C \times C|_0 : \langle W_{c_1} \rangle - \langle W_{c_2} \rangle \in \mathcal{V}\}.$$

Lemma 4.4. *$R_{\mathcal{V}}$ is a countable union of closed sets.*

We assume (4.4) for the moment and deduce (4.1). If $CH_m(X'_{\mathbb{C}})_{alg} \neq 0$, then there exists a smooth projective curve C and subvariety $W \subset C \times X$, flat over C of relative dimension m, with $R_0 \neq |C \times C|_0$. By (4.4) R_0 is a countable union of proper closed subsets of $|C \times C|_0$. Also when $\mathcal{V} \neq 0$, $R_{\mathcal{V}} \neq |C \times C|_0$, because it does not meet the diagonal. By (4.4) $R_{\mathcal{V}}$ is a countable union of proper closed subsets of $|C \times C|_0$. Now

$$|C \times C|_0 = \cup_I R_{\mathcal{V}}, \quad \text{where} \qquad I = \{\mathcal{V} \in CH_m(X_{\mathbb{C}})_{alg}/\operatorname{Ker} r : R_{\mathcal{V}} \neq \emptyset\}$$

As $|C \times C|_0$ is not the union of countably many proper closed subsets by Baire's theorem [Na, Appendix], I is uncountable. By (4.2) the quotient of $CH_m(X'_{\mathbb{C}})_{alg}$ by its torsion subgroup is uncountable. Thus $CH_m(X'_{\mathbb{C}})_{alg}$ has uncountable rank.

The proof of (4.4) uses some facts about Chow varieties which we now recall. Fix an embedding $X \subset \mathbb{P}^N$. Write $\check{\mathbb{P}}^N$ for the dual projective space, set $\Xi = \prod_{i=1}^{N-m-1} \check{\mathbb{P}}^N$, and define $\mathbf{P}_d = \mathbb{P}H^0(\Xi, \otimes_{i=1}^{N-m-1} pr_i^* \mathcal{O}(d))$. The totality of all Chow forms for cycles of dimension m and degree d whose support is contained in X form a closed subset $Chow_m^d \subset \mathbf{P}_d$. The natural map

$$H^0(\Xi, \otimes_{i=1}^{N-m-1} pr_i^* \mathcal{O}(d_1)) \otimes H^0(\Xi, \otimes_{i=1}^{N-m-1} pr_i^* \mathcal{O}(d_2)) \to H^0(\Xi, \otimes_{i=1}^{N-m-1} pr_i^* \mathcal{O}(d_1 + d_2))$$

induces a continuous, closed map of algebraic sets

$$(4.5) \qquad Chow_m^{d_1} \times Chow_m^{d_2} \to Chow_m^{d_1+d_2},$$

which on the level of cycles sends (Z_1, Z_2) to $Z_1 + Z_2$. If $p(t)$ is an integral valued polynomial with leading term $dt^m/m!$, then Mumford [Mu2, §5.4] constructs a morphism of projective schemes $Hilb^p_{\mathbb{P}^N} \to \mathbf{P}_d$ which takes a geometric point of $Hilb^p_{\mathbb{P}^N}$ to the Chow form of the corresponding cycle. This gives rise to a continuous, closed map from the the subset underlying the closed subscheme $Hilb^p_X \subset Hilb^p_{\mathbb{P}^N}$ to $Chow^d_m$. Using this map and (4.5) we can describe all the maps we need.

Proof of 4.4. Let $T = X - X'$. There is a countable collection $\{U_j\}_{j\in\mathbf{N}}$ of finite type, smooth (not necessarily connected) projective schemes and closed subschemes $V_j \subset U_j \times T$, flat of relative dimension m over U_j such that

$$\cup_j\{\langle V_{j\ u_1}\rangle - \langle V_{j\ u_2}\rangle : u_1, u_2 \in U_j(\mathbb{C})\}$$

generates $\operatorname{Ker} r$. Since $\langle V_{j\ u_1}\rangle - \langle V_{j\ u_2}\rangle \in CH_m(X_{\mathbf{C}})_{alg}$, $\deg(V_{j\ u})$ is independent of the choice of $u \in U_j(\mathbb{C})$.

