Six Spectra and Two Dimensions of an Abelian Category

Alexander L. Rosenberg

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

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Introduction.

The main (probably, the only) disadvantage of the introduced in [R3] spectrum (as well as its special case - the left spectrum of a ring [R1], [R2]) is that the spectrum of the quotient category at an open set might be larger than that open set.

The principal character of this work is the *flat spectrum* of an abelian category which is the minimal, in a certain sense, extension of the spectrum enjoying the "right properties" with respect to localizations and restrictions. The proof of these properties and the generalization onto the flat spectrum the most important constructions and facts known for the spectrum, is one of the main objects of this paper.

Another purpose here is to understand how the spectral theory presented in [R3] is related to some of the earlier attempts to create noncommutative local algebra and algebraic geometry.

Recall the major approaches to noncommutative spectral theory developed between late fifties and early eighties:

(i) The *injective spectrum* of an abelian category formed by the isomorphy classes of indecomposable injective objects [Gab].

(ii) The Goldman's spectrum of a ring which consists of prime torsion theories [G], [Gol1], [Gol2].

(iii) The spectrum of a ring understood as the set of prime ideals of this ring with Zariski topology. Affine schemes (structure sheaves) can be defined for left noetherian rings and for PI-rings [OV].

(iv) The Cohn's affine scheme of a general ring [C].

Thanks to the flat spectrum, we are able to get a new insight at the first three of the listed above spectral theories.

In more detail, the contents looks as follows.

In Section 0, we remind, for readers' convenience, some basic facts on the introduced in [R3] spectrum of an abelian category.

In Section 1, we study the *complete spectrum* of an abelian category $\mathcal{A}$, **Spec**$^\wedge\mathcal{A}$, which is defined as the set of all thick subcategories $\mathbb{T}$ of $\mathcal{A}$ such that the quotient category $\mathcal{A}/\mathbb{T}$ is local. The complete spectrum might be much bigger than one needs, and we investigate it because a number of facts is naturally formulated and proved for the complete spectrum. So that when one moves to more important subsets of $\mathbf{Spec}^{\wedge}\mathcal{A}$, it remains only to make some refinements. And this is exactly what we do.

Section 2 is concerned with the principal character of the paper - the *flat spectrum* - which is the subset of the complete spectrum formed by Serre subcategories.

In Section 3, we consider a straightforward extension of the Goldman's spectrum [G] (which is originally defined for categories of modules) on arbitrary abelian categories and show ruther easily that the Goldman's spectrum is the set of all "points" $\mathbb{T}$ of the flat spectrum such that the quotient (local) category $\mathcal{A}/\mathbb{T}$ has simple objects.

In Sections 4, we study the relation of the flat spectrum of an abelian category (which is supposed mostly to have injective hulls) with certain type of injective objects. This relation leads to a better understanding of the structure of local Grothendieck categories.

Section 5 deals with the approach to the spectral theory of abelian categories through injective objects. We introduce *the injective spectrum* of an abelian category (which contains, usually properly, the set of equivalence classes of indecomposable injective objects - *indecomposable injective spectrum*) and study some of its basic properties. In the case of a Grothendieck category, there is a natural embedding of the flat spectrum into the injective spectrum.

Section 6 is concerned with Grothendieck categories for which the Gabriel-Krull dimension is defined. In these categories, the spectrum, the flat spectrum, the Goldman's spectrum, and the injective spectrum coincide. As a consequence, we find the relation between the Krull - Gabriel dimension and the dimension defined (in an obvious way) through the flat spectrum. This relation looks pretty much the same as in the commutative case.

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0. PRELIMINARIES ON THE SPECTRUM.

A detailed exposition (with proofs) of the presented in this section facts can be found in [R5]. Here we give a sketch of basic notions and facts of [R5] for readers' convenience.

0.1. A preorder in abelian categories. Fix an abelian category $\mathcal{A}$. For any two objects, X and Y, of the category $\mathcal{A}$ we shall write $X \succ Y$ if Y is a subquotient of a coproduct of a finite number of copies of X, i.e. if, for some finite k, there exists a diagram

$$(k)X \longleftarrow U \longrightarrow Y,$$

where the left arrow is a non-zero monomorphism, and the right one is an epimorphism; $(k)X$ is a direct sum of k copies of X. One can show that

the relation $\succ$ is a preorder on $Ob\mathcal{A}$.

0.2. The spectrum of an abelian category. Let M be a nonzero object of the category $\mathcal{A}$. We write $M \in Spec\mathcal{A}$ if, for any nonzero subobject N of M, we have: $N \succ M$. Since $M \succ N$, we can say that $M \in Spec\mathcal{A}$ if and only if it is equivalent with respect to the preorder $\succ$ to any of its nonzero subobjects.

Denote by $\mathbf{Spec}\mathcal{A}$ the ordered set of equivalence classes (with respect to $\succ$) of elements of $Spec\mathcal{A}$. The set $\mathbf{Spec}\mathcal{A}$ shall be called the *spectrum of the category* $\mathcal{A}$.

0.3. Spectrum and simple objects. Clearly every simple object of the category $\mathcal{A}$ belongs to $Spec\mathcal{A}$. Moreover, we shall see in a moment that two simple objects are equivalent if and only if they are isomorphic.

0.3.1. Proposition. *Let M be a simple object of the category $\mathcal{A}$, and let N be an object of $\mathcal{A}$. Then the following conditions are equivalent:*

(a) N is isomorphic to $(k)M$ for some (finite) k;

(b) $M \succ N$.

In particular, if N and M are simple objects, then $N \succ M$ if and only if the objects M and N are isomorphic.

0.4. The spectrum and exact localizations. Recall that a full subcategory $\mathbb{S}$ of the category $\mathcal{A}$ is called *thick* if the following condition holds:

the object M in the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

belongs to $\mathbb{S}$ if and only if M' and M'' are objects of $\mathbb{S}$.

It follows from the universal property of localizations that the map

$$Q \longmapsto KerQ$$

gives a bijection of the equivalence class of exact localizations of the category $\mathcal{A}$ onto the set of thick subcategories of $\mathcal{A}$.

Here (as everywhere) $KerQ$ is the full subcategory of $\mathcal{A}$ generated by all objects which are annihilated by Q.

0.4.1. Proposition. *Let $Q: \mathcal{A} \longrightarrow \mathcal{B}$ be an exact localization of an abelian category $\mathcal{A}$. For any $P \in Spec\mathcal{A}$, either $Q(P)$ equals to zero, or $Q(P)$ belongs to $Spec\mathcal{B}$.*

For any $M \in Ob\mathcal{A}$, consider the full subcategory $\langle M \rangle$ of $\mathcal{A}$ defined as follows: $Ob\langle M \rangle$ consists of all objects N such that the relation $N \succ M$ does not hold.

0.4.2. Lemma. *For any two objects, M and M', of the category $\mathcal{A}$, the following conditions are equivalent:*

(a) $M \succ M'$;

(b) $\langle M' \rangle \subseteq \langle M \rangle$.

Thus, the map $M \longmapsto \langle M \rangle$ identifies the ordered set of equivalence classes of objects of $\mathcal{A}$ (the order is induced by $\succ$) with $(\{\langle M \rangle \mid M \in Ob\mathcal{A}\}, \supseteq)$.

For any subcategory $\mathbb{T}$ of the category $\mathcal{A}$, let $\mathbb{T}^-$ denote the full subcategory of $\mathcal{A}$ generated by all objects M such that any nonzero subquotient of M has a nonzero subobject from $\mathbb{T}$.

0.4.3. Lemma. *For any subcategory $\mathbb{T}$ of $\mathcal{A}$,*

(a) the subcategory $\mathbb{T}^-$ is thick;

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

Call a subcategory $\mathbb{T}$ of $\mathcal{A}$ a *Serre subcategory* if $\mathbb{T} = \mathbb{T}^-$.

0.4.4. Proposition. *If an object M of the category $\mathcal{A}$ belongs to $Spec\mathcal{A}$, then $\langle M \rangle$ is a Serre subcategory of $\mathcal{A}$.*

Thus, according to Proposition 0.4.4, to any point $\langle M \rangle$ of **Spec**$\mathcal{A}$, there corresponds an exact localization, $Q_{<M>}: \mathcal{A} \longrightarrow \mathcal{A}/<M>$.

0.4.5. Local abelian categories and localizations at points of the spectrum. A nonzero object M of an abelian category $\mathcal{A}$ will be called *quasifinal* if $N \succ M$ for any nonzero object N of the category $\mathcal{A}$.

In other words, a nonzero object M is quasifinal if and only if

$$<M> = \{0\} = \bigcap_{N \in Ob\mathcal{A}-\{0\}} <N>.$$

Clearly a quasifinal object of the category $\mathcal{A}$ (if any) belongs to $Spec\mathcal{A}$, and every two quasifinal objects of $\mathcal{A}$ are equivalent.

0.4.6. Definition. An abelian category $\mathcal{A}$ is called *local* if it has a quasifinal object. ■

0.4.7. Lemma. *The following properties of an abelian category* $\mathcal{A}$ *are equivalent:*

(a) $\mathcal{A}$ *is local and has simple objects;*

(b) any nonzero object of $\mathcal{A}$ *has a simple subquotient, and all simple objects of* $\mathcal{A}$ *are isomorphic one to another.*

0.4.8. Example. The category of left modules over a commutative ring k is local if and only if the ring k is local. ■

0.4.9. Proposition. *Let* $\mathcal{A}$ *be an abelian category. For any object* M *of the category* $\mathcal{A}$ *such that* $<M>$ *is a thick subcategory of* $\mathcal{A}$, *the quotient category* $\mathcal{A}/<M>$ *is local.*

In particular, for any abelian category $\mathcal{A}$ *and any object* P *from* $Spec\mathcal{A}$, *the quotient category* $\mathcal{A}/<P>$ *is local.*

0.4.10. Corollary. *If* M *is a simple object of an abelian category* $\mathcal{A}$ *then* $\mathcal{A}/<M>$ *is a local category with a unique up to isomorphism simple object.*

The last assertion follows from the fact that if $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is an exact localization and M a simple object of the category $\mathcal{A}$, then either $Q(M) = 0$, or $Q(M)$ is a simple object.

0.5. The topology τ and Zariski topology. The least requirement on the topology on $\mathbf{Spec}\mathcal{A}$ is that it should be compatible with the preorder $\succ$. This means that the closure of any point $<P> \in \mathbf{Spec}\mathcal{A}$ should contain the set

$$\mathfrak{s}(<P>) := \{<P'> \mid \ <P'> \subseteq <P>\}$$

of specializations of that point. The topology τ as the strongest among the topologies which have this property.

Call a full subcategory $\mathcal{B}$ of the category $\mathcal{A}$ *topologizing* if it contains a taken in $\mathcal{A}$ coproduct of any two of its objects and the following condition holds:

if in the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the object M belongs to $\mathcal{B}$, then M' and M'' belong to $\mathcal{B}$.

Call a full subcategory $\mathcal{B}$ of the category $\mathcal{A}$ *left closed* if it is topologizing, and the inclusion functor $\mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint functor. One can show that the subsets

$$\mathbf{Spec}\mathcal{B} = \{<P> \mid P \in Spec\mathcal{A} \cap Ob\mathcal{B}\},$$

where $\mathcal{B}$ runs through the family of all left closed subcategories of $\mathcal{A}$, is the set of closed subsets of a topology which is called (in [R5]) the *Zariski* topology and is denoted by $\mathfrak{Z}\mathfrak{t}$.

0.6. Supports. The *support* of an object M of an abelian category $\mathcal{A}$ is the set, $Supp(M)$, of all $<P> \in \mathbf{Spec}\mathcal{A}$ such that $M \succ P$.

0.6.1. Proposition. *(a) For any short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

$$Supp(M) = Supp(L) \cup Supp(N).$$

(b) For any set Ξ *of objects such that there is a coproduct* $\bigoplus_{X \in \Xi} X$,

$$Supp\Big(\bigoplus_{X \in \Xi} X\Big) = \bigcup_{X \in \Xi} Supp(X).$$

0.6.2. Proposition. *For any subset* W *of* $\mathbf{Spec}\mathcal{A}$, *the full subcategory* $\mathcal{A}(W)$ *of* $\mathcal{A}$ *generated by all objects* M *such that* $Supp(M) \subseteq W$ *is a Serre subcategory.*

0.7. The left spectrum of a ring. Let $\mathcal{A}$ be the category R-*mod* of left modules over an associative ring R with unity. Since each module from $Spec(R\text{-}mod)$ is equivalent to any of its cyclic submodules, we can take into consideration only the modules R/m, where m runs over the set I_lR of left ideals of the ring R.

The set of all left ideals p of the ring R such that R/p belongs to *SpecR-mod* is denoted by $Spec_l R$ and is called the *left spectrum* of R.

0.7.1. Lemma. *For any two left ideals* m *and* n *of the ring* R, *the relation* $R/m \succ R/n$ *is equivalent to the following condition:*

(#) there exists a finite set y *of elements of the ring* R *such that the ideal* $(m:y) := \{z \in R \mid zy \subset m\}$ *is contained in the ideal* n.

0.7.2. Corollary. *A left ideal* p *belongs to the left spectrum if and only if, for any* $x \in R\text{-}p$, *there exists a finite subset* y *of* R *such that*

$$((p:x):y) = (p:yx) \subseteq p.$$

0.7.3. Remark. If m is a two-sided ideal of the ring R, then, evidently, $R/m \succ R/m'$ if and only if m is conntained in m'. In particular, if the ring R is commutative, then the left spectrum $Spec_l R$ coincides with the set $SpecR$ of prime ideals of R. ■

0.8. Associated points. For any object M of an abelian category $\mathcal{A}$, denote by $Ass(M)$ the set of $<P> \in \mathbf{Spec}\mathcal{A}$ such that P is a subobject of M. The points of $Ass(M)$ are called *associated to* M *elements of the spectrum.*

Here we need only the very first simple facts about this notion:

0.8.1. Lemma. *For any short exact sequence,*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M'').$$

0.8.2. Corollary. *For any finite set* Ω *of objects of an abelian category* $\mathcal{A}$, *we have:*

$$Ass(\bigoplus_{X \in \Omega} X) = \bigcup_{X \in \Omega} Ass(X).$$

0.9. The relative spectrum. *The spectrum of a functor* $\mathfrak{F}$ from an abelian category $\mathcal{B}$ to an abelian category $\mathcal{A}$ is the ordered set $\mathbf{Spec}(\mathfrak{F})$ of all pairs $(<M>,<P>)$ such that there is an object M' of $\mathcal{B}$ such that $<M> = <M'>$ and $<P> \in Ass(\mathfrak{F}(M'))$. The order in $\mathbf{Spec}(\mathfrak{F})$ is induced from $\mathbf{Spec}\mathcal{B} \times \mathbf{Spec}\mathcal{A}$.

Note that, given a functor $\mathfrak{F}$, the description of $\mathbf{Spec}(\mathfrak{F})$ is reduced to the description, for any $<P> \in \mathbf{Spec}\mathcal{A}$, of the fiber of $\mathbf{Spec}(\mathfrak{F})$ over $<P>$ which is the set of all $<M> \in \mathbf{Spec}\mathcal{B}$ such that $<P> \in Ass(M)$.

1. THE COMPLETE SPECTRUM OF AN ABELIAN CATEGORY.

For an abelian category $\mathcal{A}$, denote by $\mathbf{Spec}^\wedge\mathcal{A}$ the set of all thick subcategories $\mathbb{P}$ of $\mathcal{A}$ such that the quotient category $\mathcal{A}/\mathbb{P}$ is local. We call the ordered set $(\mathbf{Spec}^\wedge\mathcal{A},\supseteq)$ *the complete spectrum of the category* $\mathcal{A}$.

1.1. Proposition. *For any thick subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$, *there is a natural embedding*

$$(\mathbf{Spec}\mathcal{A}/\mathbb{T},\supseteq) \longrightarrow (\mathbf{Spec}^\wedge\mathcal{A},\supseteq).$$

Proof. Let $Q = Q_{\mathbb{T}}$ be the localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$. Denote by $\mathbf{Spec}(\mathcal{A},\mathbb{T})$ the set of all subcategories $Q^{-1}\langle P\rangle$, where $\langle P\rangle$ runs through $\mathbf{Spec}\mathcal{A}/\mathbb{T}$. Since the functor Q is exact, $Q^{-1}\langle P\rangle$ is a thick subcategory for each $\langle P\rangle \in \mathbf{Spec}\mathcal{A}/\mathbb{T}$; and the natural functor

$$\mathcal{A}/Q^{-1}\langle P\rangle \longrightarrow (\mathcal{A}/\mathbb{T})/\langle P\rangle$$

is an equivalence of categories. Since the category $(\mathcal{A}/\mathbb{T})/\langle P\rangle$ is local, this shows that $\mathbf{Spec}(\mathcal{A},\mathbb{T})$ is a subset of $\mathbf{Spec}^\wedge\mathcal{A}$.

Clearly the map $\langle P\rangle \longmapsto Q^{-1}\langle P\rangle$ is a bijection of $\mathbf{Spec}\mathcal{A}/\mathbb{T}$ onto $\mathbf{Spec}(\mathcal{A},\mathbb{T})$. This gives the promised embedding.

Note that $Q^{-1}\langle P\rangle$ does not depend on the choice of Q. ∎

One can see that

$$\mathbf{Spec}^\wedge\mathcal{A} = \bigcup_{\mathbb{T}\,\in\,\mathbf{Thick}(\mathcal{A})} \mathbf{Spec}(\mathcal{A},\mathbb{T}) = \bigcup_{\mathbb{T}\,\in\,\mathbf{Spec}^\wedge(\mathcal{A})} \mathbf{Spec}(\mathcal{A},\mathbb{T}),$$

where $\mathbf{Thick}(\mathcal{A})$ denotes the set of thick subcategories of $\mathcal{A}$.

Indeed, let $\mathbb{T} \in \mathbf{Spec}^\wedge\mathcal{A}$, Q a localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$; and let P be a quasi-final object in the local category $\mathcal{A}/\mathbb{T}$. Then $\langle P\rangle = 0$; hence the preimage $Q^{-1}\langle P\rangle$ coincides with the subcategory $\mathbb{T}$.

1.2. The specializations of points and the topology τ. For every $\mathbb{P} \in \mathbf{Spec}^\wedge\mathcal{A}$, denote by $\mathfrak{s}(\mathbb{P})$ the set

$$\{\mathbb{P}' \in \mathbf{Spec}^\wedge\mathcal{A} \mid \mathbb{P}' \subseteq \mathbb{P}\}$$

of all specializations of the point $\mathbb{P}$.

As in the case of $\mathbf{Spec}\mathcal{A}$, only topologies on $\mathbf{Spec}^\wedge\mathcal{A}$ which are compatible with the preorder $\supseteq$ make sense. The compatiblity means that the closure of any point $\mathbb{P}$ contains the set $\mathfrak{s}(\mathbb{P})$ of its specializations.

Let τ denote the strongest topology having this property. One can see that τ has the same description as its restriction to **Spec**$\mathcal{A}$: the closure of any set $W \subseteq$ **Spec^**$\mathcal{A}$ is $\bigcup_{\mathbb{P} \in W} \mathfrak{s}(\mathbb{P})$.

This description implies immediately that the union of any family of closed in τ sets is a closed set.

1.3. Uniform subcategories and Gabriel multiplication. We call a subcategory $\mathbb{X}$ of an abelian category $\mathcal{A}$ *uniform* if it contains all subquotients of any of its objects.

1.3.1. Example. Any subcategory $\mathbb{X}$ of $\mathcal{A}$ having the property

$$M \in Ob\mathbb{X} \text{ and } M \succ L \Rightarrow L \in Ob\mathbb{X}$$

is, obviously, uniform. In particular, the subcategory $<M>$ is uniform for every object M of the category $\mathcal{A}$. ■

1.3.2. Note. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor. Then the preimage, $F^{-1}(\mathbb{X})$, of any uniform subcategory $\mathbb{X}$ of $\mathcal{B}$ is a uniform subcategory of $\mathcal{A}$. ■

1.3.3. The Gabriel multiplication. Recall that the product of two subcategories, $\mathbb{X}$ and $\mathbb{Y}$, of a category $\mathcal{A}$, is the full subcategory $\mathbb{X}\bullet\mathbb{Y}$ of $\mathcal{A}$ generated by all objects M in $\mathcal{A}$ such that there exists an exact sequence

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

such that $M' \in Ob\mathbb{X}$ and $M'' \in Ob\mathbb{Y}$. One can check that if $\mathbb{X}$ and $\mathbb{Y}$ are uniform (topologizing) subcategories, then $\mathbb{X}\bullet\mathbb{Y}$ is uniform (resp. topologizing) category.

Note that *a uniform subcategory* $\mathbb{X}$ *of an abelian category* $\mathcal{A}$ *is thick if and only if* $\mathbb{X} = \mathbb{X}\bullet\mathbb{X}$.

1.3.4. The smallest thick subcategory containing a given uniform subcategory. Let $\mathbb{P}$ be a uniform subcategory of $\mathcal{A}$. Then the smallest thick subcategory $\mathbb{P}'$ of $\mathcal{A}$ containing the subcategory $\mathbb{P}$ can be described as follows.

Set $\mathbb{P}_0 = \mathbb{P}$;

if the ordinal β equals to $\alpha + 1$, then set $\mathbb{P}_{\alpha+1} := \mathbb{P}_\alpha \bullet \mathbb{P}_\alpha$;

if β is a limit ordinal, then set $\mathbb{P}_\beta := \bigcup_{\alpha<\beta} \mathbb{P}_\alpha$.

Then the union of all subcategories $\mathbb{P}_\alpha$ coincides with $\mathbb{P}'$.

