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by

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WALL-CROSSING AND INVARIANTS OF HIGHER RANK STABLE PAIRS

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ABSTRACT. We introduce a higher rank analogue of the Pandharipande-Thomas theory of stable pairs. Given a Calabi-Yau threefold X, we define the highly frozen triples given by $O_X^{\oplus r}(-n) \to F$ where F is a pure coherent sheaf with one dimensional support, $r > 1$ and $n \gg 0$ is a fixed integer. We equip the highly frozen triples with a suitable stability condition and compute their associated invariants using Joyce-Song wall-crossing techniques in the category of weakly semistable objects.

Contents

1. INTRODUCTION

The Donaldson-Thomas theory of a Calabi-Yau threefold X is defined by Richard Thomas in [3] and [13] via integration against the virtual fundamental class of the moduli space of ideal sheaves. In [9] and [10] Pandharipande and Thomas introduced objects given by pairs (F, s) where F is a pure sheaf with one dimensional support together with fixed Hilbert polynomial and $s \in H^0(X, F)$ is given as a section of F. The authors computed

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the invariants of stable pairs using deformation theory and virtual fundamental classes. Following their work, many algebraic geometers have recently tried to study and compute the invariants of objects composed of a sheaf F and a sub-vector space of sections of F which has rank>1. We call these objects highly frozen triples. The main purpose of the current article is to construct the moduli stack of highly frozen triples equipped with Joyce-Song stability condition of pairs [7] (Definition 5.20) and compute their invariants using the wallcrossing machinery introduced in [7] and [8]; Joyce and Song in [7] compute the invariants of rank 1 pairs (rank 1 highly frozen triples in our context) using the method of wall crossing. The general philosophy is to exploit the existence of an auxiliary category $\mathcal{B}_p$ [7] (Section 13.3). The objects in $\mathcal{B}_p$ are defined similar to highly frozen triples and they are classified based on their numerical class (β, r). Here, β denotes the Chern character of F and r denotes the number of sections of F. The key strategy is to define two suitable "*weak*" stability conditions (say) $\tau^\bullet$ and $\tilde{\tau}$ for the objects of the category $\mathcal{B}_p$. The $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ are given by objects closely related to the highly frozen triples (equipped with Joyce-Song stability) and naively, (on the other side of the wall), the $\tau^\bullet$-semistable objects in $\mathcal{B}_p$ are given by simpler objects such as Gieseker semistable sheaves. Changing the weak stability condition, from $\tau^\bullet$ to $\tilde{\tau}$ and using the machinery of the Ringel-Hall algebra of stack functions discussed in [7], provides one with a wall-crossing identity in $\mathcal{B}_p$. Eventually one relates the weighted Euler characteristic of the moduli stack of $\tilde{\tau}$-(semi)stable objects to the weighted Euler characteristic of the moduli stack of $\tau^\bullet$-stable objects, which contains the Gieseker (semi)stable sheaves. In this article we use wallcrossing to discuss the computation of invariants of $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ with numerical class $(\beta, 2)$, i.e we show how to extend the calculations in [7] (Section 13.3) to rank 2. Our wallcrossing computations all take place in $\mathcal{B}_p$ using purely combinatorial calculations. We show that the invariants, $\mathbf{B}_p^{ss}(X, \beta, 2, \tilde{\tau})$, of $\tilde{\tau}$-weakly semistable objects in the category $\mathcal{B}_p$ with given numerical class $(\beta, 2)$ are computed with respect to the invariants, $\overline{DT}^{\beta_i}(\tau)$, which are the generalized Donaldson-Thomas invariants defined in [7] (Definition 5.15). Our wallcrossing identity in the category $\mathcal{B}_p$ is given by:

$$\begin{aligned}
&\mathbf{B}_p^{ss}(X, \beta, 2, \tilde{\tau}) = \\
&\sum_{1 \leq l, \beta_1 + \cdots + \beta_l = \beta} \frac{-1}{4} \cdot \left[\frac{(1)}{l!} \cdot \prod_{i=1}^{l} \left(\overline{DT}^{\beta_i}(\tau) \cdot \bar{\chi}_{\mathcal{B}_p}((\beta_1 + \cdots + \beta_{i-1}, 2), (\beta_i, 0)) \right.\right. \\
&\left.\left. \cdot (-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0)) + \sum_{i=1}^{l} \bar{\chi}_{\mathcal{B}_p}((\beta_1 + \cdots \beta_{i-1}, 2),(\beta_i, 0))} \right) \right].
\end{aligned} \tag{1.1}$$

We also exploit the relationship between $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ and stable highly frozen triples with respect to Joyce-Song stability [7] (Definition 5.20), denoted by $\hat{\tau}$-stability condition and we prove that a $\hat{\tau}$-stable highly frozen triple is associated directly to a $\tilde{\tau}$-semistable object in the category $\mathcal{B}_p$ (Lemma 8.3). Finally we define the invariants of highly frozen triples of rank 2 to be equal to the invariants on the left hand side of Equation (1.1) (Definition 8.4). The authors [7] (Equation 5.19) have given a useful formula for computation of invariants of rank 1 stable pairs in terms of the generalized Donaldson-Thomas invariants. Here our main result is to extend their calculations to the case of rank 2 highly frozen triples. Corollary 8.5 states our main result:

Corollary. *(8.5) Let $r = 2$ and the Hilbert polynomial of the sheaf F appearing in the highly frozen triples be given by P. The invariants of rank 2 $\hat{\tau}$-stable highly frozen triples are expressed in terms of generalized Donaldson-Thomas invariants:*

$$\begin{aligned}
&\mathrm{HFT}(X,P,2,\hat{\tau}) = \\
&\sum_{\beta|P_\beta=P}\ \sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4}\cdot\left[\frac{(1)}{l!}\cdot\prod_{i=1}^{l}\left(\overline{DT}^{\beta_i}(\tau)\cdot\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2),(\beta_i,0))\right.\right. \\
&\left.\left.\cdot\,(-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))}\right)\right].
\end{aligned} \tag{1.2}$$

Remark 1.1. We believe that we are making two main contributions in this article:

(1) Firstly, we show that how rigorous calculations in the auxiliary category $\mathcal{B}_p$ enables one to extend the result in [7] (Equation 5.19) to rank 2. Though we work in the framework established by Joyce and Song [7], the analysis in rank 2 is significantly more complicated in nature than rank 1. This is due to the existence of strictly $\tilde{\tau}$-semistable objects of higher rank in the category $\mathcal{B}_p$. We show that certain numerical cancellations in the level of stack functions supported over virtual decomposables magically work out in favor of obtaining elements of Hall algebra supported over virtual indecomposables which essentially induce our invariants.

(2) Toda [14] studies objects closely related to highly frozen triples of rank 2 given as $\mathcal{O}_X^{\oplus 2}\xrightarrow{\phi}F$ where F is given by a sheaf with zero dimensional support and the map ϕ is surjective. The author uses the Bridglenad type stability conditions which are more sophisticated than the weak-stability conditions. Though we study objects not quite identical to the ones in [14], it is not hard to predict that the result of our computations should relate to the ones in [14]. We

describe how to use much simpler weak stability conditions (Definition 4.7) to obtain similar results as the ones in [14] (Section 5). Whether or not the partition functions induced by our invariants satisfy integrality properties must also be possible to be investigated similar to [14] (Section 5). However we postpone that study to a subsequent article [12].

