Residues of Closed Differential Forms

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Abstract

Let $E \to \bar{M}$ be an oriented real vector bundle of even rank over a smooth compact manifold $\bar{M}$ of dimension N, with or without boundary. We show that each section f of E gives rise to a primitive of the Euler form of E away from the zero set of f. This leads to the generalized Hopf formula for vector fields on a compact manifold. A very particular case of the Hopf formula is the Gauss-Bonnet theorem on a compact manifold with boundary. The proof is similar in spirit to that of Bott and Chern (*Acta Math.*, **114** (1965)). We indicate how these techniques may be used to highlight the de Rham cohomology of $\bar{M}$ in relation to the homotopy class of the mapping $\Delta : M \times \partial\bar{M} \to \mathbf{R}^N \setminus \{0\}$, where Δ is a defining mapping for the diagonal of $\bar{M} \times \bar{M}$ and M the "interior" of $\bar{M}$. As a consequence, we derive an explicit formula for residues of closed differential forms in a shell.

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Contents

1 *Introduction*

The classical *Hopf formula* says that if $f : M \to T(M)$ is a vector field on an oriented compact closed manifold M and each critical point of f is isolated, then

$$\sum_{p \in f^{-1}(0)} \deg(f, p) = \chi(M),$$

where $\deg(f, p)$ is the local degree of f at p and $\chi(M)$ the Euler characteristic of the manifold M (see for instance Milnor [M, p.37]). Thus, the sum on the left is actually independent of the particular choice of the vector field f. We give a short proof of the Hopf formula in the appendix.

For compact manifolds with boundary this formula is no longer true, even if f has no critical points on the boundary.

Example 1.1 Let $M = B_2 \setminus \overline{B_1}$, where B_2 is a ball with center 0 in $\mathbb{R}^N$ and $B_1 \subset\subset B_2 \setminus \{0\}$ is a smaller ball. Consider the vector field

$$\begin{aligned} f(x) &= \begin{pmatrix} x_1 \\ \dots \\ x_N \end{pmatrix} \\ &= \frac{1}{2} \nabla(x_1^2 + \dots + x_N^2) \end{aligned}$$

in $\bar{M}$. Then, the only critical point of f in $\bar{M}$ is $p = 0$ and $\deg(f, 0) = 1$, while $\chi(\bar{M}) = 1 + (-1)^{N-1}$.

□

The aim of this paper is to extend the Hopf formula to smooth vector fields on a compact manifold with boundary.

Our generalization of the Hopf formula relies on the observation that the Euler characteristic of a compact closed manifold is equal to the integral of the Euler form of M. We then show that a smooth vector field f on a compact manifold $\bar{M}$ of even dimension gives rise to a primitive of the Euler form of $\bar{M}$ away from the set of critical points of f. This enables us to write the integral of the Euler form of $\bar{M}$ as the sum of two terms, the first of the two is the integral of the primitive over the boundary of $\bar{M}$, the second being the sum of local degrees of f at the critical points.

In particular, we may consider as f the gradient of a defining function ρ of the boundary of $\bar{M}$. Then, by the *Morse Inequalities*, the sum of local degrees of f at the critical points is equal to the Euler characteristic of $\bar{M}$. In this way we obtain what Bott and Chern [BCh] called the *relative Gauss-Bonnet Theorem*. Our solution falls short of providing an explicit formula for the primitive.

If M is a bounded domain in $\mathbb{R}^N$ with C^2 boundary, then the Euler form of M is zero. The integral over the boundary in the Gauss-Bonnet formula is none other than the *Poincaré index* of the unit outward normal vector to the surface $\partial\bar{M}$. With the help of this formula, we prove that the rotation on the boundary of $\bar{M}$ of the unit outward normal vector to $\partial\bar{M}$ is equal to the Euler characteristic of $\bar{M}$. Hence the same remains valid for all smooth vector fields f on $\bar{M}$ whose restrictions to the boundary of $\bar{M}$ are homotopic to the unit outward normal vector of $\partial\bar{M}$.

Starting with the Gauss-Bonnet formula, we introduce a family of double differential forms on $M \times \partial\bar{M}$ which gives rise to an explicit homotopy formula for differential forms on M.

If $\bar{M}$ is of the form of "Swiss cheese" $\bar{M}_0 \setminus (M_1 \cup \ldots \cup M_N)$, where $\bar{M}_0 \subset M'$ and $\bar{M}_i \subset M_0$ satisfy appropriate convexity conditions, the homotopy formula yields a residue formula for closed differential forms in $\bar{M}_0$ with singularities in the holes M_i, $i = 1, \ldots, I$.

We briefly sketch also the case of complex-valued vector fields indicating how these techniques result in the *Rouchet principle* for holomorphic mappings.

2 *Integration of the Euler form*

Let $E \to M$ be an oriented real vector bundle of rank $2n$ over a smooth manifold M, with or without boundary.

Fix a Euclidean metric $(\cdot,\cdot)$ on E and a connection $\partial : C^\infty_{loc}(E) \to C^\infty_{loc}(E \otimes \Lambda^1)$ on E preserving the metric, i.e., such that $d\,(u,v)_x = (\partial u, v)_x + (u, \partial v)_x$, $x \in M$, for all $u, v \in C^\infty_{loc}(E)$ (see Fedosov [F, p.27] [1]).

Denote by $\Omega = \partial \circ \partial$ the curvature of the connection ∂; this is a global section of $C^\infty_{loc}(\mathrm{Hom}\,(E,E) \otimes \Lambda^2)$.

Given any local orthonormal frame $e_1, \ldots, e_{2n}$ for E, the curvature is represented by a skew-symmetric matrix

$$\Omega = (\Omega_{ij})_{\substack{i=1,\ldots,2n\\ j=1,\ldots,2n}}$$

whose entries are 2-forms (*ibid.*, p. 28). The *Euler form* of E is the differential form of degree $2n$ on M defined by

$$\begin{aligned}\chi(E) &= \text{Pfaffian}\left(\frac{1}{2\pi}\Omega\right)\\ &= \frac{1}{2^n n!(2\pi)^n}\sum_{(i_1,j_1,\ldots,i_n,j_n)} (-1)^{\epsilon(i_1,j_1,\ldots,i_n,j_n)}\,\Omega_{i_1j_1}\ldots\Omega_{i_nj_n},\end{aligned}$$

where the sum is taken over all rearrangements $(i_1, j_1, \ldots, i_n, j_n)$ of the numbers $(1, 2, \ldots, 2n-1, 2n)$ and $\epsilon_{(i_1,j_1,\ldots,i_n,j_n)}$ is the parity of the rearrangement. In other words, $\chi(E)$ is the coefficient of proportionality in the relation

$$\frac{1}{n!}\left(\frac{1}{2}\sum_{i,j}\frac{1}{2\pi}\Omega_{ij}e_i \wedge e_j\right)^{\wedge n} = \chi(E)\,e_1 \wedge \ldots \wedge e_{2n};$$

[1] A connection with this property is called *Euclidean*.

it is known that it is independent of the choice of the local oriented orthonormal frame for E (*ibid.*, p. 29).

We are interested in evaluating the integral

$$\int_M \chi(E) \tag{2.1}$$

for compact manifolds M, with or without boundary (the dimension of M must be equal to $2n$, since otherwise the integral is zero).

The cohomology class of the Euler form $\chi(E)$ is known to be independent of the particular choice of the Euclidean connection ∂ on E (*ibid.*, p. 29). Therefore, if M is closed (i.e., compact and without boundary), then integral (2.1) is a topological invariant of M (i.e., it does not depend on the choice of the connection). It is called the *Euler number* of the bundle E. In particular, if $E = T(M)$, then integral (2.1) is equal to the Euler characteristic of M (the *Gauss-Bonnet-Chern Theorem*).

We now restrict our attention to the case where M is a compact manifold with boundary. If such is the case, it is convenient to write $\bar{M}$ rather than M, thus letting M stand for the "interior" of $\bar{M}$.

Theorem 2.1 *Suppose that E possesses a section f which vanishes at no point of $\bar{M}$. Then the Euler form of E is exact, i.e., there is a differential form $\phi(f) \in C^\infty(\Lambda^{2n-1}T^*(\bar{M}))$ such that $d\phi(f) = \chi(E)$ on $\bar{M}$.*

Proof. Let

$$e(x) = \frac{f(x)}{|f(x)|}, \quad x \in \bar{M},$$

be the section, lying on the unit sphere of E, which corresponds to f.

Consider the connection ∂_1 in the bundle E, whose value at a section $u \in C^\infty_{loc}(E)$ is

$$\partial_1 u(x) = \partial u(x) + e(x)\,(u, \partial e)_x - \partial e(x)\,(u, e)_x, \quad x \in \bar{M}$$

(it is worth mentioning that any two connections in E differ by a global one-form $\Delta\Gamma \in C^\infty_{loc}(\mathrm{Hom}\,(E, E) \otimes \Lambda^1)$.)

For each $u, v \in C^\infty_{loc}(E)$, we have

$$\begin{aligned}
(\partial_1 u, v)_x + (u, \partial_1 v)_x &= (\partial u, v)_x + (u, \partial e)_x (e, v)_x - (u, e)_x (\partial e, v)_x \\
&\quad + (u, \partial v)_x + (\partial e, v)_x (u, e)_x - (e, v)_x (u, \partial e)_x \\
&= (\partial u, v)_x + (u, \partial v)_x \\
&= d\,(u, v)_x,
\end{aligned}$$

i.e., ∂_1 is a Euclidean connection. Moreover,

$$\begin{aligned}
\partial_1 e(x) &= \partial e(x) + e(x)\,(e, \partial e)_x - \partial e(x)\,(e, e)_x \\
&= 0,
\end{aligned}$$

for $(e, e)_x \equiv 1$ and $(e, \partial e)_x = \frac{1}{2} d(e, e)_x = 0$. Hence ∂_1 vanishes at the section e.

