A class of real-analytic surfaces in the 3-Euclidean space

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Abstract. A smooth surface S in $\mathbf{R}^3$ is called *parallel curved* if there exists a plane in $\mathbf{R}^3$ parallel to a principal direction at each point of S. In this paper, the classification of the connected, complete, real-analytic, embedded, parallel curved surfaces is obtained and it is seen that the index of an isolated umbilical point on a real-analytic, parallel curved surface is equal to one or zero.

1. Introduction

Let S be a smooth surface in the 3-dimensional Euclidean space $\mathbf{R}^3$ and Π_S the set of the planes in $\mathbf{R}^3$ parallel to a principal direction at each point of S. If $\Pi_S \neq \emptyset$, then S is called *parallel curved*. For example, a plane, a cylinder and a round sphere are parallel curved. More generally, a surface of revolution is also parallel curved. The main purposes of this paper are

(1) to give the classification of the connected, complete, real-analytic, embedded, parallel curved surfaces;
(2) to study *the index of an isolated umbilical point*(see [3, pp. 137]) on a real-analytic, parallel curved surface.

A pair $\{C_1, C_2\}$ of two simple, real-analytic curves in $\mathbf{R}^3$ is called *generating* if the following hold:

(1) There exists a point $p(C_1, C_2) \in \mathbf{R}^3$ satisfying $C_1 \cap C_2 = \{p(C_1, C_2)\}$;
(2) There exist planes P_1 and P_2 such that
    (a) P_1 and P_2 are perpendicular to each other,
    (b) for $i = 1, 2$, $C_i \subset P_i$ holds.

We shall prove the following in Section 4 of this paper.

Proposition 1.1. *For a generating pair $\{C_1, C_2\}$, there exists a parallel curved surface S_0 containing a neighborhood of $p(C_1, C_2)$ in $C_1 \cup C_2$ and satisfying $P_1 \in \Pi_{S_0}$. In addition, if S_0' and S_0'' are such surfaces, then $S_0' \cap S_0''$ is also such a surface.*

The maximal surface of such surfaces as in Proposition 1.1 is denoted by S_{C_1,C_2}. We shall prove the following in Section 6 of this paper.

Theorem 1.2. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface in $\mathbf{R}^3$. Then S is homeomorphic to a sphere, a plane, a cylinder, or a torus. In addition,*

(1) *if S is homeomorphic to a sphere, then S is a surface of revolution which crosses its axis of rotation at just two points*;

(2) *if S is homeomorphic to a plane, then one of the following holds*:
  (a) *S is a surface of revolution which crosses its axis of rotation at just one point,*
  (b) *$S = S_{C_1,C_2}$ holds, where $\{C_1, C_2\}$ is a generating pair each element of which is isometric to* $\mathbf{R}$;

(3) *if S is homeomorphic to a cylinder, then $S = S_{C_1,C_2}$ holds, where $\{C_1, C_2\}$ is a generating pair such that one of C_1 and C_2 is isometric to $\mathbf{R}$ and the other isometric to a simple closed curve*;

(4) *if S is homeomorphic to a torus, then $S = S_{C_1,C_2}$ holds, where $\{C_1, C_2\}$ is a generating pair each element of which is isometric to a simple closed curve.*

Let S be a smooth surface in $\mathbf{R}^3$ with an isolated umbilical point p_0 and $\mathrm{ind}_{p_0}(S)$ the index of p_0 on S. It is known that if S is a surface with constant mean curvature, then $\mathrm{ind}_{p_0}(S) < 0$ holds([3. pp. 139]). More generally, if S is a Weingarten surface, then the same result is obtained([2]). In [1], we have obtained $\mathrm{ind}_o(\mathsf{G}_\varphi) \in \{1+i-k/2\}_{i=0}^{[k/2]}$, where φ is a homogeneous polynomial in two real variables of degree $k \geqq 3$ such that on the graph G_φ of φ, the origin $o := (0, 0, 0)$ of $\mathbf{R}^3$ is an isolated umbilical point. In addition, we have proved in [1] that each number of $\{1 + i - k/2\}_{i=0}^{[k/2]}$ may be the index of o on the graph of a homogeneous polynomial of degree k. In Section 7 of this paper, we shall prove the following.

Theorem 1.3. *Let S be a real-analytic, parallel curved surface in $\mathbf{R}^3$ with an isolated umbilical point p_0. Then $\mathrm{ind}_{p_0}(S)$ is equal to one or zero. In addition,*

(1) *if $\mathrm{ind}_{p_0}(S) = 1$, then the tangent plane $T_{p_0}(S)$ at p_0 is an element of Π_S and S is part of a surface of revolution such that p_0 lies on the axis of rotation,*

(2) *if $\mathrm{ind}_{p_0}(S) = 0$, then $T_{p_0}(S) \notin \Pi_S$ holds.*

Remark 1.4. Theorem 1.3 does not give any negative answer to Loewner's conjecture([4]).