2. Acknowledgments

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3. Main definitions

Definition 3.1. Let X be a nonsingular projective Calabi-Yau 3-fold over $\mathbb{C}$ (i.e $K_X \cong \mathcal{O}_X$ and $\pi_1(X) = 0$ which implies $\mathrm{H}^1(\mathcal{O}_X) = 0$) with a fixed polarization $\mathcal{O}_X(1)$. A holomorphic triple supported over X is given by (E, F, ϕ) consisting of a torsion-free coherent sheaf E and a pure sheaf with one dimensional support F, together with a holomorphic morphism $\phi : E \to F$. A homomorphism of triples from (E', F', ϕ') to (E, F, ϕ) is a commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\phi'} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\phi} & F \end{array}$$

Here we give definition of frozen triples and their flat families:

Definition 3.2. (*Frozen Triples*)

(1) A *frozen triple* of fixed rank r is a special case of a holomorphic triple $E \to F$ in Definition 3.1 where $E \cong \mathcal{O}_X(-n)^{\oplus r}$ for some fixed r and fixed $n \in \mathbb{Z}$. Moreover, the sheaf F is a pure sheaf with one dimensional support.

(2) One can associate a "*type*" , i.e a tuple of Hilbert polynomial and rank, (P, r) to a frozen triple of rank r by setting the Hilbert polynomial $P(F(m)) = P$ for some variable m.

(3) Given a parametrizing $\mathbb{C}$-scheme S of finite type, an S-flat family of frozen-triples is a triple $(\mathcal{E}, \mathcal{F}, \phi)$ consisting of a morphism of $\mathcal{O}_{X\times S}$ modules $\phi : \mathcal{E} \to \mathcal{F}$ such that $\mathcal{E} \cong \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{M}_S$ where $\mathcal{M}_S$ is a vector bundle of rank r on S. Moreover, $\mathcal{F}$ is given by an S-flat family of sheaves over $X \times S$ such that for all $s \in S$ $\mathcal{F} \mid_s \cong F$ (it is trivially seen that by definition $\mathcal{E} \mid_s \cong \mathcal{O}_X(-n)^{\oplus r}$). Two S-flat families of frozen-triples $(\mathcal{E}, \mathcal{F}, \phi)$ and $(\mathcal{E}', \mathcal{F}', \phi')$ are isomorphic if there exists a commutative diagram:

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\phi'"] \arrow[d, "\cong"'] & \mathcal{F}' \arrow[d, "\cong"] \\ \mathcal{E} \arrow[r, "\phi"] & \mathcal{F} \end{tikzcd}$$

Now we define "*highly frozen triples*" and their flat families:

Definition 3.3. (*Highly frozen triples*)

(1) A *highly frozen triple* of type (P, r) is a quadruple (E, F, ϕ, ψ) where (E, F, ϕ) is a frozen triple of type (P, r) as in Definition 3.2 and the extra data $\psi : E \xrightarrow{\cong} \mathcal{O}_X(-n)^{\oplus r}$ is a fixed choice of isomorphism. A morphism between highly frozen triples (E', F', ϕ', ψ') and (E, F, ϕ, ψ) is a morphism $F' \xrightarrow{\rho} F$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{O}_X(-n)^{\oplus r} \arrow[r, "\psi'^{-1}"] \arrow[d, "id"'] & E' \arrow[r, "\phi'"] \arrow[d] & F' \arrow[d, "\rho"] \\ \mathcal{O}_X(-n)^{\oplus r} \arrow[r, "\psi^{-1}"] & E \arrow[r, "\phi"] & F \end{tikzcd}$$

(2) An S-flat family of highly frozen-triples is a quadruple $(\mathcal{E}, \mathcal{F}, \phi, \psi)$ consisting of a morphism of $\mathcal{O}_{X\times S}$ modules $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ such that ψ : $\mathcal{E} \xrightarrow{\cong} \pi_X^*\mathcal{O}_X(-n) \otimes \pi_S^*\mathcal{O}_S^{\oplus r}$ is a fixed choice of isomorphism.
Two S-flat families of highly frozen-triples $(\mathcal{E}, \mathcal{F}, \phi, \psi)$ and $(\mathcal{E}', \mathcal{F}', \phi', \psi')$ are isomorphic if there exists a commutative diagram:

$$\begin{array}{ccccc}
\pi_X^*\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{O}_S^{\oplus r} & \xrightarrow{\psi'^{-1}} & \mathcal{E}' & \xrightarrow{\phi'} & \mathcal{F}' \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle\cong} \\
\pi_X^*\mathcal{O}_X(-n)\otimes\pi_S^*\mathcal{O}_S^{\oplus r} & \xrightarrow{\psi^{-1}} & \mathcal{E} & \xrightarrow{\phi} & \mathcal{F}
\end{array}$$

Definition 3.4. (*Joyce-Song stability* [7](Definition 5.20)) Use notation in Definitions 3.2 and 3.3. Given a coherent sheaf F (appearing in a frozen or highly frozen triple of type (P, r)), let p_F denote the reduced Hilbert polynomial of F with respect to the ample line bundle $\mathcal{O}_X(1)$. A highly frozen triple (E, F, ϕ, ψ) is called $\hat{\tau}$-stable if:
1. $p_{F'} \leq p_F$ for all proper subsheaves F' of F such that $F' \neq 0$.
2. If ϕ factors through F' (F' a proper subsheaf of F), then $p_{F'} < p_F$.

4. The auxiliary category $\mathcal{B}_p$

Definition 4.1. (Joyce and Song) [7] (Definition 13.1). Let X be a Calabi-Yau threefold equipped with ample line bundle $\mathcal{O}_X(1)$. Let τ denote the Gieseker stability condition on the abelian category of coherent sheaves supported over X. Define $\mathcal{A}_p$ to be the sub-category of coherent sheaves whose objects are zero sheaves and non-zero τ-semistable sheaves with fixed reduced Hilbert polynomial p.

Definition 4.2. (Joyce and Song) [7] (Definition 13.1). Fix an integer n. Now define category $\mathcal{B}_p$ to be the category whose objects are triples (F, V, ϕ), where $F \in Obj(\mathcal{A}_p)$, V is a finite dimensional $\mathbb{C}$-vector space, and $\phi : V \to \mathrm{Hom}(\mathcal{O}_X(-n), F)$ is a $\mathbb{C}$-linear map. Given (F, V, ϕ) and (F', V', ϕ') in $\mathcal{B}_p$ define morphisms $(F, V, \phi) \to (F', V', \phi')$ in $\mathcal{B}_p$ to be pairs of morphisms (f, g) where $f : F \to F'$ is a morphism in $\mathcal{A}_p$ and $g : V \to V'$ is a $\mathbb{C}$-linear map, such that the following diagram commutes:

$$\begin{array}{ccc}
V & \xrightarrow{\phi} & \mathrm{Hom}(\mathcal{O}_X(-n), F) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
V' & \xrightarrow{\phi'} & \mathrm{Hom}(\mathcal{O}_X(-n), F')
\end{array}$$

Now we define the numerical class of objects in $\mathcal{B}_p$. Joyce and Song [7] (Definition 3.1) define the Grothendieck group $K_0(\mathcal{A})$ of an abelian category $\mathcal{A}$, the Euler form $\overline{\chi}$ and the numerical Grothendieck group $K^{num}(\mathcal{A})$. Moreover they define the Grothendieck group of the category $\mathcal{B}_p$. Here, for the purpose of completeness, we include their definition

Definition 4.3. (Joyce and Song) [7] (Definition 3.1)

(1) Define the Grothendieck group $\mathcal{K}(\mathcal{B}_p) = \mathcal{K}(\mathcal{A}_p) \oplus \mathbb{Z}$ where $\mathcal{K}(\mathcal{A}_p)$ is given by the image of $\mathcal{K}_0(\mathcal{A}_p)$ in $\mathcal{K}(\mathrm{Coh}(X)) = \mathcal{K}^{num}(\mathrm{Coh}(X))$ defined in [7] (Definition 13.1). Given $(F, V, \phi) \in \mathcal{B}_p$, we write $[(F, V, \phi)] = ([F], \dim(V))$.