Now, we consider the homotopy between ∂ and ∂_1, i.e., the family of connections in the bundle E

$$\begin{aligned}\partial_t &= (1-t)\,\partial + t\,\partial_1 \\ &= \partial + t\,(e\,(\cdot,\partial e) - \partial e\,(\cdot,e)) \qquad (2.2)\end{aligned}$$

depending on the parameter $t \in [0,1]$. Analysis similar to the above shows that ∂_t is a Euclidean connection in E.

Letting $\Omega_t = \partial_t \circ \partial_t$ denote the curvature of the connection, we get, for each $u \in C^\infty_{loc}(E)$,

$$\begin{aligned}\Omega_t u &= \partial\partial u + t\,(\partial e\,(u,\partial e) + ed(u,\partial e)_x - \partial\partial e\,(u,e)_x + \partial e \wedge d(u,e)_x) \\ &\quad -te(\partial u,\partial e)_x + t^2 e\,((e,\partial e)_x \wedge (u,\partial e)_x + (\partial e,\partial e)_x(u,e)_x) \\ &\quad -t\partial e \wedge (\partial u, e)_x - t^2 \partial e \wedge ((e,e)_x(u,\partial e)_x + (\partial e, e)(u,e)_x) \\ &= \Omega u + (2t-t^2)\,\partial e \wedge (u,\partial e)_x + t\,(e(\Omega e,u)_x - \Omega e\,(e,u)_x)\,, \qquad (2.3)\end{aligned}$$

since $(e,\partial e)_x = 0$ and $(\partial e,\partial e)_x = 0$.

Denote by $\chi_t(E) = \text{Pfaffian}\left(\frac{1}{2\pi}\,\Omega_t\right)$ the corresponding Euler form of the bundle E. As

$$\begin{aligned}\Omega_1 e &= \partial_1\,(\partial_1 e) \\ &\equiv 0,\end{aligned}$$

we can assert that $\chi_1 = \text{Pfaffian}\left(\frac{1}{2\pi}\,\Omega_1\right) \equiv 0$.

Indeed, complete the section e to a local orthonormal frame $e_1,\ldots,e_{2n}$ for E, with $e_1 = e$. In this frame, the curvature Ω_1 is represented by a skew-symmetric matrix (Ω_{ij}) with $\Omega_{1j} = \Omega_{i1} \equiv 0$ for all $i,j = 1,\ldots,2n$. From this, the desired conclusion follows.

Thus,

$$\begin{aligned}\chi(E) &= \chi_0(E) \\ &= -\int_0^1 \left(\frac{d}{dt}\,\chi_t(E)\right) dt.\end{aligned}$$

Our next goal is to find a primitive for $\frac{d}{dt}\,\chi_t(E)$. To this end, we invoke the *variation formula* for the curvature (cf. Proposition 1.2.4 in Fedosov [F]), according to which $\frac{d}{dt}\,\Omega_t = \partial_t\dot\Gamma$, where

$$\dot\Gamma = e\,(\cdot,\partial e)_x - \partial e\,(\cdot,e)_x. \qquad (2.4)$$

Hence it follows that

$$\begin{aligned}&\frac{d}{dt}\,\text{Pfaffian}\left(\frac{1}{2\pi}\,\Omega_t\right) e_1 \wedge \ldots \wedge e_{2n} \\ &= \frac{1}{(n-1)!}\left(\frac{1}{2}\sum_{i,j}\frac{1}{2\pi}\left(\frac{d}{dt}\Omega_t\right)_{ij} e_i \wedge e_j\right) \wedge \left(\frac{1}{2}\sum_{i,j}\frac{1}{2\pi}\,(\Omega_t)_{ij}\,e_i \wedge e_j\right)^{\wedge(n-1)} \\ &= \frac{1}{(n-1)!}\left(\frac{1}{2}\sum_{i,j}\frac{1}{2\pi}\,(\partial_t\dot\Gamma)_{ij} e_i \wedge e_j\right) \wedge \left(\frac{1}{2}\sum_{i,j}\frac{1}{2\pi}\,(\Omega_t)_{ij} e_i \wedge e_j\right)^{\wedge(n-1)}. \qquad (2.5)\end{aligned}$$

Note that the equality (2.5) remains valid for each oriented orthonormal frame $e_1, \ldots, e_{2n}$. In particular, for each fixed $(t,x) \in [0,1] \times \bar{M}$, we may choose a proper basis. Namely, given any point $(t,x) \in [0,1] \times \bar{M}$, take $e_1, \ldots, e_{2n}$ such that $\partial_t e_j = 0$ at this point. Then, the covariant derivative ∂_t coincides with the exterior derivative d at this point. By *Bianchi's identity*, we obtain

$$\begin{aligned} d\Omega_t \,|_{(t,x)} &= \partial_t \Omega_t \,|_{(t,x)} \\ &= 0, \end{aligned}$$

so equality (2.5) implies

$$\begin{aligned} &\frac{d}{dt} \operatorname{Pfaffian}\left(\frac{1}{2\pi}\Omega_t\right) \\ &= \frac{1}{2^n (n-1)! (2\pi)^n} \sum_{(i_1,j_1,\ldots,i_n,j_n)} (-1)^{\epsilon(i_1,j_1,\ldots,i_n,j_n)}\, d\left((\dot{\Gamma})_{i_1 j_1} (\Omega_t)_{i_2 j_2} \ldots (\Omega_t)_{i_n j_n}\right). \end{aligned}$$

Combining this with the above expression for $\chi(E)$, we derive the desired primitive for the Euler form in the form

$$\begin{aligned} \phi(f) = &- \int_0^1 \frac{1}{2^n (n-1)! (2\pi)^n} \\ &\times \sum_{(i_1,j_1,\ldots,i_n,j_n)} (-1)^{\epsilon(i_1,j_1,\ldots,i_n,j_n)} (\dot{\Gamma})_{i_1 j_1} (\Omega_t)_{i_2 j_2} \ldots (\Omega_t)_{i_n j_n}\, dt, \end{aligned} \tag{2.6}$$

where $\dot{\Gamma}$ is given by formula (2.4) and Ω_t by formula (2.3). This completes the proof. □

It follows from Theorem 2.1 that, if E possesses a section f vanishing at no point of $\bar{M}$, then integral (2.1) can be reduced to an integral over the boundary of $\bar{M}$. Namely, we have, by Stokes' formula,

$$\begin{aligned} \int_{\bar{M}} \chi(E) &= \int_{\bar{M}} d\,\phi(f) \\ &= \int_{\partial\bar{M}} \phi(f). \end{aligned} \tag{2.7}$$

3 *Kronecker formula*

In the sequel, we restrict our attention to the case where E is the tangent bundle of $\bar{M}$, i.e., $E = T(\bar{M})$. In this case, f is a vector field on $\bar{M}$ vanishing nowhere in $\bar{M}$.

Consider the particular case where M is a bounded domain in $\mathbb{R}^{2n}$ with a smooth boundary $\partial\bar{M}$ (possibly consisting of several connected components).

Identifying $T(\bar{M})$ with $\bar{M} \times \mathbb{R}^n$ and taking $\partial = d$ as the original connection on E, we see at once that

$$\begin{aligned} \Omega_t &= (2t - t^2)\, de \wedge (\cdot, de)_x, \\ \dot{\Gamma} &= e\,(\cdot, de)_x - de\,(\cdot, e)_x. \end{aligned}$$

Writing $f(x) = \sum_{j=1}^{2n} f_j(x)\,\partial/\partial x_j$, with f_j a C^∞ function on $\bar{M}$, we get

$$\begin{aligned}(\Omega_t)_{ij} &= (2t-t^2)\,de_i \wedge de_j,\\ (\dot{\Gamma})_{ij} &= e_i\,de_j - e_j\,de_i,\end{aligned}$$

where $e_j(x) = \frac{f_j(x)}{|f(x)|}$, $j = 1,\dots,2n$. Thus, the primitive $\phi(f)$ becomes

$$\begin{aligned}\phi(f) &= -\frac{1}{2^{n-1}(n-1)!(2\pi)^n}\int_0^1 (2t-t^2)^{n-1}\,dt\\ &\quad\times \sum_{(i_1,j_1,\dots,i_n,j_n)} (-1)^{\epsilon(i_1,j_1,\dots,i_n,j_n)}\, e_{i_1} de_{j_1}\wedge de_{i_2}\wedge de_{j_2}\wedge\dots\wedge de_{i_n}\wedge de_{j_n}\\ &= \frac{(-1)}{\sigma_{2n}}\sum_{j=1}^{2n}(-1)^{j-1}\,e_j\,de[j],\end{aligned}$$

where σ_N is the area of the unit sphere in $\mathbb{R}^N$ and $de[j]$ is the exterior product of the differentials $de_1,\dots,de_{2n}$ one after another except de_j.

Since $\Omega \equiv 0$, it follows that $\chi(T(\bar{M}) \equiv 0$. Hence, formula (2.7) means that

$$\int_{\partial\bar{M}} \phi(f) = 0, \tag{3.1}$$

as a matter of course.