In fact, it follows from *(i)* that all the subcategories $\mathbb{P}_\alpha$ are topologizing. It remains to check that $\mathbb{P}'$ is closed under extensions; i.e. if in the exact sequence

$$0 \longrightarrow M' \xrightarrow{\ u\ } M \xrightarrow{\ v\ } M'' \longrightarrow 0$$

M' and M'' are in $\mathbb{P}'$, then so is M. But M', M'' are in $\mathbb{P}'$ iff they belong to $\mathbb{P}_\alpha$ for some α; in that case $M \in Ob\mathbb{P}_{\alpha+1}$.

1.4. The complete spectrum of a topologizing category. It is shown in [R3] (Lemma 5.3.1) that, $\mathbf{Spec}\mathcal{A} \cap |\mathbb{T}| = \mathbf{Spec}\mathcal{A}$ for any topologizing subcategory of an abelian category $\mathcal{A}$. Here, we shall prove an analog of this statement for the complete spectrum.

First we need the following Lemma.

1.4.1. Lemma. *Let* $\mathbb{T}$ *be a topologizing subcategory; and let* $\mathbb{S}$ *be a thick subcategory of an abelian category* $\mathcal{A}$. *Then* $\mathbb{T} \cap \mathbb{S}$ *is a thick subcategory of the category* $\mathbb{T}$; *and, given localizations*

$$Q\colon \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S} \quad \text{and} \quad Q'\colon \mathbb{T} \longrightarrow \mathbb{T}/(\mathbb{T} \cap \mathbb{S}),$$

there is unique functor $J\colon \mathbb{T}/(\mathbb{T} \cap \mathbb{S}) \longrightarrow \mathcal{A}/\mathbb{S}$ *such that the diagram*

$$\begin{array}{ccccc}
\mathbb{T} \cap \mathbb{S} & \longrightarrow & \mathbb{T} & \xrightarrow{\ Q'\ } & \mathbb{T}/(\mathbb{T} \cap \mathbb{S}) \\
{\scriptstyle J'}\downarrow & & {\scriptstyle J_{\mathbb{T}}}\downarrow & & \vdots\downarrow{\scriptstyle J} \\
\mathbb{S} & \longrightarrow & \mathcal{A} & \xrightarrow{\ Q\ } & \mathcal{A}/\mathbb{S}
\end{array}$$

is commutative. The functor J *is an embedding which establishes an equivalence between* $\mathbb{T}/(\mathbb{T} \cap \mathbb{S})$ *and a topologizing subcategory of the category* $\mathcal{A}/\mathbb{S}$.

Proof. Clearly $\mathbb{T} \cap \mathbb{S}$ is a thick subcategory of the category $\mathbb{T}$. Since $Q \circ J_{\mathbb{T}}$ is an exact functor which annihilates $\mathbb{T} \cap \mathbb{S}$, by the universal property of (exact) localizations, there is unique functor J such that $J \circ Q' = Q \circ J_{\mathbb{T}}$.

And $Ker(J) = 0$; i.e. the functor J is faithful.

It remains to show that the full subcategory $\mathbb{T}'$ of $\mathcal{A}/\mathbb{S}$ generated by all objects M which are isomorphic to some object from the image of J is topologizing.

Assume for convenience that the quotient categories $\mathcal{A}/\mathbb{S}$ and $\mathbb{T}/(\mathbb{T} \cap \mathbb{S})$, and localizations Q and Q' are chosen canonically; i.e. $Ob\mathcal{A}/\mathbb{S} = Ob\mathcal{A}$, $Ob\mathbb{T}/(\mathbb{T} \cap \mathbb{S}) = Ob\mathbb{T}$, and the functors Q, Q' map objects identically.

Let

$$0 \longrightarrow M' \xrightarrow{\mathfrak{u}} M \xrightarrow{\mathfrak{v}} M'' \longrightarrow 0$$

be an exact sequence in $\mathcal{A}/\mathbb{S}$ such that $M = Q(M) \in Ob\mathbb{T}$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\mathfrak{u}} & M & \xrightarrow{\mathfrak{v}} & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & QL' & \xrightarrow{Q\mathfrak{u}^\wedge} & QL & \xrightarrow{Q\mathfrak{v}^\wedge} & QL'' & \longrightarrow & 0 \end{array}$$

where all the vertical arrows are isomorphisms, and the diagram

$$0 \longrightarrow L' \xrightarrow{\mathfrak{u}^\wedge} L \xrightarrow{\mathfrak{v}^\wedge} L'' \longrightarrow 0 \qquad (1)$$

is exact ([Gab], Corollary III.1.1). Moreover, the object L in (1) belongs to the subcategory $\mathbb{T}$.

In fact, the morphism $\mathfrak{u}$ is the image of an $\mathfrak{u}' \in \mathcal{A}(K',M/K)$, where K' is a subobject of M', and the objects M'/K' and K belong to $\mathbb{S}$. Since $Q\mathfrak{u}'$ is a monoarrow, $Ker(\mathfrak{u}') \in Ob\mathbb{S}$. Now we take $L' = Coim(\mathfrak{u}')$, $L = M/K$, $\mathfrak{u}^\wedge$ = the induced by $\mathfrak{u}$ arrow, $L'' = Coker(\mathfrak{u}^\wedge)$, $\mathfrak{v}^\wedge$ = the canonical epimorphism. The formulas for the vertical isomorphisms are left to the reader.

Note that $L = M/K$, being a quotient of an object from $\mathbb{T}$, belongs to $\mathbb{T}$. This, in turn, implies that both L' and L'' are objects of $\mathbb{T}$. This proves that the objects M' and M'' belong to the subcategory $\mathbb{T}'$. It is clear that $\mathbb{T}'$ contains with every pair of objects their product (since $\mathbb{T}$ has this property). ■

For a topologizing subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$, denote by $U^\wedge(\mathbb{T})$ the set $\{\mathbb{S} \in \mathbf{Spec}^\wedge\mathcal{A} \mid \mathbb{T} \subseteq \mathbb{S}\}$ and by $V^\wedge(\mathbb{T})$ its complement:

$$V^\wedge(\mathbb{T}) := \mathbf{Spec}^\wedge\mathcal{A} - U^\wedge(\mathbb{T}).$$

1.4.2. Proposition. *For any topologizing subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$, *the map* $\iota_{\mathbb{T}}: \mathbb{S} \longmapsto \mathbb{S} \cap \mathbb{T}$ *is a bijection of the set* $V^\wedge(\mathbb{T})$ *onto* $\mathbf{Spec}^\wedge\mathbb{T}$.

Proof. a) Pick an arbitrary $\mathbb{S} \in V^\wedge(\mathbb{T})$. According to Lemma 1.4.1, the quotient category $\mathbb{T}/(\mathbb{T} \cap \mathbb{S})$ is equivalent to a non-zero topologizing subcategory $\mathbb{T}'$ of the local category $\mathcal{A}/\mathbb{S}$. Being topologizing implies that if $\mathbb{T}'$ contains an object M, then it contains also all objects X such that $M \succ X$. In particular, all quasi-final objects of the category $\mathcal{A}$ belong to $\mathbb{T}'$ which implies that $\mathbb{T}'$ itself is a local category. Therefore $\mathbb{T}/(\mathbb{T} \cap \mathbb{S})$ is local.

b) Injectivity of $\iota_{\mathbb{T}}$. Let $\mathbb{S}$, $\mathbb{S}'$ be elements of $V^{\wedge}(\mathbb{T})$ such that $\mathbb{T} \cap \mathbb{S} = \mathbb{T} \cap \mathbb{S}'$. Replacing $\mathcal{A}$ by $\mathcal{A}/(\mathbb{S} \cap \mathbb{S}')$ and $\mathbb{T}$ by $\mathbb{T}/(\mathbb{T} \cap \mathbb{S} \cap \mathbb{S}')$, we shall assume that $\mathbb{S} \cap \mathbb{S}' = 0$ and (thanks to *a)*) $\mathbb{T}$ is a local (topologizing) subcategory of $\mathcal{A}$ such that $\mathbb{T} \cap \mathbb{S} = 0 = \mathbb{T} \cap \mathbb{S}'$. If $\mathbb{S} - \mathbb{S}'$ is nonempty, then the image, $\mathbb{S}''$, of $\mathbb{S}$ under the localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S}'$ is a thick nonzero subcategory in $\mathcal{A}/\mathbb{S}'$. In particular, it has nonzero intersection with the image of $\mathbb{T}$ in $\mathcal{A}/\mathbb{S}'$. But, this implies that $\mathbb{T} \cap \mathbb{S} \neq 0$ which contradicts to the initial hypothesis.

Thus, $\mathbb{S} \subseteq \mathbb{S}'$, and by symmetry, $\mathbb{S}' \subseteq \mathbb{S}$.

c) It remains to show that the map $\iota_{\mathbb{T}}: V^{\wedge}(\mathbb{T}) \longrightarrow \mathbf{Spec}^{\wedge}\mathbb{T}$ is surjective.

(i) Let $\mathbb{P}$ be any thick subcategory of $\mathbb{T}$; and let $\mathbb{P}'$ denote the smallest thick subcategory of $\mathcal{A}$ containing $\mathbb{P}$. Then $\mathbb{P}' \cap \mathbb{T} = \mathbb{P}$.

Indeed, thanks to 1.3.4, it suffices to show that $\mathbb{P}_{\alpha} \cap \mathbb{T} = \mathbb{P}$ for every ordinal α (see 1.3.4 for the definition of $\mathbb{P}_{\alpha}$).

$\iota)$ It is so by definition if $\alpha = 0$: $\mathbb{P}_0 = \mathbb{P}$.

$\beta)$ Suppose that $\mathbb{P}_{\alpha} \cap \mathbb{T} = \mathbb{P}$ for all $\alpha < \beta$.

If $\beta = \alpha + 1$ for some α, then $\mathbb{P}_{\beta} = \mathbb{P}_{\alpha} \bullet \mathbb{P}_{\alpha}$ (cf. 1.3.3). Take any M from $Ob(\mathbb{P}_{\beta} \cap \mathbb{T})$. By definition, there is an exact sequence

$$0 \longrightarrow M' \xrightarrow{\ u\ } M \xrightarrow{\ v\ } M'' \longrightarrow 0 \qquad (1)$$

where M', M'' are objects of $\mathbb{P}_{\alpha}$. Since $\mathbb{T}$ is topologizing and $M \in Ob\mathbb{T}$, both M' and M'' are objects of $\mathbb{T}$; i.e. M' and M'' belong to $\mathbb{P}_{\alpha} \cap \mathbb{T}$ which, by the induction hypothesis, coincides with $\mathbb{P}$. Thus, the exact sequence (1) lies entirely in $\mathbb{T}$ and its ends, M' and M'', belong to $\mathbb{P}$ which is a thick subcategory in $\mathbb{T}$. Hence M belongs to $\mathbb{P}$.

If β is a limit ordinal, then $\mathbb{P}_{\beta} := \bigcup_{\alpha<\beta} \mathbb{P}_{\alpha}$; hence

$$\mathbb{T} \cap \mathbb{P}_{\beta} = \bigcup_{\alpha<\beta} (\mathbb{T} \cap \mathbb{P}_{\alpha}) = \bigcup_{\alpha<\beta} \mathbb{P} = \mathbb{P}$$

by the induction hypothesis.

(ii) We are ready to finish the proof; i.e. to show that, *for any* $\mathbb{P}$ from

$\mathbf{Spec}^\wedge\mathbb{T}$, *there is an* $\mathbb{S} \in \mathbf{Spec}^\wedge\mathcal{A}$ *such that* $\mathbb{P} = \mathbb{T} \cap \mathbb{S}$.

Let $\mathbb{P}'$ be the smallest thick subcategory of the category $\mathcal{A}$ containing $\mathbb{P}$. Replacing $\mathcal{A}$ by the quotient category $\mathcal{A}/\mathbb{P}'$ and $\mathbb{T}$ by $\mathbb{T}/\mathbb{P}$ (and using the equality $\mathbb{P} = \mathbb{T} \cap \mathbb{P}'$; c.f. *(i)*), we assume that $\mathbb{T}$ is a local topologizing subcategory of $\mathcal{A}$.

Consider the family Ω of all uniform subcategories of $\mathcal{A}$ (cf. 1.3) which have trivial intersection with $\mathbb{T}$. One can see that Ω is closed under the Gabriel multiplication (cf. 1.3.3); i.e. for any pair of categories $\mathbb{X}$, $\mathbb{Y}$ from Ω, their product, $\mathbb{X}\bullet\mathbb{Y}$, belongs to Ω. Since $\mathbb{X} \subseteq \mathbb{X}\bullet\mathbb{Y} \supseteq \mathbb{Y}$, this implies that Ω is directed with respect to the inclusion. Therefore, the union, $\Omega^\wedge$, of all categories from Ω is the largest topologizing subcategory of $\mathcal{A}$ having zero intersection with $\mathbb{T}$. Since $\Omega^\wedge\bullet\Omega^\wedge$ is in Ω, and $\Omega^\wedge \subseteq \Omega^\wedge\bullet\Omega^\wedge$, we have: $\Omega^\wedge = \Omega^\wedge\bullet\Omega^\wedge$; i.e. the subcategory $\Omega^\wedge$ is thick.

We claim that $\Omega^\wedge \in \mathbf{Spec}^\wedge\mathcal{A}$; *i.e.* $\mathcal{A}/\Omega^\wedge$ *is a local category.*

Let P be a quasi-final object of the local subcategory $\mathbb{T}$ regarded as an object of $\mathcal{A}$. Clearly $\langle P\rangle \cap \mathbb{T} = 0$; i.e. $\langle P\rangle \in \Omega$ which implies the inclusion $\langle P\rangle \subseteq \Omega^\wedge$.

Note that the inverse inclusion, $\Omega^\wedge \subseteq \langle P\rangle$, also holds.

Indeed, if there is an object M in $\Omega^\wedge$ such that $M \succ P$, then, since the category $\Omega^\wedge$ is topologizing, $P \in Ob\Omega^\wedge$ which contradicts to the equality $\Omega^\wedge \cap \mathbb{T} = 0$.

Thus we have proved that the thick subcategory $\Omega^\wedge$ coincides with $\langle P\rangle$ which implies that $\mathcal{A}/\Omega^\wedge = \mathcal{A}/\langle P\rangle$ is a local category (cf. [R3], Note 2.6.3). This finishes the proof. ∎

1.4.3. Lemma. *Let* $\mathbb{T}$ *be a topologizing subcategory of an abelian category* $\mathcal{A}$; *and let* $\mathbb{T}'$ *be the smallest thick subcategory of* $\mathcal{A}$ *containing* $\mathbb{T}$. *Then* $V^\wedge(\mathbb{T}) = V^\wedge(\mathbb{T}')$, *and the map*

$$\mathbb{P} \longmapsto \mathbb{T} \cap \mathbb{P}$$

is a bijection of $\mathbf{Spec}\mathbb{T}'$ *onto* $\mathbf{Spec}\mathbb{T}$.

Proof. Clearly $V^\wedge(\mathbb{T}) \subseteq V^\wedge(\mathbb{T}')$.

On the other hand, if $\mathbb{P}$ is a subcategory from $\mathbf{Spec}^\wedge\mathcal{A}$ such that $\mathbb{T} \subseteq \mathbb{P}$, then $\mathbb{T}' \subseteq \mathbb{P}$ (since $\mathbb{P}$ is thick) which proves the inverse inclusion, $V^\wedge(\mathbb{T}') \subseteq V^\wedge(\mathbb{T})$.

The second assertion follows now from Proposition 1.4.2. ∎

1.4.4. Decompositions of the complete spectrum. Let $\mathbb{T}$ be a thick subcategory of an abelian category $\mathcal{A}$. One of the advantages of the complete spectrum is the following decomposition formula:

$$\mathbf{Spec}^\wedge\mathcal{A} \simeq \mathbf{Spec}^\wedge\mathbb{T} \cup \mathbf{Spec}^\wedge\mathcal{A}/\mathbb{T}. \tag{1}$$

The decomposition (1) comes from the decomposition

$$\mathbf{Spec}^\wedge\mathcal{A} = V^\wedge(\mathbb{T}) \cup U^\wedge(\mathbb{T}),$$

the bijection

$$V^\wedge(\mathbb{T}) \longrightarrow \mathbf{Spec}^\wedge\mathbb{T}, \quad \mathbb{P} \longmapsto \mathbb{P} \cap \mathbb{T},$$

of Proposition 1.4.2, and the map

$$\sigma = \sigma_{\mathbb{T}} : U^\wedge(\mathbb{T}) \longrightarrow \mathbf{Spec}^\wedge\mathcal{A}/\mathbb{T}$$

which assigns to a 'point' $\mathbb{P}$ of $U^\wedge(\mathbb{T})$ the subcategory $\mathbb{P}/\mathbb{T}$ of the quotient category $\mathcal{A}/\mathbb{T}$.

Since, for any $\mathbb{P}' \in \mathbf{Spec}\mathcal{A}/\mathbb{T}$, the canonical functor

$$\mathcal{A}/\sigma^{-1}(\mathbb{P}') \longrightarrow (\mathcal{A}/\mathbb{T})/\mathbb{P}'$$

is an equivalence of categories, the map $\mathbb{P}' \longmapsto \sigma^{-1}(\mathbb{P}')$ takes values in the set $U^\wedge(\mathbb{T})$ and is, evidently, inverse to the map $\sigma = \sigma_{\mathbb{T}}$.

Note that the corresponding decomposition for $\mathbf{Spec}\mathcal{A}$ fails in general. Of course, we have a part of it: the bijection of

$$V(\mathbb{T}) := V^\wedge(\mathbb{T}) \cap \mathbf{Spec}\mathcal{A} = \{\langle P\rangle \in \mathbf{Spec}\mathcal{A} \mid P \in Ob\mathbb{T}\}$$

onto $\mathbf{Spec}\mathbb{T}$, and the injection of

$$U(\mathbb{T}) := U^\wedge(\mathbb{T}) \cap \mathbf{Spec}\mathcal{A} = \{\langle P\rangle \in \mathbf{Spec}\mathcal{A} \mid \mathbb{T} \subseteq \langle P\rangle\}$$

into $\mathbf{Spec}\mathcal{A}/\mathbb{T}$. However the latter map is usually not surjective.

1.5. Topologies. The defined in 1.2 topology τ can be obtained as follows: a subset $U \subseteq \mathbf{Spec}^\wedge\mathcal{A}$ is open with respect to τ iff $U = U^\wedge(\mathbb{T}) \simeq \mathbf{Spec}^\wedge\mathbb{T}$ for some topologizing category $\mathbb{T}$. Besides, we have:

$$U^\wedge(\mathbb{S}\bullet\mathbb{T}) = U^\wedge(\mathbb{S}) \cap U^\wedge(\mathbb{T}) \tag{1}$$

for any pair $\mathbb{S}$, $\mathbb{T}$ of topologizing categories, and

$$U^\wedge(\bigcap_{\mathbb{T}'\in\Omega} \mathbb{T}') = \bigcup_{\mathbb{T}'\in\Omega} U^\wedge(\mathbb{T}') \tag{2}$$

for any family Ω of topologizing subcategories.

Note that, for any thick subcategory $\mathbb{T}$ of the category $\mathcal{A}$, the maps

$$\mathbf{Spec}^\wedge\mathcal{A}/\mathbb{T} \longrightarrow U^\wedge(\mathbb{T}) \quad \text{and} \quad V^\wedge(\mathbb{T}) \longrightarrow \mathbf{Spec}^\wedge\mathbb{T}$$

(c.f. 1.4.4) are homeomorphisms with respect to the topology τ.

Since any other compatible with specializations topology on $\mathbf{Spec}^\wedge\mathcal{A}$ is weaker than τ, its open sets are of the form $U^\wedge(\mathbb{T})$, where $\mathbb{T}$ runs through some set of topologizing subcategories. So, a way to obtain a topology on $\mathbf{Spec}^\wedge\mathcal{A}$ is to choose a set, say Ξ, of topologizing subcategories and declare the set $\{U^\wedge(\mathbb{S}) \mid \mathbb{S} \in \Xi\}$ a base of open sets of the topology in question.

1.6. Complete supports. Define *the complete support* of an object M of an abelian category $\mathcal{A}$ as the set $Supp^\wedge(M)$ of all points $\mathbb{P}$ of $\mathbf{Spec}^\wedge\mathcal{A}$ such that $M \notin \mathbb{P}$. Clearly, $Supp^\wedge(M)$ is closed in the topology τ for any object M.

1.6.1. Lemma. *For any exact short sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$
$$Supp^\wedge(M) = Supp^\wedge(M') \cup Supp^\wedge(M'').$$

Proof. The argument goes along the same lines as that of the first assertion of Proposition 5.2.2 in [R3]:

If $M \in Ob\mathbb{P}$, then both M' and M'' belong $\mathbb{P}$ which is equivalent to the inclusion $Supp^\wedge(M') \cup Supp^\wedge(M'') \subseteq Supp^\wedge(M)$.

Note that $\mathbb{P} \in Supp^\wedge(M)$ iff $Q_{\mathbb{P}}(M) \neq 0$. Since the sequence

$$0 \longrightarrow Q_{\mathbb{P}}(M') \longrightarrow Q_{\mathbb{P}}(M) \longrightarrow Q_{\mathbb{P}}(M'') \longrightarrow 0$$

is exact, $Q_{\mathbb{P}}(M) \neq 0$ implies that $Q_{\mathbb{P}}(M') \oplus Q_{\mathbb{P}}(M'') \neq 0$. ∎

For any subset W of $\mathbf{Spec}^\wedge\mathcal{A}$, denote by $\mathcal{A}^\wedge(W)$ the full subcategory of $\mathcal{A}$ generated by all objects M of $\mathcal{A}$ such that $Supp^\wedge(M) \subseteq W$.

1.6.2. Proposition. *(a) For any* $W \subseteq \mathbf{Spec}^\wedge\mathcal{A}$, *we have:*

$$\mathcal{A}^\wedge(W) = \bigcap_{\mathbb{P} \in W\perp} \mathbb{P},$$

where $W\perp = \mathbf{Spec}^\wedge\mathcal{A} - W$.