(2) (Joyce and Song) [7] (Definition. 13.5). Define the positive cone of $\mathcal{B}_p$ by:

$$\mathcal{C}(\mathcal{B}_p) = \{(\beta, d) \mid \beta \in \mathcal{C}(\mathcal{A}_p) \text{ and } d \geq 0 \text{ or } \beta = 0 \text{ and } d > 0\}.$$

We state the following results by Joyce and Song without proof:

Lemma 4.4. *(Joyce and Song)* [7] *(Lemma 13.2). The category $\mathcal{B}_p$ is abelian and $\mathcal{B}_p$ satisfies the condition that If $[F] = 0 \in \mathcal{K}(\mathcal{A}_p)$ then $F \cong 0$. Moreover, $\mathcal{B}_p$ is noetherian and artinian and the moduli stacks $\mathfrak{M}_{\mathcal{B}_p}^{(\beta,d)}$ are of finite type $\forall (\beta, d) \in C(\mathcal{B}_p)$.*

Remark 4.5. The category $\mathcal{A}_p$ embeds as a full and faithful sub-category in $\mathcal{B}_p$ by $F \to (F, 0, 0)$. Moreover, it is shown by Joyce and Song in [7] that every object (F, V, ϕ) sits in a short exact sequence.

$$0 \to (F, 0, 0) \to (F, V, \phi) \to (0, V, 0) \to 0 \tag{4.1}$$

Next we recall the definition of weak (semi)stability from [7] for a general abelian category $\mathcal{A}$.

Definition 4.6. (Joyce and Song)[7](Definition. 3.5). Let $\mathcal{A}$ be an abelian category, $\mathcal{K}(\mathcal{A})$ be the quotient of $\mathcal{K}_0(\mathcal{A})$ by some fixed group. Let $C(\mathcal{A})$ be the positive cone of $\mathcal{A}$. Suppose $(T, \leq)$ is a totally ordered set and $\tau : C(\mathcal{A}) \to T$ a map. We call $(\tau, T, \leq)$ a stability condition on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$ or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$ or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. We call $(\tau, T, \leq)$ a weak stability condition on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) \leq \tau(\beta) \leq \tau(\gamma)$ or $\tau(\alpha) \geq \tau(\beta) \geq \tau(\gamma)$. For such $(\tau, T, \leq)$, we say that a nonzero object E in $\mathcal{A}$ is

(1) τ-semistable if $\forall S \subset E$ where $S \ncong 0$, we have $\tau([S]) \leq \tau([E/S])$

(2) τ-stable if $\forall S \subset E$ where $S \ncong 0$, we have $\tau([S]) < \tau([E/S])$

(3) τ-unstable if it is not τ-semistable.

Now we apply the definition of weak stability conditions to the category $\mathcal{B}_p$:

Definition 4.7. (Joyce and Song) [7] (Definition. 13.5). Define the weak stability conditions $\tau^{\bullet}$, $\tilde{\tau}$ and τ^n in $\mathcal{B}_p$ by:

(1) $\tau^{\bullet}(\beta, d) = 0$ if $d = 0$ and $\tau^{\bullet}(\beta, d) = -1$ if $d > 0$.

(2) $\tilde{\tau}(\beta, d) = 0$ if $d = 0$ and $\tilde{\tau}(\beta, d) = 1$ if $d > 0$.

(3) $\tau^n(\beta, d) = 0\ \forall(\beta, d)$.

5. MODULI STACK OF OBJECTS IN $\mathcal{B}_p$

In this section we describe the moduli stack of weakly semistable objects in $\mathcal{B}_p$. We construct this moduli stack of for the $\tilde{\tau}$-(weak)semistability condition. The constructions are similar for the case of the $\tau^{\bullet}$-(weak)semistability. In order to construct the moduli stack we give the definition of a new set of objects called the *rigidified* objects in $\mathcal{B}_p$. Our goal is to show that the moduli stack of objects in $\mathcal{B}_p$ is given by a stacky quotient of the moduli stack of rigidified objects in $\mathcal{B}_p$.

Remark 5.1. By [7] (Page 185) there exists a natural embedding functor $\mathfrak{F} : \mathcal{B}_p \to D(X)$ which takes $(F, V, \phi_V) \in \mathcal{B}_p$ to an object in the derived category given by $\cdots \to 0 \to V \otimes \mathcal{O}_X(-n) \to F \to 0 \to \cdots$ where $V \otimes \mathcal{O}_X(-n)$ and F sit in degree -1 and 0. Assume that $\dim(V) = r$. In that case $V \otimes \mathcal{O}_X(-n) \cong \mathcal{O}_X(-n)^{\oplus r}$. Hence one may view an object $(F, V, \phi_V) \in \mathcal{B}_p$ as a complex $\phi : E \to F$ such that $E \cong \mathcal{O}_X(-n)^{\oplus r}$ (note the similarity between the objects in $\mathcal{B}_p$ and frozen triples in Definition 3.2).

Definition 5.2. Fix a parametrizing scheme of finite type S. Let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ denote the natural projections. Use the natural embedding functor $\mathfrak{F} : \mathcal{B}_p \to D(X)$ in Remark 5.1. Define the S-flat family of objects in $\mathcal{B}_p$ of type (β, r) as a complex

$$\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$$

sitting in degree -1 and 0 such that $\mathcal{F}$ is given by an S-flat family of semistable sheaves with fixed reduced Hilbert polynomial p with $\mathrm{Ch}(F) = \beta$ and M is a vector bundle of rank r over S. A morphism between two such S-flat families is given by a morphism between the complexes $\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^* M' \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{tikzcd}
\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \arrow[r, "\psi_S"] \arrow[d] & \mathcal{F} \arrow[d] \\
\pi_S^* M' \otimes \pi_X^* \mathcal{O}_X(-n) \arrow[r, "\psi'_S"] & \mathcal{F}'.
\end{tikzcd}$$

Moreover an isomorphism between two such S-flat families in $\mathcal{B}_p$ is given by an isomorphism between the associated complexes $\pi_S^* M \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^* M' \otimes \pi_X^* \mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{tikzcd} \pi_S^*M\otimes\pi_X^*\mathcal{O}_X(-n) \arrow[r, "\psi_S"] \arrow[d, "\cong"'] & \mathcal{F} \arrow[d, "\cong"] \\ \pi_S^*M'\otimes\pi_X^*\mathcal{O}_X(-n) \arrow[r, "\psi'_S"] & \mathcal{F}'. \end{tikzcd}$$

Note the similarity between definition of isomorphism between S-flat families of objects of type (β,r) in $\mathcal{B}_p$ and the isomorphism between two S-flat families of frozen triples of type (P,r) in Definition 3.2. From now on by objects in $\mathcal{B}_p$ we mean the objects which lie in the image of the natural embedding functor $\mathfrak{F}:\mathcal{B}_p\rightarrow D(X)$ in Remark 5.1. Moreover, by the S-flat family of objects in $\mathcal{B}_p$, their morphisms (or isomorphisms) we mean the corresponding definitions as in Definition 5.2.