Equality (3.1) can be easily extended to the case where f is a vector field on $\bar{M}$ with isolated critical points (if such is the case, the number of critical points is finite). For this purpose, we cut out small neighborhoods $B(p,\epsilon)$ of the points $p \in f^{-1}(0)$ and apply formula (3.1) to the domain $M \setminus \cup_{p\in f^{-1}(0)} \overline{B(p,\epsilon)}$. This yields

$$\int_{\partial\bar{M}\setminus\cup_{p\in f^{-1}(0)}B(p,\epsilon)} \phi(f) = \sum_{p\in f^{-1}(0)} \int_{\bar{M}\cap\partial B(p,\epsilon)} \phi(f) \tag{3.2}$$

for each $\epsilon > 0$ small enough.

Corollary 3.1 *Suppose M is a bounded domain with smooth boundary in $\mathbb{R}^{2n}$ and f is a smooth vector field on $\bar{M}$ whose critical points are isolated and do not meet $\partial\bar{M}$. Then,*

$$\int_{\partial\bar{M}} \phi(f) = -\sum_{p\in f^{-1}(0)} \deg(f,p) \tag{3.3}$$

Proof. Indeed, if $p \in M$ is an "interior" critical point of f, then we may choose $\epsilon > 0$ such that the ball $B(p,\epsilon)$ lies entirely in M and contains no critical points of f different from p. Then, by the *Poincaré formula*,

$$\begin{aligned}\int_{\bar{M}\cap\partial B(p,\epsilon)} \phi(f) &= -\int_{\partial B(p,\epsilon)} \frac{1}{\sigma_{2n}}\sum_{j=1}^{2n}(-1)^{j-1}\frac{f_j}{|f|}\,d\frac{f}{|f|}[j]\\ &= -\deg(f,p)\end{aligned}$$

(cf. for instance Example 6.1.11 in Tarkhanov [T2]). Thus, if f has no critical points on the boundary of $\bar{M}$, then equality (3.2) reduces to (3.3), as required.

□

The equality (3.3) is known as the *Kronecker formula* (*ibid.*, Remark 6.1.8). We emphasize that it remains valid also for domains in an odd-dimensional space $\mathbb{R}^N$.

4 The generalized Hopf formula

We are now in a position to extend the Hopf formula to the case of compact manifolds with boundary.

Theorem 4.1 *Suppose f is a vector field of class C^2 on an oriented compact manifold $\bar{M}$ (with or without boundary), whose critical points are isolated and do not meet the boundary. Then,*

$$\int_{\bar{M}} \chi(T(\bar{M})) = \int_{\partial \bar{M}} \phi(f) + \sum_{p\in f^{-1}(0)} \deg(f,p). \tag{4.1}$$

Proof. Choose an $\epsilon_0 > 0$ such that the balls $B(p,\epsilon_0)$, $p \in f^{-1}(0)$, lie in the interior of $\bar{M}$ and do not meet each other. Applying formula (2.7 to the manifold with boundary $\bar{M} \setminus \cup_{p\in f^{-1}(0)} B(p,\epsilon)$, we obtain

$$\int_{\bar{M}\setminus \cup_{p\in f^{-1}(0)} B(p,\epsilon)} \chi(T(\bar{M})) = \int_{\partial \bar{M}} \phi(f) - \sum_{p\in f^{-1}(0)} \int_{\partial B(p,\epsilon)} \phi(f),$$

for all $\epsilon < \epsilon_0$.

We are going to pass to the limit in both sides of this equality, when $\epsilon \to 0$. Since the Euler form $\chi(T(\bar{M}))$ is smooth on the whole manifold $\bar{M}$, the limit of the left-hand side does exist and is equal to $\int_{\bar{M}} \chi(T(\bar{M}))$.

Moreover, each summand $\int_{\partial B(p,\epsilon)} \phi(f)$ has a limit, when $\epsilon \to 0$. Indeed, if $0 < \epsilon' < \epsilon'' < \epsilon_0$, then we get, by Stokes' formula,

$$\begin{aligned} \left| \int_{\partial B(p,\epsilon'')} \phi(f) - \int_{\partial B(p,\epsilon')} \phi(f) \right| &= \left| \int_{\partial (B(p,\epsilon'')\setminus B(p,\epsilon'))} \phi(f) \right| \\ &= \left| \int_{B(p,\epsilon'')\setminus B(p,\epsilon')} \chi(T(\bar{M})) \right| \\ &\le \sup |\chi(T(\bar{M}))| \operatorname{meas}(B(p,\epsilon'')). \end{aligned}$$

Hence it follows that $\left(\int_{\partial B(p,\epsilon)} \phi(f)\right)_{\epsilon<\epsilon_0}$ is a Cauchy net, as $\epsilon \to 0$.

The same argument shows that the limit $\lim_{\epsilon\to 0} \int_{\partial B(p,\epsilon)} \phi(f)$ is actually independent on the particular choice of the base $(B(p,\epsilon))_{\epsilon<\epsilon_0}$ for the neighborhood system of the point p. In particular, we may require $B(p,\epsilon)$ to be the ball in local coordinates at p.

From what has already been proved, it follows that

$$\int_{\bar{M}} \chi(T(\bar{M})) = \int_{\partial \bar{M}} \phi(f) - \sum_{p\in f^{-1}(0)} \lim_{\epsilon\to 0} \int_{\partial B(p,\epsilon)} \phi(f). \tag{4.2}$$

We are thus left with the task of identifying the individual limits in the right-hand side of (4.2).

To this end, we recall that the definition of the form $\phi(f)$ includes a Euclidean connection ∂ on the tangent bundle $T(\bar{M})$ (cf. $\Omega = \partial^2$ in (2.3)). In order to evaluate the limit

$$\lim_{\epsilon\to 0} \int_{\partial B(p,\epsilon)} \phi(f), \tag{4.3}$$

we only need to know the connection ∂ in a neighborhood of p. We claim that the particular choice of this *local connection* does not affect the limit (4.3). As each local Euclidean connection can be extended to a global one, what we have to show is that, if ∂_1 and ∂_2 are two connections on $T(\bar{M})$, which agree close to the boundary of $\bar{M}$, and ϕ_1 and ϕ_2 are the corresponding differential forms (2.6), then

$$\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}(\phi_2-\phi_1)=0.$$

Indeed, since $\partial_1=\partial_2$ close to the boundary of $\bar{M}$, we deduce from (2.6), (2.4) and (2.3) that $\phi_1=\phi_2$ near $\partial\bar{M}$. Moreover, letting χ_1 and χ_2 denote the corresponding Euler forms of $T(\bar{M})$, we have $\chi_2-\chi_1=d\phi$, where ϕ is a smooth differential form of degree $n-1$ on $\bar{M}$, vanishing close to the boundary of $\bar{M}$. Let us fix a smooth function χ with a compact support in the ball $B(p,\epsilon_0)$, such that $\chi=1$ in a neighborhood of the point p. Then,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}(\phi_2-\phi_1) &= \lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}\chi\,(\phi_2-\phi_1)\\ &= -\lim_{\epsilon\to 0}\int_{\bar{M}\setminus B(p,\epsilon)}d\chi\wedge(\phi_2-\phi_1)+\chi\,d(\phi_2-\phi_1)\\ &= -\int_{\bar{M}}d\chi\wedge(\phi_2-\phi_1)+\chi\,(\chi_2-\chi_1),\end{aligned}$$

for $d\chi$ vanishes near the singular point p.

Since $\operatorname{supp}\chi\subset\subset M$, we get, by Stokes' formula,

$$\begin{aligned}\int_{\bar{M}}\chi\,(\chi_2-\chi_1) &= \int_{\bar{M}}\chi\,d\phi\\ &= -\int_{\bar{M}}d\chi\wedge\phi,\end{aligned}$$

whence

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}(\phi_2-\phi_1) &= -\int_{\bar{M}}d\chi\wedge(\phi_2-\phi_1-\phi)\\ &= -\int_{\bar{M}}d(\chi-1)\wedge(\phi_2-\phi_1-\phi)\\ &= -\int_{\partial\bar{M}}(\chi-1)\,(\phi_2-\phi_1-\phi)+\int_{\bar{M}}(\chi-1)\wedge d(\phi_2-\phi_1-\phi)\\ &= 0\end{aligned}$$

because $(\phi_2-\phi_1-\phi)=0$ on $\partial\bar{M}$ and $d(\phi_2-\phi_1-\phi)=0$ on $\bar{M}$. This is the desired conclusion.

Finally, we can assume, by decreasing ϵ_0 if necessary, that the restriction of the bundle $T(\bar{M})$ to $B(p,\epsilon_0)$ is isomorphic to $B(p,\epsilon_0)\times\mathbb{R}^n$. Consider the local connection ∂ in $T(\bar{M})$ given by the exterior derivative in $B(p,\epsilon_0)$. Then, as in Section 3, the form $\phi(f)$ in $B(p,\epsilon_0)$ becomes $\frac{(-1)}{\sigma_{2n}}\sum_{j=1}^{2n}(-1)^{j-1}e_j\,de[j]$, whence

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}\phi(f) &= -\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}\frac{1}{\sigma_{2n}}\sum_{j=1}^{2n}(-1)^{j-1}\frac{f_j}{|f|}\,d\frac{f}{|f|}[j]\\ &= -\deg(f,p),\end{aligned}$$

which gives (4.1) when substituted in (4.2).

□

Formula (4.1) contains the Hopf formula as a very particular case. Indeed, if M is closed, then the integral over the boundary in the right-hand side of (4.1) disappears and we deduce that the integral of the Euler form is equal to the algebraic sum of the local degrees of the vector field f at its critical points.