In particular, the subcategory $\mathcal{A}^\wedge(W)$ *is thick.*

(b) $\mathbf{V}^\wedge(\mathcal{A}(W)) = W$ *if and only if the set* W *is closed in the topology* τ.

Proof. (a) The inclusion $M \in \bigcap_{\mathbb{P} \in W\perp} \mathbb{P}$ means exactly that

$$Supp^\wedge(M) \subseteq \mathbf{Spec}^\wedge\mathcal{A} - W\perp = W.$$

(b) The set $\mathbf{V}^\wedge(\mathcal{A}(W))$ consists of all $\mathbb{P} \in \mathbf{Spec}^\wedge\mathcal{A}$ such that $\mathcal{A}(W)$ is not a subcategory of $\mathbb{P}$; i.e. there is an object M in $\mathcal{A}$ such that $Supp^\wedge(M) \subseteq W$ and $M \notin Ob\mathbb{P}$. The latter means that $\mathbb{P} \in Supp^\wedge(M)$. Hence $\mathbf{V}^\wedge(\mathcal{A}(W)) \subseteq W$.

Suppose now that W is closed in the topology τ. Then we claim the inverse inclusion: $W \subseteq \mathbf{V}^\wedge(\mathcal{A}(W))$. This inclusion means that, for every $\mathbb{P} \in W$, there is an object M such that

$$Supp^\wedge(M) \subseteq W \text{ and } \mathbb{P} \in Supp^\wedge(M).$$

Take an object M such that $Q_{\mathbb{P}}(M)$ is a quasi-final object of the (local) category $\mathcal{A}/\mathbb{P}$. Note now that

$$Supp^\wedge(M) = \mathfrak{s}(\mathbb{P}) := \{\mathbb{P}' \in \mathbf{Spec}^\wedge\mathcal{A} \mid \mathbb{P}' \subseteq \mathbb{P}\}$$

is the set of specializations of the point $\mathbb{P}$; i.e. $Supp^\wedge(M)$ is the closure of $\mathbb{P}$ in the topology τ.

In fact, it is clear that $\mathbb{P} \in Supp^\wedge(M)$; therefore $\mathfrak{s}(\mathbb{P})$ is a subset of $Supp^\wedge(M)$ (since the latter set is closed in τ).

Let $\mathbb{P}'$ be an arbitrary point of $Supp^\wedge(M)$. Since M does not belong to $\mathbb{P}'$, its localization at $\mathbb{P}$ does not belong to the subcategory $\mathbb{P}'/(\mathbb{P} \cap \mathbb{P}')$ of $\mathcal{A}/\mathbb{P}$. But, since $Q_{\mathbb{P}}(M)$ is a quasi-final object of $\mathcal{A}/\mathbb{P}$ and the subcategory $\mathbb{P}'/(\mathbb{P} \cap \mathbb{P}')$ is thick, this means that $\mathbb{P}'/(\mathbb{P} \cap \mathbb{P}') = 0$; i.e. $\mathbb{P}' \subseteq \mathbb{P}$. ■

1.7. Associated points. For any object M of an abelian category $\mathcal{A}$, denote by $Ass^\wedge(M)$ the set of $\mathbb{P} \in \mathbf{Spec}^\wedge\mathcal{A}$ for which there exists a subobject X of M such that its localization, $Q_{\mathbb{P}}(X)$, is a quasi-final object of $\mathcal{A}/\mathbb{P}$.

Clearly $Ass^\wedge(M) \subseteq Supp^\wedge(M)$.

1.7.1. Lemma. *Let* $\mathbb{P} \in$ **Spec^**$\mathcal{A}$; *and let* M *be a* $\mathbb{P}$-*torsion free object of the category* $\mathcal{A}$ *such that* $Q_{\mathbb{P}}(M)$ *is quasi-final.*

Then $Ass^\wedge(M) = \{\mathbb{P}\}$.

Proof. As it was shown in the argument proving Proposition 1.6.2, $Supp^\wedge(M) = \mathfrak{s}(\mathbb{P})$. Thus, we have: $\mathbb{P} \in Ass^\wedge(M) \subseteq \mathfrak{s}(\mathbb{P})$.

Suppose that $\mathbb{P}' \in Ass^\wedge(M)$ (hence $\mathbb{P}' \subseteq \mathbb{P}$); and let X be a subobject of M such that $Q_{\mathbb{P}'}(X)$ is a quasi-final object of $\mathcal{A}/\mathbb{P}'$. If $\mathbb{P}' \neq \mathbb{P}$, then $\mathbb{P}/\mathbb{P}'$, being a nonzero thick subcategory of the local category $\mathcal{A}/\mathbb{P}'$, contains $Q_{\mathbb{P}'}(X)$. This means that X belongs to $\mathbb{P}$. But, this contradicts to the fact that X is a nonzero subobject of a $\mathbb{P}$-torsion free object. Therefore $\mathbb{P} = \mathbb{P}'$. ∎

1.7.2. Proposition. *For any short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Ass^\wedge(M') \subseteq Ass^\wedge(M) \subseteq Ass^\wedge(M') \cup Ass^\wedge(M'').$$

Proof. Clearly $Ass^\wedge(M') \subseteq Ass^\wedge(M)$.

Let $\mathbb{P} \in Ass^\wedge(M)$; and let X be a subobject of M such that $Q_{\mathbb{P}}(X)$ is a quasi-final object in $\mathcal{A}/\mathbb{P}$. There are only two alternatives: either $X \cap M' \notin Ob\mathbb{P}$, or $X \cap M' \in Ob\mathbb{P}$.

If $X' := X \cap M' \notin Ob\mathbb{P}$, then $Q_{\mathbb{P}}(X')$ is a nonzero subobject of $Q_{\mathbb{P}}(X)$; hence $Q_{\mathbb{P}}(X')$ is also a quasi-final object in $\mathcal{A}/\mathbb{P}$ which means that $\mathbb{P}$ belongs to $Ass^\wedge(M')$.

If $X' = X \cap M' \in Ob\mathbb{P}$, then $X'' := X/X'$ is a subobject of M'' and the localization $Q_{\mathbb{P}}$ sends the projection $X \longrightarrow X''$ to an isomorphism. In particular, $Q_{\mathbb{P}}(X'')$ is a quasi-final object of $\mathcal{A}/\mathbb{P}$. therefore $\mathbb{P} \in Ass^\wedge(M'')$. ∎

1.7.3. Corollary. *For any finite family* Ω *of objects of* $\mathcal{A}$,

$$Ass^\wedge\Big(\bigoplus_{M \in \Omega} M\Big) = \bigcup_{M \in \Omega} Ass^\wedge(M).$$

Proof. It suffices to check the equality for a set consisting of two objects: $\Omega = \{M, L\}$. According to Proposition 1.7.2, we have:

$$Ass^\wedge(M) \cup Ass^\wedge(L) \subseteq Ass^\wedge(M \oplus L) \subseteq Ass^\wedge(M) \cup Ass^\wedge(L). \quad ∎$$

1.7.4. Corollary. *Let* Ξ *be a finite family of subobjects of an object* M *such that* $\bigcap_{X \in \Xi} X = 0$. *Then* $Ass^{\wedge}(M) \subseteq \bigcup_{X \in \Xi} Ass^{\wedge}(M/X)$.

Proof. The assertion is a consequence of Corollary 1.7.3 and of the monomorphness of the canonical arrow $M \longrightarrow \bigoplus_{X \in \Xi} M/X$. ■

1.7.5. Associate points and exact localizations. It is shown in [R3], Section 8.5.4, that any exact localization Q induces an injection of the set

$$\{<P> \in Ass(M) \mid KerQ \subseteq <P>\} \text{ into } Ass(Q(M)).$$

The set $Ass^{\wedge}(M)$ has a better property:

1.7.5.1. Proposition. *Let* $Q: \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ *be a localization at a thick subcategory* $\mathbb{T}$. *For any object* M *of* $\mathcal{A}$, *the localization* Q *induces a bijection of* $Ass^{\wedge}(M) \cap \mathbf{U}^{\wedge}(\mathbb{T})$ *onto* $Ass^{\wedge}(Q(M))$.

Proof. a) It is shown in 1.4.4 that the localization $Q = Q_{\mathbb{T}}$ induces a bijection of the open (in the topology τ) set $\mathbf{U}^{\wedge}(\mathbb{T})$ onto $\mathbf{Spec}^{\wedge}\mathcal{A}/\mathbb{T}$.

Let $\mathbb{P} \in Ass^{\wedge}(M) \cap \mathbf{U}^{\wedge}(\mathbb{T})$; and let X be a subobject of M such that $Q_{\mathbb{P}}(X)$ is a quasi-final object of $\mathcal{A}/\mathbb{P}$. Then $Q_{\mathbb{T}}(X)$ is a subobject of $Q_{\mathbb{T}}(M)$, and $Q_{\mathbb{P}/\mathbb{T}}(Q_{\mathbb{T}}(X)) \simeq Q_{\mathbb{P}}(X)$ (we identify the quotient category $(\mathcal{A}/\mathbb{T})/(\mathbb{P}/\mathbb{T})$ with $\mathcal{A}/\mathbb{P}$) is a quasi-final object. This shows that the canonical bijection

$$\mathbf{U}^{\wedge}(\mathbb{T}) \longrightarrow \mathbf{Spec}^{\wedge}\mathcal{A}/\mathbb{T}$$

induces an injection $Ass^{\wedge}(M) \cap \mathbf{U}^{\wedge}(\mathbb{T}) \longrightarrow Ass^{\wedge}(Q_{\mathbb{T}}(M))$.

b) Conversely, let $\mathbb{P}'$ be an arbitrary point of $Ass^{\wedge}(Q(M))$; and let

$$f: X' \longrightarrow Q(M)$$

be a monomorphism such that $Q_{\mathbb{P}'}(X')$ is a quasi-final object. The morphism f is the image of an element $f'' \in \mathcal{A}(Y, M/L)$, where X'/Y and L belong to $\mathbb{T}$. Denote by X'' the pullback of the arrows

$$Y \longrightarrow M/L \longleftarrow M.$$

Let X denote the image of the projection $X'' \longrightarrow M$; and let

$$\gamma : X \longrightarrow M$$

be the canonical monoarrow. One can see that the subobject

$$Q\gamma : Q(X) \longrightarrow Q(M)$$

is isomorphic to $f: X' \longrightarrow Q(M)$. Therefore the object $Q_{\mathbb{P}}(X)$, where $\mathbb{P} \in \mathbf{U}^\wedge(\mathbb{T})$ is the preimage of $\mathbb{P}'$ in $\mathcal{A}$, is quasi-final. This shows that the map

$$\mathbf{U}^\wedge(\mathbb{T}) \cap Ass^\wedge(M) \longrightarrow Ass^\wedge(Q(M))$$

is surjective. ∎

1.8. Fields of fractions. With a local category $\mathcal{B}$, we associate the category $\mathcal{K}(\mathcal{B})$ - "the residue category of $\mathcal{B}$" which is by definition (given in [R3], 5.4) the full subcategory of the category $\mathcal{B}$ generated by all those objects of $\underline{B}$ which are supremums of the family of their quasi-final subobjects.

Recall that if the category $\mathcal{B}$ has simple objects, then the residue category of $\mathcal{B}$ is equivalent to the category $K(\mathcal{B})\text{-}Vec$ of vector spaces over the skew field $K(\mathcal{B})$ of endomorphisms of a simple object of the category $\mathcal{B}$ (cf. Lemma 5.4.1 in [R3]). Since the category $\mathcal{B}$ is local, this simple object is unique up to isomorphism; hence the skew field $K(\mathcal{B})$ is defined uniquely up to isomorphism.

Thus, given a general abelian category $\mathcal{A}$, we can assign to each point $\mathbb{P}$ of $\mathbf{Spec}^\wedge\mathcal{A}$ the residue category $\mathcal{K}_{\mathbb{P}} := \mathcal{K}(\mathcal{A}/\mathbb{P})$ of the point $\mathbb{P}$. And if the category $\mathcal{A}/\mathbb{P}$ has simple objects, then the category $\mathcal{K}_{\mathbb{P}} := \mathcal{K}(\mathcal{A}/\mathbb{P})$ is equivalent to the category of vector spaces over the *residue skew field* $K_{\mathbb{P}} := K(\mathcal{A}/\mathbb{P})$ of the point $\mathbb{P}$.

1.9. Complete spectrum and the center. Consider the center $\mathfrak{z}(\mathcal{A}) := End(Id_{\mathcal{A}})$ of an abelian category $\mathcal{A}$. Any localization Q of the category $\mathcal{A}$ maps the center of $\mathcal{A}$ to the center of the quotient category. In particular, the localization at any point $\mathbb{P} \in \mathbf{Spec}^\wedge\mathcal{A}$ provides a ring homomorphism

$$\lambda_{\mathbb{P}} : \mathfrak{z}(\mathcal{A}) \longrightarrow \mathfrak{z}(\mathcal{A}/\mathbb{P}).$$

According to Proposition 2.5.1 in [R3], $\mathfrak{z}(\mathcal{A}/\mathbb{P})$ is a local ring. Denote by $(\mathbb{P})$ the preimage, $\lambda_{\mathbb{P}}^{-1}(m_{\mathbb{P}})$, of the (unique) maximal ideal $m_{\mathbb{P}}$ of the ring $\mathfrak{z}(\mathcal{A}/\mathbb{P})$. Thus, we have a map

$$\varphi^\wedge = \varphi^\wedge_{\mathcal{A}} : \mathbf{Spec}^\wedge\mathcal{A} \longrightarrow Spec\,\mathfrak{z}(\mathcal{A}), \quad \mathbb{P} \longmapsto (\mathbb{P}).$$

There is the following analogue of Lemma 7.1.1 in [R3]:

1.9.1. Lemma. *Suppose that* $\mathbb{P} \in$ **Spec**$^\wedge\mathcal{A}$ *has the property: there is a* $\mathbb{P}$*-torsion free object* X *such that* $Q_{\mathbb{P}}(X)$ *is a quasi-final object in* $\mathcal{A}/\mathbb{P}$ *(which is the case if* $\mathbb{P} \in$ **Spec**$\mathcal{A}$, *or if the subcategory* $\mathbb{P}$ *is coreflective, i.e. the inclusion functor* $\mathbb{P} \longrightarrow \mathcal{A}$ *has a right adjoint). Then*

(a) For any $\xi \in \mathfrak{z}(\mathcal{A})$, *either* $\xi(X)$ *is a monomorphism, or* $\xi(X)$ *is zero.*

(b) The ideal $(\mathbb{P})$ *consists of all* $\xi \in \mathfrak{z}(\mathcal{A})$ *for which* $\xi(X) = 0$.

Proof. (a) Suppose that $Ker\xi(X) \neq 0$. Then, since the object X is $\mathbb{P}$-torsion free, such is its subobject $Ker\xi(X)$. Thanks to the (left) exactness of the localization $Q = Q_{\mathbb{P}}$, the canonical arrow

$$Q(Ker\xi(X)) \longrightarrow KerQ\xi(Q(X))$$

is an isomorphism. Since

$$Q\xi(KerQ\xi(Q(X))) = 0 \quad \text{and} \quad KerQ\xi(Q(X)) \rightarrowtail Q(X),$$

we have: $Q\xi(Q(X)) = 0$ (cf. the proof of Lemma 7.1.1 in [R3]). Finally, since X is $\mathbb{P}$-torsion free, the equality $Q\xi(Q(X)) = 0$ is equivalent to the equality $\xi(X) = 0$.

(b) On the other hand, the equality $Q\xi(Q(X)) = 0$ means that $Q\xi$ belongs to the unique maximal ideal of the local ring $\mathfrak{z}(\mathcal{A}/\mathbb{P})$ (cf. the argument of Proposition 2.5.1 in [R3]). ∎

Lemma 1.9.1 shows that the map

$$\varphi^\wedge_{\mathcal{A}}: \mathbf{Spec}^\wedge\mathcal{A} \longrightarrow Spec\mathfrak{z}(\mathcal{A}), \quad \mathbb{P} \longmapsto (\mathbb{P})$$

is a natural extension of the defined in [R3], Section 7.1 map

$$\varphi_{\mathcal{A}}: \mathbf{Spec}\mathcal{A} \longrightarrow Spec\mathfrak{z}(\mathcal{A}), \quad <P> \longmapsto \{\xi \in \mathfrak{z}(\mathcal{A}) \mid \xi(P) = 0\}$$

(cf. [R3], Corollary 7.1.2).

We define the *central topology* $\tau_{\mathfrak{z}}$ on **Spec**$^\wedge\mathcal{A}$ exactly like we have defined the central topology on **Spec**$\mathcal{A}$ in [R3], Section 7.1.

Namely, $\tau_{\mathfrak{z}}$ is the weakest topology for which the map $\varphi^\wedge_{\mathcal{A}}$ is continuous.

2. THE FLAT SPECTRUM.

Now we shall take a step back, and consider the subset $\mathbf{Spec}^{-}\mathbb{A}$ of the complete spectrum $\mathbf{Spec}^{\wedge}\mathbb{A}$ formed by all *Serre* subcategories $\mathbb{P}$ of $\mathbb{A}$ such that $\mathbb{A}/\mathbb{P}$ is a local category. We call the subset $\mathbf{Spec}^{-}\mathbb{A}$ *the flat spectrum of the category* $\mathbb{A}$.

Recall that a subcategory $\mathbb{T}$ of $\mathbb{A}$ is *coreflective* if the inclusion functor $\mathbb{T} \longrightarrow \mathbb{A}$ has a right adjoint. In other words, if every object M of $\underline{A}$ has a subobject maximal among all the subobjects of M which belong to $\mathbb{T}$.

2.1. Lemma. *Let* $\mathbb{T}$ *be a coreflective thick subcategory of an abelian category* $\mathbb{A}$; *and let* Q *be a localization* $\mathbb{A} \longrightarrow \mathbb{A}/\mathbb{T}$.

For any Serre subcategory $\mathbb{X}$ *of* $\mathbb{A}/\mathbb{T}$, *its preimage* $Q^{-1}(\mathbb{X})$ *is a Serre subcategory of the category* $\mathbb{A}$.

Proof. Let M be an arbitrary object of $Q^{-1}(\mathbb{X})^{-}$; i.e. any nonzero subquotient of M has a nonzero subobject which belongs to $Q^{-1}(\mathbb{X})$. We claim that $Q(M)$ belongs to $\mathbb{X}$.

Indeed, for any nonzero epimorphism $f: Q(M) \longrightarrow L$, there is a commutative diagram

$$\begin{array}{ccc} Q(M) & \xrightarrow{\quad e \quad} & L \\ \downarrow & & \downarrow \\ Q(M') & \xrightarrow{\ Qe'\ } & Q(L') \end{array}$$

in which M' is a subobject of M; the both vertical arrows are invertible, $e': M' \longrightarrow L'$ is an epimorphism, and the object L' is $\mathbb{T}$-torsion free (the last property is available thanks to the coreflectiveness of $\mathbb{T}$). Since the object M', being a subobject of M, belongs to $Q^{-1}(\mathbb{X})^{-}$, there is a nonzero monoarrow

$$\mathrm{i} : K \longrightarrow L'$$

such that $K \in ObQ^{-1}(\mathbb{X})$; or equivalently, $Q(K) \in \mathbb{X}$. Note that, $Q(K) \neq 0$, because K is nonzero and $\mathbb{T}$-torsion free. And, since $Q\mathrm{i}$ is a monoarrow, $Q(K)$ is a subobject of L.

This shows that $Q(M) \in Ob\mathbb{X}^{-} = Ob\mathbb{X}$ (since, by hypothesis, $\mathbb{X}$ is a Serre subcategory; i.e. $\mathbb{X}^{-} = \mathbb{X}$). Or, equivalently M is an object of $Q^{-1}(\mathbb{X})$. Since M had been chosen arbitrarily, we have proved that $Q^{-1}(\mathbb{X}) = Q^{-1}(\mathbb{X})^{-}$. ∎

2.2. Proposition. *For any coreflective thick subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$*, the canonical embedding*

$$\mathbf{Spec}\mathcal{A}/\mathbb{T} \longrightarrow \mathbf{Spec}^{\wedge}\mathcal{A}.$$

of Proposition 1.1 *induces an embedding*

$$\mathbf{Spec}\mathcal{A}/\mathbb{T} \longrightarrow \mathbf{Spec}^{-}\mathcal{A}.$$

Proof. By Proposition 3.3 in [R3], $<P>$ is a Serre subcategory of $\mathcal{A}/\mathbb{T}$ for any $P \in Spec\mathcal{A}/\mathbb{T}$. According to Lemma 2.1, $Q^{-1}(<P>)$ is a Serre subcategory of the category $\mathcal{A}$. But, $Q^{-1}(<P>)$ is the image of $<P>$ under the embedding

$$\mathbf{Spec}\mathcal{A}/\mathbb{T} \longrightarrow \mathbf{Spec}^{\wedge}\mathcal{A}. \quad \blacksquare$$

Although Proposition 2.2 looks somewhat restrictive, we still can represent $\mathbf{Spec}^{-}\mathcal{A}$ as the union of the images of the $\mathbf{Spec}\mathcal{A}/\mathbb{T}$, where $\mathbb{T}$ runs through the set $\mathbf{Serre}(\mathcal{A})$ of Serre subcategories of the category $\mathcal{A}$:

2.3. Proposition. *For any abelian category* $\mathcal{A}$,

$$\mathbf{Spec}^{-}\mathcal{A} = \bigcup_{\mathbb{T} \in \mathbf{Serre}(\mathcal{A})} \mathbf{Spec}(\mathcal{A},\mathbb{T}) = \bigcup_{\mathbb{T} \in \mathbf{Spec}^{-}(\mathcal{A})} \mathbf{Spec}(\mathcal{A},\mathbb{T}),$$

where $\mathbf{Spec}(\mathcal{A},\mathbb{T})$ *is the image of* $\mathbf{Spec}\mathcal{A}/\mathbb{T}$ *in* $\mathbf{Spec}^{\wedge}\mathcal{A}$ *(cf.* 1.1*).*

Proof. Recall that $\mathbf{Spec}(\mathcal{A},\mathbb{T})$ consists of all subcategories $Q^{-1}<P>$, where $<P>$ runs through $\mathbf{Spec}\mathcal{A}/\mathbb{T}$.