2. Preliminaries

Let f be a smooth function of two real variables x, y and G_f the graph of f. We may consider the natural coordinates (x, y) on the xy-plane as coordinates on G_f. We set $p_f := \dfrac{\partial f}{\partial x}, q_f := \dfrac{\partial f}{\partial y}$, and

$$E_f := 1 + p_f^2, \qquad F_f := p_f q_f, \qquad G_f := 1 + q_f^2.$$

The first fundamental form of G_f is the symmetric tensor field I_f on G_f of type $(0, 2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{I}_f := E_f dx^2 + 2F_f dx dy + G_f dy^2,$$

where

$$dx^2 := dx \otimes dx, \quad dxdy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

We set $r_f := \dfrac{\partial^2 f}{\partial x^2}, s_f := \dfrac{\partial^2 f}{\partial x \partial y}, t_f := \dfrac{\partial^2 f}{\partial y^2}$ and

$$L_f := \frac{r_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathrm{I}_f)}},$$

where $\det(\mathrm{I}_f) := E_f G_f - F_f^2$. *The second fundamental form of* G_f is the symmetric tensor field II_f on G_f of type $(0,2)$ represented in terms of the coordinates (x,y) as

$$\mathrm{II}_f := L_f dx^2 + 2M_f dxdy + N_f dy^2.$$

Let $T_{(x_0,y_0)}(\mathsf{G}_f)$ be the tangent plane to G_f at (x_0,y_0) and $U_{(x_0,y_0)}(\mathsf{G}_f)$ the subset of $T_{(x_0,y_0)}(\mathsf{G}_f)$ defined by

$$U_{(x_0,y_0)}(\mathsf{G}_f) := \left\{\mathbf{u} \in T_{(x_0,y_0)}(\mathsf{G}_f) \ ; \ \mathrm{I}_{f,(x_0,y_0)}(\mathbf{u},\mathbf{u}) = 1\right\},$$

and $L_{(x_0,y_0)}(\mathsf{G}_f)$ the set of the one-dimensional subspaces of $T_{(x_0,y_0)}(\mathsf{G}_f)$. Let $\nu_{f,(x_0,y_0)}$ be the function on $U_{(x_0,y_0)}(\mathsf{G}_f)$ defined by

$$\nu_{f,(x_0,y_0)}(\mathbf{u}) := \mathrm{II}_{f,(x_0,y_0)}(\mathbf{u},\mathbf{u})$$

for any $\mathbf{u} \in U_{(x_0,y_0)}(\mathsf{G}_f)$. Suppose that $\nu_{f,(x_0,y_0)}$ attains an extremum at $\mathbf{u}_0 \in U_{(x_0,y_0)}(\mathsf{G}_f)$. Then the extremum $\nu_{f,(x_0,y_0)}(\mathbf{u}_0)$ is called *a principal curvature of* G_f *at* (x_0,y_0) and the element of $L_{(x_0,y_0)}(\mathsf{G}_f)$ determined by $\mathbf{u}_0$ is called *a principal direction of* G_f *at* (x_0,y_0). The set of the principal directions at (x_0,y_0) is represented by $PD_{(x_0,y_0)}(\mathsf{G}_f)$. *The Weingarten map of* G_f is the tensor field W_f on G_f of type $(1,1)$ satisfying

$$\left[\mathrm{W}_f\left(\frac{\partial}{\partial x}\right), \mathrm{W}_f\left(\frac{\partial}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] W_f,$$

where

$$W_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

By Lagrange's method of indeterminate coefficients, we obtain

Proposition 2.1. *The principal curvatures and the principal directions of* G_f *at* (x_0,y_0) *are given by the eigenvalues and the eigenspaces of* $\mathrm{W}_{f,(x_0,y_0)}$, *respectively.*

Gaussian curvature $K_f(x_0,y_0)$ *of* G_f *at* (x_0,y_0) is given by $K_f(x_0,y_0) := \det\left(\mathrm{W}_{f,(x_0,y_0)}\right)$. We obtain

$$K_f = \frac{\det(\mathrm{II}_f)}{\det(\mathrm{I}_f)},$$

where $\det(\mathrm{II}_f) := L_f N_f - M_f^2$.

Let PD_f be the symmetric tensor field on G_f of type $(0,2)$ represented in terms of the coordinates (x,y) as

$$\mathrm{PD}_f := \frac{1}{\sqrt{\det(\mathrm{I}_f)}}\{A_f dx^2 + B_f dxdy + C_f dy^2\},$$

where

$$A_f := F_f L_f - E_f M_f, \quad B_f := G_f L_f - E_f N_f, \quad C_f := G_f M_f - F_f N_f.$$

For $\mathbf{v}_1$, $\mathbf{v}_2 \in T_{(x_0,y_0)}(\mathsf{G}_f)$, the following holds:

$$\frac{1}{2}\{\mathrm{W}_{f,(x_0,y_0)}(\mathbf{v}_1) \wedge \mathbf{v}_2 + \mathrm{W}_{f,(x_0,y_0)}(\mathbf{v}_2) \wedge \mathbf{v}_1\}$$
$$= \mathrm{PD}_{f,(x_0,y_0)}(\mathbf{v}_1, \mathbf{v}_2)\frac{1}{\sqrt{\det\left(\mathrm{I}_{f,(x_0,y_0)}\right)}}\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)_{(x_0,y_0)}.$$