Let $(c_1, c_2) \in |C \times C|_0$. A rational equivalence between $[W_{c_1}] + [V_{j\ u_1}]$ and $[W_{c_2}] + [V_{j\ u_2}]$ is given by a collection of closed subschemes, $\Gamma_1, ..., \Gamma_r \subset \mathbb{P}^1 \times X$, which are flat of relative dimension m over $\mathbb{P}^1$ and satisfy

$$(4.6) \qquad \sum_{1\le i\le r} [\Gamma_{i0}] = [W_{c_1}] + [V_{j\ u_1}] + Z, \qquad \sum_{1\le i\le r} [\Gamma_{i\infty}] = [W_{c_2}] + [V_{j\ u_2}] + Z.$$

In other words, a rational equivalence results from a morphism of schemes,

$$(4.7) \qquad F : \mathbb{P}^1 \to \prod_{1\le i\le r} Hilb^{p_i}_X,$$

where p_i is the Hilbert polynomial for the fiber of Γ_i over $\mathbb{P}^1$.

For a fixed finite sequence of natural numbers $\vec{j} = (j_1, ..., j_s)$ let $d_{\vec{j}} = \sum_{i=1}^s \deg V_{j_i\ u_i}$. Write d_W for the degree of W_c, d for the sum of the degrees of the fibers of the Γ_i's, and set $d_0 = d - d_W - d_{\vec{j}}$. There are continuous, closed maps of algebraic sets

$$\prod_{1\le i\le r} Hilb^{p_i}_X \xrightarrow{\xi} Chow^d_m \xleftarrow{\phi_{\vec{j}}} C \times Chow^{d_0}_m \times \prod_{i=1}^s U_{j_i}$$

$$\phi_{\vec{j}}(c, Z, u_1, ..., u_s) = [W_c] + \sum_{i=1}^s [V_{j_i\ u_i}] + Z$$

By (4.6) we are interested in those F which satisfy

$$(4.8) \qquad (\xi(F(0)), \xi(F(\infty))) \in \phi_1 \times \phi_2(C \times C \times \Delta_{Chow^{d_0}_m} \times \prod_{i=1}^s U_{j_i}) \subset (Chow^d_m)^2.$$

For each integer N the set of morphisms (4.7) which satisfy (4.8) and

$$(4.9) \qquad deg(F^*\mathcal{O}(1)) \le N$$

is a closed subset of projective space, denoted $\Sigma_{N,p_1,\dots,p_r,\vec{j}}$. Thus the map

$$\Sigma_{N,p_1,\dots,p_r,\vec{j}} \xrightarrow{\psi} (Chow_m^d)^2, \qquad \psi(F) = (\xi(F(0)), \xi(F(\infty))).$$

is closed. The projection $pr_{C\times C} : (C \times Chow_m^{d_0} \times \prod_{i=1}^{s} U_{j_i})^2 \to C \times C$ is also closed. Hence

$$pr_{C\times C}((\phi_1 \times \phi_2)^{-1}(\psi(\Sigma_{N,p_1,\dots,p_r,\vec{j}})))$$

is a closed set. For each (c_1, c_2) in this set, $\langle W_{c_1}\rangle - \langle W_{c_2}\rangle \in \mathcal{V}$. The union over all tuples of Hilbert polynomials, $p_1, \dots, p_r$, over all $\vec{j}$, and over all N is $R_{\mathcal{V}}$. This proves (4.4).