Let $\mathbb{T} \in \mathbf{Spec}^{-}\mathcal{A}$, Q a localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$; and let P be a quasi-final object in the local category $\mathcal{A}/\mathbb{T}$. Then, since $<P> = 0$, the subcategory $Q^{-1}<P>$ coincides with the Serre subcategory $\mathbb{T}$. $\blacksquare$

For a topologizing subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$, denote by $U^{-}(\mathbb{T})$ the set $\{\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A} \mid \mathbb{T} \subseteq \mathbb{P}\}$ and by $V^{-}(\mathbb{T})$ its complement:

$$V^{-}(\mathbb{T}) := \mathbf{Spec}^{-}\mathcal{A} - U^{-}(\mathbb{T}).$$

2.4. Proposition. 1) *For any topologizing subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$*, the map* $\mathbb{P} \longmapsto \mathbb{P} \cap \mathbb{T}$ *defines a bijection,* $^{-}\iota_{\mathbb{T}}$*, of the set* $V^{-}(\mathbb{T})$ *onto* $\mathbf{Spec}^{-}\mathbb{T}$.

2) If $\mathbb{T}$ *is a coreflective thick subcategory, then the map*

$$\mathbf{Spec}\mathcal{A}/\mathbb{T} \longrightarrow \mathbf{Spec}^{-}\mathcal{A}$$

of Proposition 2.2 unduces a bijection of $\mathbf{Spec}\mathcal{A}/\mathbb{T}$ *onto* $U^{-}(\mathbb{T})$.

Proof. 1) *(i)* The intersection of any Serre subcategory $\mathbb{S}$ of the category $\mathcal{A}$ with $\mathbb{T}$ is a Serre subcategory of $\mathbb{T}$. In particular, $\mathbb{P} \cap \mathbb{T}$ is a Serre subcategory of $\mathbb{T}$ for every point $\mathbb{P}$ of $\mathbf{Spec}^{-}\mathcal{A}$. Therefore, by Proposition 1.4.2, the subcategory $\mathbb{P} \cap \mathbb{T}$ belongs to $\mathbf{Spec}^{-}\mathbb{T}$.

The injectivity of $^{-}\iota_{\mathbb{T}}$: $V^{-}(\mathbb{T}) \longrightarrow \mathbf{Spec}^{-}\mathbb{T}$ follows from the injectivity of $\iota_{\mathbb{T}}$: $V^{\wedge}(\mathbb{T}) \longrightarrow \mathbf{Spec}^{\wedge}\mathbb{T}$ (cf. Proposition 1.4.2).

(ii) It remains to prove the surjectivity of $^{-}\iota_{\mathbb{T}}$.

Let $\mathbb{P} \in \mathbf{Spec}^{-}\mathbb{T}$. By Proposition 1.4.2, there exists a unique subcategory $\mathbb{P}'$ from $\mathbf{Spec}^{\wedge}\mathcal{A}$ such that $\mathbb{T} \cap \mathbb{P}' = \mathbb{P}$.

a) Note that $\mathbb{T} \cap \mathbb{P}'^{-} = \mathbb{P}$.

In fact, since the subcategory $\mathbb{T}$ is topologizing and $\mathbb{P}$ is a Serre subcategory in $\mathbb{T}$, i.e. $\mathbb{P}^{-} \cap \mathbb{T} = \mathbb{P}$, we have:

$$\mathbb{T} \cap \mathbb{P}'^{-} = (\mathbb{T} \cap \mathbb{P}')^{-} \cap \mathbb{T} = \mathbb{P}^{-} \cap \mathbb{T} = \mathbb{P}.$$

b) The equality $\mathbb{T} \cap \mathbb{P}'^{-} = \mathbb{P}$ means that the intersection of the subcategories $\mathbb{P}'^{-}/\mathbb{P}'$ and $\mathbb{T}/\mathbb{P}$ of the local category $\mathcal{A}/\mathbb{P}'$ is trivial. Since $\mathbb{T}/\mathbb{P}$ contains the quasi-final object of $\mathcal{A}/\mathbb{P}'$, this implies that $\mathbb{P}'^{-}/\mathbb{P}' = 0$; i.e. $\mathbb{P}'^{-} = \mathbb{P}'$.

2) The second assertion follows from Proposition 2.2. ∎

2.5. Remarks about coreflective thick subcategories. The following Proposition is relevant to Lemma 2.1.

2.5.1. Proposition. *Let* $\mathbb{S}$ *and* $\mathbb{T}$ *be thick subcategories in* $\mathcal{A}$. *If the subcategory* $\mathbb{T}$ *is coreflective, then the canonical functor*

$$J': \mathbb{T}/(\mathbb{T} \cap \mathbb{S}) \longrightarrow \mathcal{A}/\mathbb{S}$$

has a right adjoint functor; i.e. $\mathbb{T}/(\mathbb{T} \cap \mathbb{S})$ *is a coreflective subcategory of* $\mathcal{A}/\mathbb{S}$.

Proof. Denote for convenience $\mathbb{S} \cap \mathbb{T}$ by $\mathbb{X}$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{T}/\mathbb{X} & \xrightarrow{\quad J' \quad} & \mathcal{A}/\mathbb{S} \\ {\scriptstyle Q=Q_{\mathbb{X}}}\Big\uparrow & & \Big\uparrow{\scriptstyle Q_{\mathbb{S}}=Q'} \\ \mathbb{T} & \xrightarrow{\quad J \quad} & \mathcal{A} \end{array}$$

(i) Note that the functor $Q_{\mathbb{X}}\circ J^{\wedge}$ inverts all morphisms which are inverted by the localization $Q_{\mathbb{S}}$.

In fact, take an arrow s such that $Q's$ is an isomorphism. This means that the objects X and Y in the exact sequence

$$0 \longrightarrow X = Ker(s) \xrightarrow{\ i\ } M \xrightarrow{\ s\ } L \xrightarrow{\ \varepsilon\ } Y = Cok(s) \longrightarrow 0$$

belong to $\mathbb{S}$.

Since the functor $J^{\wedge}$ - the right adjoint to $J\colon \mathbb{T} \longrightarrow \mathcal{A}$ - is left exact, the sequence

$$0 \longrightarrow J^{\wedge}X \xrightarrow{\ J^{\wedge}i\ } J^{\wedge}M \xrightarrow{\ J^{\wedge}s\ } J^{\wedge}L \xrightarrow{\ J^{\wedge}\varepsilon\ } J^{\wedge}Y$$

is exact. Since the cokernel of $J^{\wedge}s$ $(= Im(J^{\wedge}\varepsilon))$ is a subobject of the object $J^{\wedge}Y$ of $\mathbb{T} \cap \mathbb{S} = Ob\mathbb{X}$, it is also an object of $\mathbb{X}$; as well as the kernel of $J^{\wedge}s$ (which is isomorphic to $J^{\wedge}X$). Therefore the functor $Q = Q_{\mathbb{X}}$ inverts the arrow $J^{\wedge}s$.

(ii) It follows from *(i)* that, due to the universal property of the localization $Q = Q_{\mathbb{S}}$, there exists unique functor

$$\Psi : \mathcal{A}/\mathbb{S} \longrightarrow \mathbb{T}/\mathbb{X}$$

such that $Q\circ J^{\wedge} = \Psi\circ Q'$. We have:

$$(\Psi\circ J')\circ Q = \Psi\circ Q'\circ J = Q\circ J^{\wedge}\circ J \xrightarrow{\ Q\varepsilon\ } Q \tag{1}$$

$$(J'\circ\Psi)\circ Q' = J'\circ Q\circ J^{\wedge} = Q'\circ J\circ J^{\wedge} \xleftarrow{\ Q'\gamma\ } Q' \tag{2}$$

Here ε and γ are the adjunction arrows

$$Id_{\mathbb{T}} \longrightarrow J^{\wedge}\circ J \quad \text{and} \quad J\circ J^{\wedge} \longrightarrow Id_{\mathcal{A}}$$

respectively.

Thanks to the universality of Q and Q', there is unique isomorphism

$$\delta'\colon Id_{\mathbb{T}/\mathbb{X}} \longrightarrow \Psi\circ J'$$

and a unique functor morphism

$$\gamma': J'\circ\Psi \longrightarrow Id_{\mathcal{A}/\mathcal{S}}$$

such that $Q\varepsilon = \delta' Q$ and $Q'\gamma = \gamma' Q'$.

We have:

$$(\gamma' J'\circ J'\delta')Q = \gamma' J'Q\circ J'\delta' Q = \gamma' Q'J\circ J'Q\varepsilon = Q'\gamma J\circ Q'J\varepsilon = Q'(\gamma J\circ J\varepsilon) =$$

$$Q'id_J = id_{J'}Q$$

and

$$(\Psi\gamma'\circ\delta'\Psi)Q' = \Psi\gamma' Q'\circ\delta'\Psi Q' = \Psi Q'\gamma\circ\delta' QJ^{\wedge} = QJ^{\wedge}\gamma\circ Q\varepsilon J^{\wedge} = Q(J^{\wedge}\gamma\circ\varepsilon J^{\wedge}) =$$

$$Qid_{J^{\wedge}} = id_{\Psi}Q'.$$

Thanks to the universality of Q and Q', the equalities

$$(\gamma' J'\circ J'\delta')Q = id_{J'}Q \quad \text{and} \quad (\Psi\gamma'\circ\delta'\Psi)Q' = id_{\Psi}Q'$$

imply the equlities

$$\gamma' J'\circ J'\delta' = id_{J'} \quad \text{and} \quad \Psi\gamma'\circ\delta'\Psi = id_{\Psi}$$

which mean exactly that δ' and γ' are adjunction morphisms. ∎

Recall that if the category $\mathcal{A}$ has the property *(sup)* (cf. 0.4.3.2), then every Serre subcategory of $\mathcal{A}$ is coreflective (Lemma 2.4.4 in [R3]).

And if $\mathcal{A}$ has injective hulls, then every coreflective thick subcategory is localizing (cf. [Gab], Corollary III.3.3).

In particular, if an abelian category $\mathcal{A}$ has both the property *(sup)* and injective hulls (e.g. $\mathcal{A}$ is a Grothendieck category), then every Serre subcategory of $\mathcal{A}$ is localizing. Note that, in this case, the name *flat spectrum* becomes meaningful:

for any $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ *the localization at* $\mathbb{P}$ *is flat.*

2.6. Flat supports. Define the *flat support* of an object M of an abelian category $\mathcal{A}$ as the set $Supp^{-}(M)$ of all points $\mathbb{P}$ of $\mathbf{Spec}^{-}\mathcal{A}$ such that $M \notin \mathbb{P}$.

In other words, $Supp^{-}(M) = Supp^{\wedge}(M) \cap \mathbf{Spec}^{-}\mathcal{A}$.

2.6.1. Lemma. *(a) For any exact short sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Supp^{-}(M) = Supp^{-}(M') \cup Supp^{-}(M'').$$

(b) Suppose that $\mathcal{A}$ *has the property (sup). Then, for any object* M *which is a supremum of an ascending family* Ξ *of its subobjects, we have the equality:*

$$\bigcup_{X \in \Xi} Supp^{-}(X) = Supp^{-}(M).$$

Proof. (a) is a corollary of Lemma 1.5.1.

(b) The inclusion $\bigcup_{X \in \Xi} Supp^{-}(X) \subseteq Supp^{-}(M)$ follows from *(a)*. So, we need to show that the inverse inclusion holds; i.e. if a point $\mathbb{P}$ of $\mathbf{Spec}^{-}\mathcal{A}$ does not belong to $\bigcup_{X \in \Xi} Supp^{-}(X)$, then it does not belong to $Supp^{-}(M)$.

The relation $\mathbb{P} \notin \bigcup_{X \in \Xi} Supp^{-}(X)$ means exactly that the set Ξ is contained in $Ob\mathbb{P}$. But then, due to the property *(sup)*, $M = sup(\Xi) \in Ob\mathbb{P}$; i.e. $\mathbb{P}$ does not belong to $Supp^{-}(M)$. ∎

For any subset W of $\mathbf{Spec}^{-}\mathcal{A}$, set $\mathcal{A}^{-}(W) := \mathcal{A}^{\wedge}(W) \cap \mathbf{Spec}^{-}\mathcal{A}$. In other words, $\mathcal{A}^{-}(W)$ is the full subcategory of $\mathcal{A}$ generated by all objects M of $\underline{A}$ such that $Supp^{-}(M) \subseteq W$.

2.6.2. Proposition. *(a) For any* $W \subseteq \mathbf{Spec}^{-}\mathcal{A}$, *we have:*

$$\mathcal{A}^{-}(W) = \bigcap_{\mathbb{P} \in W\perp} \mathbb{P},$$

where $W\perp = \mathbf{Spec}^{-}\mathcal{A} - W$.

In particular, $\mathcal{A}^{-}(W)$ *is a Serre subcategory.*

(b) $\mathbf{V}^{-}(\mathcal{A}(W)) = W$ *iff the subset* W *is closed in the topology* τ.

Proof. The assertion follows from Proposition 1.6.2. ∎

2.7. Associated points. For any object M of an abelian category $\mathcal{A}$, denote by $Ass^{-}(M)$ the set of $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ for which there exists a subobject X of M such that the localization, $Q_{\mathbb{P}}(X)$, of X at $\mathbb{P}$ is a quasi-final object of $\mathcal{A}/\mathbb{P}$. In other words,

$$Ass^{-}(M) = Ass^{\wedge}(M) \cap \mathbf{Spec}^{-}\mathcal{A}.$$

Clearly $Ass^{-}(M) \subseteq Supp^{-}(M)$.

2.7.1. Lemma. *Let* $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$; *and let* M *be a* $\mathbb{P}$*-torsion free object of the category* $\mathcal{A}$ *such that* $Q_{\mathbb{P}}(M)$ *is quasi-final.*

Then $Ass^{-}(M) = \{\mathbb{P}\}$.

Proof. According to Lemma 1.7.1 that $Ass^{\wedge}(M) = \{\mathbb{P}\}$. Since $\mathbb{P} \in$ **Spec**$^{-}\mathcal{A}$, the set $Ass^{\wedge}(M)$ coincides with $Ass^{-}(M)$. ∎

2.7.2. Proposition. *(a) For any short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Ass^{-}(M') \subseteq Ass^{-}(M) \subseteq Ass^{-}(M') \cup Ass^{-}(M'').$$

(b) Suppose that $\mathcal{A}$ *has the property* (sup)*. And let an object* M *of* $\mathcal{A}$ *is a supremum of an ascending family* Ξ *of its subobjects. Then*

$$\bigcup_{X \in \Xi} Ass^{-}(X) = Ass^{-}(M).$$

Proof. (a) The assertion *(a)* follows from Proposition 1.7.2.

(b) The inclusion $\bigcup_{X \in \Xi} Ass^{-}(X) \subseteq Ass^{-}(M)$ follows from *(a)*.

Let now $\mathbb{P} \in Ass^{-}(M)$; and let L be a subobject of M such that $Q_{\mathbb{P}}(L)$ is a quasi-final object in $\mathcal{A}/\mathbb{P}$.

Thanks to the property *(sup)*, there is an subobject $X \in \Xi$ such that $L' = L \cap X$ does not belong to $\mathbb{P}$ (since otherwise the object $sup\{L \cap X \mid X \in \Xi\} \simeq L$ would belong to $\mathbb{P}$). Then L' is a subobject of X such that $Q_{\mathbb{P}}(L')$ is a quasi-final object in $\mathcal{A}/\mathbb{P}$; i.e. $\mathbb{P} \in Ass^{-}(M)$. ∎

2.7.3. Corollary. *Suppose that* $\mathcal{A}$ *has the property (sup). Then, for any family* Ω *of objects of* $\mathcal{A}$ *such that the direct sum of* Ω *exists,*

$$Ass^{-}\Big(\bigoplus_{M \in \Omega} M\Big) = \bigcup_{M \in \Omega} Ass^{-}(M).$$

Proof. The truth of the assertion for a finite family Ω is a consequence of Proposition 1.7.2 (without any restrictions on the category $\mathcal{A}$). Since the object $\bigoplus_{M \in \Omega} M$ is the supremum of coproducts of finite subfamilies of Ω, the fact follows from the assertion *(b)* of Proposition 2.7.2. ∎

2.7.4. Corollary. *Let* Ξ *be a finite family of subobjects of an object* M *such that* $\bigcap_{X \in \Xi} X = 0$. *Then* $Ass^{-}(M) \subseteq \bigcup_{X \in \Xi} Ass^{-}(M/X)$.

Proof. This follows from Corollary 1.7.3. ∎

2.7.5. Associate points and exact localizations. The following assertion is a consequence of Proposition 1.7.5.1 and the second assertion of Proposition 2.4.

2.7.5.1. Proposition. *Let* $Q: \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ *be a localization at a coreflective thick subcategory* $\mathbb{T}$. *For any object* M *of* $\mathcal{A}$, *the localization* Q *induces a bijection of the set* $Ass^{-}(M) \cap \mathrm{U}^{-}(\mathbb{T})$ *onto* $Ass^{-}(Q(M))$.

3. THE GOLDMAN'S SPECTRUM.

Fix an abelian category $\mathcal{A}$.

3.1. Lemma. *For any object* M *of* $\mathcal{A}$, *there exists the biggest subcategory,* $\chi(M)$, *among Serre subcategories* $\mathbb{T}$ *such that the object* M *is* $\mathbb{T}$*-torsion free.*

Proof. Consider the set $\Xi(M)$ of all thick subcategories $\mathbb{T}$ of the category $\mathcal{A}$ such that M is $\mathbb{T}$-torsion free.

(a) The set $\Xi(M)$ *is directed with respect to the inclusion.*

(i) Take any two subcategories, $\mathbb{T}$ and $\mathbb{S}$, from $\Xi(M)$. Note that the object M is $\mathbb{T}\bullet\mathbb{S}$-torsion free.

In fact, suppose that X is a subobject of M which belongs to $\mathbb{T}\bullet\mathbb{S}$. The latter means that X has a subobject, Y, from $\mathbb{T}$ such that $X/Y \in Ob\mathbb{S}$. Since Y is a subobject of M and M is $\mathbb{T}$-torsion free, $Y = 0$. But, then $X \simeq X/Y \in Ob\mathbb{S}$. Hence $X = 0$.

(ii) Now note that if $\mathbb{X}$ is a topologizing subcategory such that M is $\mathbb{X}$-torsion free, then M is $\mathbb{X}^{-}$-torsion free.

Suppose that $g: L \longrightarrow M$ is a nonzero monoarrow such that $L \in Ob\mathbb{X}^{-}$. But then L should contain a nonzero subobject from $\mathbb{X}$ which contradicts to the hypothesis that M is $\mathbb{X}$-torsion free.

(iii) It follows from *(i)* and *(ii)* that, for any $\mathbb{S}$, $\mathbb{T}$ from $\Xi(M)$, the subcategory $(\mathbb{T}\bullet\mathbb{S})^{-}$ belongs to $\Xi(M)$. But, since $\mathbb{T}\bullet\mathbb{S}$ is topologizing, its 'closure', $(\mathbb{T}\bullet\mathbb{S})^{-}$ is a Serre subcategory of $\mathcal{A}$ which contains both $\mathbb{T}$ and $\mathbb{S}$ (one can see that $(\mathbb{T}\bullet\mathbb{S})^{-}$ is the smallest among Serre subcategories containing $\mathbb{T}$ and $\mathbb{S}$).

(b) Since $\Xi(M)$ is directed with respect to $\subseteq$, the union, Ω, of all subcategories from $\Xi(M)$ is also a thick subcategory from $\Xi(M)$. Since the inclusion $\Omega \in \Xi(M)$ implies that Ω^{-} also belongs to $\Xi(M)$, $\Omega = \Omega^{-}$; i.e. Ω is a Serre subcategory of the category $\mathcal{A}$. ■

3.2. Example. Let $M \in Spec\mathcal{A}$. Then $\chi(M) = \langle M\rangle$.

In fact, if $\mathbb{X}$ is a topologizing subcategory of $\mathcal{A}$ which does not contain M, then $\mathbb{X} \subseteq \langle M \rangle$. Because, if $V \notin \langle M \rangle$ for some $V \in Ob\mathbb{X}$, i.e. $V \succ M$, then $M \in \mathbb{X}$. ■

An object M of the category $\mathcal{A}$ is called *critical* if the cokernel of any nonzero monoarrow $L \longrightarrow M$ belongs to $\chi(M)$.

The Goldman's spectrum, **Speg**$\mathcal{A}$, of the category $\mathcal{A}$ is the set of all Serre subcategories $\chi(M)$, where M runs through the class of all critical objects of $\mathcal{A}$.

3.3. Proposition. *a) The Goldman's spectrum* **Speg**$\mathcal{A}$ *of any abelian category* $\underline{A}$ *is contained in* **Spec**$^{-}\mathcal{A}$.

b) If the category $\mathcal{A}$ *has the property (sup), then* **Speg**$\mathcal{A}$ *consists of all* $\mathbb{P} \in$ **Spec**$^{-}\mathcal{A}$ *such that the quotient category* $\mathcal{A}/\mathbb{P}$ *has simple objects.*

Proof. a) Let M be a critical object in $\mathcal{A}$. Clearly being critical implies that the localization M' of M at $\chi(M)$ is a simple object of the quotient category $\mathcal{A}/\chi(M)$. In particular, M' belongs to **Spec**$\mathcal{A}/\chi(M)$; hence $\langle M' \rangle$ is a Serre subcategory of $\mathcal{A}/\chi(M)$. Let $\mathbb{P}$ be the preimage of $\langle M' \rangle$ in $\mathcal{A}$.

Clearly $\mathbb{P}$ is a thick subcategory which contains $\chi(M)$.