5 *The relative Gauss-Bonnet Theorem*

Let $\bar{M}$ be an oriented compact manifold with boundary, of dimension $N = 2n$ and let $\bar{M}$ be embedded in a differentiable manifold M' without boundary, of the same dimension.

We denote by M the interior of $\bar{M}$ and by $\partial\bar{M}$ the boundary of $\bar{M}$. We assume that $\partial\bar{M}$ is a submanifold of codimension 1 and class C^2 in M' (*hypersurface*).

There exists a real-valued function $\rho \in C^2_{loc}(M')$ such that $M = \{x \in M' : \rho(x) < 0\}$ and $d\rho(x) \neq 0$ for $x \in \partial\bar{M}$.

If ρ_1 and ρ_2 are two functions with these properties, then there is a positive function $k \in C^1_{loc}(M')$ such that $\rho_2 = k\rho_1$ on M'. In this way we obtain what is referred to as the *defining function* of the oriented hypersurface $\partial\bar{M}$.

By a *stationary point* of ρ is meant a point $x^0 \in M'$ such that $d\rho(x^0) = 0$. Recall that a number $v \in \mathbb{R}$ is said to be a *critical value* of ρ if the preimage $\rho^{-1}(v)$ contains no stationary points of ρ.

Lemma 5.1 *Let $[a,b]$, $a < b$, be a closed interval containing no critical values of ρ and let $\rho^{-1}[a,b]$ be compact. Then, the hypersurface $\rho^{-1}(a)$ is C^2 diffeomorphic to $\rho^{-1}(b)$ and the manifold with boundary $\rho^{-1}(-\infty,a]$ is C^2 diffeomorphic to $\rho^{-1}(-\infty,b]$.*

Proof. The idea of the proof is to push $\rho^{-1}(b)$ down to $\rho^{-1}(a)$ along the orthogonal trajectories of the hypersurfaces $\rho(x) = \text{constant}$. See Milnor [M, p.12].

□

Choose a Riemannian metric on M' (i.e., an inner product in the fibers $T_x(M')$ of the tangent bundle); and let $(f',f'')_x$ denote the inner product of two tangent vectors, as determined by this metric. The *gradient* of ρ is the vector field $\nabla\rho$ on M' which is characterized by the identity $(f,\nabla\rho)_x = f(\rho)$ (= directional derivative of ρ along f) for any vector field f. In classical notation, in terms of local coordinates $x = (x_1,\dots,x_N)$, the gradient has components $\sum_{j=1}^N g^{ij}\frac{\partial\rho}{\partial x_j}$, where (g^{ij}) is the inverse to the matrix (g_{ij}) of the Riemannian metric. This vector field $\nabla\rho$ vanishes precisely at the stationary points of ρ.

When applied to the vector field $f = \nabla\rho$ on $\bar{M}$, Theorem 4.1 yields the following result.

Theorem 5.2 *Let $\bar{M} \subset M'$ be an oriented compact manifold with C^2 boundary and let $\rho \in C^2_{loc}(M')$ be a defining function of $\partial\bar{M}$. Then,*

$$\int_{\bar{M}} \chi(T(\bar{M})) = \int_{\partial\bar{M}} \phi(\nabla\rho) + \chi(\bar{M}). \tag{5.1}$$

If $\bar{M}$ is a compact closed manifold, then (5.1) becomes $\int_{\bar{M}} \chi(T(\bar{M})) = \chi(\bar{M})$ which is the content of the Gauss-Bonnet-Chern Theorem, as above. For this reason, we call Theorem 5.2 the relative Gauss-Bonnet Theorem.

To prove Theorem 5.2 we recall several results of the Morse theory. The best general reference here is the book of Milnor [M].

By a *Morse function* on M' is meant a smooth function u all of whose stationary points are non-degenerate, i.e., the Hessian $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ is non-singular at those points. It can be checked directly that non-degeneracy does not depend on the coordinate system.

According to the *Morse Lemma* (cf. *ibid.*, p. 6), if p is a non-degenerate stationary point of u, then there are local coordinates $x = (x_1, \ldots, x_N)$ in some neighborhood of p, in terms of which the function u is given in that neighborhood by $u(x) = u(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_N^2$.

The set of Morse functions is known to be dense in the space of all smooth functions on M' in the topology of uniform convergence along with a finite number of derivatives on compact subsets of M'.

The number $i = \text{index}\,(u, p)$ is called the *index* of the stationary point; obviously, it is independent of the local coordinates. If by pushing $\rho^1(b)$ down to $\rho^{-1}(a)$ we cross a stationary point of ρ of index i, then the set $\rho^{-1}(-\infty, b]$ has the homotopy type of $\rho^{-1}(-\infty, a]$ with an i-cell attached (*ibid.*, p. 14).

There is a close connection between the number of stationary points of a function u on a compact closed manifold M' and certain topological invariants of the manifold. In particular, the number

$$\sum_{i \geq 0} (-1)^i \, \#\{p \in \nabla u^{-1}(0) : \text{index}\,(u, p) = i\}$$

is actually independent of the function u, coinciding as it does with the Euler characteristic of M' (*Weak Morse Inequalities*, cf. *ibid.*, p. 29).

We are now in a position to deduce Theorem 5.2 as a consequence of Theorem 4.1 and the Morse theory.

Proof. We first observe that the equality (5.1) depends only on the restriction of the function ρ to an infinitesimal neighborhood of the boundary of $\bar{M}$. By assumption, ρ has no stationary points near $\partial\bar{M}$. Therefore, we can assume without loss of generality that ρ is a Morse function on M', for if not, we correct ρ away from a neighborhood of $\partial\bar{M}$ in M'.

If ρ is a Morse function on M', then the critical points of the vector field $\nabla\rho$ are non-degenerate and do not meet $\partial\bar{M}$. Hence it follows, by Theorem 4.1, that

$$\int_{\bar{M}} \chi(T(\bar{M})) = \int_{\partial\bar{M}} \phi(\nabla\rho) + \sum_{p \in \nabla\rho^{-1}(0) \cap M} \deg(\nabla\rho, p). \tag{5.2}$$

If $\nabla\rho(p) = 0$, then p is a stationary point of the function ρ. Let i be the index of p. Then there are local coordinates $x = (x_1, \ldots, x_N)$ with center at p, such that $\rho(x) = \rho(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_N^2$.

To evaluate the local degree of $\nabla\rho$ at p, we invoke formula (3.3) according to which

$$\deg(\nabla\rho,p)=\int_{|x|=\epsilon}\frac{1}{\sigma_N}\sum_{j=1}^{N}(-1)^{j-1}\frac{\frac{\partial\rho}{\partial x_j}}{|\nabla\rho|}\,d\frac{\nabla\rho}{|\nabla\rho|}\,[j],$$

where ϵ is small enough. Thus,

$$\begin{aligned}\deg(\nabla\rho,p) &= (-1)^i\int_{|x|=\epsilon}\frac{1}{\sigma_N}\sum_{j=1}^{N}(-1)^{j-1}\frac{x_j}{|x|}\,d\frac{x}{|x|}\,[j]\\ &= (-1)^i.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{p\in\nabla\rho^{-1}(0)}\deg(\nabla\rho,p) &= \sum_{i\geq 0}(-1)^i\,\#\{p\in\nabla\rho^{-1}(0)\cap M:\ \mathrm{index}\,(\rho,p)=i\}\\ &= \sum_{i\geq 0}(-1)^i\,\#\{i\text{-cells in }\bar{M}\}\\ &= \chi(\bar{M}),\end{aligned}$$

the last equality being a consequence of Theorem 3.2 in Milnor [M, p.14].

When substituted into (5.2), this gives (5.1), which completes the proof.

□

6 *Poincare index*

In this section we briefly discuss a particular case of the relative Gauss-Bonnet theorem, when M is a bounded domain in $\mathbb{R}^N$ with a boundary of class C^2. We need not assume that N is even.

Fix a defining function $\rho\in C^2_{loc}(\mathbb{R}^N)$ of the surface $\partial\bar{M}$, so that $M=\{x\in\mathbb{R}^N:\rho(x)<0\}$ and $d\rho|_{\partial\bar{M}}\neq 0$. As described in Section 3, we consider the differential form

$$\phi(\nabla\rho)=\frac{(-1)}{\sigma_N}\sum_{j=1}^{N}(-1)^{j-1}\frac{\frac{\partial\rho}{\partial x_j}}{|\nabla\rho|}\,d\frac{\nabla\rho}{|\nabla\rho|}\,[j].$$

The integral $-\int_{\partial\bar{M}}\phi(\nabla\rho)$ is called the *rotation* of the vector field $\nabla\rho$ on the boundary of $\bar{M}$. This is known also as the *Poincaré index* of the cycle $(\nabla\rho)_\sharp\partial\bar{M}$ with respect to the point 0. A simple verification shows that the integral is actually independent of the particular choice of the defining function ρ for $\partial\bar{M}$. Thus, it can be referred to as the rotation of the unit outward normal vector on $\partial\bar{M}$.

Corollary 6.1 *For each domain $M\subset\subset\mathbb{R}^N$ with C^2 boundary, the rotation of the unit outward normal vector on $\partial\bar{M}$ is equal to the Euler characteristic of $\bar{M}$.*

Proof. The assertion of the corollary amounts to the fact that

$$-\int_{\partial\bar{M}}\phi(\nabla\rho)=\chi(\bar{M}).$$

For even N, this follows from (5.1) because $\chi(T(\bar{M})) = 0$. For odd N, the proof runs just as in Theorem 5.2, with Theorem 4.1 replaced by Corollary 3.1.