Therefore from Proposition2.1, we obtain

Proposition 2.2. *A tangent vector* $\mathbf{v}_0 \in T_{(x_0,y_0)}(\mathsf{G}_f)$ *is in an element of* $PD_{(x_0,y_0)}(\mathsf{G}_f)$ *if and only if* $\mathrm{PD}_{f,(x_0,y_0)}(\mathbf{v}_0,\ \mathbf{v}_0) = 0$ *holds.*

A point (x_0, y_0) is called *an umbilical point of* G_f if $\nu_{f,(x_0,y_0)}$ is a constant function. By Proposition 2.2, we obtain

Proposition 2.3. *The following are mutually equivalent*:

(1) (x_0, y_0) *is an umbilical point of* G_f,
(2) $PD_{(x_0,y_0)}(\mathsf{G}_f) = L_{(x_0,y_0)}(\mathsf{G}_f)$,
(3) $A_f(x_0, y_0) = B_f(x_0, y_0) = C_f(x_0, y_0) = 0$.

3. A parallel curved surface, I

Let $Pl(\mathbf{R}^3)$ be the set of the planes in $\mathbf{R}^3$ and $L(\mathbf{R}^3)$ the set of the lines in $\mathbf{R}^3$. For $X_1, X_2 \in Pl(\mathbf{R}^3) \sqcup L(\mathbf{R}^3)$, $X_1 \parallel X_2$ means that X_1 and X_2 are parallel in $\mathbf{R}^3$ to each other, and $X_1 \perp X_2$ means that X_1 and X_2 are perpendicular in $\mathbf{R}^3$ to each other. Let S be a connected, real-analytic, embedded, parallel curved surface in $\mathbf{R}^3$ and Π_S the set of the elements of $Pl(\mathbf{R}^3)$ parallel to a principal direction at each point of S. For $P \in \Pi_S$, we set

$$\Xi_{S,P} := \{p \in S \ ;\ T_p(S) \nparallel P\}.$$

Suppose that there exists an element P_0 of Π_S satisfying $\Xi_{S,P_0} = \emptyset$. Then we see that S is part of an element of $Pl(\mathbf{R}^3)$. In the following, suppose that for any $P \in \Pi_S$, $\Xi_{S,P} \neq \emptyset$ holds. Then $\Xi_{S,P}$ is an open set of S and therefore $\Xi_{S,P}$ is also a surface in $\mathbf{R}^3$.

For $P \in \Pi_S$ and for $p \in \Xi_{S,P}$, let $P_p^{\perp}(S,P)$ be the element of $Pl(\mathbf{R}^3)$ satisfying $p \in P_p^{\perp}(S,P)$, $P_p^{\perp}(S,P) \perp P$ and $P_p^{\perp}(S,P) \perp T_p(\Xi_{S,P})$, and $l_p(S,P)$ the connected component of $\Xi_{S,P} \cap P_p^{\perp}(S,P)$ containing p. We shall prove

Proposition 3.1. *The following hold*:

(1) *The set* $l_p(S,P)$ *is a real-analytic curve;*
(2) *For any* $q \in l_p(S,P)$*, the tangent line* $T_q(l_p(S,P))$ *to* $l_p(S,P)$ *at* q *satisfies* $T_q(l_p(S,P)) \in PD_q(\Xi_{S,P})$ *and* $T_q(l_p(S,P)) \nparallel P$*;*
(3) *For* $q \in l_p(S,P)$*, let* $T_q^{\perp}(l_p(S,P))$ *be the element of* $PD_q(\Xi_{S,P})$ *satisfying* $T_q^{\perp}(l_p(S,P)) \parallel P$*. Then for* $q_1, q_2 \in l_p(S,P)$*,* $T_{q_1}^{\perp}(l_p(S,P)) \parallel T_{q_2}^{\perp}(l_p(S,P))$ *holds.*

Proof. For orthogonal coordinates (x,y,z) on $\mathbf{R}^3$, the x-axis(resp. the y-axis, the z-axis) and the xy-plane(resp. the yz-plane, the xz-plane) are denoted by A_x(resp. A_y, A_z) and by P_{xy}(resp. P_{yz}, P_{xz}), respectively. Suppose that (x,y,z) satisfies $p = o$, $A_x \subset T_o(S)$ and $P_{xz} \parallel P$. Then we obtain $A_z \cap T_o(S) = \{o\}$. Therefore there exist two positive numbers $x_0, y_0 > 0$ and a real-analytic function f on a neighborhood $U_{x_0,y_0} := (-x_0, x_0) \times (-y_0, y_0)$ of o in P_{xy} such that G_f is a neighborhood of o in $\Xi_{S,P}$. The function f satisfies $f(0,0) = p_f(0,0) = 0$. We see that at each point of G_f, the tangent vector $\dfrac{\partial}{\partial x}$ is in a principal direction. Then from Proposition 2.2 we obtain

$$s_f\left(1 + p_f^2\right) = p_f q_f r_f \tag{1}$$

on U_{x_0,y_0}. We may represent f as

$$f(x,y) := \sum_{i,j=0}^{\infty} a_{ij} x^i y^j, \tag{2}$$

where $a_{ij} \in\mathbf{R}$ and where $a_{00} = a_{10} = 0$. Then at $(0,y) \in U_{x_0,y_0}$, we may rewrite (1) into

$$\begin{aligned} &\left(\sum_{j=0}^{\infty}(j+1)a_{1j+1}y^j\right) \times \left(1 + \left(\sum_{j=0}^{\infty} a_{1j}y^j\right)^2\right) \\ &= 2\left(\sum_{j=0}^{\infty} a_{1j}y^j\right) \times \left(\sum_{j=0}^{\infty}(j+1)a_{0j+1}y^j\right) \times \left(\sum_{j=0}^{\infty} a_{2j}y^j\right). \end{aligned} \tag{3}$$