Now we define the *rigidified* objects in $\mathcal{B}_p$. These are the analog of the highly frozen triples in Definition 3.3. We give the category of these objects a new name $\mathcal{B}_p^{\mathbf{R}}$;

Definition 5.3. Fix a positive integer r and define the category $\mathcal{B}_p^{\mathbf{R}}$ to be the category of rigidified objects in $\mathcal{B}_p$ of rank r to be the category whose objects are defined by tuples $(F,\mathbb{C}^{\oplus r},\rho)$ where F is a coherent sheaf with reduced Hilbert polynomial p and $\mathrm{Ch}(F)=\beta$ and $\rho:\mathbb{C}^r\rightarrow\mathrm{Hom}(\mathcal{O}_X(-n),F)$. Given two rigidified objects of fixed given type (β,r) as $(F,\mathbb{C}^{\oplus r},\rho)$ and $(F',\mathbb{C}^{\oplus r},\rho')$ in $\mathcal{B}_p^{\mathbf{R}}$ define morphisms $(F,\mathbb{C}^{\oplus r},\rho)\rightarrow(F',\mathbb{C}^{\oplus r},\rho')$ to be given by a morphism $f:F\rightarrow F'$ in $\mathcal{A}_p$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^{\oplus r} \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & \mathrm{Hom}(\mathcal{O}_X(-n),F) \arrow[d, "f"] \\ \mathbb{C}^{\oplus r} \arrow[r, "\rho'"] & \mathrm{Hom}(\mathcal{O}_X(-n),F'). \end{tikzcd}$$

Remark 5.4. There exists a natural embedding functor $\mathfrak{F}^{\mathbf{R}}:\mathcal{B}_p^{\mathbf{R}}\rightarrow D(X)$ which takes $(F,\mathbb{C}^{\oplus r},\rho)\in\mathcal{B}_p^{\mathbf{R}}$ to an object in the derived category given by $\cdots\rightarrow 0\rightarrow\mathbb{C}^{\oplus r}\otimes\mathcal{O}_X(-n)\rightarrow F\rightarrow 0\rightarrow\cdots$ where $\mathbb{C}^{\oplus r}\otimes\mathcal{O}_X(-n)$ sits in degree -1 and F sits in degree 0. One may view an object in $\mathcal{B}_p^{\mathbf{R}}$ as a complex $\phi:E\rightarrow F$ with an additional structure such that $\psi:E\cong\mathcal{O}_X(-n)^{\oplus r}$ is a fixed choice of isomorphism (note the similarity between the objects in $\mathcal{B}_p^{\mathbf{R}}$ and highly frozen triples in Definition 3.3).

Definition 5.5. Fix a parametrizing scheme of finite type S. Use the natural embedding functor $\mathfrak{F}^{\mathbf{R}}:\mathcal{B}_p^{\mathbf{R}}\rightarrow D(X)$ in Remark 5.4. An S-flat family of objects of type (β,r) in $\mathcal{B}_p^{\mathbf{R}}$ is given by a complex

$$\pi_S^*\mathcal{O}_S^{\oplus r}\otimes\pi_X^*\mathcal{O}_X(-n)\xrightarrow{\psi_S}\mathcal{F}$$

sitting in degree -1 and 0 such that $\mathcal{F}$ is given by an S-flat family of semistable sheaves with fixed reduced Hilbert polynomial p with $\mathrm{Ch}(\mathcal{F}_s) = \beta$ for all $s \in S$. A morphism between two such S-flat families in $\mathcal{B}_p^{\mathbf{R}}$ is given by a morphism between the complexes $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc} \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\quad \psi_S \quad} & \mathcal{F} \\ {\scriptstyle \mathrm{id}_{\mathcal{O}_{X\times S}}}\downarrow & & \downarrow \\ \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\quad \psi'_S \quad} & \mathcal{F}'. \end{array}$$

Moreover an isomorphism between two such S-flat families in $\mathcal{B}_p^{\mathbf{R}}$ is given by an isomorphism between the associated complexes $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi_S} \mathcal{F}$ and $\pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) \xrightarrow{\psi'_S} \mathcal{F}'$:

$$\begin{array}{ccc} \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\quad \psi_S \quad} & \mathcal{F} \\ {\scriptstyle \mathrm{id}_{\mathcal{O}_{X\times S}}}\downarrow & & \downarrow{\scriptstyle \cong} \\ \pi_S^*\mathcal{O}_S^{\oplus r} \otimes \pi_X^*\mathcal{O}_X(-n) & \xrightarrow{\quad \psi'_S \quad} & \mathcal{F}'. \end{array}$$

Note the similarity between definition of isomorphism between S-flat families of objects of type (β, r) in $\mathcal{B}_p^{\mathbf{R}}$ and the isomorphism between two S-flat families of highly frozen triples of type (P, r) in definition 3.3. Similar to the way that we treated objects in $\mathcal{B}_p$, from now on by objects in $\mathcal{B}_p^{\mathbf{R}}$ we mean the objects which lie in the image of the natural embedding functor $\mathfrak{F}^{\mathbf{R}} : \mathcal{B}_p^{\mathbf{R}} \to D(X)$ in Remark 5.4. Moreover by the S-flat family of objects in $\mathcal{B}_p^{\mathbf{R}}$, their morphisms (or isomorphisms) we mean the corresponding definitions as in Definition 5.5.

We equip the categories $\mathcal{B}_p^{\mathbf{R}}$ and $\mathcal{B}_p$ with $\tilde{\tau}$-semistability (or $\tau^\bullet$-stability respectively). Now we show that there exists a moduli functor $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau}) : Sch/\mathbb{C} \to \mathrm{Sets}$ which sends a $\mathbb{C}$-scheme S to an S-flat family of $\tilde{\tau}$-semsistable objects of type (β, r) in $\mathcal{B}_p$. Moreover we show that this moduli functor (as a functor with groupoid sections) is equivalent to a quotient stack. Finaly we show that the moduli stack $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ is given by a stacky quotient of $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^{\mathbf{R}},ss}(\tilde{\tau})$ (which itself is defined as the moduli stack of $\tilde{\tau}$-semistable objects of type (β, r) in $\mathcal{B}_p^{\mathbf{R}}$).