□

7 *Double differential forms* $K_q(F)$

Given N-dimensional vectors $v_1, \ldots, v_\nu$ with entries in an algebra, we denote by $D_{N_1,\ldots,N_\nu}(v_1, \ldots, v_\nu)$ the determinant of order N whose first N_1 columns are v_1, the second N_2 columns are v_2 etc., the last N_ν columns are v_ν, where $N_1 + \ldots + N_\nu = N$. The determinant is calculated by columns, i.e., $\det(v_{ij}) = \sum_I (-1)^{\varepsilon_I} v_{i_1 1} \ldots v_{i_N N}$ where ε_I denotes the parity of the rearrangement $I = (i_1, \ldots, i_N)$ of the integers $(1, \ldots, N)$.

For a smooth mapping $F = F(x, y, t)$ of an open set $\Omega \subset M' \times M' \times \mathbb{R}^1$ to $\mathbb{R}^N$, we consider the double differential forms $K_q(F)$, $1 \le q \le N$, of degree $q-1$ in x and of degree $N-q$ in y and t, given by

$$K_q(F) = \frac{(-1)^q}{\sigma_N (q-1)!(N-q)!} D_{1,q-1,N-q} \left(\begin{pmatrix} \frac{F_1}{|F|} \\ \cdots \\ \frac{F_N}{|F|} \end{pmatrix}, \begin{pmatrix} d_x \frac{F_1}{|F|} \\ \cdots \\ d_x \frac{F_N}{|F|} \end{pmatrix}, \begin{pmatrix} (d_y + d_t) \frac{F_1}{|F|} \\ \cdots \\ (d_y + d_t) \frac{F_N}{|F|} \end{pmatrix} \right). \tag{7.1}$$

Moreover, we set $K_0 = K_{N+1} = 0$.

It is worth mentioning that these double forms were first introduced by the third author in [T1] (see also Tarkhanov [T2, 6.1.8]).

Lemma 7.1 *The form $K_q(F(x, y, t))$ is the component of degree $q-1$ in x of the "pull-back" $F^\sharp K_1(z)$ times $(-1)^{(N-1)(q-1)}$ where we first have to take full differentials in x, y and t, then to place all the differentials in x after the differentials in y and t, and finally to declare the form to be a double one.*

Proof. The proof is straightforward.

□

Recall that two continuous mappings $f_0, f_1 : T_1 \to T_2$ of topological spaces are said to be *homotopic* if there exists a continuous mapping $f_t : T_1 \times [0, 1] \to T_2$ which coincides with f_0 for $t = 0$ and with f_1 for $t = 1$.

If f_0 and f_1 are homotopic and differentiable, then one can choose the homotopy f_t to be differentiable too, roughly speaking, of the same class as f_0 and f_1 (see Theorem 8 of Pontryagin [P, p.64]).

From the topological point of view, if smooth mappings $f_0, f_1 : \partial\bar{M} \to \mathbb{R}^N \setminus \{0\}$ are homotopic, then they have the same rotations on $\partial\bar{M}$ (see Krasnosel'skii and Zabreiko [KZ, p.16]).

From the analytical point of view, it follows from the Poincaré formula that $\int_{\partial\bar{M}} f_1^\sharp K_1(z) - f_0^\sharp K_1(z) = 0$. Hence the difference $f_1^\sharp K_1(z) - f_0^\sharp K_1(z)$ is, by the *de Rham Theorem*, exact on $\partial\bar{M}$.

Our next objective is to extend this obvious observation to the double forms $K_q(F(x, y, t))$.

Lemma 7.2 *Let $F = F(x,y,t)$ be a mapping of an open set $\Omega \subset M' \times M' \times \mathbb{R}^1$ to $\mathbb{R}^N$, of class C^2. Then the following equality holds on the set $\Omega \setminus F^{-1}(0)$:*

$$d_x K_q(F) + (-1)^{q+1}(d_y + d_t)\, K_{q+1}(F) = 0. \tag{7.2}$$

Proof. Using Lemma 7.1 and the equality $dK_1(z) = -\delta(z)dz$ in $\mathbb{R}^N$ [2], we obtain

$$\begin{aligned} &d_x\, K_q(F) + (-1)^{q+1}\, (d_y + d_t)\, K_{q+1}(F) \\ &\quad = (-1)^{N-q} d_x \left((-1)^{(N-1)(q-1)} F^\sharp K_1(z)\right) + (-1)^{q+1}(d_y + d_t) \left((-1)^{(N-1)q} F^\sharp K_1(z)\right) \\ &\quad = (-1)^{Nq+1}\, d\, F^\sharp\, K_1(z) \\ &\quad = (-1)^{Nq}\, F^\sharp\, \delta(z)dz \\ &\quad = 0, \end{aligned}$$

for $(x,y,t) \in \Omega \setminus F^{-1}(0)$, as desired.

□

For another proof of Lemma 7.2, see the proof of Lemma 1.2 in the book of Aizenberg and Dautov [AD].

Lemma 7.3 *If $f_0(x,y)$, $f_1(x,y)$ are homotopic mappings of the set $M \times \partial\bar{M}$ to $\mathbb{R}^N \setminus \{0\}$, of class C^2, and $f_t(x,y)$, $t \in [0,1]$, is a C^2 homotopy between them, then, for every $0 \le q \le N-1$, we have on $M \times \partial\bar{M}$:*

$$\begin{aligned} &K_{q+1}(f_1) - K_{q+1}(f_0) \\ &\quad = d_x \left((-1)^q \int_0^1 dt \rfloor K_q(f_t)\; dt\right) - (-1)^q\, d_y \left((-1)^{q+1} \int_0^1 dt \rfloor K_{q+1}(f_t)\; dt\right). \end{aligned} \tag{7.3}$$

Proof. It suffices to integrate equality (7.2) over $t \in [0,1]$ and to take into account that

$$d_y \int_0^1 dt \rfloor K_{q+1}(f_t)\; dt = -\int_0^1 dt \rfloor\, d_y K_{q+1}(f_t)\; dt.$$

□

8 *Homotopy formula*

Denote by [diagonal $(M' \times M')$] the current of integration over the diagonal submanifold of $M' \times M'$.

Example 8.1 If $M' = \mathbb{R}^N$, then

$$[\text{diagonal}\,(M' \times M')] = (-1)^N\, \Delta^\sharp\, \delta(z)dz,$$

where $z = \Delta(x,y)$ is the mapping of $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, given by $\Delta(x,y) = y - x$.

□

[2]Here, $\delta(z)$ stands for the Dirac functional in $\mathbb{R}^N$.

We call a C^2 mapping $z = \Delta(x,y)$ of $M' \times M' \to \mathbb{R}^N$ *defining* for the diagonal of $M' \times M'$, if $\Delta^\sharp \delta(z)dz = (-1)^N\,[\text{diagonal}\,(M' \times M')]$.

For a mapping $\Delta : M' \times M' \to \mathbb{R}^N$, being defining is equivalent to the following conditions:

- $\text{diagonal}\,(M' \times M') = \{(x,y) \in M' \times M' : \Delta(x,y) = 0\}$;
- $\det \frac{\partial \Delta}{\partial y}(x,y) > 0$ for each $(x,y) \in \Delta$, -

as well as a number of similar conditions on other maximal minors of the *Jacobi matrix* $\frac{\partial \Delta(x,y)}{\partial(x,y)}$.

Example 8.2 Let M' be an open set in $\mathbb{R}^N$ and

$$\Delta(x,y) = \begin{pmatrix} \phi_1(x,y)\,(y_1 - x_1) \\ \dots \\ \phi_N(x,y)\,(y_N - x_N) \end{pmatrix},$$

where $\phi_j \in C^2_{loc}(M' \times M')$, $j = 1, \dots, N$. Then,

$$\det \frac{\partial \Delta}{\partial y}(x,x) = \phi_1(x,x) \dots \phi_N(x,x),$$

so Δ is a defining mapping for the diagonal of $M' \times M'$ if and only if

$$\phi_1(x,x) \dots \phi_N(x,x) > 0 \quad \text{for all} \quad x \in M'.$$

□

When working with holomorphic mappings of Stein manifolds, we are not able to ensure the existence of such a mapping in general. Moreover, if M' is not *parallelizable*, one can deduce from the results of Schneider [S] that it is even impossible to satisfy only the second condition.

To obtain a defining mapping for the diagonal of $M' \times M'$ in case M' is parallelizable, we may use the following simple construction (cf. Khenkin and Leiterer [KhL, 4.2.4]).

Lemma 8.3 *There exists a smooth mapping $\Delta : M' \times M' \to T(M')$ such that the following conditions are fulfilled:*

1) $\Delta(x,y) \in T_x(M')$ for all $(x,y) \in M' \times M'$, i.e., Δ is a section of the pull-back of the bundle $T(M')$ by the mapping $(x,y) \mapsto x$ of $M' \times M' \to M'$; and

2) for every fixed $x \in M'$, we have $\Delta(x,x) = 0$ and the mapping $\Delta(x,y)$ from M' to $T_x(M')$ is a diffeomorphism in some neighborhood of $y = x$.