Since $a_{10} = 0$, we obtain $a_{11} = 0$. Generally, by (3) we see that if each element of $\{a_{1k}\}_{k=0}^{j-1}$ for a natural number $j \in\mathbf{N}$ is equal to zero, then a_{1j} is also equal to zero. Therefore we obtain $a_{1j} = 0$ for any $j \in\mathbf{N}\cup\{0\}$. Then for any $y \in (-y_0, y_0)$, $p_f(0,y) = 0$ holds. This implies $T_{(0,y)}(\mathsf{G}_f) \parallel A_x$. Hence we obtain (3) of Proposition 3.1. Noticing $P_o^{\perp}(S,P) = P_{yz}$, we see that for any $y \in (-y_0, y_0)$, $T_{(0,y)}(\mathsf{G}_f) \perp P_o^{\perp}(S,P)$ holds. Therefore we obtain (1) of Proposition 3.1. Since $p_f(0,y) = s_f(0,y) = 0$ hold for any $y \in (-y_0, y_0)$, we see by Proposition 2.2 that $\left(\dfrac{\partial}{\partial y}\right)_{(0,y)}$ is in an element of $PD_{(0,y)}(\mathsf{G}_f)$. Then we obtain (2) of Proposition 3.1. $\square$

By (2) of Proposition 3.1, we obtain

Corollary 3.2. *Let f be a real-analytic function on an open set U of $\mathbf{R}^2$ such that G_f is a parallel curved surface satisfying $P_{xy} \in \Pi_{\mathsf{G}_f}$. Then $A_f = -C_f$ holds at each point of U.*

4. A generating pair

Let P be a plane in $\mathbf{R}^3$ and Γ_P the set of the simple, real-analytic curves in P, and for $C \in \Gamma_P$ and for a point $p \in C$, let $L_p^{\perp}(C)$ be the line in P normal to C at p. Then for $C, \tilde{C} \in \Gamma_P$, we write $C \sim \tilde{C}$ if and only if there exists a continuous bijective map $\alpha_{C,\tilde{C}}$ from C onto $\tilde{C}$ satisfying $L^{\perp}_{\alpha_{C,\tilde{C}}(p)}(\tilde{C}) = L_p^{\perp}(C)$ for any $p \in C$. It is seen that $\sim$ is an equivalence relation in Γ_P. We denote by Γ_C the equivalence class of $C \in \Gamma_P$, and by $\mathrm{Reg}(C)$ the connected component of the set $\bigcup_{C' \in \Gamma_C} C'$ containing C. We immediately obtain

Lemma 4.1. *For $C \in \Gamma_P$ and for each $p \in C$, there exists a neighborhood U_p of p in C such that $\mathrm{Reg}(U_p)$ is a domain of P .*

We denote by Proj_P the map from $\mathbf{R}^3$ onto P such that if $l \in L(\mathbf{R}^3)$ satisfies $l \perp P$, then $\mathrm{Proj}_P(l)$ consists of the only one point of $l \cap P$.

Proof of Proposition 1.1. Let $\{C_1, C_2\}$ be a generating pair and P_1, P_2 planes satisfying $P_1 \perp P_2$ and $C_i \subset P_i$ for $i = 1, 2$, and P_3 the plane satisfying $P_1 \perp P_3$, $P_2 \perp P_3$ and $p(C_1, C_2) \in P_3$. Then by Lemma 4.1, we see that for $i = 1, 2$, there exists a neighborhood U_i of $p(C_1, C_2)$ in C_i such that

(1) $\mathrm{Proj}_{P_1}(U_2) \subset \mathrm{Reg}(U_1)$ holds,
(2) Proj_{P_3} embeds each connected component of $U_2 \setminus \{p(C_1, C_2)\}$ into P_3.

Then there exists the only one surface S_0 such that if $P \in Pl(\mathbf{R}^3)$ satisfies $P \parallel P_1$ and $P \cap U_2 \neq \emptyset$, then each connected component of $\mathrm{Proj}_{P_1}(S_0 \cap P)$ is an element of Γ_{U_1}. For each $p \in S_0 \backslash U_1$, let (x, y, z) be orthogonal coordinates on $\mathbf{R}^3$ satisfying $p = o$, $A_x \subset T_o(S_0)$ and $P_{xz} \parallel P_1$. Then referring to the proof of Proposition 3.1, we obtain $A_x \subset PD_o(S_0)$. Therefore we see that S_0 is a parallel curved surface containing $U_1 \cup U_2$ and satisfying $P_1 \in \Pi_{S_0}$. It is clear that if S_0' and S_0'' are parallel curved surfaces containing a neighborhood of $p(C_1, C_2)$ in $C_1 \cup C_2$ and satisfying $P_1 \in \Pi_{S_0}$, then $S_0' \cap S_0''$ is also such a surface. Hence we have proved Proposition 1.1. □

5. A parallel curved surface, II

Suppose that for $P \in \Pi_S$, there exists a point p_0 of $S \setminus \Xi_{S,P}$. The purpose of this section is to prove the following two propositions.