Remark 5.6. According to Definition 4.2 an object in the category $\mathcal{A}_p$ consists of semistable sheaves with fixed Hilbert polynomial p. As discussed in [4] (Theorem 3.37), the family of τ-semistable (i.e Gieseker semistable) sheaves F on X such that F has a fixed Hilbert polynomial is bounded. Hence a given family of τ-semistable sheaves F on X with Hilbert polynomial $P_F(t) = \frac{k}{d!} \cdot p(t)$ for some $k = 0, 1, \cdots, N$ is also bounded. Suppose that we fix polynomial $P(t) = \frac{N}{d!} p(t)$ for some $N > 0$. As we will see later, based on results of Joyce and Song [7], Equation (1.2) can be proved via considering only finitely many values of k, namely $k = 0, 1, \cdots, N$ such that $P_F(t) = \frac{k}{d!} \cdot p(t)$. Our analysis inherits this finiteness property directly from applying [7] (Proposition 13.7). In what follows we will construct the moduli space of rigidified objects $[\mathcal{O}_X^{\oplus r}(-n) \to F]$ of type (β, r). We use the fact that by the Grothendieck-Riemann-Roch theorem fixing the Chern character of a pure sheaf with one dimensional support is equivalent to fixing its Hilbert polynomial. Hence in constructing our moduli spaces, we assume that the sheaf F appearing in the corresponding rigidified objects has a fixed Hilbert polynomial and moreover, there are only finitely many possible fixed Hilbert polynomials [7] (Proposition 13.7) for which this construction needs to be carried out.

5.1. **The underlying parameter scheme.** Fix some Hilbert polynomial $P(t)$ (in short P). In order to construct a parametrizing scheme of rigidified objects $[\mathcal{O}_X^{\oplus r}(-n) \to F]$ in $\mathcal{B}_p$ where $P_F(t) = P$ one uses the boundedness property of the family of τ-semistable coherent sheaves F with given fixed Hilbert polynomial. We denote by $\mathcal{F}$ the family as a coherent $\mathcal{O}_{X\times S}$-module and by F we mean the fiber of this family over a geometric point of S. By construction, the family of coherent sheaves F appearing in a $\tilde{\tau}$-semistable rigidified object is bounded (since the sheaves themselves are τ-semistable with fixed Hilbert polynomial) and moreover $F(m')$ is globally generated for all $m' \geq m$. Fix such m' and let V be a complex vector space of dimension $d = P(m')$ given as $V = \mathrm{H}^0(F \otimes \mathcal{O}_X(m'))$. Twisting the sheaf F by the fixed large enough integer m' would ensure one to get a surjective morphism of coherent sheaves $V \otimes \mathcal{O}_X(-m') \to F$. One can construct a scheme parametrizing the flat quotients of $V \otimes \mathcal{O}_X(-m')$ with fixed given Hilbert polynomial. This by usual arguments provides us with Grothendieck's Quot-scheme. Here to shorten the notation we use $\mathcal{Q}$ to denote $\mathrm{Quot}_P(V \otimes \mathcal{O}_X(-m'))$. Now consider a sub-locus $\mathcal{Q}^{ss} \subset \mathcal{Q}$ which parametrizes the Gieseker semistable sheaves F with fixed Hilbert polynomial P.

Definition 5.7. Fix some integer $n \gg m'$. Define $\mathcal{P}$ over $\mathcal{Q}^{ss}$ to be the bundle whose fibers parametrize $\mathrm{H}^0(F(n))$. The fibers of the bundle $\mathcal{P}^{\oplus r}$ parametrize $\mathrm{H}^0(F(n))^{\oplus r}$. In other words the fibers of $\mathcal{P}^{\oplus r}$ parametrize the maps $\mathcal{O}_X^{\oplus r}(-n) \to F$ (which define the complexes representing the objects in $\mathcal{B}_p^{\mathbf{R}}$). Now let $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau}) \subset \mathcal{P}^{\oplus r}$ be given as an open subscheme of $\mathcal{P}^{\oplus r}$ whose fibers parametrize $\tilde{\tau}$-semistable objects in $\mathcal{B}_p^{\mathbf{R}}$.

There exists a right action of $\mathrm{GL}(V)$ (where V is as above) on the Quot scheme $\mathcal{Q}$ which induces an action on $\mathcal{Q}^{ss}$ after restriction to the open subscheme of τ-semistable sheaves. It is trivially seen that the action of $\mathrm{GL}(V)$ on $\mathcal{Q}^{ss}$ induces a right action on $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$. Moreover there exists, an action of $\mathrm{GL}_r(\mathbb{C})$ on $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$; let $[\mathcal{O}_X^{\oplus r}(-n) \xrightarrow{\phi} F]$ be given as a point in $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$. Let $\psi \in \mathrm{GL}_r(\mathbb{C})$ be the map given by $\psi : \mathcal{O}_X(-n)^{\oplus r} \to \mathcal{O}_X(-n)^{\oplus r}$. The action of $\mathrm{GL}_r(\mathbb{C})$ on $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$ is defined via precomposing the sections of F with ψ as shown in the diagram below:

$$\begin{tikzcd} \mathcal{O}_X^{\oplus r}(-n) \arrow[d, "\psi"'] & \\ \mathcal{O}_X^{\oplus r}(-n) \arrow[r, "\phi"] & F, \end{tikzcd} \tag{5.1}$$

To construct the moduli space of rigidified objects in $\mathcal{B}_p$ the usual strategy is to send the objects parametrized by $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$ to their associated equivalence classes via taking the quotient of $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$ by the action of the group $G := \mathrm{GL}(V) \times \mathrm{GL}_r(\mathbb{C})$ which acts on $\mathfrak{S}_{ss}^{P,r}(\tilde{\tau})$. Note that here, in order to avoid dealing with issues such as getting a coarse moduli space and so on, we take the quotients in the stacky sense rather than using GIT quotients. These constructions are done with further detail in [11] (Section 3.3). However, in order to keep completeness we review them briefly in the next section.

5.2. **The Artin stacks $\mathfrak{M}_{ss,\mathcal{B}_p}^{(\beta,r)}(\tilde{\tau})$ and $\mathfrak{M}_{ss,\mathcal{B}_p^{\mathbf{R}}}^{(\beta,r)}(\tilde{\tau})$.** By definitions 5.2 and 5.5 and because of the similarity between objects in $\mathcal{B}_p$ and $\mathcal{B}_p^{\mathbf{R}}$ with frozen and highly frozen triples respectively, the construction of their corresponding moduli stacks are given as the constructions given for moduli stacks of frozen and highly frozen triples in [11] (Section 5);

Theorem 5.8. *Let $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ be the underlying scheme in Definition 5.7 parametrizing $\tilde{\tau}$-semistable rigidified objects of type (β, r). Let $G := \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}(V)$ where V is as in Section 5.1. Let $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{G}\right]$ be the stack theoretic quotient of $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ by G. There exists an isomorphism of groupoids*

$$\mathfrak{M}_{ss,\mathcal{B}_p}^{(\beta,r)}(\tilde{\tau}) \cong \left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{G}\right].$$

In particular $\mathfrak{M}_{ss,\mathcal{B}_p}^{(\beta,r)}(\tilde{\tau})$ is an Artin stack.