Proof. According to the *Whitney Theorem*, M' can be embedded as a submanifold into a space $\mathbb{R}^{N'}$ for some large but finite N'. In other words, there exists a smooth one-to-one mapping $f : M' \to \mathbb{R}^{N'}$ such that, for all $x \in M'$, the Jacobi matrix of f (with respect to local coordinates) has maximal rank. Let $Tf : T(M') \to T(\mathbb{R}^{N'})$ stand for the tangent mapping of f which is locally defined by the Jacobi matrix of f. Then Tf is injective, and so there exists a smooth bundle homomorphism $Tf^{(-1)} : f(M') \times \mathbb{R}^{N'} \to T(M')$ with the property that $Tf^{(-1)}Tf$ is the identity mapping of $T(M')$. Set

$$\Delta(x,y) = Tf^{(-1)}\left(f(x), f(y) - f(x)\right), \quad (x,y) \in M' \times M'.$$

Then it is clear that condition 1) is fulfilled. Moreover, it follows from the Taylor formula that 2) is satisfied.

□

The double differential forms $K_q(\Delta)$ provide us with reproducing kernels for differential forms on M'.

Lemma 8.4 *If M is a relatively compact open subset of M' with a piecewise smooth boundary and $u \in C^1(\Lambda^q T^*(\bar{M}))$, then*

$$\begin{aligned} &-\int_{\partial \bar{M}} u(y) \wedge K_{q+1}(\Delta(x,y)) \\ &\quad + \int_{\bar{M}} du(y) \wedge K_{q+1}(\Delta(x,y)) + d\int_{\bar{M}} u(y) \wedge K_q(\Delta(x,y)) = \begin{cases} u(x), & x \in M, \\ 0, & x \in M' \setminus \bar{M}. \end{cases} \end{aligned} \tag{8.1}$$

Proof. From the invariance of the exterior derivative under a differentiable change of variables it follows that the current $\Delta^\sharp K_1(z)$ satisfies the *fundamental equation*

$$d\, \Delta^\sharp K_1(z) = (-1)^N \left[\text{diagonal}\,(M' \times M')\right]$$

on $M' \times M'$. Combining this with Lemma 7.1, we deduce that the family of double forms $K_q(\Delta)$, $q = 0, 1, \ldots, N$, is a fundamental solution of the de Rham complex on M'. To complete the proof, it suffices to use Corollary 2.5.1 in Tarkhanov [T2].

□

Having disposed of this preliminary step, we can now invoke Lemma 7.3 to modify formula (8.1) thus arriving at a homotopy formula with an arbitrary "barrier function."

Theorem 8.5 *Let $\bar{M}$ be a compact manifold with a boundary of class C^2 and let $f = f(x,y)$ be a C^2 mapping of $M \times \partial\bar{M}$ to $\mathbb{R}^N \setminus \{0\}$. If f is homotopic to the mapping Δ on $M \times \partial\bar{M}$ and f_t, $t \in [0,1]$, is a C^2 homotopy between f and Δ, then, for every form $u \in C^1(\Lambda^q T^*(\bar{M}))$, we have*

$$u(x) = -\int_{\partial\bar{M}} u(y) \wedge K_{q+1}(f(x,y)) + hdu + dhu, \quad x \in M, \tag{8.2}$$

where

$$h_q u\,(x) = -\int_{\partial\bar{M}\times[0,1]} u(y) \wedge ((-1)^q K_q(f_t(x,y))) + \int_{\bar{M}} u(y) \wedge K_q(\Delta(x,y)).$$

Proof. Setting $f_1 = \Delta$ and $f_0 = f$, we decompose the kernel $K_{q+1}(\Delta(x, y))$ on the set $M \times \partial \bar{M}$ in accordance with equality (7.3). Substituting this decomposition into the integral over the boundary in (8.1) and using Stokes' formula, we derive (8.2), as desired.

□

Thus, any smooth deformation of the mapping $\Delta(x, y)$ on $M \times \partial\bar{M}$ leads to an integral representation of differential forms in $\bar{M}$.

Example 8.6 Suppose $f = f(x, y)$ is a continuous mapping of $M \times \partial\bar{M}$ to $\mathbb{R}^N \setminus \{0\}$ such that $\langle f, \Delta\rangle_{(x,y)} > -|f||\Delta|$ on this set. Then, f is homotopic to Δ. Indeed, consider the continuous mapping $f_t = t\Delta + (1-t)f$ of $M \times \partial\bar{M} \times [0, 1]$ to $\mathbb{R}^N$. In order to prove that $f_t \neq 0$, we write

$$\begin{aligned} |f_t|^2 &= t^2|\Delta|^2 + 2t(1-t)\langle\Delta, f\rangle_{(x,y)} + (1-t)^2|f|^2 \\ &> (t|\Delta| - (1-t)|f|)^2 \\ &\geq 0, \end{aligned}$$

whence the desired conclusion follows.

□

9 *"Convex" manifolds*

In this section, we consider a sample application of Theorem 8.5. To begin, we introduce a class of manifolds to be considered.

Definition 9.1 *A manifold $\bar{M}$ is said to be* convex *if there exists a C^1 mapping $b : \partial\bar{M} \to \mathbb{R}^N$ such that $\langle b(y), \Delta(x, y)\rangle_{(x,y)} > 0$ for all $(x, y) \in M \times \partial\bar{M}$.*

The important point to note here is that, in contrast to what was required in Sections 7 and 8, we are going to apply the above results with $f(x, y) = b(y)$ which is of merely class C^1. However, the same arguments still go when f is of class C^1, provided it depends on only one variable x or y.

Suppose $\bar{M}$ is convex and b is a mapping guaranteed by Definition 9.1. Using Example 8.6 we conclude that the mapping $b : M \times \partial\bar{M} \to \mathbb{R}^N \setminus \{0\}$ is homotopic to the mapping Δ on the set $M \times \partial\bar{M}$. Moreover the homotopy constructed above is $f_t(x, y) = t\Delta(x, y) + (1-t)b(y)$ which is smooth. Applying now Theorem 8.5 we derive the following series of formulas valid for $u \in C^1(\Lambda^q T^*(\bar{M}))$:

$$u(x) = -\int_{\partial\bar{M}} u(y) \wedge K_{q+1}(b(y)) + hdu + dhu, \quad x \in M. \tag{9.1}$$

Since b does not depend on x, the kernels $K_{q+1}(b(y))$ vanish provided $q > 0$. Therefore the equalities (9.1) all together mean that the operator h is a fundamental solution, at positive degrees, of the de Rham complex on $\bar{M}$.

Example 9.2 Let M be a convex domain in $\mathbb{R}^N$ with twice differentiable boundary. Choose a real-valued function ρ of class C^2 in a neighborhood of $\bar{M}$, such that $M = \{x : \rho(x) < 0\}$ and $d\rho(y) \neq 0$ for $y \in \partial\bar{M}$. Since the tangent hyperplane to $\partial\bar{M}$ at a point $y \in \partial\bar{M}$ does not meet M, we have $\langle \nabla\rho(y), y - x\rangle_{(x,y)} > 0$ on the set $M \times \partial\bar{M}$, where $\nabla\rho$ is the gradient of ρ. Thus, taking $b(y) = \nabla\rho(y)$, we see that $\bar{M}$ meets Definition 9.1.

□

Thus Theorem 8.5 shows a way to find new fundamental solutions of the de Rham complex on a manifold with boundary.

10 *Residue formula*

We now consider another method for achieving $K_{q+1}(f(x,y)) = 0$. If $f(x,y)$ is independent of y, then $K_{q+1}(f(x,y)) = 0$ for $q \leq N - 2$! This works well for *concave* boundaries by simply switching the variables x and y for convex manifolds. We discuss in detail the simple case of convex domains in $\mathbb{R}^N$ with convex "holes."

Suppose $\bar{M} = \bar{M}_0 \setminus (M_1 \cup \ldots \cup M_I)$, where M_0 and $M_i \subset\subset M_0$ are convex domains with C^2 boundaries in $\mathbb{R}^N$.

For each $i = 0, 1, \ldots, I$, we choose a convex functions $\rho_i \in C^2_{loc}(\mathbb{R}^N)$ such that $M_i = \{x \in \mathbb{R}^N : \rho_i(x) < 0\}$ and $d\rho_i(x) \neq 0$ for $x \in \partial\bar{M}_i$. Consider the mapping $f : M \times \partial\bar{M} \to \mathbb{R}^N$ given by

$$f(x,y) = \begin{cases} \nabla\rho_0(y) & \text{if} \quad (x,y) \in M \times \partial\bar{M}_0; \\ -\nabla\rho_1(x) & \text{if} \quad (x,y) \in M \times \partial\bar{M}_1; \\ \ldots & \ldots \quad \ldots \\ -\nabla\rho_I(x) & \text{if} \quad (x,y) \in M \times \partial\bar{M}_I. \end{cases}$$

By assumption, the tangent hyperplane to $\partial\bar{M}_0$ at a point $y \in \partial\bar{M}_0$ does not intersect M_0. Moreover, for each $i = 1, \ldots, I$, the tangent hyperplane to the hypersurface $\rho_i^{-1}(\rho_i(x))$ does not meet $\partial\bar{M}_i$ provided $x \notin \bar{M}_i$. Hence it follows that $\langle f(x,y), y - x\rangle_{(x,y)} > 0$ on the set $M \times \partial\bar{M}$. Using Example 8.6 we deduce that the mapping $f : M \times \partial\bar{M} \to \mathbb{R}^N \setminus \{0\}$ is homotopic to the mapping Δ on the set $M \times \partial\bar{M}$. Moreover the homotopy constructed above is $f_t(x,y) = t(y-x) + (1-t)f(x,y)$ which is smooth.