Proposition 5.1. *Suppose that $T_{p_0}(S) \cap V_{p_0} \neq \{p_0\}$ holds for any neighborhood V_{p_0} of p_0 in S. Then the following hold:*

(1) *There exist a neighborhood* U_{p_0} *of* p_0 *in* S *and a real-analytic curve* C_{p_0} *in* U_{p_0} *satisfying* $T_{p_0}(S) \cap U_{p_0} = C_{p_0}$;

(2) *For* $p \in C_{p_0}$, *let* $P_p^\perp(C_{p_0})$ *be the plane normal to* C_{p_0} *at* p. *Then there exists a neighborhood* U_p *of* p *in* S *such that*

 (a) $P_p^\perp(C_{p_0}) \cap U_p$ *is a real-analytic curve,*

 (b) *for each point* q *of* $P_p^\perp(C_{p_0}) \cap U_p$, $T_q(S) \perp P_p^\perp(C_{p_0})$ *holds.*

Proposition 5.2. *If for a neighborhood* V_{p_0} *of* p_0 *in* S, $T_{p_0}(S) \cap V_{p_0} = \{p_0\}$ *holds, then* S *is part of a surface of revolution such that* p_0 *lies on its axis of rotation.*

Let (x, y, z) be orthogonal coordinates in $\mathbf{R}^3$ satisfying $p_0 = o$ and $T_o(S) = P_{xy}$. Then there exist two positive numbers $x_0, y_0 > 0$ and a real-analytic function f on a neighborhood U_{x_0,y_0} of o in P_{xy} such that G_f is a neighborhood of o in S. At each point of U_{x_0,y_0}, the following holds:

$$\mathrm{PD}_f(\mathbf{Grad}_f^\perp, \mathbf{Grad}_f^\perp) = 0, \tag{4}$$

where $\mathbf{Grad}_f^\perp$ is the vector field on G_f defined by $\mathbf{Grad}_f^\perp := -q_f \dfrac{\partial}{\partial x} + p_f \dfrac{\partial}{\partial y}$. We set

$$\mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}, \quad \mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \mathrm{grad}_f^\perp := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}.$$

Let $\langle\ ,\ \rangle$ be the scalar product in $\mathbf{R}^2$. Then we see that (4) holds at each point if and only if f is a solution of the following differential equation:

$$\left\langle \mathrm{Hess}_f \, \mathrm{grad}_f, \mathrm{grad}_f^\perp \right\rangle = 0. \tag{5}$$

For any $\phi \in \mathbf{R}$, we set $u_\phi(x, y) := (\cos\phi)x + (\sin\phi)y$. A function f is said to be *of one-variable* if there exist a number $\phi_0 \in \mathbf{R}$ and a real-analytic function $F_f^{(1)}$ on a neighborhood of 0 in $\mathbf{R}$ such that $f(x, y) = F_f^{(1)}(u_{\phi_0}(x, y))$ holds on U_{x_0,y_0}. A function f is said to be *radial* if there exists a real-analytic function $F_f^{(2)}$ on a neighborhood of 0 in $\mathbf{R}$ such that $f(x, y) = F_f^{(2)}(r^2)$ holds on U_{x_0,y_0}, where $r := \sqrt{x^2 + y^2}$. We shall prove

Lemma 5.3. *Let* φ *be a homogeneous polynomial. Then* φ *is a solution of equation* (5) *if and only if* φ *is of one-variable or radial.*

Proof. For a homogeneous polynomial φ, the following holds:

$$\left\langle \mathrm{Hess}_\varphi \, \mathrm{grad}_\varphi, \mathrm{grad}_\varphi^\perp \right\rangle = \frac{\det(\mathrm{Hess}_\varphi)}{\deg(\varphi) - 1}(x q_\varphi - y p_\varphi).$$

Since $\det(\mathrm{Hess}_\varphi) \equiv 0$(resp. $x q_\varphi \equiv y p_\varphi$) holds if and only if φ is of one-variable(resp. radial), we obtain Lemma 5.3. $\square$

For homogeneous polynomials $\varphi_1, \varphi_2, \varphi_3$, we set

$$F_{\varphi_1,\varphi_2,\varphi_3} := \left\langle \mathrm{Hess}_{\varphi_1} \mathrm{grad}_{\varphi_2},\ \mathrm{grad}^{\perp}_{\varphi_3} \right\rangle,$$
$$\Phi_{\varphi_1,\varphi_2,\varphi_3} := \sum_{\{j_1,j_2,j_3\}=\{1,2,3\}} F_{\varphi_{j_1},\varphi_{j_2},\varphi_{j_3}}.$$

We shall prove

Lemma 5.4. *Suppose that φ_1 and φ_2 are radial. Then φ_3 is also radial if and only if $\Phi_{\varphi_1,\varphi_2,\varphi_3} \equiv 0$ holds.*