5. 1-cycles on cubic hypersurfaces.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree 3 defined over an algebraicly closed field, k. If $n \leq 2$, $CH_1(X)_{alg} = 0$. If $n = 3$ and the characteristic of k is not 2, then $CH_1(X)_{alg}$ is naturally isomorphic to the k-rational points of an abelian variety (see [Mur] and and use the divisiblity of $CH_1(X)_{alg}$). If $n = 4$, $CH_1(X_{\mathbb{C}})_{alg}$ is not representable (0.5). The purpose of this section is to prove

Theorem 5.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree 3 defined over an algebraicly closed subfield of $\mathbb{C}$. If $n \geq 6$, then $CH_1(X)_{alg} = 0$.*

In preparation for the proof of the theorem we recall some facts about singular cubic hypersurfaces. Suppose first that X has an isolated singular point, p_0, of multiplicity 2. The intersection of X with the tangent cone to X at p_0 is a cone over a complete intersection, F, of multi-degree (2,3) in $\mathbb{P}^n$. Projection from p_0 induces a birational morphsim, $\phi : X - p_0 \to \mathbb{P}^n$. The inverse map is given by the linear system of cubics in $\mathbb{P}^n$ through F. These cubics generate the ideal sheaf of F. Thus ϕ^{-1} is the blow up of F followed by contracting the unique quadric containing F to the singular point p_0. The behaviour of Chow groups under a monoidal transformation with center a complete intersection is well understood [F, 6.7, 3.3b]. Since $CH_0(F)_{hom} = 0$ when $n \geq 5$ [R3, Thm 4.2], one deduces easily that $CH_1(X) \simeq \mathbb{Z}$ with a line through p_0 as generator.

Proof of 5.1. Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of dimension $n \geq 6$. We shall assume that the base field is the complex numbers. The general case follows from this special case by the injectivity of the pull back map $CH_1(X)_{alg} \to CH_1(X_{\mathbb{C}})_{alg}$ (1.9) and [L]. Write $\check{X} \subset \check{\mathbb{P}}^{n+1}$ for the dual hypersurface in the dual projective space and $I = \{(x, H) \in X \times \check{X} : x \in H\}$ for the total space of the family of singular hyperplane sections of X. It is known that $\check{X}$ is a hypersurface in $\check{\mathbb{P}}^{n+1}$ and that non-singular points correspond to hyperplane sections with exactly one isolated ordinary double point. In fact the locus of hyperplane sections with only isolated double point singularities is an open subset $\dot{X} \subset \check{X}$ and the complement has codimension at least 2. Let $\bar{C} \subset \dot{X}$ be a complete, irreducible curve, with normalization $\nu : C \to \bar{C}$. Define $Y = C \times_{\check{X}} I$. There are tautological maps

$$\begin{array}{ccc} Y & \xrightarrow[q]{} & X \\ {\scriptstyle p}\downarrow & & \\ C & & \end{array}$$

with p flat and q projective and surjective. Let η denote the generic point of C. There is a short exact sequence

$$\oplus_{c\in|C|_0} CH_1(Y_c) \xrightarrow{\oplus i_{c*}} CH_1(Y) \to CH_0(Y_\eta) \to 0. \tag{5.2}$$

Apply Roitman's Theorem [R3, Thm 4.2] and (1.9) to the cubic hypersurface, Y_η to conclude that $CH_0(Y_\eta)$ has rank 1. The image of $q_* \circ (\oplus i_{c*})$ is generated by lines.

Lemma 5.3. *All lines on a smooth cubic hypersurface $X_{\mathbf{C}} \subset \mathbb{P}_{\mathbf{C}}^{n+1}$ of dimension $n \geq 5$ are rationally equivalent.*

Proof. By (2.3) and [A-K, Prop. 1.8] the parameter space of lines on $X_{\mathbf{C}}$ is a smooth projective variety whose anti-canonical bundle is ample. The assertion now follows from [Ca].

It follows that $CH_1(X_{\mathbf{C}})$ has finite rank. By (4.1) $CH_1(X_{\mathbf{C}})_{alg} = 0$.

Remark 5.4. The question as to whether $CH_1(X_{\mathbf{C}})_{alg}$ is representable when $n = 5$ remains open. Since $CH_1(X_{\mathbf{C}})_{alg} = 0$ for cubic 5-folds with one ordinary double point, one is tempted to suspect that $CH_1(X_{\mathbf{C}})_{alg} = 0$ might hold for smooth cubic 5-folds as well.

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