Proof. For $F \in Coh(X))$ fixing β is equivalent to fixing the Hilbert polynomial P. Now replace τ'-stability and $\mathfrak{S}_{ss}^{P,r}(\tau')$ in [11] (Section 5) with

$\tilde{\tau}$-stability and $\mathfrak{S}_{ss}^{\beta,r}(\tilde{\tau})$ respectively. The rest of the proof follows directly from [11] (Corollary 6.4). $\square$

Corollary 5.9. *Apply the proof of Theorem 5.8 to* $\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})$ *and* $G = \mathrm{GL}(V)$ *and obtain a natural isomorphism between* $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^R,ss}(\tilde{\tau})$ *and* $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{\mathrm{GL}(V)}\right]$. *One may use this natural isomorphism in order to obtain an alternative definition of the moduli stack of* $\tilde{\tau}$*-semistable rigidified objects of type* (β, r) *as the quotient stack* $\left[\frac{\mathfrak{S}_{ss}^{(\beta,r)}(\tilde{\tau})}{\mathrm{GL}(V)}\right]$

Corollary 5.10. *By Theorem 5.8 and Corollary 5.9 it is true that:*

$$\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_{p,ss}}(\tilde{\tau}) = \left[\frac{\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p^R}(\tilde{\tau})}{\mathrm{GL}_r(\mathbb{C})}\right]$$

Proposition 5.11. *The moduli stack,* $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^R,ss}(\tilde{\tau})$*, is a* $\mathrm{GL}_r(\mathbb{C})$*-torsor over* $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})$*. It is true that locally in the flat topology,* $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau}) \cong \mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^R,ss}(\tilde{\tau}) \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$*. This isomorphism does not hold true globally unless* $r = 1$*.*

Proof. Replace τ'-stability and $\mathfrak{S}_{ss}^{P,r}(\tau')$ in [11] (Proposition 5.5) with $\tilde{\tau}$-stability and $\mathfrak{S}_{ss}^{\beta,r}(\tilde{\tau})$ respectively. The rest of the proof follows directly from [11] (Proposition 5.5). $\square$

Via replacing $\tilde{\tau}$ with $\tau^{\bullet}$ stability one constructs $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tau^{\bullet})$ similarly.

Proposition 5.12. *(a).* $\forall(\beta, d) \in C(\mathcal{B}_p)$ *we have natural stack isomorphisms* $\mathfrak{M}^{(\beta,0)}_{ss,\mathcal{B}_p}(\tau^{\bullet}) \cong \mathfrak{M}^{\beta}_{ss}(\tau)$ *(τ stands for Gieseker semistability condition and* $\mathfrak{M}^{\beta}_{ss}(\tau)$ *stands for moduli stack of Gieseker semistable coherent sheaves with* $\mathcal{K}$*-theory class* β*.) which is obtained by identifying* $(F, 0, 0)$ *with* F*. Moreover,* $\mathfrak{M}^{(0,1)}_{ss,\mathcal{B}_p}(\tau^{\bullet}) \cong [Spec(\mathbb{C})/\mathbb{G}_m]$ *with the unique point given by* $(0, \mathbb{C}, 0)$*. Furthermore,* $\mathfrak{M}^{(\beta,2)}_{ss,\mathcal{B}_p}(\tau^{\bullet}) = \varnothing$ *for* $\beta \neq 0$*.*
(b). $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^{\bullet}) \cong [\mathrm{Spec}(\mathbb{C}) / \mathrm{GL}_2(\mathbb{C})]$ *with the unique point given by* $(0, \mathbb{C}^2, 0)$*.*

Proof. The first two parts of part (a) of Proposition 5.12 are proved in [7] (Prop. 15.6). We start by proving the last part of (a). We know that every object $[(F, V, \phi)] = (\beta, 2)$ fits in a short exact sequence

$$0 \to (F, 0, 0) \to (F, V, \phi) \to (0, V, 0) \to 0,$$

here $[(F, 0, 0)] = (\beta, 0)$ and $[(0, V, 0)] = (0, 2)$. By Definition 4.7 $\tau^{\bullet}(F, 0, 0) = 0 > \tau^{\bullet}(0, V, 0) = -1$ therefore $(F, 0, 0)$ $\tau^{\bullet}$-destabilizes (F, V, ϕ) $\forall[(F, V, \phi)] = (\beta, 2)$ and this finishes the proof of last part of (a).

(b). Note that $(0,\mathbb{C}^2,0)$ is a unique point in $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet)$ which is made of two copies of $(0,\mathbb{C},0)$ which is the unique object in $\mathfrak{M}^{(0,1)}_{ss,\mathcal{B}_p}(\tau^\bullet)$. Moreover, the only nonzero sub-object that can destabilize $(0,\mathbb{C}^2,0)$ is $(0,\mathbb{C},0)$. There exists a short exact sequence:

$$0 \to (0,\mathbb{C},0) \to (0,\mathbb{C}^2,0) \to (0,\mathbb{C},0) \to 0. \tag{5.2}$$

It is easily seen that $\tau^\bullet(0,\mathbb{C},0) = \tau^\bullet(0,\mathbb{C}^2,0) = -1$ and therefore the sub-object $(0,\mathbb{C},0)$ does not destabilize $(0,\mathbb{C}^2,0)$ and $(0,\mathbb{C}^2,0)$ is weak $\tau^\bullet$-semistable. Since the automorphisms of $(0,\mathbb{C}^2,0)$ are given by $\mathrm{GL}_2(\mathbb{C})$ then $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^\bullet) \cong [\mathrm{Spec}(\mathbb{C})/\,\mathrm{GL}_2(\mathbb{C})]$. □

6. Stack function identities in the Ringel Hall algebra

Let $\mathfrak{M}$ be a $\mathbb{C}$-stack with affine geometric stabilizers. Recall that by [7] (Definition 2.16) the space of stack functions $\underline{\mathrm{S}}\overline{\mathrm{F}}(\mathfrak{M},\chi,\mathbb{Q})$ is given by the $\mathbb{Q}$-vector space generated by equivalence classes of pairs $[(\mathfrak{R},\rho)] := [\mathfrak{R} \xrightarrow{\rho} \mathfrak{M}]$ with the following relations imposed:

(1) Given a closed substack $(\mathfrak{G},\rho\mid_{\mathfrak{G}}) \subset (\mathfrak{R},\rho)$ we have $[(\mathfrak{R},\rho)] = [(\mathfrak{G},\rho\mid_{\mathfrak{G}})] + [(\mathfrak{R}/\mathfrak{G},\rho\mid_{\mathfrak{R}/\mathfrak{G}})]$

(2) Let $\mathfrak{R}$ be a $\mathbb{C}$-stack of finite type with affine geometric stabilizers and let $\mathcal{U}$ denote a quasi-projective $\mathbb{C}$-variety and $\pi_{\mathfrak{R}} : \mathfrak{R}\times U \to \mathfrak{R}$ the natural projection and $\rho : \mathfrak{R} \to \mathfrak{M}$ a 1-morphism. Then $[(\mathfrak{R}\times \mathcal{U},\rho\circ\pi_{\mathfrak{R}})] = \chi([\mathcal{U}])[(\mathfrak{R},\rho)]$.

(3) Assume $\mathfrak{R} \cong [X/G]$ where X is a quasiprojective $\mathbb{C}$-variety and G a very special algebraic $\mathbb{C}$-group acting on X with maximal torus T^G, then we have

$$[(\mathfrak{R},\rho)] = \sum_{Q\in\mathcal{Q}(G,T^G)} F(G,T^G,Q)[([X/Q],\rho\circ\iota^Q)],$$

where the rational coefficients $F(G,T^G,Q)$ have a complicated definition explained in [6] (Section 6.2). Here $\mathcal{Q}(G,T^G)$ is the set of closed $\mathbb{C}$-subgroups Q of T^G such that $Q = T^G \cap C_G(Q)$ where $C_G(Q) = \{g\in G : sg = gs \text{ for all } s\in Q\}$ and $\iota^Q : [X/Q] \to \mathfrak{R} \cong [X/G]$ is the natural projection 1-morphism, where $C(G)$ denotes the center of the group G. Similarly, one defines $\overline{\mathrm{SF}}(\mathfrak{M},\chi,\mathbb{Q})$ by restricting the 1-morphisms ρ to be representable.