Note that M is $\mathbb{P}$-torsion free. Since, if it is not, there is a nonzero subobject $L \longrightarrow M$ such that $L \in Ob\mathbb{P}$. Since M is critical, $M/L \in Ob\chi(M) \subseteq Ob\mathbb{P}$. This implies that $M \in \mathbb{P}$ which contradicts to the fact that the localization of M' at $\langle M' \rangle$ is nonzero.

Thus, $\mathbb{P} = \chi(M)$, and $\mathcal{A}/\mathbb{P}$ is a local category with a simple object. In particular, $\mathbb{P}$ is a Serre subcategory.

b) Suppose now that the category $\mathcal{A}$ has the property *(sup)*. And let $\mathbb{P}$ be a Serre subcategory of $\mathcal{A}$ such that $\mathcal{A}/\mathbb{P}$ is a local category with a simple object, say M'. Let $M' \simeq Q_{\mathbb{P}}(M)$ for an object M of $\mathcal{A}$. Thanks to the property *(sup)*, any Serre subcategory of $\mathcal{A}$ is coreflective. Replacing M by $M/\mathbb{P}M$, where $\mathbb{P}M$ is a $\mathbb{P}$-torsion of M, we assume that M is $\mathbb{P}$-torsion free. In particular, $\mathbb{P} \subseteq \chi(M)$.

We claim that $\mathbb{P} = \chi(M)$.

In fact, let $\mathbb{T}$ be the image of $\chi(M)$ in $\mathcal{A}/\mathbb{P}$. Clearly $\mathbb{T}$ is a thick subcategory in $\mathcal{A}/\mathbb{P}$ such that the quasi-final (simple) object M' is $\mathbb{T}$-torsion free. But this implies that $\mathbb{T} = 0$ (if not, i.e. there is a nonzero object X in $\mathbb{T}$, then the relation $X \succ M$ implies that $M \in \mathbb{T}$). In other words, $\mathbb{P}$ coincides with $\chi(M)$. ∎

3.4. Corollary. *For any abelian category* $\mathcal{A}$, *the intersection* $\mathbf{Speg}\mathcal{A} \cap \mathbf{Spec}\mathcal{A}$ *consists of all* $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ *such that the local category* $\mathcal{A}/\langle P\rangle$ *has simple objects.*

For any topologizing subcategory $\mathbb{T}$ of $\mathcal{A}$, set

$$V_g(\mathbb{T}) := V^-(\mathbb{T}) \cap \mathbf{Speg}\mathcal{A} = \{\mathbb{P} \in \mathbf{Speg}\mathcal{A} \mid \mathbb{P} \cap \mathbb{T} \neq \mathbb{T}\}$$

and

$$U_g(\mathbb{T}) := \mathbf{Speg}\mathcal{A} \cap U^-(\mathbb{T}) = \{\mathbb{P} \in \mathbf{Speg}\mathcal{A} \mid \mathbb{T} \subseteq \mathbb{P}\}.$$

3.5. Proposition. *For any coreflective thick subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$, *the map*

$$V^-(\mathbb{T}) \longrightarrow \mathbf{Spec}^-\mathbb{T}, \quad \mathbb{P} \longmapsto \mathbb{P} \cap \mathbb{T},$$

of Proposition 2.4 induces a bijection $V_g(\mathbb{T})$ *onto* $\mathbf{Speg}\mathbb{T}$.

Proof. It is clear (from the argument of Propositions 2.4 and 1.4.2) that the subcategory $\mathbb{T}/(\mathbb{P} \cap \mathbb{T})$ of the local category $\mathcal{A}/\mathbb{P}$ contains quasi-final objects of $\mathcal{A}/\mathbb{P}$. Since $\mathbb{T}/(\mathbb{P} \cap \mathbb{T})$ is thick, these quasi-final objects are semisimple in $\mathbb{T}/(\mathbb{P} \cap \mathbb{T})$ iff they are semisimple in $\mathcal{A}/\mathbb{P}$.

The assertion follows now from Proposition 3.3. ∎

3.6. Residue skew fields at points of $\mathbf{Speg}\mathcal{A}$. By Proposition 3.3, the category $\mathcal{A}/\mathbb{P}$ is local and has simple objects for all points $\mathbb{P}$ of $\mathbf{Speg}\mathcal{A}$. Therefore, for every point $\mathbb{P} \in \mathbf{Speg}\mathcal{A}$, the residue category $\mathcal{K}_{\mathbb{P}} := \mathcal{K}(\mathcal{A}/\mathbb{P})$ is equivalent to the category $K(\mathbb{P})\text{-}Vec$ of vector spaces over the residue skew field of $\mathbb{P}$ (cf. 1.8).

4. THE FLAT SPECTRUM AND INJECTIVE OBJECTS.

In what follows, $\mathcal{A}$ is an abelian category.

4.0. Preliminaries: the correspondence between Serre subcategories and classes of injective objects. For any object M of $\mathcal{A}$, denote by $\mathbb{S}(M)$ the full subcategory of $\mathcal{A}$ formed by all the objects X such that $\mathcal{A}(X,M) = 0$.

4.0.1. Lemma. *1) For any $M \in Ob\mathcal{A}$, the category $\mathbb{S}(M)$ has the following properties:*

(a) If $D\colon \mathcal{D} \longrightarrow \mathcal{A}$ is a (small) diagram with values in $\mathbb{S}(M)$, then its colimit (if any) also belongs to $\mathcal{A}$.

(b) If the objects Y and Y' in the exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Y' \longrightarrow 0$$

belong to $\mathbb{S}(M)$, then X is also from $\mathbb{S}(M)$.

If $X \in Ob\mathbb{S}(M)$, then $Y' \in Ob\mathbb{S}(M)$.

2) If the object M is injective, then $\mathbb{S}(M)$ is a Serre subcategory of the category $\mathcal{A}$.

Proof. 1) The assertion *1)* follows from the corresponding properties of the functor $\mathcal{A}(\ ,M)$.

2) Let now M be an injective object. Then any object X of $\mathbb{S}(M)$ contains all its subobjects. Because, the injectivity of M means exactly, that the map

$$\mathcal{A}(\mathrm{i}, id_M)\colon \mathcal{A}(X,M) \longrightarrow \mathcal{A}(Y,M)$$

is injective for any monomorphism $\mathrm{i} : Y \longrightarrow X$. This, together with the assertion *1)*, proves the thickness of $\mathbb{S}(M)$.

Let $L \in Ob\mathbb{S}(M)^{-}$. And suppose that there is a nonzero arrow f from L to M. By condition, there is a nonzero subobject of the image of f which belongs to $\mathbb{S}(M)$. But, this nonzero subobject is also a subobject of M which is a contradiction. Thus, L is an object of $\mathbb{S}(M)$. ∎

4.0.2. A preorder among injective objects. Define a relation $\gg$ among objects of a category $\mathcal{A}$ as follows:

$$M \gg L \text{ if } \mathcal{A}(M,L) \text{ is integral;}$$

i.e. for any two distinct arrows $f, g\colon X \longrightarrow M$ there exists an arrow φ from M to L such that $\varphi \circ f \neq \varphi \circ g$.

Clearly the relation $\twoheadrightarrow$ is transitive. We are interested in the restriction of the preorder $\twoheadrightarrow$ to the full subcategory $\mathfrak{Inj}\mathcal{A}$ of $\mathcal{A}$ generated by injective objects.

4.0.2.1. Lemma. *Suppose that* M' *is an injective object of an abelian category* $\mathcal{A}$, *and* M *an arbitrary object of* $\mathcal{A}$. *Then*

$$M \twoheadrightarrow M' \quad \text{if and only if} \quad \mathbb{S}(M') \subseteq \mathbb{S}(M).$$

Proof. Clearly $\mathbb{S}(M') \subseteq \mathbb{S}(M)$ if and only if, for any nonzero subobject Y of M, there exists a nonzero arrow f' from Y to M'. Since M' is injective, the morphism $f' = f \circ j$ for a morphism $f: M \longrightarrow M'$. This implies that the set of all arrows from M to M' is integral.

Conversely, if $\mathcal{A}(M,M')$ is integral, then, without even any requirements on M', we have the inclusion $\mathbb{S}(M') \subseteq \mathbb{S}(M)$. ∎

4.0.2.2. Corollary. *Suppose that* $\mathcal{A}$ *is a category with (small) products. And let* M, M' *be injective objects. Then* $\mathbb{S}(M')$ *is a subcategory of* $\mathbb{S}(M)$ *if and only if* M *is a retract of the product of a set of the copies of* M'.

Let $\approx$ denote the induced by $\twoheadrightarrow$ equivalence relation.

4.0.2.3. Corollary. *(a) The map* $M \longmapsto \mathbb{S}(M)$ *induces an injection of the ordered set* $(Ob\mathfrak{Inj}\mathcal{A}/\approx, \twoheadrightarrow)$ *of equivalence classes of injective objects into the ordered set* $(\mathbf{Serre}\mathcal{A}, \supseteq)$ *of Serre subcategories of the category* $\mathcal{A}$

(b) If $\mathcal{A}$ *is a Grothendieck category (or, more generally,* $\mathcal{A}$ *is a category with injective hulls and the property (sup)*; cf. 0.4.3.2*), then the map*

$$(Ob\mathfrak{Inj}\mathcal{A}/\approx, \twoheadrightarrow) \longrightarrow (\mathbf{Serre}\mathcal{A}, \supseteq), \quad M \longmapsto \mathbb{S}(M),$$

is bijective.

Proof. (a) follows straightforwardly from Lemma 4.0.2.1.

(b) The assertion *(b)* follows from the well known fact that, under the conditions of *(b)*,

any Serre subcategory of $\mathcal{A}$ *is of the form* $\mathbb{S}(M)$ *for some injective object* M.

For the reader's convenience, we sketch the proof.

Let $\mathfrak{L}$ be an arbitrary Serre subcategory of the category $\mathcal{A}$, and

$$Q: \mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{L}$$

a localization at $\mathfrak{L}$. Fix an injective cogenerator, M', in the category $\mathcal{A}/\mathfrak{L}$. Since the functor Q is exact, the object $M = Q^\wedge(M')$, where $Q^\wedge$ is a right adjoint to Q, is injective (cf. 6.3 and 6.4 in [BD]). The claim is that $\mathfrak{L}$ coincides with $\mathfrak{S}(M)$.

Clearly, $\mathfrak{L}$ is contained in $\mathfrak{S}(M)$. To prove the inverse inclusion, we should show that $Q(V) = 0$ for all $V \in Ob\mathfrak{S}(M)$.

Suppose that $V \notin Ob\mathfrak{L}$; i.e. $Q(V) \neq 0$; or, equivalently, $Q^\wedge Q(V) \neq 0$. Since M' is a cogenerator of $\mathcal{A}/\mathfrak{L}$, the set of all arrows from $Q(V)$ to M' is integral. The functor $Q^\wedge$, being a right adjoint functor, respects integral families. In particular the family $\mathcal{A}(Q^\wedge Q(V),M)$ is integral. This implies the composition of the adjunction arrow $V \longrightarrow Q^\wedge Q(V)$ with some arrow g from $Q^\wedge Q(V)$ to M is nonzero; i.e. V does not belong to $\mathfrak{S}(M)$. ∎

4.1. Injective objects of a local category. Suppose now that the category $\mathcal{A}$ is local; and let V be a quasi-final object in $\mathcal{A}$.

4.1.1. Lemma. *The injective hull $h(V)$ of the quasifinal object V is a cogenerator of the local category $\mathcal{A}$.*

Proof. We ought to show that, for any nonzero object X of the category $\mathcal{A}$, there exists a nonzero morphism $X \longrightarrow h(V)$.

In fact, there is a diagram

$$(I)X \xleftarrow{\;\mathfrak{i}\;} K \xrightarrow{\;\mathfrak{e}\;} V \xrightarrow{\;\mathfrak{v}\;} h(V),$$

where $\mathfrak{i}$ and $\mathfrak{v}$ are monoarrows and $\mathfrak{e}$ is an epimorphism. Since the object $h(V)$ is injective, there is an arrow

$$g: (I)X \longrightarrow h(V)$$

such that $g \circ \mathfrak{i} = \mathfrak{v} \circ \mathfrak{e}$. Therefore, since $\mathfrak{v} \circ \mathfrak{e}$ is nonzero, g is nonzero which implies that the composition of g with one of the canonical embeddings

$$X \longrightarrow (I)X$$

is nonzero. ∎

4.1.2. Corollary. *Let $\mathcal{A}$ be a category with simple objects and with injective hulls of simple objects. Then the category $\mathcal{A}$ is local if and only if it has an* *indecomposable injective cogenerator.*

Proof. Only if. Let the category $\mathcal{A}$ be local, and let M be a simple object of $\mathcal{A}$. Since M is a quasi-final object of $\mathcal{A}$, it follows from Lemma 4.1.1 that the injective hull, $h(M)$, of the object M is a cogenerator. Since M is simple, the object $h(M)$ is indecomposable.

If. Let M be a simple object of the category $\mathcal{A}$; and let E be the indecomposable injective cogenerator. Since E is a cogenerator, there exists a nonzero arrow from M to E which is a monomorphism thanks to the simplicity of M. Therefore E is the injective hull of M.

By assumption, for any nonzero object X of the category $\mathcal{A}$, there exists a nonzero arrow $g: X \longrightarrow E$. Since the object E is indecomposable and M is simple, the intersection of M with $Im(g)$ is isomorphic to M. Thus, we have the diagram

$$X \xleftarrow{\iota} N \xrightarrow{\varepsilon} M$$

where the monoarrow ι is the preimage of $M \longrightarrow E$ and ε is the natural epimorphism. ∎

4.2. A characterization of local Grothendieck categories. Let V be an object of an abelian category $\mathcal{A}$. Denote by $\mathcal{A}_V$ the ring $\mathcal{A}(V,V)$ and by $\mathfrak{S}_V$ the functor from the dual to $\mathcal{A}$ category $\mathcal{A}^{op}$ to the category $mod\text{-}\mathcal{A}_V$ of the right $\mathcal{A}_V$-modules which assigns to every object X the $\mathcal{A}_V$-module $\mathfrak{S}_V := (\mathcal{A}(X,V),c)$, where the right action c of the ring $\mathcal{A}_V$ is the composition; $\mathfrak{S}_V(f) = \mathcal{A}(f, id_V)$ for any morphism f.

4.2.1. Lemma. *The object V is a cogenerator if and only if the functor $\mathfrak{S}_V$ is faithful.*

Proof is an easy exercise. ∎

4.2.2. Lemma. *Suppose that the object V of the abelian category $\mathcal{A}$ is such that there exists a product, $[J]V$, of any (small) family J of copies of V. Then the functor $\mathfrak{S}_V$ has a left adjoint*

$$\Sigma_V: mod\text{-}\mathcal{A}_V \longrightarrow \mathcal{A}^{op}.$$

Proof. 1) The first step is to define the functor Σ on the full subcategory $Free\text{-}\mathcal{A}_V$ formed by free modules. Since

$$mod\text{-}\mathcal{A}_V(\mathcal{A}_V, \mathfrak{S}_V X) \simeq \mathcal{A}(X, V)$$

for every object X of the category $\mathcal{A}$, there is an isomorphism

$$J(X)\colon\ mod\text{-}\mathcal{A}_V((J)\mathcal{A}_V, \mathfrak{S}_V X) \simeq \mathcal{A}(X, [J]V)$$

which depends functorially on X. This implies that any morphism

$$u\colon\ (J)\mathcal{A}_V \longrightarrow (I)\mathcal{A}_V$$

induces a functor morphism

$$u'\colon\ \mathcal{A}(-,[I]V) \longrightarrow \mathcal{A}(-,[J]V).$$

By the Ioneda's lemma, $u' = \mathcal{A}(-, \mathfrak{T}u)$ for a uniquely defined arrow

$$\mathfrak{T}u\colon\ [I]V \longrightarrow [J]V.$$

2) Now we define the functor $\mathfrak{T}_V\colon\ \mathcal{A}^{op} \longrightarrow mod\text{-}\mathcal{A}_V$ as the left derived of the functor $\mathfrak{T}$. This means, that, for every right $\mathcal{A}_V$-module M, we choose an exact sequence

$$(J)\mathcal{A}_V \xrightarrow{\ \mathfrak{v}\ } (I)\mathcal{A}_V \xrightarrow{\ \mathfrak{u}\ } M \longrightarrow 0 \qquad (1)$$

and set $\mathfrak{T}_V(M) := Ker\mathfrak{T}\mathfrak{v}$. It follows from the commutative diagram

$$\begin{array}{ccccc}
mod\text{-}\mathcal{A}_V(M, \mathfrak{S}_V(X)) & \rightarrowtail & mod\text{-}\mathcal{A}_V((I)\mathcal{A}_V, \mathfrak{S}_V(X)) & \longrightarrow & mod\text{-}\mathcal{A}_V((I)\mathcal{A}_V, \mathfrak{S}_V(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}(X, \mathfrak{T}_V M) & \xrightarrow{\ \mathcal{A}(X,\, ker\mathfrak{T}\mathfrak{v})\ } & \mathcal{A}\ (X, [I]V) & \xrightarrow{\ \mathcal{A}(X, \mathfrak{T}\mathfrak{v})\ } & \mathcal{A}(X, [J]V)
\end{array}$$

where the upper row is induced by the exact sequence (1), that the functor $\mathfrak{T}_V$ is left adjoint to the functor $\mathfrak{S}_V$. ■

4.2.3. Lemma. *In the notations of Lemma 4.2.2, the adjunction morphism*

$$\varphi(P)\colon\ P \longrightarrow \mathfrak{S}_V \circ \mathfrak{T}_V(P)$$

is an isomorphism for each projective right $\mathcal{A}_V$*-module* P.

Proof. It follows from the construction of the functor $\mathfrak{T}_V$ that $\varphi(P)$ is an isomorphism for every free module P. Since projective modules are retracts of free modules, this implies the statement. ■

4.2.4. Proposition. *Let* $\mathcal{A}$ *be an abelian category with simple objects, injective hulls of simple objects, and products.*

Then the following conditions are equivalent:

(a) The category $\mathcal{A}$ *is local.*

(b) There is a local ring R *and an exact faithful functor*

$$\mathfrak{S}: \mathcal{A}^{op} \longrightarrow mod\text{-}R$$

such that

(i) $\mathfrak{S}$ *has a left adjoint functor,* $\mathfrak{T}: mod\text{-}R \longrightarrow \mathcal{A}^{op}$*;*

(ii) The adjunction morphism $\varphi(P): P \longrightarrow \mathfrak{S}\circ\mathfrak{T}(P)$ *is an isomorphism for every projective module* P*.*

Proof. (a) Let the category $\mathcal{A}$ be local, and let V be an indecomposable injective cogenerator in $\mathcal{A}$. Set $R = \mathcal{A}(V,V)$, $\mathfrak{S} = \mathfrak{S}_V$ and $\mathfrak{T} = \mathfrak{T}_V$ (in the notations of Lemma 4.2.2). By Lemma 4.2.3, the adjunction arrow $P \longrightarrow \mathfrak{S}\circ\mathfrak{T}(P)$ is invertible for every projective R-module P. Since V is a cogenerator, the functor $\mathfrak{T}$ is faithful; the injectivity of V means that $\mathfrak{T}$ is exact. Finally, since V is an indecomposable injective object, the ring R is local.

(b) Let the conditions *(b)* hold. Set $V := \mathfrak{T}(R)$. Note that

1) The object V *is a cogenerator in* $\mathcal{A}$*; i.e.* $\mathcal{A}(X,V) = 0$ *if and only if* $X = 0$*.*

In fact, $\mathcal{A}(X,V) := \mathcal{A}(X,\mathfrak{T}(V)) \simeq mod\text{-}R(R,\mathfrak{S}(X)) \simeq \mathfrak{S}(X)$. Hence $\mathcal{A}(X,V) = 0$ if and only if $\mathfrak{S}(X) = 0$. But, thanks to the faithfulness of $\mathfrak{S}$, $\mathfrak{S}(X) = 0$ if and only if $X = 0$.

2) The object V *is injective.*

It follows from the dual version of Proposition 4.1.1 that, since the functor $\mathfrak{S}$ is exact, the functor $\mathfrak{T}$ sends projective objects of the category *mod-R* into projective objects of the category $\mathcal{A}^{op}$. But projective objects of $\mathcal{A}^{op}$ are injective objects of $\mathcal{A}$.

3) The object V *is indecomposable.*

(i) Note that, being local the ring R does not contain nontrivial idempotents.

In fact, if e is an idempotent, then either $e = 1$, or e is not invertible. In the last case, since the ring R is local, the idempotent element $1 - e$ is invertible; hence $e = 0$.

(ii) The absence of nontrivial idempotents means that the projective right module R is indecomposable.

Since the adjunction morphism $P \longrightarrow \mathfrak{S}\circ\mathfrak{T}(P)$ is an isomorphism for every projective module P (Lemma 4.2.3), and the functor $\mathfrak{S}$ is faithful, the indecomposability of the right module R implies that of the object V.

4) By Lemma 4.1.2, the category $\mathcal{A}$ is local. ■

In conclusion of this section, we shall make one more step to the better understanding of the structure of injective objects of a local Grothendieck category.

4.2.5. Lemma. *Let V be an object of the abelian category $\mathcal{A}$ which satisfies the conditions of Lemma 4.2.2; and let ΣV be the full subcategory of the category $\mathcal{A}$ generated by the objects which are isomorphic to the objects $\Sigma_V(P)$, where P runs through the class of projective right $\mathcal{A}_V$-modules.*

1) The functors $\mathfrak{S}_V$, Σ_V induce duality between the category $Proj\text{-}\mathcal{A}_V$ of the projective right $\mathcal{A}_V$-modules and the category ΣV (i.e. an equivalence of the categories $Proj\text{-}\mathcal{A}_V$ and ΣV^{op}).

2) If the object V is injective, then all objects of the subcategory ΣV are injective objects of the category $\mathcal{A}$.