Proof. If we set $\varphi_j = r^{2k_j}$ for $j = 1, 2$, then we obtain

$$\begin{aligned}
F_{\varphi_{j_1},\varphi_{j_2},\varphi_3} &= -4k_1k_2(2k_{j_1} - 1)r^{2(k_1+k_2-2)}(xq_{\varphi_3} - yp_{\varphi_3}),\\
F_{\varphi_{j_1},\varphi_3,\varphi_{j_2}} &= 4k_1k_2r^{2(k_1+k_2-2)}(xq_{\varphi_3} - yp_{\varphi_3}),\\
F_{\varphi_3,\varphi_{j_1},\varphi_{j_2}} &= 4k_1k_2(\deg(\varphi_3) - 1)r^{2(k_1+k_2-2)}(xq_{\varphi_3} - yp_{\varphi_3}),
\end{aligned}$$

where $\{j_1, j_2\} = \{1, 2\}$. Therefore we see that $\Phi_{\varphi_1,\varphi_2,\varphi_3} \equiv 0$ holds if and only if φ_3 is radial. $\square$

We may represent f as $f(x, y) := \sum_{i \geqq 2} f^{(i)}(x, y)$, where

$$f^{(i)}(x, y) := \frac{1}{i!} \sum_{j=0}^{i} {}_iC_j \frac{\partial^i f}{\partial x^j \partial y^{i-j}}(0, 0)x^j y^{i-j}.$$

We call $f^{(i)}$ *the i-th component of f*. We set

$$I_f := \left\{ i_0 \in \mathbf{N}\ ;\ f^{(i_0)} \neq 0 \right\}, \qquad \deg(f) := \min I_f.$$

Then we may represent $\left\langle \mathrm{Hess}_f \mathrm{grad}_f, \mathrm{grad}^{\perp}_f \right\rangle$ as

$$\left\langle \mathrm{Hess}_f \mathrm{grad}_f, \mathrm{grad}^{\perp}_f \right\rangle = \sum_{j_1,j_2,j_3 \in I_f} F_{f^{(j_1)},f^{(j_2)},f^{(j_3)}}.$$

We see that the $(3\deg(f) - 4)$-th component of $\left\langle \mathrm{Hess}_f \mathrm{grad}_f, \mathrm{grad}^{\perp}_f \right\rangle$ is given by $F_{f^{(\deg(f))},f^{(\deg(f))},f^{(\deg(f))}}$. Therefore by Lemma 5.3, we see that if f satisfies (5), then $f^{(\deg(f))}$ is of one-variable or radial. If $I_f = \{\deg(f)\}$, then we obtain Proposition 5.1 and Proposition 5.2.

Suppose that $I_f \neq \{\deg(f)\}$ and that I_f is a finite set. Then set $n := \sharp I_f$ and let $i_1, \dots, i_n$ be the integers satisfying $i_1 < \cdots < i_n$ and $I_f = \{i_j\}_{j=1}^n$. If $f^{(i_1)}, \dots, f^{(i_j)}$ are radial for $j \in \{1, \dots, n-1\}$, then we see by Lemma 5.4 that $f^{(i_{j+1})}$ is also radial. Therefore we see that if $f^{(\deg(f))}$ is radial, then f is also radial. If I_f is an infinite set, then we obtain the same result. Hence if $f^{(\deg(f))}$ is radial, then $T_o(S) \cap \mathsf{G}_f = \{o\}$ holds and S is part of a surfece of revolution such that o lies on its axis of rotation.

Suppose that $I_f \neq \{\deg(f)\}$ and that $f^{(\deg(f))}$ is of one-variable. Then we may suppose that $f^{(\deg(f))}(x, y) = x^{\deg(f)}$. We shall prove

Lemma 5.5. *Suppose that f is not of one-variable. Then for any $(u_0, v_0) \in \mathsf{G}_f$, there exists the only one element $P_{(u_0,v_0)} \in Pl(\mathbf{R}^3)$ such that*

(1) $(u_0, v_0) \in P_{(u_0,v_0)}$,
(2) $P_{(u_0,v_0)} \perp P_{xy}$,
(3) *for each point $(u, v) \in$ of $P_{(u_0,v_0)} \cap \mathsf{G}_f$, $P_{(u_0,v_0)} \perp T_{(u,v)}(\mathsf{G}_f)$ holds.*