Applying now Theorem 8.5 we derive the following series of formulas valid for $u \in C^1(\Lambda^q T^*(\bar{M}))$:

$$\begin{aligned} u(x) = &- \int_{\partial\bar{M}_0} u(y) \wedge K_{q+1}(\nabla\rho_0(y)) \\ &- \sum_{i=1}^{I} \int_{\partial\bar{M}_i} u(y) \wedge K_{q+1}(\nabla\rho_i(x)) + hdu + dhu, \quad x \in M. \end{aligned} \tag{10.1}$$

If $q > 0$, then $K_{q+1}(\nabla\rho_0(y)) = 0$ for $(x,y) \in M \times \partial\bar{M}_0$. Thus, (10.1) yields the following "residue formula."

Theorem 10.1 *Suppose $1 \le q \le N$. Then, for each closed differential form $u \in C^1(\Lambda^q T^*(\bar{M}))$, we have*

$$u(x) = -\sum_{i=1}^{I} \int_{\partial \bar{M}_i} u(y) \wedge K_{q+1}(\nabla \rho_i(x)) + dhu, \quad x \in M. \tag{10.2}$$

We emphasize that, if $q < N-1$, then $K_{q+1}(\nabla \rho_i(x)) = 0$ for all $i = 1, \dots, I$. Thus, for such q, the sum on the right-hand side of (10.2) vanishes. On the other hand, for $q = N-1$ this sum becomes

$$-\sum_{i=1}^{I} \left(\int_{c_i} u \right) K_N(\nabla \rho_i(x)),$$

where c_i is an arbitrary $(N-1)$-dimensional cycle in M surrounding $\bar{M}_i$ with multiplicity 1.

Corollary 10.2 *If, for a closed form $u \in C^1(\Lambda^{N-1}T^*(\bar{M}))$, all the periods $\int_{c_i} u$, $i = 1, \dots, I$, are zero, then $u = dhu$ in M.*

Proof. This follows from (10.2).

$\square$

11 *Rouchet principle*

We endow the real space $\mathbb{R}^{2n}$ with the complex structure $z_j = x_j + \sqrt{-1}x_{n+j}$, $j = 1, \dots, n$, thus obtaining a complex space $\mathbb{C}^n$. Under this structure, one considers the complex derivatives $\partial/\partial z_j = \frac{1}{2}\left(\partial/\partial x_j - \sqrt{-1}\partial/\partial x_{n+j}\right)$, for $j = 1, \dots, n$.

Let M be a bounded domain in $\mathbb{C}^n$ with a boundary of class C^2. As above, we write $M = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ with a suitable real-valued function $\rho \in C^2_{loc}(\mathbb{C}^n)$ satisfying $d\rho(z) \neq 0$ for $z \in \partial \bar{M}$.

For a point $\zeta \in \partial \bar{M}$, we have

$$\langle \nabla \rho(y), x - y \rangle_{(x,y)} = 2 \operatorname{Re} \langle \nabla_{\mathbb{C}} \rho(\zeta), z - \zeta \rangle_{(z,\zeta)},$$

where $\zeta_j = y_j + \sqrt{-1}y_{n+j}$, $j = 1, \dots, n$, and

$$\nabla_{\mathbb{C}} \rho\,(\zeta) = \begin{pmatrix} \partial\rho/\partial\zeta_1 \\ \dots \\ \partial\rho/\partial\zeta_n \end{pmatrix}.$$

Theorem 11.1 *Let F be a holomorphic mapping of $M \to \mathbb{C}^n$, of class $C^1(\bar{M})$. If the image of $\partial \bar{M}$ by the mapping $\zeta \mapsto \langle \nabla_{\mathbb{C}}\rho, F\rangle_\zeta$ does not separate 0 from ∞ in the complex plane, then the number of zeroes of F in M, along with their multiplicities, is equal to $\chi(\bar{M})$.*

Proof. It follows from the hypothesis of the theorem that F has no zeroes on the boundary of $\bar{M}$. Hence all zeroes of F in M are isolated and the number of zeroes, along with their multiplicities, is equal to

$$N = \int_{\partial\bar{M}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{F}_j}{|F|^{2n}} dF \wedge d\bar{F}[j] \tag{11.1}$$

(cf. Theorem 2.4 in Aizenberg and Yuzhakov [AYu]).

Since the image of $\partial\bar{M}$ by the mapping $\zeta \mapsto \langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta$ does not separate 0 from ∞ in the complex plane and $|\nabla_{\mathbf{C}}\rho(\zeta)| \geq c > 0$ for $\zeta \in \partial\bar{M}$, it follows that the image of $\partial\bar{M}$ by the mapping

$$\zeta \mapsto \frac{\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta}{|\nabla_{\mathbf{C}}\rho(\zeta)|^2} \tag{11.2}$$

does not separate 0 from ∞ in the complex plane. Consequently, there exists a path $p : [0,1) \to \mathbb{C}$ such that

$$\begin{aligned} p(0) &= 0, \\ \lim_{t\to 1} |p(t)| &= \infty \end{aligned}$$

and $p(t)$ does not meet the image of $\partial\bar{M}$ by the mapping (11.2). We can certainly assume that $p(t) \neq 1$ for all $t \in [0,1)$. Consider the mapping $F_t : \partial\bar{M} \times [0,1] \to \mathbb{C}^n$ given by

$$F_t(\zeta) = \frac{-1}{p(t)-1} F(\zeta) + \frac{p(t)}{p(t)-1} \bar{\nabla}_{\mathbf{C}}\rho(\zeta),$$

where

$$\bar{\nabla}_{\mathbf{C}}\rho\,(\zeta) = \begin{pmatrix} \partial\rho/\partial\bar{\zeta}_1 \\ \dots \\ \partial\rho/\partial\bar{\zeta}_n \end{pmatrix}.$$

We have $F_0(\zeta) = F(\zeta)$ and $F_1(\zeta) = \bar{\nabla}_{\mathbf{C}}\rho(\zeta)$. Moreover, an easy verification shows that

$$\langle F_t, \nabla_{\mathbf{C}}\rho\rangle_\zeta = \frac{-1}{p(t)-1} \langle F, \nabla_{\mathbf{C}}\rho\rangle_\zeta + \frac{p(t)}{p(t)-1} |\nabla_{\mathbf{C}}\rho(\zeta)|^2$$

vanishes only if $p(t) = \frac{\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta}{|\nabla_{\mathbf{C}}\rho(\zeta)|^2}$ for some $\zeta \in \partial\bar{M}$, which is impossible. From what has already been proved we see that $F_t : \partial\bar{M} \times [0,1] \to \mathbb{C}^n \setminus \{0\}$ is a smooth homotopy between F and $\bar{\nabla}_{\mathbf{C}}\rho$.

As the integral on the right-hand side of (11.1) is an integer number (see, for instance, Lemma 2.6 in [AYu]), we get

$$N = \int_{\partial\bar{M}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\frac{\partial\rho}{\partial\zeta_j}}{|\nabla_{\mathbf{C}}\rho|^{2n}} d\,\bar{\nabla}_{\mathbf{C}}\rho \wedge d\,\nabla_{\mathbf{C}}\rho[j].$$

To complete the proof it remains to use the *Poincaré index* together with the observation that the differential form under integration is equal to $K_1(\nabla\rho)$ up to an exact form on the boundary of $\bar{M}$. Indeed, for *each* C^1 mapping F from an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$, we have, away from $F^{-1}(0)$,

$$\begin{aligned}&\frac{(n-1)!}{(2\pi\sqrt{-1})^n}\sum_{j=1}^{n}(-1)^{j-1}\frac{\bar{F}_j}{|F|^{2n}}\,dF\wedge d\bar{F}[j]\\ &\quad= -K_1(f)+\sqrt{-1}\,d\left(\frac{1}{\sigma_{2n}(2n-2)}\sum_{j=1}^{n}\frac{1}{|f|^{2n-2}}\,df[j,n+j]\right),\end{aligned}$$

where $F_j = f_j + \sqrt{-1}f_{n+j}$, $j = 1,\ldots,n$ [3].

□

Obviously, if $\operatorname{Re}\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta \neq 0$ for all $\zeta\in\partial\bar{M}$, then the cycle $\{\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta : \zeta\in\partial\bar{M}\}$ in the complex plane does not separate 0 from ∞. Hence the number of zeroes of F in M, along with their multiplicities, is equal to the Euler characteristic of $\bar{M}$.

The condition "$\operatorname{Re}\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta \neq 0$ for all $\zeta\in\partial\bar{M}$" cannot be relaxed to the condition "$\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta \neq 0$ for each $\zeta\in\partial\bar{M}$."

Example 11.2 For the mapping $F(z) = z^2$ of the closed unit disc in $\mathbb{C}^1$, we have

$$\begin{aligned}\langle\nabla_{\mathbf{C}}\rho, F\rangle_\zeta &= \langle\frac{\partial}{\partial\zeta}(\sqrt{\zeta\bar{\zeta}}-1),\zeta^2\rangle_\zeta\\ &= \frac{1}{2}\zeta\\ &\neq 0\end{aligned}$$

for $|\zeta| = 1$, while F vanishes at the origin with multiplicity 2.

□

We mention a corollary of Theorem 11.1 which concerns the so-called linearly convex domains in complex analysis (cf. [AYu, §24]).

Corollary 11.3 *If there is a point $z^0\in M$ with the property that the cycle $\{\langle\nabla_{\mathbf{C}}\rho(\zeta), z^0-\zeta\rangle_\zeta : \zeta\in\partial\bar{M}\}$ in the complex plane does not separate 0 from ∞, then $\chi(\bar{M}) = 1$.*

Proof. Apply Theorem 11.1 to $F(z) = z - z^0$.