Proof. 1) The first statement follows from Lemma 4.2.2.

2) The injectivity of the object V means that the functor $\mathfrak{S}_V$ is exact. Therefore, by Proposition 4.1.1, the functor Σ_V assigns to projective right $\mathcal{A}_V$-modules injective objects of the category $\mathcal{A}$. ■

4.2.6. Proposition. *Let $\mathcal{A}$ be a local Grothendieck category with simple objects; and let V be an indecomposable injective object in $\mathcal{A}$. Then every object of the subcategory ΣV (cf. Lemma 4.2.5) is isomorphic to the product $[J]V$ of a family J of copies of the injective object V.*

Proof. This statement is the corollary of Theorem 4.2.4 and the Kaplanski's theorem:

every projective module over a local ring is free. ■

5. INJECTIVE SPECTRA.

5.1. Definitions. Fix an abelian category $\mathcal{A}$.

Define *$ISpec\mathcal{A}$* as the class of all nonzero injective objects E of the category $\mathcal{A}$ such that $E \twoheadrightarrow E'$ (i.e. $\mathcal{A}(E,E')$ is an integral set of arrows) for any nonzero injective subobject E' of E. And let **$\mathbf{ISpec}\mathcal{A}$** be the set $\{\mathfrak{S}(E) \mid E \in ISpec\mathcal{A}\}$ of Serre subcategories of $\mathcal{A}$.

We call the ordered set *$(\mathbf{ISpec}\mathcal{A}, \supseteq)$ the injective spectrum of $\mathcal{A}$.*

One can see that any nonzero injective subobject of an object E from *$ISpec\mathcal{A}$* belongs to *$ISpec\mathcal{A}$* and its image in **$\mathbf{ISpec}\mathcal{A}$** is the same as the image of E, i.e. $s(E)$.

Clearly the class $I^{\wedge}Spec\mathcal{A}$ of all indecomposable injective objects of $\underline{A}$ is contained in $ISpec\mathcal{A}$.

We denote the image of $I^{\wedge}Spec\mathcal{A}$ in $\mathbf{ISpec}\mathcal{A}$ by $\mathbf{I^{\wedge}Spec}\mathcal{A}$ and call it *indecomposable injective spectrum of* $\mathcal{A}$.

5.2. Lemma. *Let* $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ *be such that the localization* $Q = Q_{\mathbb{P}}$ *at* $\mathbb{P}$ *is flat (i.e. has a right adjoint functor,* $Q^{\wedge}$*), and the quotient category* $\mathcal{A}/\mathbb{P}$ *has injective hulls (which is the case if* $\mathcal{A}$ *has injective hulls).*

Let M' *be an injective hull of a quasi-final object* X *of the category* $\mathcal{A}/\mathbb{P}$*. Then* $M := Q^{\wedge}(M') \in ISpec\mathcal{A}$.

Proof. Let $\mathrm{i} : E \longrightarrow M$ be a nonzero injective subobject of M. Since E is injective, there is a morphism $\mathrm{e} : M \longrightarrow E$ such that $\mathrm{e} \circ \mathrm{i} = id_E$ which implies that $Q\mathrm{e} \circ Q\mathrm{i} = id_{Q(M)}$. Hence the object $Q(E)$, being a retract of an injective object $Q(M) = QQ^{\wedge}(M') \simeq M'$ is injective as well. Since E is $\mathbb{P}$-torsion free, $Q(E) \neq 0$. Therefore the intersection $Y := X \cap Q(E)$ is nonzero (we are using the fact that M' is an injective hull of X). Being a nonzero subobject of a quasi-final object, Y is also quasi-final. This implies that

$$s(Q^{\wedge}Q(E)) = s(M).$$

Note now that the adjunction arrow $\eta : E \longrightarrow Q^{\wedge}Q(E)$ is an isomorphism. This is a corollary of Proposition III.3.6 in [Gab]. Actually, one can see this fact immediately taking into consideration that E is a direct summand of $M = Q^{\wedge}(M')$. ∎

5.3. Proposition. *Let* $\mathcal{A}$ *be a category with injective hulls and with property (sup). Then the flat spectrum* $\mathbf{Spec}^{-}\mathcal{A}$ *is a subset of the injective spectrum. And the Goldman's spectrum of* $\mathcal{A}$ *is a subset of* $\mathbf{I^{\wedge}Spec}\mathcal{A}$*. So that we have the diagram of inclusions:*

$$\begin{array}{ccc} \mathbf{Spec}^{-}\mathcal{A} & \longrightarrow & \mathbf{ISpec}\mathcal{A} \\ \uparrow & & \uparrow \\ \mathbf{Speg}\mathcal{A} & \longrightarrow & \mathbf{I^{\wedge}Spec}\mathcal{A} \end{array}$$

Proof. Under the assumptions, the localization at every point $\mathbb{P}$ of $\mathbf{Spec}^{-}\mathcal{A}$ is flat, and $\mathcal{A}/\mathbb{P}$ is a category with injective hulls. So, the inclusion

$\mathbf{Spec}^{-}\mathcal{A} \subseteq \mathbf{ISpec}\mathcal{A}$ follows from Lemma 5.2.

Suppose now that $\mathbb{P} \in \mathbf{Speg}\mathcal{A}$. By Proposition 3.2, this means that the local category $\mathcal{A}/\mathbb{P}$ has simple objects. Let M' be an injective hull of a simple object of $\mathcal{A}/\mathbb{P}$. Then $Q^{\wedge}(M')$ (where $Q^{\wedge}$ is a right adjoint to a localization $Q\colon \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{P}$) is an indecomposable injective object of $\mathcal{A}$, and $\mathbb{P} = \mathfrak{S}(Q^{\wedge}(M'))$. ∎

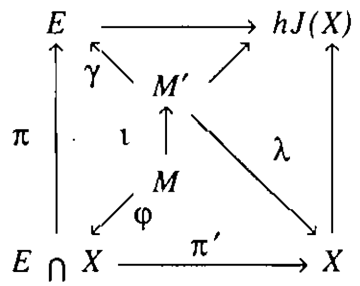
5.4. Lemma. *Let $\mathcal{A}$ be an abelian category with injective hulls. Let $\mathbb{T}$ be a topologizing subcategory of $\mathcal{A}$ and J the inclusion functor $\mathbb{T} \longrightarrow \mathcal{A}$. The map which assigns to an object X of $\mathbb{T}$ the injective hull, $hJ(X)$, of its image in $\mathcal{A}$ induces morphisms of ordered sets*

$$\mathbf{ISpec}\mathbb{T} \longrightarrow \mathbf{ISpec}\mathcal{A} \quad \textit{and} \quad \mathbf{I^{\wedge}Spec}\mathbb{T} \longrightarrow \mathbf{I^{\wedge}Spec}\mathcal{A}.$$

Proof. Suppose that $X \in \mathit{ISpec}\mathbb{T}$; and let E be a nonzero injective subobject of $hJ(X)$. Then $E \cap X$ is a nonzero object in $\mathbb{T}$.

(a) We claim that $E \cap X$ is an injective object in $\mathbb{T}$.

Indeed, any diagram $E \cap X \xleftarrow{\varphi} M \xrightarrow{\iota} M'$, where ι is a monoarrow, can be included into a commutative diagram



(1)

with arrows γ and λ due to the injectivity of the object X in $\mathbb{T}$ and E in $\mathcal{A}$. Since the outer square of the diagram (1) is cartesian, there is unique morphism $\iota'\colon M \longrightarrow E \cap X$ such that $\gamma = \pi\circ\iota'$ and $\lambda = \pi'\circ\iota'$. The equalities

$$\pi\circ(\iota'\circ\iota) = \gamma\circ\iota = \pi\circ\varphi, \quad \pi'\circ(\iota'\circ\iota) = \lambda\circ\iota = \pi'\circ\varphi$$

imply, thanks to the universal property of a cartesian square, the required equality $\iota'\circ\iota = \varphi$.

(b) Since $X \in \mathit{ISpec}\mathbb{T}$ and $E \cap X$ is a nonzero injective (in $\mathbb{T}$) subobject of X, we have $X \succ E \cap X$; i.e. $\mathcal{A}(X, E \cap X)$ is an integral family of ar-

rows which implies, of course, that $\mathcal{A}(X,E)$ is an integral family of arrows. But, the integrality of $\mathcal{A}(X,E)$ implies that of $\mathcal{A}(hJ(X),E)$.

Indeed, if Y is a nonzero subobject of $hJ(X)$, then, since $hJ(X)$ is just the injective hull of X in $\mathcal{A}$, the intersection $X \cap Y$ is nonzero. Thanks to the integrality of $\mathcal{A}(X,E)$, there is an arrow $s: X \longrightarrow E$ such that the composition of s and the monoarrow $X \cap Y \xrightarrow{\iota} X$ is nonzero. Since E is injective, there exists a morphism $Y \xrightarrow{\mathrm{i}} E$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{i}} & E \\ \uparrow & & \uparrow{\scriptstyle s} \\ X \cap Y & \xrightarrow{\iota} & X \end{array}$$

is commutative. Clearly ι is nonzero. This finishes the proof. ∎

5.5. Lemma. *Let Q be a localization of an abelian category $\mathcal{A}$ at a thick subcategory $\mathbb{T}$; and let E be a $\mathbb{T}$-torsion free injective object in $\mathcal{A}$. Then the object $Q(E)$ is injective.*

Proof. Let we have a diagram

$$Q(M) \xleftarrow{\mathrm{i}} Q(X) \xrightarrow{\mathrm{f}} Q(E) \qquad (1)$$

in $\mathcal{A}/\mathbb{T}$ such that i is a monoarrow. The arrows f and i are the images of some elements $\mathrm{f}' \in \mathcal{A}(X',E/V)$ and $\mathrm{i}' \in \mathcal{A}(X'',M/M'')$ respectively, where X' and X'' are subobjects of X such that X/X' and X/X'' belong to $\mathbb{T}$, as well as M/M'' and E/V.

This means that the diagram (1) is represented by the diagram

$$\begin{array}{ccccccccccc} M & & & & & & X & & & & E \\ & \overset{\mathfrak{u}}{\searrow} & & & & \overset{\mathfrak{v}}{\nearrow} & & \overset{\mathfrak{u}'}{\nwarrow} & & \overset{\mathfrak{v}'}{\swarrow} & \\ & & M' & \xleftarrow{\mathrm{i}'} & X'' & & & & X' \xrightarrow{\mathrm{f}'} V & & \end{array} \qquad (2)$$

in which all diagonal arrows are send to invertible ones by Q.

The diagram (2) can be included into the diagram

$$
\begin{array}{ccccccccc}
 & & & & Y \underset{X}{\times} Y' & & & & \\
 & & & \swarrow & & \searrow & & & \\
M & \xleftarrow{\pi'} & Y & & X & & Y' & \longrightarrow & E \\
{\scriptstyle \mathfrak{u}}\searrow & & & \searrow \;\; \nearrow {\scriptstyle \mathfrak{v}} & & {\scriptstyle \mathfrak{u}'}\nwarrow \;\; \swarrow & & & \swarrow {\scriptstyle \mathfrak{v}'} \\
 & M' & \xleftarrow{\mathrm{i}'} & X'' & & X' & \xrightarrow{\mathrm{f}'} & V &
\end{array}
\qquad (3)
$$

where $Y = M \underset{M'}{\times} X''$, $Y' = X' \underset{V}{\times} E$, and all diagonal dot arrows are inverted by the localization Q. This shows that the diagram (1) can be included into a commutative diagram of the form

$$
\begin{array}{ccccc}
 & {\scriptstyle \mathrm{i}}\swarrow & Q(X) & \searrow {\scriptstyle \mathrm{f}} & \\
Q(M) & & \mathfrak{w} \uparrow & & Q(E) \\
 & {\scriptstyle Q\iota}\nwarrow & Q(W) & \nearrow {\scriptstyle Q\phi} &
\end{array}
$$

where $\mathfrak{w}$ is an isomorphism.

Since $Ker(\iota) \in Ob\mathbb{T}$ and the object E is $\mathbb{T}$-torsion free, ϕ annihilates $Ker(\iota)$. So, we can assume that ι is a monomorphism which implies, thanks to the injectivity of E the existence of a morphism $\sigma: M \longrightarrow E$ such that $\phi = \sigma \circ \iota$. Clearly $Q\sigma$ is what we are looking for. Because we have:

$$Q\sigma \circ \mathrm{i} \circ \mathfrak{w} = Q\sigma \circ Q\iota = Q\phi = \mathrm{f} \circ \mathfrak{w};$$

and the equality $Q\sigma \circ \mathrm{i} \circ \mathfrak{w} = \mathrm{f} \circ \mathfrak{w}$ implies that $Q\sigma \circ \mathrm{i} = \mathrm{f}$. ■

For any topologizing subcategory $\mathbb{T}$ of the category $\mathcal{A}$, set
$\mathbf{VI}(\mathbb{T}) := \{\mathfrak{S}(E) \mid E \in ISpec\mathcal{A}$ and E is not $\mathbb{T}$-torsion free$\}$,

$$\mathbf{VI}^{\wedge}(\mathbb{T}) := \mathbf{VI}(\mathbb{T}) \cap \mathbf{I^{\wedge}Spec}\mathcal{A}.$$

and let

$$\mathbf{UI}(\mathbb{T}) := \mathbf{ISpec}\mathcal{A} - \mathbf{VI}(\mathbb{T}), \quad \mathbf{UI}^{\wedge}(\mathbb{T}) := \mathbf{UI}(\mathbb{T}) \cap \mathbf{I^{\wedge}Spec}\mathcal{A}.$$

5.6. Proposition. *Let $\mathcal{A}$ be an abelian category with injective hulls. Then, for every coreflective thick subcategory $\mathbb{T}$ of $\mathcal{A}$,*

(i) the map which assigns to any object of $\mathbb{T}$ its injective hull in $\mathcal{A}$ induces bijections

$$\mathbf{ISpec}\mathbb{T} \longrightarrow \mathbf{VI}(\mathbb{T}) \quad \textit{and} \quad \mathbf{I^{\wedge}Spec}\mathbb{T} \longrightarrow \mathbf{VI}^{\wedge}(\mathbb{T}).$$

(ii) The localization $Q = Q_{\mathbb{T}}$ *induces bijections*

$$\mathbf{UI}(\mathbb{T}) \longrightarrow \mathbf{ISpec}\mathcal{A}/\mathbb{T} \quad and \quad \mathbf{UI}^{\wedge}(\mathbb{T}) \longrightarrow \mathbf{I}^{\wedge}\mathbf{Spec}\mathcal{A}/\mathbb{T}.$$

Proof. (i) Since the subcategory $\mathbb{T}$ is thick, the inclusion functor

$$J: \mathbb{T} \longrightarrow \mathcal{A}$$

is exact which implies that its right adjoint, $J^{\wedge}$, sends injective objects of $\mathcal{A}$ into injective objects of $\mathbb{T}$ (c.f. [BD], Proposition VI.6.3).

(a) First note that, for any injective object X of $\mathbb{T}$, the canonical morphism $\sigma = \sigma(X): X \longrightarrow J^{\wedge}hJ(X)$ is an isomorphism.

Clearly σ is a monomorphism. Since X is injective, this implies that $J^{\wedge}hJ(X) \simeq X \oplus Y$ for some subobject Y. Since every nonzero subobject of $hJ(X)$ should have a nonzero intersection with X, the direct summand Y is zero.

(b) We claim that $J^{\wedge}(E) \in ISpec\mathbb{T}$ for every $E \in \mathbf{VI}(\mathbb{T})$.

Let Y be a subobject of $J^{\wedge}(E)$ which is annihilated by all morphisms from $J^{\wedge}(E)$ to X. Since the object X is injective, the induced by the mono-arrow $Y \longrightarrow J^{\wedge}(E)$ map

$$\mathbb{T}(J^{\wedge}(E),X) \longrightarrow \mathbb{T}(Y,X)$$

is surjective. This means that $\mathbb{T}(Y,X) = 0$.

Now, we have the following commutative diagram

$$\begin{array}{ccc} 0 = \mathbb{T}(Y,X) & \longleftarrow & \mathbb{T}(Y,J^{\wedge}hJ(X)) \\ \downarrow & & \downarrow \\ \mathcal{A}(J(Y),J(X)) & \longleftarrow & \mathcal{A}(J(Y),hJ(X)) \end{array} \qquad (1)$$

in which both vertical arrows are (canonical) bijections, and the upper horizontal arrow is bijective too, as we have showed in *(a)*. Hence all arrows in the diagram (1) are bijective which implies that $\mathcal{A}(J(Y),hJ(X)) = 0$. But, $\mathcal{A}(J(Y),E)$ is nonzero, since $J(Y)$ is a subobject of E; and $\mathcal{A}(E,hJ(X))$ is an integral family of arrows. So, if $Y \neq 0$, then there is an arrow,

$$g: E \longrightarrow hJ(X),$$

such that the composition of g with the embedding $J(Y) \longrightarrow E$ is nonzero.

This proves that Y is zero. Hence $J^\wedge(E) \in ISpec\mathbb{T}$.

(c) Combining *(a)* and *(b)*, we see that the map

$$E \longmapsto J^\wedge(E) \quad \text{from} \quad VI(\mathbb{T}) := \{E \in ISpec\mathcal{A} \mid J^\wedge(E) \neq 0\} \quad \text{to} \quad ISpec\mathbb{T}$$

is left inverse to the map

$$X \longmapsto hJ(X) \quad \text{from} \quad ISpec\mathbb{T} \quad \text{to} \quad VI(\mathbb{T}).$$

In particular, the induced by $J^\wedge$ map

$$\mathbf{VI}(\mathbb{T}) \longrightarrow \mathbf{ISpec}\mathbb{T}$$

is left inverse to the induced by hJ map $\mathbf{ISpec}\mathbb{T} \longrightarrow \mathbf{VI}(\mathbb{T})$ of Lemma 5.4.

Note that $hJJ^\wedge(E)$ is a nonzero injective subobject of E which means (since $E \in ISpec\mathcal{A}$) that $hJJ^\wedge(E)$ is equivalent to E; i.e. $\mathfrak{S}(E) = \mathfrak{S}(hJJ^\wedge(E))$. This shows that the induced by hJ map $\mathbf{ISpec}\mathbb{T} \longrightarrow \mathbf{VI}(\mathbb{T})$ is left inverse to the map induced by the functor $J^\wedge$. Therefore these two maps are mutually inverse.

(d) The bijectivity of $\mathbf{I^\wedge Spec}\mathbb{T} \longrightarrow \mathbf{VI}^\wedge(\mathbb{T})$ follows from the bijectivity of $\mathbf{ISpec}\mathbb{T} \longrightarrow \mathbf{VI}(\mathbb{T})$. The details are left to the reader.

(ii) Let now E be an object of $ISpec\mathcal{A}/\mathbb{T}$. Then $Q^\wedge(E)$ is a $\mathbb{T}$-torsion free injective object of $\mathcal{A}$. We claim that the object $Q^\wedge(E)$ belongs to $ISpec\mathcal{A}$.

In fact, let X be a nonzero injective subobject of $Q^\wedge(E)$. Since the object X is $\mathbb{T}$-torsion free, $\mathbb{T} \subseteq \mathfrak{S}(X)$. This shows that $\mathfrak{S}(Q^\wedge(E)) = S(X)$ if and only if $S(QQ^\wedge(E)) = \mathfrak{S}(Q(X))$.

Since the object X is a retract of $Q^\wedge(E)$, its image, $Q(X)$, is a retract of $QQ^\wedge(E) \simeq E$. Therefore $Q(X)$ is a nonzero injective subobject of E which implies (since $E \in ISpec\mathcal{A}/\mathbb{T}$) that $\mathfrak{S}(Q(X)) = \mathfrak{S}(E)$.

Conversely, let E' be a $\mathbb{T}$-torsion free object from $ISpec\mathcal{A}$. According to Lemma 5.5, $Q(E')$ is an injective object in $\mathcal{A}/\mathbb{T}$. Let $\iota: X \longrightarrow Q(E')$ be a nonzero injective subobject. Consider the diagram

$$Q^\wedge(X) \xrightarrow{Q^\wedge\iota} Q^\wedge Q(E') \xleftarrow{\eta(E')} E', \tag{2}$$

where η denotes the adjunction arrow. Since E' is $\mathbb{T}$-torsion free, the arrow $\eta(E')$ is a monomorphism. Therefore

$$Q^\wedge Q(E') \simeq E' \oplus Y \tag{3}$$

for some $\mathbb{T}$-torsion free object Y. Since the functor morphism $Q\eta$ is an isomorphism, it follows from (3) that $Q(Y) = 0$. Therefore $Y = 0$; or, equivalently, $\eta(E')$ is an isomorphism. Thus, $Q^\wedge(X)$ is an injective subobject of E'. Since $E' \in ISpec\mathcal{A}$, $Q^\wedge(X)$ is equivalent to E'; i.e.

$$\mathbb{S}(Q^\wedge(X)) = \mathbb{S}(E'). \tag{4}$$

Since $\mathbb{T} \subseteq \mathbb{S}(E')$, the equality (4) is equivalent to the equality

$$\mathbb{S}(X) = \mathbb{S}(QQ^\wedge(X)) = \mathbb{S}(E').$$

Here we use the (adjunction) isomorphism $QQ^\wedge(X) \simeq X$. ■

6. The Gabriel-Krull dimension.

6.0. Preliminaries. We fix an abelian category $\mathcal{A}$ satisfying the property *(sup)* (cf. 0.4.3.2).

The *Gabriel filtration of* $\mathcal{A}$ assigns to every ordinal α a Serre subcategory $\mathcal{A}_\alpha$ of $\mathcal{A}$ which is constructed as folows:

Set $\mathcal{A}_0 := 0$.

If α is not a limit ordinal, then $\mathcal{A}_\alpha$ is the smallest Serre subcategory of $\mathcal{A}$ containing all objects M such that the localization $Q_{\alpha-1}(M)$ of M at $\mathcal{A}_{\alpha-1}$ has a finite length.