Proof. We represent f as in (2). Then by (5), we obtain $a_{n1} = 0$ for any $n \in \mathbf{N} \cup \{0\}$. Therefore we see that for $(0,0) \in U_{x_0,y_0}$, P_{xz} satisfies $(1) \sim (3)$ in Lemma 5.5. Then noticing Proposition 3.1, we see that for any $(u_0, v_0) \in U_{x_0,y_0}$, there exists an element $P \in Pl(\mathbf{R}^3)$ satisfying $(1) \sim (3)$ in Lemma 5.5. In addition, we see that at any point $(u_0, v_0) \in \mathsf{G}_f \cap \Xi_{S,P}$, $P^{\perp}_{(u_0,v_0)}(\mathsf{G}_f, P_{xy})$ is the only one element of $Pl(\mathbf{R}^3)$ satisfying $(1) \sim (3)$ in Lemma 5.5. Therefore we have only to show that P_{xz} is the only one element of $Pl(\mathbf{R}^3)$ satisfying $(1) \sim (3)$ in Lemma 5.5 at $(0,0)$. We easily see that for any $\phi \in (-\pi/2, \pi/2) \setminus \{0\}$, the element of $Pl(\mathbf{R}^3)$ determined by $u_\phi(x, y) = 0$ does not satisfy $(1) \sim (3)$ in Lemma 5.5 at $(0,0)$. Suppose that P_{yz} satisfies $(1) \sim (3)$ in Lemma 5.5 at $(0,0)$. Then for any $(u_0, v_0) \in \mathsf{G}_f \cap \Xi_{S,P}$, $P_{(u_0,v_0)} \cap \mathsf{G}_f$ intersects $\mathsf{G}_f \cap P_{xz}$ or $\mathsf{G}_f \cap P_{yz}$ orthogonally. Therefore $P_{(u_0,v_0)} \parallel P_{xz}$ or $P_{(u_0,v_0)} \parallel P_{yz}$ holds and each of $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ is in a principal direction at each point of G_f. Then we see that f is of one-variable. Hence we have proved Lemma 5.5. □

We may suppose that for any $u_0 \in (-x_0, x_0) \setminus \{0\}$, $(u_0, 0) \in \Xi_{S,P}$ holds. Then by Lemma 5.5, we see that for any $u_0 \in (-x_0, x_0) \setminus \{0\}$, there exists a real-analytic curve C_{u_0} on G_f such that

(1) $(u_0, 0) \in C_{u_0}$,
(2) C_{u_0} is contained in the plane determined by $z = f(u_0, 0)$,
(3) $\mathrm{Reg}\left(\mathrm{Proj}_{P_{xy}}(C_{u_0})\right)$ is a neighborhood of o in P_{xy}.

Therefore we see that $V_o \cap P_{xy} \neq \{o\}$ holds for any neighborhood V_o of o in S and that at o, (1) and (2) of Proposition 5.1 hold. Hence we have proved Proposition 5.1 and Proposition 5.2.

By Proposition 3.1, Proposition 5.1 and by Proposition 5.2, we obtain

Proposition 5.6. *For a connected, real-analytic, embedded, parallel curved surface S in $\mathbf{R}^3$, there exists a generating pair $\{C_1, C_2\}$ satisfying $S \subset S_{C_1,C_2}$.*

6. Classification

In this section, let S be complete.

Suppose that there exists an element P_0 of Π_S satisfying $\Xi_{S,P_0} = S$. Then from Proposition 3.1, we see that for any $p \in S$, $l_p(S, P_0)$ is isometric to $\mathbf{R}$. For any $p \in S$, there exists the element $P \in \Pi_S$ satisfying $p \in P$ and $P \parallel P_0$. Then $S \cap P$ is isometric to $\mathbf{R}$ or to a simple closed curve. Therefore we obtain

Proposition 6.1. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface in $\mathbf{R}^3$ satisfying $\Xi_{S,P_0} = S$ for some $P_0 \in \Pi_S$. Then there exists a generating pair $\{C_1, C_2\}$ such that*

(1) *$P_1 \parallel P_0$ holds,*
(2) *C_1 is isometric to $\mathbf{R}$ or to a simple closed curve,*
(3) *C_2 is isometric to $\mathbf{R}$,*
(4) *$S = S_{C_1,C_2}$.*

Then S is homeomorphic to a plane or to a cylinder.

Suppose that $\Xi_{S,P} \neq S$ and $\Xi_{S,P} \neq \emptyset$ hold for any $P \in \Pi_S$. Then for any $P \in \Pi_S$ and for any $p \in \Xi_{S,P}$, we see from Proposition 3.1, Proposition 5.1 and from Proposition 5.2 that the connected component of $P_p^{\perp}(S,P) \cap S$ containing p is a real-analytic curve isometric to $\mathbf{R}$ or to a simple closed curve. There exists the element P' of $Pl(\mathbf{R}^3)$ satisfying $p \in P'$ and $P' \parallel P$. Then by Proposition 5.1, we see that the connected component of $S \cap P'$ containing p is a real-analytic curve isometric to $\mathbf{R}$ or to a simple closed curve.

Suppose that $P_p^{\perp}(S,P) \cap S$ and $S \cap P'$ cross at two points q_1, q_2. Then we may suppose that on the domain $I_{P_p^{\perp}(S,P)\cap S}$(resp. $I_{S\cap P'}$) of $P_p^{\perp}(S,P) \cap S$(resp. $S \cap P'$) determined by q_1 and q_2, there exists no point of $P_p^{\perp}(S,P) \cap S \cap P'$. Then by Proposition 3.1, we see that there exists the only one point p_0 of $I_{P_p^{\perp}(S,P)\cap S}$ such that for any $q \in I_{S\cap P'}$, $P_q^{\perp}(S,P) \cap I_{P_p^{\perp}(S,P)\cap S} = \{p_0\}$ holds. Then $p_0 \notin \Xi_{S,P}$ holds. By Proposition 5.1 and by Proposition 5.2, we see that S is a surface of revolution and that p_0 lies on its axis of rotation. Therefore we obtain

Proposition 6.2. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface satisfying $\Xi_{S,P} \neq S$ and $\Xi_{S,P} \neq \emptyset$ for any $P \in \Pi_S$. Then one of the following holds*:

(1) *S is a surface of revolution with an axis of rotation A_S satisfying $\sharp S \cap A_S \in \{1, 2\}$. Then S is homeomorphic to a plane or to a sphere,*
(2) *There exists a generating pair $\{C_1, C_2\}$ such that*
  (a) *for $i = 1, 2$, C_i is isometric to $\mathbf{R}$ or to a simple closed curve,*
  (b) *$S = S_{C_1,C_2}$.*
  Then S is homeomorphic to a plane, a cylinder or to a torus.