□

If M is a *strictly pseudoconvex* domain in $\mathbb{C}^n$ with C^2 boundary, then the *Levi determinant* of the function ρ

$$\mathcal{L}(\rho) = -\det\begin{pmatrix} 0 & \frac{\partial\rho}{\partial\zeta_1} & \cdots & \frac{\partial\rho}{\partial\zeta_n}\\ \frac{\partial\rho}{\partial\zeta_1} & \frac{\partial^2\rho}{\partial\zeta_1\partial\zeta_1} & \cdots & \frac{\partial^2\rho}{\partial\zeta_n\partial\zeta_1}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\rho}{\partial\zeta_n} & \frac{\partial^2\rho}{\partial\zeta_1\partial\zeta_n} & \cdots & \frac{\partial^2\rho}{\partial\zeta_n\partial\zeta_n}\end{pmatrix}$$

[3] If $n = 1$, one must replace $\frac{1}{2n-2}\frac{1}{|f|^{2n-2}}$ by $\log|f|$.

is positive at each point $\zeta \in \partial \bar{M}$ (cf. [AYu, §8]). Then, instead of comparing F with the complex gradient of ρ, we may compare it with the mapping of $\bar{M} \to \mathbb{C}^n$ whose components are complementary minors of the elements $\partial\rho/\partial\zeta_j$, $j = 1, \dots, 2$, in the Levi determinant $\mathcal{L}(\rho)$. In this way we obtain the following result.

Theorem 11.4 *Suppose M is a strictly pseudoconvex domain with twice differentiable boundary in $\mathbb{C}^n$ and F is a holomorphic mapping of $M \to \mathbb{C}^n$, of class $C^1(\bar{M})$. If the image of $\partial\bar{M}$ by the mapping*

$$\zeta \mapsto \det \begin{pmatrix} 0 & F_1 & \dots & F_n \\ \frac{\partial\rho}{\partial\zeta_1} & \frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1} & \dots & \frac{\partial^2\rho}{\partial\zeta_n\partial\bar{\zeta}_1} \\ \dots & \dots & \dots & \dots \\ \frac{\partial\rho}{\partial\zeta_n} & \frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_n} & \dots & \frac{\partial^2\rho}{\partial\zeta_n\partial\bar{\zeta}_n} \end{pmatrix}$$

does not separate 0 from ∞ in the complex plane, then the number of zeroes of F in M, along with their multiplicities, is equal to $\chi(\bar{M})$.

A *A proof of the classical Hopf formula*

Let $f : M \to T(M)$ be a vector field on a oriented compact closed manifold M of dimension N. Suppose all critical points of f are non-degenerate.

Recall that each vector field f on M generates a *flow* on M, that is a family of diffeomorphisms $F_t(x)$ of M, parametrized by $t \in [0, T)$. For a point $x \in M$, it is obtained by solving the system of ordinary differential equations $\frac{d}{dt}F_t(x) = f(x)$ with the initial value $F_0(x) = x$. (In local coordinates near x, write $f(x) = \sum_{j=1}^N f_j(x)\, \partial/\partial x_j$, then the system becomes $\frac{d}{dt}(F_t)_j(x) = f_j(x)$, $j = 1, \dots, N$.) It is well-known from the theory of ordinary differential equations that the $F_t(x)$ exists for each x, provided T is small enough, and depends smoothly on x varying over a compact set (see Milnor [M, p.10] for more details).

We now invoke the general *Lefschetz Fixed Point Theorem* for CW-complexes (cf. Dold [D]). Namely, each smooth mapping $F : M \to M$ is known to induce a homomorphism ("pull-back") $F^\sharp$ of the cohomology $H^\cdot(M)$ of M [4]. Thus, for each $i = 0, 1, \dots, N$, the trace $\operatorname{tr} F^\sharp|_{H^i(M)}$ is well-defined. On the other hand, if σ is a closed subset of M and U a neighborhood of σ, then F induces a smooth mapping of the pair $(F^{-1}(U), F^{-1}(U) \setminus F^{-1}(\sigma)) \to (U, U \setminus \sigma)$ and consequently a homomorphism of the relative cohomology

$$F^\sharp : H^\cdot(F^{-1}(U), F^{-1}(U) \setminus F^{-1}(\sigma)) \to H^\cdot(U, U \setminus \sigma).$$

In particular, if σ is invariant under F, i.e., $F(\sigma) \subset \sigma$, this gives rise to a homomorphism $F^\sharp$ of $H^\cdot(U, U \setminus \sigma)$. Then, the Lefschetz Fixed Point Theorem states that if

[4]Here, M need not be closed.

the set of fixed points of F is discrete, then

$$\sum_{i=0}^{N}(-1)^i \operatorname{tr} F^\sharp|_{H^i(M)} = \sum_{p=F(p)} \left(\sum_{i=0}^{N}(-1)^i \operatorname{tr} F^\sharp|_{H^i(U,U\setminus p)} \right). \tag{A.1}$$

In case p is an interior point of M the contribution $\sum_{i=0}^{N}(-1)^i \operatorname{tr} F^\sharp|_{H^i(U,U\setminus p)}$ of p reduces to the local degree of $1 - \frac{\partial F}{\partial x}$ at p.

We are going to apply (A.1) to the diffeomorphism F_t. Granting that the set of fixed points of F_t is discrete, we obtain

$$\sum_{i=0}^{N}(-1)^i \operatorname{tr} F_t^*|_{H^i(M)} = \sum_{p=F_t(p)} \deg\left(1 - \frac{\partial F_t}{\partial x}, p\right),$$

The left-hand side of this equality is independent of t small enough, for homotopic mappings of $M \to M$ induce the same homomorphism of the cohomology. Therefore, letting $t \to 0$, we deduce that

$$\begin{aligned}\sum_{i=0}^{N}(-1)^i \operatorname{tr} F_t^*|_{H^i(M)} &= \sum_{i=0}^{N}(-1)^i \operatorname{tr} F_0^*|_{H^i(M)} \\ &= \chi(M)\end{aligned}$$

for all $t \in [0,T)$.

On the other hand, as the solution of the local Cauchy problem for ordinary differential equations is unique, we can assert that a point $p \in M$ is a fixed point of F_t if and only if p is a critical point of f, i.e., $f(p) = 0$. Moreover, we claim that

$$\deg\left(1 - \frac{\partial F_t}{\partial x}, p\right) = (-1)^N \ \deg(f,p).$$

Indeed, choose a local chart on M with center at p, then

$$f_j(x) = \sum_{k=1}^{N} c_{jk}x_k + O(|x|^2), \qquad j = 1, \ldots, N,$$

for x close to p. Hence it follows that

$$(F_t)_j(x) = x_j + \sum_{k=1}^{N} c_{jk}x_k t + O(t^2 + |x|^2), \qquad j = 1, \ldots, N,$$

provided $t^2 + |x|^2 \ll 1$. Thus,

$$\begin{aligned}\deg\left(1 - \frac{\partial F_t}{\partial x}, p\right) &= \operatorname{sign}\ \det\left(1 - \frac{\partial F_t}{\partial x}\right)(p) \\ &= \operatorname{sign}\ \det(-c_{jk}) \\ &= (-1)^N \ \deg(f,p),\end{aligned}$$

as desired.

Summarizing, we obtain $\sum_{p\in f^{-1}(0)} \deg(f,p) = (-1)^N\chi(M)$. We now apply this argument again, with f replaced by $-f$, to obtain $\sum_{p\in f^{-1}(0)} \deg(f,p) = \chi(M)$. These formulas are certainly equivalent for, by the Poincaré duality, the Euler characteristic of an odd-dimensional manifold is equal to zero (this follows also from our argument). The proof is complete.

References

[AD] L. A. Aizenberg and Sh. A. Dautov. *Differential Forms Orthogonal to Holomorphic Functions or Forms, and Their Properties.* AMS, Providence, R.I., 1983.

[AYu] L. A. Aizenberg and A. P. Yuzhakov. *Integral Representations and Residues in Many-Dimensional Complex Analysis.* Nauka, Novosibirsk, 1979.

[BCh] Raoul Bott and S. S. Chern. Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. *Acta Math.*, 114: 71–112, 1965.

[D] A. Dold. *Lectures on Algebraic Topology.* Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[F] B. V. Fedosov. *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1995.

[KhL] G. M. Khenkin and J. Leiterer. *Theory of Functions on Complex Manifolds.* Birkhäuser, Basel et al., 1984.

[KZ] M. A. Krasnosel'sky and P. P. Zabreiko. *Geometrical Methods of Non-Linear Analysis.* Nauka, Moscow, 1975.

[M] J. W. Milnor. *Morse Theory.* Princeton Univ. Press, Princeton, 1969.

[P] L. S. Pontryagin. *Smooth Manifolds and Their Applications in the Theory Homotopies.* Nauka, Moscow, 1976.

[S] M. Schneider. Vollständige Durchschnitte in Steinschen Mannigfaltigkeiten. *Math. Ann.*, 186: 191–200, 1970.

[T1] N. N. Tarkhanov. A characteristic property of closed exterior differential forms. In: *On Holomorphic Functions of Several Complex Variables*, Inst. Fiz. SO Akad. Nauk SSSR, Krasnoyarsk, 1976, 203–209 (Russian).

[T2] N. N. Tarkhanov. *Complexes of Differential Operators.* Kluwer Academic Publishers, Dordrecht, NL, 1995.