If β is a limit ordinal, then $\mathcal{A}_\beta$ is the smallest Serre subcategory containing all subcategories $\mathcal{A}_\alpha$ for $\alpha < \beta$.

Let $\mathcal{A}_\omega$ denote the smallest Serre subcategory containing all the subcategories $\mathcal{A}_\alpha$. Clearly the quotient category $\mathcal{A}/\mathcal{A}_\omega$ has no simple objects.

An object M is said to have the *Gabriel dimension* β,

$$Gdim(M) = \beta,$$

if β is the smallest ordinal such that M belongs to $\mathcal{A}_\beta$.

The following assertion follows from the definitions:

6.0.1. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}_\omega$. *Then*

$$sup(Gdim(M'), Gdim(M'')) \leq Gdim(M) \leq Gdim(M') + Gdim(M'').$$

If $\mathcal{A} = \mathcal{A}_\omega$, then the smallest ordinal α such that $\mathcal{A} = \mathcal{A}_\alpha$ is called *the Gabriel dimension* of the category $\mathcal{A}$: $Gdim(\mathcal{A}) = \alpha$.

Given a Serre subcategory $\mathbb{S}$ of $\mathcal{A}$, the Gabriel filtration $\{\mathcal{A}_\alpha\}$ induces the filtration $\{\mathcal{A}_\alpha/(\mathcal{A}_\alpha \cap \mathbb{S})\}$ of the category $\mathcal{A}/\mathbb{S}$ in which some of the consecutive subcategories may coincide. It is clear that $\mathcal{A}_\alpha/(\mathcal{A}_\alpha \cap \mathbb{S}) \subseteq (\mathcal{A}/\mathbb{S})_\alpha$ for any ordinal α. This shows that if $\mathcal{A} = \mathcal{A}_\omega$, then $\mathbb{S} = \mathbb{S}_\omega$ and $(\mathcal{A}/\mathbb{S}) = (\mathcal{A}/\mathbb{S})_\omega$.

Thus, we have the following Proposition (IV.1.1 in [Gab]):

6.0.2. Proposition. *Let* $\mathbb{S}$ *be a Serre subcategory of* $\mathcal{A}$. *Then*

$$\mathcal{A} = \mathcal{A}_\omega \text{ if and only if } \mathbb{S}_\omega = \mathbb{S} \text{ and } (\mathcal{A}/\mathbb{S})_\omega = \mathcal{A}/\mathbb{S}.$$

In this case,

$$sup(Gdim(\mathbb{S}), Gdim(\mathcal{A}/\mathbb{S})) \leq Gdim(\mathcal{A}) \leq Gdim(\mathbb{S}) + Gdim(\mathcal{A}/\mathbb{S}).$$

Proof. (a) The assertion *(a)* is already proved.

(b) Denote by $\mathbb{S}_\alpha$ the preimage of the subcategory $(\mathcal{A}/\mathbb{S})_\alpha$ in $\mathcal{A}$. We claim that if $\mathbb{S} = \mathbb{S}_\omega$, then $\mathbb{S}_\alpha = (\mathbb{S}_\alpha)_\omega$ for any α.

It is true, of course, for $\alpha = 0$.

Suppose it is true for all $\alpha < \beta$.

(i) If β is a limit ordinal, then $\mathbb{S}_\beta = (\bigcup_{\alpha<\beta} \mathbb{S}_\alpha)^-$; and since $\mathbb{S}_\alpha \subseteq \mathcal{A}_\omega$ for any $\alpha < \beta$, the same holds for $\mathbb{S}_\beta$.

(ii) Suppose now that β is not a limit ordinal.

By the induction hypothesis, $\mathbb{S}_{\beta-1} \subseteq \mathcal{A}_\omega$. Therefore there is an ordinal γ such that $\mathbb{S}_{\beta-1} \subseteq \mathcal{A}_\gamma$, but $\mathbb{S}_{\beta-1}$ is not contained in $\mathcal{A}_\alpha$ for any $\alpha < \gamma$. Clearly $\mathbb{S}_\beta \subseteq \mathcal{A}_{\gamma+1}$. ■

6.0.3. Corollary. *Let* $\mathcal{A}$ *be an abelian category which has Gabriel dimension; i.e.* $\mathcal{A} = \mathcal{A}_\omega$. *Then, for any proper Serre subcategory* $\mathbb{T}$ *of* $\mathcal{A}$, *the quotient category* $\mathcal{A}/\mathbb{T}$ *has simple objects.*

6.0.4. Corollary. *If* $\mathcal{A} = \mathcal{A}_\omega$, *then any Serre subcategory* $\mathbb{T}$ *of the category* $\mathcal{A}$ *coincides with the intersection of all* $\mathbb{P} \in$ **Speg**$\mathcal{A}$ *containing* $\mathbb{T}$.

Proof. Let $\mathbb{S}$ denote the intersection of all $\mathbb{P} \in$ **Speg**$\mathcal{A}$ which contain $\mathbb{T}$. If $\mathbb{S} \neq \mathbb{T}$, then $\mathbb{S}/\mathbb{T}$ is a nonzero Serre subcategory of the category $\mathcal{A}/\mathbb{T}$. Therefore the sucategory $\mathbb{S}/\mathbb{T}$ has Gabriel dimension (cf. Proposition 6.0.2). By Corollary 6.0.3, $\mathbb{S}/\mathbb{T}$ has simple objects. Let M denote one of them; and let

$\mathbb{P}$ be the preimage of $<M> \in \mathbf{Spec}\mathcal{A}/\mathbb{T}$. Since M is simple, its image in $\mathcal{A}/<M>$ is a simple object; i.e. $<M> \in \mathbf{Speg}\mathcal{A}/\mathbb{T}$ which implies that $\mathbb{P} \in \mathbf{Speg}\mathcal{A}$. Now note that $\mathbb{S}$ is not contained in $\mathbb{P}$, since the object M (of $\mathbb{S}/\mathbb{T}$) does not belong to $\mathbb{P}/\mathbb{T} = <M>$. Thus, we have run into a contradiction which shows that $\mathbb{S}$ should coincide with $\mathbb{T}$. ■

6.0.5. Locally noetherian categories. An object M of an abelian category $\mathcal{A}$ is called *noetherian* if any increasing sequence of subobjects of M stabilizes.

An object M of $\mathcal{A}$ is called *locally noetherian* if M is the supremum of a family of its noetherian subobjects.

An abelian category $\mathcal{A}$ is called *noetherian* if all objects of $\mathcal{A}$ are noetherian.

An abelian category $\mathcal{A}$ is called *locally noetherian* if it has the property *(sup)* and every object of $\mathcal{A}$ is the supremum of a family of its noetherian subobjects.

6.0.5.1. Remarks. *(a)* Clearly, any noetherian category has the property *(sup)*. In particular, any noetherian category is locally noetherian.

(b) The given above definition of a locally noetherian category is not an exact copy of the conventional one (cf. [Gab] or [BD]). In [Gab], $\mathcal{A}$ is required to be a Grothendieck category. Thus, a nonzero noetherian category cannot be locally noetherian in the conventional sense.

(c) Noetherian objects of any abelian categoy $\mathcal{B}$ generate a thick subcategory, $\mathfrak{N}\mathcal{B}$, of $\mathcal{B}$. In other words, the coproduct of any two noetherian objects is a noetherian object, as well as any subquotient of a noetherian object. This implies, in particular, that every object of a locally noetherian category is the supremum of an *increasing* family of its noetherian subobjects.

Clearly $\mathfrak{N}\mathcal{B}$ is the biggest thick noetherian subcategory of $\mathcal{B}$.

(d) Suppose that $\mathcal{B}$ is an abelian category with the property *(sup)*. Then the smallest Serre subcategory $(\mathfrak{N}\mathcal{B})^-$ containing $\mathfrak{N}\mathcal{B}$ coincides with the full subcategory of $\mathcal{B}$ generated by all locally noetherian objects of $\mathcal{B}$. ■

6.0.5.2. Lemma. *Let* $\mathbb{S}$ *be a thick subcategory of an abelian category* $\mathcal{A}$, *and* Q *a localization* $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{S}$. *Then, for any noetherian object* M *of* $\mathcal{A}$, *the object* $Q(M)$ *is noetherian.*

Proof. a) First note that M has a maximal subobject, $\mathbb{S}M$, among the subobjects which belong to $\mathbb{S}$. So that the object $M/\mathbb{S}M$ is $\mathbb{S}$-torsion free. Replacing M by $M/\mathfrak{S}M$, we assume that M is $\mathfrak{S}$-torsion free.

b) Let $f: Q(X) \longrightarrow Q(M)$ be a monoarrow. By the definition of a quotient category, the arrow f is the image of an arrow

$$f: X' \longrightarrow M/M',$$

where X' is a subobject of X such that $X/X' \in Ob\mathfrak{S}$ and M' is a subobject of M which belongs to $\mathfrak{S}$. But, M is assumed to be $\mathfrak{S}$-torsion free; hence $M' = 0$. Replacing X' by its image in M, we see that the subobject f is isomorphic to the subobject Qf'' for some monoarrow $f'': X'' \longrightarrow M$.

c) To prove that the object $Q(M)$ is noetherian, one need to show that, for any family Ω of subobjects of $Q(M)$, there is a finite subset Ξ of Ω such that, for any bigger finite subset, Ξ', of Ω, the canonical arrow

$$sup(\Xi) \longrightarrow sup(\Xi')$$

is an isomorphism which implies that $sup(\Xi) = sup(\Omega)$).

Let Ω be any family of subobjects of the object $Q(M)$. And let Ω' be the corresponding family of subobjects of M (cf. the part *b)* of the argument). Since M is a noetherian object, there is a finite subset Ξ' of Ω' such that $sup(\Xi') = sup(\Omega')$.

Since the localization $Q: \mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{S}$ is an exact functor, this implies that $sup(Q(\Xi')) = sup(\Omega)$. ∎

6.0.5.3. Corollary. *Let* $\mathcal{A}$ *be a (locally) noetherian category. Then, for any Serre subcategory* $\mathfrak{S}$ *of* $\mathcal{A}$, *the quotient category,* $\mathcal{A}/\mathfrak{S}$, *is (locally) noetherian.*

6.0.5.4. Corollary. *Every locally noetherian category has Gabriel dimension.*

Proof. According to Corollary 6.0.5.3, the category $\mathcal{A}/\mathcal{A}_\omega$ is locally noetherian. So, if $\mathcal{A} \neq \mathcal{A}_\omega$, i.e. $\mathcal{A}/\mathcal{A}_\omega \neq 0$, then there are nonzero noetherian objects in the category $\mathcal{A}/\mathcal{A}_\omega$. Since any nonzero noetherian object has a maximal proper subobject, $\mathcal{A}/\mathcal{A}_\omega$ has simple objects which cannot happen. ∎

6.1. The spectra of a category which has Gabriel dimension. Fix an abelian category $\mathcal{A}$ with the property *(sup)*.

6.1.1. Proposition. *If* $\mathcal{A} = \mathcal{A}_\omega$ *then* $\mathbf{Spec}^-\mathcal{A} = \mathbf{Speg}\mathcal{A}$.

Proof. The assertion *(a)* is a straightforward consequence of Corollary 6.0.3 and Proposition 3.3. ∎

6.1.2. Proposition. *Let* $\mathcal{A}$ *be an abelian category with injective hulls and the property (sup). Suppose that* $\mathcal{A} = \mathcal{A}_\omega$. *Then*

(a) the embedding $\mathbf{Speg}\mathcal{A} \longrightarrow \mathbf{I^\wedge Spec}\mathcal{A}$ *is a bijection;*

(b) $\mathbf{I^\wedge Spec}\mathcal{A} = \mathbf{ISpec}\mathcal{A}$.

Proof. (a) Let $E \in I^\wedge Spec\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}_\omega$, there exists an ordinal α such that E has a nonzero subobject from $\mathcal{A}_{\alpha+1}$, but is $\mathcal{A}_\alpha$-torsion free. Clearly E contains a subobject M such that the localization $Q_\alpha(M)$ of M at $\mathcal{A}_\alpha$ is a simple object of the category $\mathcal{A}/\mathcal{A}_\alpha$. In particular, the preimage of $<Q_\alpha(M)>$ in $\mathcal{A}$ belongs to $\mathbf{Speg}\mathcal{A}$.

On the other hand, this preimage coincides with $\mathfrak{S}(E)$. This follows from the fact that E is the injective hull of M.

Indeed, if $\mathcal{A}(X,E) = 0$, then $Q_\alpha(X) \in <Q_\alpha(M)>$.

Since E is an $\mathcal{A}_\alpha$-torsion free injective object, the functor Q_α maps $\mathcal{A}(X,E)$ bijectively onto $\mathcal{A}/\mathcal{A}_\alpha(Q_\alpha(X),Q_\alpha(E))$ for any X. In particular, $Q_\alpha(E)$ is an injective object in $\mathcal{A}/\mathcal{A}_\alpha$.

An easy way to see this, is to use the adjunction isomorphism

$$\mathcal{A}/\mathcal{A}_\alpha(Q_\alpha(X),Q_\alpha(E)) \simeq \mathcal{A}_\alpha(X,Q_\alpha{}^\wedge Q_\alpha(E))$$

and the isomorphness of adjunction arrow $E \longrightarrow Q_\alpha{}^\wedge Q_\alpha(E))$ (cf. the end of the argument of Lemma 5.2). Now we have, for any object X of $\mathcal{A}$, the following implications:

$$\mathcal{A}(X,E) = 0 \iff \mathcal{A}/\mathcal{A}_\alpha(Q_\alpha(X),Q_\alpha(E)) = 0 \qquad (1)$$

and

$$\mathcal{A}/\mathcal{A}_\alpha(Q_\alpha(X),Q_\alpha(E)) = 0 \iff Q_\alpha(X) \in <Q_\alpha(M)>. \qquad (2)$$

To prove (2), note that if $Q_\alpha(X) \succ Q_\alpha(M)$, then, due to the injectivity of $Q_\alpha(E)$ and the existence of a (mono)morphism from $Q_\alpha(M)$ to $Q_\alpha(E)$, there is a nonzero arrow from $Q_\alpha(X)$ to $Q_\alpha(E)$ (cf. the proof of Lemma 4.1.1).

Conversely, if $g: Q_\alpha(X) \longrightarrow Q_\alpha(E)$ is a nonzero morphism, then $im(g)$ has a nonzero intersection with $Q_\alpha(M)$ (since $Q_\alpha(E)$ is the injective hull of $Q_\alpha(M)$. Since $Q_\alpha(M)$ is simple, $im(g)$ contains $Q_\alpha(M)$. This implies that

$Q_\alpha(X) \succ Q_\alpha(M)$.

(b) Let now E be an object of *ISpec*$\mathcal{A}$. Take an ordinal α such that E is $\mathcal{A}_\alpha$-torsion free, and has a nonzero subobject M from $\mathcal{A}_{\alpha+1}$. This implies that E has a subobject M such that $Q_\alpha(M)$ is a simple object of the category $\mathcal{A}/\mathcal{A}_\alpha$. One can see that the injective hull, $h(M)$, of the object M is an indecomposable injective subobject of E. Since $E \in$ *ISpec*$\mathcal{A}$, $\mathbb{S}(E) = \mathbb{S}(h(M))$; i.e. E defines the same element of **ISpec**$\mathcal{A}$ as the object $h(M)$ of *I^Spec*$\mathcal{A}$. This means that **ISpec**$\mathcal{A}$ = **I^Spec**$\mathcal{A}$. ■

6.2. Spec and Spec¯. In general, **Spec**$\mathcal{A}$ is a pretty meagre proper subset of **Spec¯**$\mathcal{A}$. But, if $\mathcal{A}$ has the Gabriel-Krull dimension, then **Spec**$\mathcal{A}$ is ample, as the following lemma shows.

6.2.1. Lemma. *Suppose that* $\mathcal{A} = \mathcal{A}_\omega$. *Then* $Supp(M) \neq \varnothing$ *for any nonzero object* M *of* $\mathcal{A}$.

Proof. In fact, the full subcategory of $\mathcal{A}$ generated by all objects M for which $Supp(M) = \varnothing$ coincides with $S\mathcal{A} = \bigcap_{<P> \in \mathbf{Spec}\mathcal{A}} <P>$; in particular, $S\mathcal{A}$ is a Serre subcategory.

According to Proposition 6.0.2, the equality $\mathcal{A} = \mathcal{A}_\omega$ implies that $S\mathcal{A} = (S\mathcal{A})_\omega$. Therefore either $S\mathcal{A}$ has simple objects, or it is a zero subcategory. Since $S\mathcal{A}$ cannot have simple objects, it equals to zero. ■

6.2.2. Corollary. *Suppose that* $\mathcal{A} = \mathcal{A}_\omega$. *Then, for any Serre subcategory* $\mathbb{S}$ *of* $\mathcal{A}$, $Supp(M) \neq \varnothing$ *for any nonzero object of* $\mathbb{S}$ *or* $\mathcal{A}/\mathbb{S}$.

Proof. The fact follows from Lemma 6.2.1 and Proposition 6.0.2 (= Proposition IV.1.1 in [Gab]). ■

6.2.3. Proposition. *Suppose that* $\mathcal{A} = \mathcal{A}_\omega$. *Then the following conditions are equivalent:*

(a) **Spec**$\mathcal{A}$ = **Spec¯**$\mathcal{A}$;

(b) for any non-limit ordinal α,

$$\mathcal{A}_\alpha = \mathcal{A}(\mathbf{Spec}\mathcal{A}_\alpha);\ \text{i.e.}\ Ob\mathcal{A}_\alpha = \{M \in Ob\mathcal{A} \mid Supp(M) \subseteq \mathbf{Spec}\mathcal{A}_\alpha\}.$$

In other words, any object M *such that*

$$M \succ P,\ P \in Spec\mathcal{A},\ \Rightarrow\ P \in Ob\mathcal{A}_\alpha$$

belongs to $\mathcal{A}_\alpha$.

Proof. (b) $\Rightarrow$ *(a).* We begin with the following observation:

(i) Let $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ *and* $\mathbb{P}$ *does not contain* $\mathcal{A}_{\alpha}$ *for some ordinal* α. *Then* $\mathbb{P} \in \mathbf{Spec}\mathcal{A}$ *if and only if* $\mathbb{P} \cap \mathcal{A}_{\alpha} \in \mathbf{Spec}\mathcal{A}_{\alpha}$.

This follows from Lemma 5.3.1 in [R3] and Proposition 2.4.

(ii) Thanks to *(i)*, it suffices to show that, for any ordinal α,

$$\mathbf{Spec}^{-}\mathcal{A}_{\alpha} = \mathbf{Spec}\mathcal{A}_{\alpha}. \tag{1}$$

Clearly the statement is true for $\alpha = 0$:

$$\mathbf{Spec}^{-}\mathcal{A}_{0} = \mathbf{Simple}\mathcal{A} = \mathbf{Spec}\mathcal{A}_{0}. \quad \text{and} \quad \mathcal{A}_{0} = \mathcal{A}(\mathbf{Simple}\mathcal{A}).$$

Suppose that the equalities (1) holds for all $\alpha < \beta$.

1) If β is a limit ordinal, then we have:

$$\mathbf{Spec}^{-}\mathcal{A}_{\beta} = \bigcup_{\alpha<\beta} \mathbf{Spec}^{-}\mathcal{A}_{\alpha} = \bigcup_{\alpha<\beta} \mathbf{Spec}\mathcal{A}_{\alpha} = \mathbf{Spec}\mathcal{A}_{\beta}.$$

and

$$\mathcal{A}_{\beta} = \mathcal{A}(\mathbf{Spec}^{-}\mathcal{A}_{\beta}) = \mathcal{A}(\mathbf{Spec}\mathcal{A}_{\beta}).$$

Here we identify $\mathbf{Spec}^{-}\mathcal{A}_{\alpha}$ (and $\mathbf{Spec}\mathcal{A}_{\alpha}$) with the set of all $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ (resp. $\mathbb{P} \in \mathbf{Spec}\mathcal{A}_{\alpha}$) which do not contain $\mathcal{A}_{\alpha}$ (cf. *(i)* and Proposition 2.4).

2) Suppose now that β is not a limit ordinal. Let $\mathbb{P}$ be an arbitrary element of $\mathbf{Spec}^{-}\mathcal{A}_{\beta}$. If $\mathcal{A}_{\beta-1}$ is not contained in $\mathbb{P}$, then

$$\mathbb{P} \cap \mathcal{A}_{\beta-1} \in \mathbf{Spec}^{-}\mathcal{A}_{\beta-1},$$

and $\mathbf{Spec}^{-}\mathcal{A}_{\beta-1}$ is equal to $\mathbf{Spec}\mathcal{A}_{\beta-1}$ by induction hypothesis. So, we shall assume that $\mathcal{A}_{\beta-1} \subseteq \mathbb{P}$.

Let M be an object of $\mathcal{A}$ such that its localization $Q_{\beta-1}(M)$ is a simple object of $\mathcal{A}_{\beta}/\mathcal{A}_{\beta-1} - \mathbb{P}/\mathcal{A}_{\beta-1}$. By induction hypothesis,

$$Supp(M) \text{ is not contained in } \mathbf{Spec}\mathcal{A}_{\beta-1}.$$

So, there exists an element $<P> \in Supp(M) - \mathbf{Spec}\mathcal{A}_{\beta-1}$; i.e. $M \succ P$ and $P \notin Ob\mathcal{A}_{\beta-1}$. This means that

$$Q_{\beta-1}(M) \succ Q_{\beta-1}(P) \neq 0. \tag{3}$$

Since $Q_{\beta-1}(M)$ is a simple object, the relation (3) implies that $Q_{\beta-1}(P)$ is equivalent (in the sense of the relation $\succ$) to $Q_{\beta-1}(M)$ (according to Proposition 1.3.1 in [R3], the object $Q_{\beta-1}(P)$ is the direct sum of a finite number of copies of $Q_{\beta-1}(M)$).