By Proposition 6.1 and by Proposition 6.2, we may construct Theorem 1.2.

7. An isolated umbilical point of a parallel curved surface

Firstly, we shall give examples of parallel curved surfaces with an isolated umbilical point.

Example 7.1. Let f be a real-analytic, radial function on a neighborhood of o in $\mathbf{R}^2$. Then the graph G_f of f is a parallel curved surface satisfying

$P_{xy} \in \Pi_{\mathsf{G}_f}$. If G_f is not part of a round sphere, then o is an isolated umbilical point of G_f satisfying $\mathrm{ind}_o(\mathsf{G}_f) = 1$. Therefore Theorem 1.3 says that any parallel curved surface with an isolated umbilical point of index one is given by such a function as f.

Example 7.2. Let C_1 be the curve in P_{xz} defined by $z = x^4$ and C_2 the curve in P_{yz} defined by $z = -y^4$. Then $\{C_1, C_2\}$ is a generating pair and S_{C_1,C_2} is homeomorphic to a plane. For each $p \in S_{C_1,C_2}$, we see that each of the closed half spaces of $\mathbf{R}^3$ determined by $T_p(S_{C_1,C_2})$ does not contain any neighborhood of p in S_{C_1,C_2}. Therefore Gaussian curvature is non-positive on S_{C_1,C_2}. Let f be the real-analytic function on P_{xy} satisfying $\mathsf{G}_f = S_{C_1,C_2}$. Then we see that $\deg(f) = 4$ holds and that there exists a real number a_{22} satisfying $f^{(4)}(x,y) = x^4 + a_{22}x^2y^2 - y^4$. In addition, noticing $K_f \leqq 0$ on P_{xy}, we obtain $a_{22} = 0$. Then we see that there exists a neighborhood U_o of o in P_{xy} such that at each point of $U_o \setminus \{o\}$, $B_f \neq 0$ holds. Therefore we see that o is an isolated umbilical point of S_{C_1,C_2}. We immediately obtain $\mathrm{ind}_o(S_{C_1,C_2}) = 0$ and $T_o(S_{C_1,C_2}) \notin \Pi_{S_{C_1,C_2}}$.

Proof of Theorem 1.3. Let S be a real-analytic, parallel curved surface in $\mathbf{R}^3$ with an isolated umbilical point p_0. Suppose that $T_{p_0}(S) \in \Pi_S$ holds. Let (x, y, z) be orthogonal coordinates on $\mathbf{R}^3$ satisfying $p_0 = o$ and $T_o(S) = P_{xy}$, and f a real-analytic function on a neighborhood U_o of o in P_{xy} such that G_f is a neighborhood of o in S. In Section 5 we have shown that $f^{(\deg(f))}$ is of one-variable or radial. If $f^{(\deg(f))}$ is radial, then Proposition 5.2 says that S is part of a surface of revolution such that o lies on the axis of rotation. Then $\mathrm{ind}_o(S) = 1$ holds. Suppose that $f^{(\deg(f))}$ is of one-variable. Then we may suppose that $f^{(\deg(f))}(x,y) = x^{\deg(f)}$ holds. In addition, noticing Proposition 3.1, Corollary 3.2 and Proposition 5.1, we may suppose that $B_f \neq 0$ holds on $U_o \setminus \{o\}$. We represent f as in (2). Suppose that $a_{0n} = 0$ holds for any $n \in \mathbf{N} \cup \{0\}$. Then noticing Proposition 5.1, we may suppose that any zero point of f on U_o lies on A_y. Then f is of one-variable and o is not an isolated umbilical point. Suppose that $a_{0n} \neq 0$ holds for some $n \in \mathbf{N} \cup \{0\}$. The minimum of such integers as n is denoted by n_0. Then $n_0 \geqq 3$ holds. Since $B_f \neq 0$ holds on $U_o \setminus \{o\}$, we see that $\deg(f)$ and n_0 are even and that $a_{0n_0} < 0$ holds. However these contradict Proposition 5.1. Therfore $f^{(\deg(f))}$ is not of one-variable. Hence if $T_{p_0}(S) \in \Pi_S$, then $\mathrm{ind}_{p_0}(S) = 1$ holds and S is part of a surface of revolution such that p_0 lies on the axis of rotation.

If $p_0 \in \Xi_{S,P}$ holds for any $P \in \Pi_S$, then by Proposition 3.1 we obtain $\mathrm{ind}_{p_0}(S) = 0$.

Hence we have proved Theorem 1.3. $\square$

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