Remark 6.1. (1) There exist the notions of multiplication, pullback, push-forward of stack functions in $\underline{\mathrm{S}}\overline{\mathrm{F}}(\mathfrak{M},\chi,\mathbb{Q})$ and $\overline{\mathrm{SF}}(\mathfrak{M},\chi,\mathbb{Q})$. For further discussions look at (Joyce and Song) [7] (Definitions. 2.6, 2.7) and (Theorem. 2.9).

(2) Joyce and Song in [7] (Section 13.3) define the notion of characteristic stack functions $\bar{\delta}_{ss}^{(\beta,d)}(\tilde{\tau}) \in \overline{\mathrm{SF}}(\mathfrak{M}_{\mathcal{B}_p}(\tilde{\tau}),\chi,\mathbb{Q})$ and $\bar{\delta}_{ss}^{(\beta,d)}(\tau^{\bullet}) \in \overline{\mathrm{SF}}(\mathfrak{M}_{\mathcal{B}_p}(\tau^{\bullet}),\chi,\mathbb{Q})$. Moreover, in the instance where the moduli stack contains strictly semistable objects, the authors define the "*logarithm*" of the moduli stack by the stack function $\bar{\epsilon}^{(\beta,d)}(\tilde{\tau})$ given as an element of the Hall-algebra of stack functions supported over virtual indecomposables.

Proposition 6.2. *(Joyce and Song)* [7] *(Proposition 13.7). For all (β,d) in $C(\mathcal{B}_p)$, the following identity holds in the Ringel Hall algebra of $\mathcal{B}_p$:*

$$\bar{\epsilon}^{(\beta,d)}(\tilde{\tau}) = \sum_{n\geq 1} \sum_{\substack{((\beta_1,d_1),\cdots,(\beta_n,d_n))\in C(\mathcal{B}_p)^n:\\ (\beta_1,d_1)+\cdots+(\beta_n,d_n)=(\beta,d)}} U\Bigg((\beta_1,d_1),\cdots(\beta_n,d_n);\tau^{\bullet},\tilde{\tau}\Bigg)$$
$$\cdot\,\bar{\epsilon}^{(\beta_1,d_1)}(\tau^{\bullet}) * \cdots\cdots * \bar{\epsilon}^{(\beta_n,d_n)}(\tau^{\bullet}). \tag{6.1}$$

There are only finitely many choices of $n \geq 1$ as well as $(\beta_i,d_i) \in C(\mathcal{B}_p)$ for which the coefficients $U\Bigg((\beta_1,d_1),\cdots(\beta_n,d_n);\tau^{\bullet},\tilde{\tau}\Bigg)$ do not vanish.

Now we recall the definition of the function U in Proposition 6.2 from [7] (Definition 3.8);

Definition 6.3. [7] (Definition 3.8). Let $n \geq 1$ and

$$(\beta_1,d_1),\cdots,(\beta_n,d_n) \in C(\mathcal{B}_p).$$

We define a number, $S((\beta_1,d_1),\cdots,(\beta_n,d_n);\tau^{\bullet},\tilde{\tau})$ as follows: If for all $i = 1,\cdots,n$ we have either:

(a) $\tau^{\bullet}(\beta_i,d_i) \leq \tau^{\bullet}(\beta_{i+1},d_{i+1})$ and

$$\tilde{\tau}((\beta_1,d_1)+\cdots+(\beta_i,d_i)) > \tilde{\tau}((\beta_{i+1},d_{i+1})+\cdots+(\beta_n,d_n)).$$

or

(b) $\tau^{\bullet}(\beta_i,d_i) > \tau^{\bullet}(\beta_{i+1},d_{i+1})$ and

$$\tilde{\tau}((\beta_1,d_1)+\cdots+(\beta_i,d_i)) \leq \tilde{\tau}((\beta_{i+1},d_{i+1})+\cdots+(\beta_n,d_n)),$$

then define $S((\beta_1,d_1),\cdots,(\beta_n,d_n);\tau^{\bullet},\tilde{\tau}) = (-1)^r$, where r is the number of times that for all $i = 1,\cdots,n-1$ condition (a) is satisfied and otherwise if for some $i = 1,\cdots,n-1$ neither (a) nor (b) is true, then set $S = 0$. Given $n \geq 1$ and $(\beta_1,d_1),\cdots,(\beta_n,d_n)$ as above, choose two numbers l and m such that $1 \leq l \leq m \leq n$. Now for this choice choose numbers $0 = a_0 < a_1 < \cdots < a_m$ and $0 = b_0 < b_1 < \cdots < b_l = m$. Given such m and $a_1,\cdots,a_m$, define elements $\theta_1,\cdots,\theta_m \in C(\mathcal{B}_p)$ by $\theta_i = (\beta_{a_{i-1}+1},d_{a_{i-1}+1})+\cdots(\beta_{a_i},d_{a_i})$

(To add two pairs just add them coordinate-wise in $C(\mathcal{B}_p)$). Also given such $l, b_1, \cdots, b_l$ define elements $\gamma_1, \cdots, \gamma_l \in C(\mathcal{B}_p)$ by $\gamma_i = \theta_{b_{i-1}+1} + \cdots \theta_{b_i}$. Let Λ denote the set of choices $(l, m, a_1, \cdots, a_m, b_1, \cdots, b_l)$ for which the two following conditions are satisfied:

(1) $\tau^\bullet(\theta_i) = \tau^\bullet(\beta_j, d_j)$ for $i = 1, \cdots, m$ and $a_{i-1} < j \leq a_i$.

(2) $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\beta, d)$ for $i = 1, \cdots l$ (here $\beta = \sum_i \beta_i$ and $d = \sum_i d_i$).

Now define:

$$U\bigg((\beta_1, d_1), \cdots, (\beta_n, d_n); \tau^\bullet, \tilde{\tau}\bigg) =$$

$$\sum_{\Lambda} \frac{(-1)^{l-1}}{l} \prod_{i=1}^{l} S(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots, \theta_{b_i}; \tau^\bullet, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}. \tag{6.2}$$