Thus, $\mathbb{P} \cap \mathcal{A}_\beta$ is the preimage of the subcategory $<Q_{\beta-1}(P)>$ under the co-restriction of $Q_{\beta-1}$ to $\mathcal{A}_\beta/\mathcal{A}_{\beta-1}$. This implies that $\mathbb{P} \cap \mathcal{A}_\beta = <P> \cap \mathcal{A}_\beta$. Therefore, according to Proposition 2.4, $\mathbb{P}$ coincides with $<P>$.

(a) $\Rightarrow$ *(b)*. Let now that $\mathbf{Spec}\mathcal{A} = \mathbf{Spec}^{-}\mathcal{A}$. And suppose there exists an object M of $\mathcal{A}$ such that $Supp(M) \subseteq \mathbf{Spec}\mathcal{A}_{\beta-1}$ for some non-limit ordinal β, but $M \notin Ob\mathcal{A}_{\beta-1}$. Since the image, M', of M in the quotient category $\mathcal{A}/\mathcal{A}_{\beta-1}$ is a nonzero object, and the category $\mathcal{A}/\mathcal{A}_{\beta-1}$ has Gabriel-Krull dimension, the support of M' is nonempty. In other words, there exists $\mathbb{P} \in \mathbf{Spec}\mathcal{A}$ such that

$$M \notin Ob\mathbb{P}, \quad \text{and} \quad \mathcal{A}_{\beta-1} \subseteq \mathbb{P}. \qquad (1)$$

By assumption, $\mathbb{P} = <P>$ for some $P \in Spec\mathcal{A}$. And (1) means exactly that

$$<P> \in Supp(M), \quad \text{but} \quad <P> \notin \mathbf{Spec}\mathcal{A}_{\beta-1}$$

which contradicts to the initial assumption. ∎

6.2.4. Corollary. *Under the equivalent conditions of Proposition 6.1.4, any Serre subcategory* $\mathbb{S}$ *of* $\mathcal{A}$ *is the intersection of all* $<P> \in \mathbf{Spec}\mathcal{A}$ *such that* $\mathbb{S} \subseteq <P>$.

Proof. Every Serre subcategory $\mathbb{S}$ of $\mathcal{A}$ is the intersection of all $\mathbb{P} \in \mathbf{Speg}\mathcal{A}$ such that $\mathbb{S} \subseteq \mathbb{P}$ (cf. Corollary 6.0.4). But

$$\mathbf{Speg}\mathcal{A} = \mathbf{Spec}^{-}\mathcal{A} = \mathbf{Spec}\mathcal{A}$$

according to Propositions 6.1.2 and 6.1.4. ∎

The following, very basic, example shows that one should not restrict oneself to the categories satisfying the conditions of Proposition 6.2.3.

6.2.5. Example: Proj. Fix a $\mathbb{Z}_+$-graded ring $R = \bigoplus_{n\geq 0} R_n$. And consider the category $\mathcal{A} := \mathrm{gr}_{\mathbb{Z}}R\text{-}mod$ of $\mathbb{Z}$-graded R-modules. Let $\mathbb{T}$ denote the full subcategory of $\mathcal{A}$ generated by all graded R-modules M such that $Ann(M)$ contains the two-sided ideal $R_+ := \bigoplus_{n\geq 1} R_n$.

Note that $\mathbf{Spec}\mathcal{A} = \mathbf{Spec}\mathbb{T} \simeq \bigcup_{\mathbb{Z}\text{ copies}} \mathbf{Spec}R_0\text{-}mod$.

In fact, for any graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and any $n \in \mathbb{Z}$, the sum

$$M[n] := \bigoplus_{i\geq n} M_i$$

is a submodule in M. Clearly $M[n]$ is equivalent to M (with resp. to $\succ$) iff

$M_i = 0$ for all $i < n$. Therefore, if $P \in Spec\mathcal{A}$, then there exists m such that $P_n = 0$ if $n \neq m$, and the R_0-module P_m belongs to $SpecR_0\text{-}mod$.

Note that **Spec**$\mathcal{A}$ is ample; i.e. if $Supp(M) = \varnothing$, then $M = 0$.

This follows from the observation that, for any integer m, the subquotient $M\{m+1\}/M\{m\}$ belongs to the subcategory $\mathbb{T}$. Clearly $M\{m+1\}/M\{m\}$ is isomorphic to a module V such that $V_n = 0$ if $n \neq m$, and $V_m = M_m$.

Define **Proj**(R) as the quotient category $\mathcal{A}/\mathbb{T}^{-}$.

It is not difficult to show that $\mathbb{T}^{-} = \mathcal{T}^{-}$, where $\mathcal{T}$ is a thick subcategory of $\mathcal{A}$ generated by all R-modules M such that the set $\{n \mid M_n \neq 0\}$ is finite. If the ring R is commutative, then, by a Serre's theorem, the category **Proj**(R) is equivalent to the category of quasi-coherent sheaves on the projective spectrum, $Proj(R)$, of the ring R.

Thus, **Spec**(**Proj**(R)) has nothing to do with **Spec**$\mathcal{A}$. At the same time, if the ring R is noetherian, then **Proj**(R) is a noetherian category. In particular, it has Gabriel-Krull dimension (cf. Corollary 6.0.5.4); hence, the spectrum of **Proj**(R) is ample by Lemma 6.2.1.

Note that **Spec**$^{-}$(**Proj**(R)) is an open subset (with respect to the topology τ) of **Spec**$^{-}\mathcal{A}$ = **Speg**$\mathcal{A}$.

6.3. Dimensions. Fix an abelian category $\mathcal{A}$. To every ordinal α, we assign a subset $\mathfrak{E}_\alpha$ of **Spec**$^{-}\mathcal{A}$ defined as follows.

$\mathfrak{E}_0 = \varnothing$;

if α is not a limit ordinal, $\mathfrak{E}_\alpha$ consists of all elements $\mathbb{P} \in$ **Spec**$^{-}\underline{\mathbf{A}}$ such that any $\mathbb{P}' \in$ **Spec**$^{-}\mathcal{A}$ for which $\mathbb{P}' \subset \mathbb{P}$, but $\mathbb{P}$ is distinct of $\mathbb{P}'$, belongs to $\mathfrak{E}_{\alpha-1}$;

if β is a limit ordinal, then $\mathfrak{E}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{E}_\beta$.

Clearly $\mathfrak{E}_\alpha \subseteq \mathfrak{E}_\beta$ if $\alpha \leq \beta$ which implies that the set $\mathfrak{E}_\alpha$ is closed in the topology τ.

Denote by **Spec**$_\omega^{-}\mathcal{A}$ the union of all subsets $\mathfrak{E}_\alpha$. Since the union of any family of closed in τ subsets is closed, **Spec**$_\omega^{-}\mathcal{A}$ is a closed subset of the topological space $(\mathbf{Spec}^{-}\mathcal{A}, \tau)$.

For any $\mathbb{P} \in$ **Spec**$_\omega^{-}\mathcal{A}$, there is the biggest ordinal, $\mathfrak{ht}^{-}(\mathbb{P})$, among the ordinals α such that $\mathbb{P} \notin \mathfrak{E}_\alpha$. We call the ordinal $\mathfrak{ht}^{-}(\mathbb{P})$ *the height of* $\mathbb{P}$.

Thus, $\mathfrak{E}_\alpha$ is the set $\{\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A} \mid \mathfrak{ht}^{-}(\mathbb{P}) \leq \alpha\}$.

We define *the flat dimension of* $\mathcal{A}$ (or f-dimension) as the supremum of all

$\mathfrak{ht}^-(\mathbb{P})$, where $\mathbb{P}$ runs through $\mathbf{Spec}_\omega^-\mathcal{A}$.

The notation: $\mathfrak{fdim}\mathcal{A}$.

For any ordinal α, set $\mathcal{A}_\alpha^- := \mathcal{A}(\mathfrak{E}_\alpha)$; i.e. $\mathcal{A}(\mathfrak{E}_\alpha)$ is the full subcategory of the category $\mathcal{A}$ generated by all objects M such that $Supp^-(M) \subseteq \mathfrak{E}_\alpha$.

Set $\mathcal{A}_\omega^- := \mathcal{A}(\mathbf{Spec}_\omega^-\mathcal{A})$.

According to Proposition 2.6.2, $\mathcal{A}_\alpha^-$ and $\mathcal{A}_\omega^-$ are Serre subcategories of $\mathcal{A}$, and

$$\mathbf{Spec}^-\mathcal{A}(\mathfrak{E}_\alpha) = \mathfrak{E}_\alpha \text{ and } \mathbf{Spec}^-(\mathcal{A}_\omega^-) = \mathbf{Spec}_\omega^-\mathcal{A}.$$

In particular,

$$\mathfrak{dim}^-\mathcal{A} = \mathfrak{dim}^-(\mathcal{A}_\omega^-).$$

6.3.1. Remark. Note that $\mathbf{Spec}_\omega^-\mathcal{A} = \varnothing$ if and only if $\mathbf{Spec}^-\mathcal{A}$ has no closed (in the topology τ) points.

Clearly $\mathbf{Spec}^-(\mathcal{A}/\mathcal{A}_\omega^-)$ has no closed points. ∎

6.3.2. Proposition. *The following conditions on an element* $\mathbb{P}$ *of* $\mathbf{Spec}^-\mathcal{A}$ *are equivalent:*

(a) $\mathfrak{ht}^-(\mathbb{P})$ *is a finite number.*

(b) There is the maximal integer, n, *among nonnegative integers* m *such that there exists a chain*

$$\mathbb{P} \supset \mathbb{P}_1 \supset \mathbb{P}_2 \supset \ldots \supset \mathbb{P}_m$$

of distinct elements of $\mathbf{Spec}^-\mathcal{A}$.

The number n *in (b) is equal to* $\mathfrak{ht}^-(\mathbb{P})$.

Proof. Clearly, if there exists a chain

$$\mathbb{P} \supset \mathbb{P}_1 \supset \mathbb{P}_2 \supset \ldots \supset \mathbb{P}_n \tag{1}$$

of distinct elements of $\mathbf{Spec}^-\mathcal{A}$, then $\mathfrak{ht}^-(\mathbb{P}) \geq n$.

Therefore the assertion shall be proved if we show that, for any $\mathbb{P} \in \mathbf{Spec}^-\mathcal{A}$ such that $\mathfrak{ht}^-(\mathbb{P}) = n$, there exists a chain (1) of distinct elements of $\mathbf{Spec}^-\mathcal{A}$. But, the latter statement follows almost immediately from the definition of the height.

In fact, since $\mathfrak{ht}^-(\mathbb{P}) = n$, there exists an element $\mathbb{P}_1$ in $\mathbf{Spec}^-\mathcal{A}$ of the height $n - 1$ and such that $\mathbb{P}_1 \subset \mathbb{P}$ (otherwise the height of $\mathbb{P}$ would be less than n). So, we can use the (finite) induction. ∎

Proposition 6.3.2 shows that our definition of the height and dimension are generalizations of the conventional ones.

One can repeat all this constructions replacing $\mathbf{Spec}^{-}\mathcal{A}$ by any other of the considered here spectra. Thus we obtain different notions of height and dimension:

- $\mathfrak{iht}$, $\mathfrak{idim}$ (corresponding to $\mathbf{ISpec}\mathcal{A}$);
- $\mathfrak{i}^{\wedge}\mathfrak{ht}$, $\mathfrak{i}^{\wedge}\mathfrak{dim}$ (corresponding to $\mathbf{I}^{\wedge}\mathbf{Spec}\mathcal{A}$);
- $\mathfrak{ght}$, $\mathfrak{gdim}$ (corresponding to $\mathbf{Speg}\mathcal{A}$);
- and, finally, we shall write simply $\mathfrak{ht}$ and $\mathfrak{dim}$ for the height and dimension corresponding to $\mathbf{Spec}\mathcal{A}$.

6.4. The case of a category which has Gabriel dimension. Fix an abelian category $\mathcal{A}$. For any $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}_{\omega}$, denote by $\mathfrak{h}(\mathbb{P})$ the biggest ordinal α such that $\mathbb{P}$ contains the subcategory $\mathcal{A}_{\alpha}$.

On the other hand, $\mathfrak{h}(\mathbb{P}) = Gdim(M)$, where M is an object of $\mathcal{A}$ such that the localization of M at $\mathbb{P}$ is a quasi-final object of $\mathcal{A}/\mathbb{P}$.

6.4.1. Proposition. *Let* $\mathcal{A} = \mathcal{A}_{\omega}$. *Then, for any* $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$, *the height of* $\mathbb{P}$ *coincides with* $\mathfrak{h}(\mathbb{P})$: $\mathfrak{ht}^{-}(\mathbb{P}) = \mathfrak{h}(\mathbb{P})$.

Proof. Denote temporarily the set $\{\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A} \mid \mathfrak{h}(\mathbb{P}) < \alpha\}$ by $\mathfrak{Q}_{\alpha}$. Clearly $\mathfrak{Q}_{0} = \varnothing$; hence $\mathfrak{Q}_{0} = \mathfrak{C}_{0}$.

Assume that $\mathfrak{Q}_{\alpha} = \mathfrak{C}_{\alpha}$ for all $\alpha < \beta$ for some ordinal β.

(a) If β is a limit ordinal, then $\mathfrak{Q}_{\beta} = \bigcup_{\alpha<\beta} \mathfrak{Q}_{\alpha}$ which implies that $\mathfrak{Q}_{\beta}$ is equal to $\mathfrak{C}_{\beta}$.

(b) Consider now the case when β is not a limit ordinal.

For any Serre subcategory $\mathbb{S}$ and any ordinal α, denote by $\mathbb{S}(\alpha)$ the intersection $\mathbb{S} \cap \mathcal{A}_{\alpha}$.

(i) Let $\mathbb{P} \in \mathbf{Spec}^{-}\mathcal{A}$ be such that $\mathfrak{h}(\mathbb{P}) = \beta$. And let $\mathbb{P}'$ be a specialization of $\mathbb{P}$; i.e. $\mathbb{P}' \subset \mathbb{P}$. Clearly $\mathfrak{h}(\mathbb{P}') \le \mathfrak{h}(\mathbb{P})$. We claim that $\mathfrak{h}(\mathbb{P}') = \mathfrak{h}(\mathbb{P})$ *if and only if* $\mathbb{P}' = \mathbb{P}$.

In fact, if $\mathfrak{h}(\mathbb{P}') = \mathfrak{h}(\mathbb{P})$, then $\mathbb{P}' \supset \mathcal{A}_{\beta}$, and $\mathbb{P}'(\beta+1)/\mathcal{A}_{\beta}$ is a specialization of $\mathbb{P}(\beta+1)/\mathcal{A}_{\beta} \in \mathbf{Spec}^{-}(\mathcal{A}_{\beta+1}/\mathcal{A}_{\beta})$. Since

$$\mathbf{Spec}^{-}(\mathcal{A}_{\beta+1}/\mathcal{A}_{\beta}) = \mathbf{Spec}(\mathcal{A}_{\beta+1}/\mathcal{A}_{\beta}) = \mathbf{Simple}(\mathcal{A}_{\beta+1}/\mathcal{A}_{\beta}),$$

this implies that

$$\mathbb{P}(\beta+1)/\mathcal{A}_{\beta} = \mathbb{P}'(\beta+1)/\mathcal{A}_{\beta}. \tag{1}$$

According to Proposition 2.4, the equality (1) is equivalent to the equality $\mathbb{P}/\mathcal{A}_{\beta} = \mathbb{P}'/\mathcal{A}_{\beta}$ which, in turn, means that $\mathbb{P}' = \mathbb{P}$.

(ii) Thus, if $\mathbb{P}' \subset \mathbb{P}$ and $\mathbb{P} \neq \mathbb{P}'$, then $\mathfrak{h}(\mathbb{P}') < \mathfrak{h}(\mathbb{P})$; i.e. $\mathbb{P}' \in \mathfrak{Q}_{\beta-1}$. And $\mathfrak{Q}_{\beta-1} = \mathfrak{E}_{\beta-1}$ by induction hypothesis. This shows that $\mathfrak{Q}_{\beta} \subseteq \mathfrak{E}_{\beta}$.

(iii) Conversely, let $\mathbb{P} \in \mathfrak{E}_{\beta}$, and let $\mathfrak{h}(\mathbb{P}) = \gamma$. We claim that $\gamma \leq \beta$.

Suppose that, on the contrary, $\beta < \gamma$. By Proposition 6.1.3, $\mathbf{Spec}^{-}\mathcal{A} = \mathbf{Spec}\mathcal{A}$. In particular, $\mathbb{P} = \langle P \rangle$ for some object P from $Spec\mathcal{A}$. Since $\mathbb{P}$ does not contain $\mathcal{A}_{\gamma+1}$, there exists a simple object M in the category $\mathcal{A}_{\gamma+1}/\mathcal{A}_{\gamma}$ which does not belong to the subcategory $\mathbb{P}/\mathcal{A}_{\gamma}$. Since $\mathbb{P} = \langle P \rangle$, the subcategory $\mathbb{P}/\mathcal{A}_{\gamma}$ equals to $\langle Q_{\gamma}(P) \rangle$. So, the relation $M \notin Ob\mathbb{P}/\mathcal{A}_{\gamma}$ means that $M \succ Q_{\gamma}(P)$; i.e. $Q_{\gamma}(P)$ is the direct sum of a finite number of copies of M. Replacing P by an appropriate subquotient of P, we assume that $Q_{\gamma}(P)$ is a simple object.

(i) There exists a subobject M' of P such that

$$Q_{\gamma}(P/M') = 0 \quad \text{and} \quad Q_{\beta}(P/M') \neq 0. \tag{2}$$

In fact, if there is no subobject M' with properties (1), then $Q_{\beta}(P)$ is a simple object. In particular, $P \in Ob\mathcal{A}_{\beta+1} \subseteq Ob\mathcal{A}_{\gamma}$ which contradicts to the choice of P.

(ii) Let M' be a subobject satisfying (2). Clearly

$$\beta \leq \sigma := Gdim(P/M') + 1 < \gamma.$$

The object P/M' has a subquotient L of dimension σ such that $Q_{\sigma}(L)$ is a simple object of $\mathcal{A}/\mathcal{A}_{\sigma}$.

The preimage $\mathbb{P}'$ of $\langle Q_{\sigma}(L) \rangle$ in $\mathcal{A}$ is a point of $\mathbf{Spec}^{-}\mathcal{A}$ such that $\mathfrak{h}(\mathbb{P}') \geq \beta$ and $\mathbb{P}'$ is properly contained in $\mathbb{P}$.

The latter follows from the relation $Q_{\sigma}(P) \succ Q_{\sigma}(L)$ (which is a consequence of the relation $P \succ L$ and the exactness of the functor Q_{σ}) which is equivalent to the inclusion

$$\langle Q_{\sigma}(L) \rangle \subseteq \langle Q_{\sigma}(P) \rangle.$$

But, by the assumption, every element $\mathbf{P}' \in \mathbf{Spec}^{-}\mathcal{A}$ such that $\mathbf{P}'$ is a proper specialization of $\mathbb{P}$, belongs to $\mathfrak{E}_{\alpha}$ for certain $\alpha < \beta$, and, by the induction hypothesis, $\mathfrak{E}_{\alpha} = \mathfrak{Q}_{\alpha}$ if $\alpha < \beta$. Thus, we have run into a contradiction. ∎

6.4.2. Corollary. *Suppose that* $\mathcal{A}$ *has Gabriel dimension. Then*

$$\mathfrak{dim}^{-}\mathcal{A} \text{ is finite if and only if } Gdim(\mathcal{A}) \text{ is finite,}$$

and in this case $\mathfrak{dim}^{-}\mathcal{A} = Gdim(\mathcal{A})$; *and* $\mathfrak{dim}^{-}\mathcal{A}$ *coincides with the maximal length of the chains*

$$\mathbb{P} \supset \mathbb{P}_1 \supset \ldots \supset \mathbb{P}_n$$

of distinct elements of $\mathbf{Spec}^{-}\mathcal{A}$.

Proof. The assertion is a consequence of Propositions 6.4.1 and 6.3.2. ∎

7. Quasi-schemes. Having six spectra might create a confusion. So, probably, some of readers are interested to know how the author places them. I would like to begin with a 'politically correct' statement: all these spectra are natural and, therefore, each of them should be useful for something.

However, I give a preference to **Spec** considering the other spectra (in particular, **Spec**$^{-}$ and **Speg**) as a background one should keep in mind and be ready to use.

The priority of **Spec** is due to the following reasons:

a) **Spec** is the smallest among six, if the category has Gabriel-Krull dimension (and most of categories of interest do have Gabriel-Krull dimension).

b) It is, usually, much easier to describe the **Spec** of concrete categories than their other spectra.

Of course, the latter is an experimental fact (cf. [R4] – [R6]). But, the experiments were so convincing that I would like to commit myself futher by giving the following definition:

Call an abelian category $\mathcal{A}$ satisfying the property (sup) *a quasi-scheme* if, for any nonzero object M of $\mathcal{A}$, the support $Supp(M)$ of M is nonempty.

Clearly any topologizing subcategory of a quasi-scheme is a quasi-scheme.

Note also that quasi-schemes stand localizations at open sets of any reasonable topology, to begin with topology τ.

In fact, let $\mathcal{U}$ be any subset of $\mathbf{Spec}\mathcal{A}$; and let M be a nonzero object of the quotient category $\mathcal{A}/\langle\mathcal{U}\rangle$. Here $\langle\mathcal{U}\rangle := \bigcap_{\langle P\rangle \in \mathcal{U}} \langle P\rangle$. Let M' be a preimage of M in $\mathcal{A}$. Clearly $M' \succ P$ for some $\langle P\rangle \in \mathcal{U}$; hence $Q_{\langle\mathcal{U}\rangle}P \in Supp(M)$.

This argument shows also that closed points of $\mathcal{A}/\langle\mathcal{U}\rangle$ are images of points of $\mathcal{U}$.

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