7. Wallcrossing computations for objects of rank 2 in $\mathcal{B}_p$

Our main goal is to compute the wall-crossing identity for the invariants of objects of type $(\beta, 2)$ in $\mathcal{B}_p$ by changing the weak stability condition from $\tau^\bullet$ to $\tilde{\tau}$. One needs to first write the class $(\beta, 2)$ with respect to irreducible classes. Therefore, break $d = 2$ into smaller dimensions and then decompose β. The only two possible ways to break $d = 2$ is to write $2 = 2 + 0$ and $2 = 1 + 1$. Now for each choice of decomposition of d one decomposes β into irreducible classes β_i. For example for the case $2 = 2 + 0$, the decomposition of β produces elements in $C(\mathcal{B}_p)$ of type $(\beta_1, d_1), \cdots (\beta_n, d_n)$ where $\beta_1 + \cdots + \beta_n = \beta$ and $d_1 + \cdots + d_n = 2$, hence there exists a tuple in this sequence which is of type $(\beta_i, 2)$ and the remaining objects are of type $(\beta_j, 0)$. Now use Proposition 5.12 and note that $\mathfrak{M}^{(\beta_i,2)}_{ss}(\tau^\bullet) = \varnothing$ unless $\beta_i = 0$. Hence the corresponding sequence of numerical classes is given as $(\beta_1, 0), \cdots, (0, 2), \cdots, (\beta_n, 0)$. Similarly for the decomposition of type $2 = 1 + 1$ one obtains elements of type $(\beta_1, 0), \cdots, (\beta_{k-1}, 0), (0, 1), \cdots, (\beta_{m-1}, 0), (0, 1), \cdots, (\beta_n, 0)$ for $1 \leq k \neq m \leq n$. In order to ease the bookkeeping we use a re-parameterization of (β_i, d_i) which is consistent with work of Joyce and Song. For a decomposition $2 = 2 + 0$ define $(\psi_i, d_i) = (\beta_i, 0)$ for $i \leq k - 1$, and $(\psi_i, d_i) = (\beta_{i+1}, 0)$ for $i \geq k$. For decomposition of type $2 = 1 + 1$ define $(\psi_i, d_i) = (\beta_i, 0)$ for $i \leq k - 1$, $(\psi_i, d_i) = (\beta_{i+1}, 0)$ for $k \leq i \leq m - 1$ and $(\psi_i, d_i) = (\beta_{i+2}, 0)$ for $i \geq m$.

Definition 7.1. Consider Definition 6.3.

(1) Fix some k such that $1 \leq k \leq n$. Given a sequence of numerical classes in $C(\mathcal{B}_p)$:

$$(\psi_1, 0), \cdots (\psi_{k-1}, 0), (0, 2), (\psi_k, 0), \cdots, (\psi_{n-1}, 0),$$

define

$$U_k = U\bigg((\psi_1, 0), \cdots (\psi_{k-1}, 0), (0,2), (\psi_k, 0), \cdots, (\psi_{n-1}, 0); \tau^\bullet, \tilde{\tau}\bigg)$$

(2) Similarly, fix some k, m such that $1 \leq k \leq m \leq n$. Given a sequence $(\psi_1, 0), \cdots, (\psi_{k-1}, 0), (0,1), (\psi_k, 0), \cdots, (\psi_{m-1}, 0), (0,1), (\psi_m, 0) \cdots, (\psi_{n-2}, 0)$

define

$$U_{k,m} = U\bigg((\psi_1, 0), \cdots, (\psi_{k-1}, 0), (0,1), (\psi_k, 0), \cdots, (\psi_{m-1}, 0), (0,1), (\psi_m, 0), \cdots \cdots, (\psi_{n-2}, 0); \tau^\bullet, \tilde{\tau}\bigg)$$

Now Equation 6.2 for the case of $(\beta, 2)$ is written as:

$$\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = \left[\sum_{1 \leq k \leq n} U_k \cdot \bar{\epsilon}^{(\psi_1,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\psi_{k-1},0)}(\tau^\bullet) * \bar{\epsilon}^{(0,2)}(\tau^\bullet) * \bar{\epsilon}^{(\psi_k,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\psi_{n-1},0)}(\tau^\bullet)\right] + \left[\sum_{\substack{k,m: \\ 1 \leq k \neq m \leq n}} U_{k,m} \cdot \bar{\epsilon}^{(\psi_1,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\psi_{k-1},0)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(\psi_k,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\psi_{m-1},0)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(\psi_m,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\psi_{n-2},0)}(\tau^\bullet)\right] \tag{7.1}$$

Let $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively denote the first and second brackets on the right hand side of (7.1).

7.1. **Computation of $\mathbf{E}_1$.** By (7.1) and (6.2) U_k is given by:

$$U_k = U\bigg((\psi_1, 0), \cdots (\psi_{k-1}, 0), (0,2), (\psi_k, 0), \cdots, (\psi_{n-1}, 0); \tau^\bullet, \tilde{\tau}\bigg) = \sum_{\Lambda} \frac{(-1)^{l-1}}{l} \cdot \prod_{i=1}^{l} S_{\mathbf{E}_1}(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots \theta_{b_i}; \tau^\bullet, \tilde{\tau}) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}. \tag{7.2}$$

Here we compute U_k. Apply Definition 6.3 and obtain the following conditions:

(1) In order to have $\tilde{\tau}(\gamma_i) = \tilde{\tau}(\beta, 2)$ for all $i = 1, \cdots, l$ one should set $l = 1$, [7] (Proposition 15.8). Therefore the set Λ reduces to the set of choices of m where $1 \leq m \leq n$.

(2) It is clear that the only way that $\tau^{\bullet}(\theta_i)=\tau^{\bullet}(\beta_j,d_j)$ for $i=1,\cdots,m$ and $a_{i-1}<j\leq a_i$ is that there exists some $p=1,\cdots,m$ where $a_{p-1}=k-1$ and $a_p=k$ ($k=$location of $(0,2)$).

In (7.2) $\tau^{\bullet}(\theta_i)=0$ for $i<p$ and $\tau^{\bullet}(\theta_p)=-1$ and $\tau^{\bullet}(\theta_i)=0$ for $i>p$, therefore the following hold true:

(1) $\tau^{\bullet}(\theta_i)=\tau^{\bullet}(\theta_{i+1})=0$ and $\tilde{\tau}(\theta_1+\cdots+\theta_i)\not>\tilde{\tau}(\theta_{i+1}+\cdots+\theta_n)$ for $i<p-1$

(2) $0=\tau^{\bullet}(\theta_i)>\tau^{\bullet}(\theta_{i+1})=-1$ and $0=\tilde{\tau}(\theta_1+\cdots+\theta_i)\leq\tilde{\tau}(\theta_{i+1}+\cdots+\theta_n)=1$ for $i=p-1$

(3) $\tau^{\bullet}(\theta_i)\leq\tau^{\bullet}(\theta_{i+1})$ and $\tilde{\tau}(\theta_1+\cdots+\theta_i)>\tilde{\tau}(\theta_{i+1}+\cdots+\theta_n)$ for $i\geq p$

From this analysis one concludes that in (7.2) for $i<p-1$ neither condition (a) nor (b) are satisfied for $i=p-1$ condition (b) is satisfied and for $i\geq p$ condition (a) is satisfied (this implies $p=1$ or $p=2$). Moreover, $p=1$ when $k=1$ and $p>1$ when $k>1$ and $S_{\mathbf{E}_1}=0$ for $p>2$. By the above computations when $p=1$ we have

$$(U_k)\mid_{p=1}=\sum_{\substack{1\leq m\leq n,\\ 1=a_1<a_2<\cdots<a_m}}(-1)^{m-1}\cdot\prod_{i=2}^{m}\frac{1}{(a_i-a_{i-1})!},$$

and for $p=2$ and each fixed k such that $1<k\leq n$ we have

$$(U_k)\mid_{p=2}=\sum_{1\leq m\leq n,k=a_2<a_3<\cdots<a_m}(-1)^{m-2}\cdot\prod_{i=3}^{m}\frac{1}{(a_i-a_{i-1})!}.$$

Now we can compute $\mathbf{E}_1$ as follows:

$$\begin{aligned}\mathbf{E}_1=&\sum_{\substack{1\leq m\leq n,\\ 1=a_1<a_2<\cdots<a_m}}(-1)^{m-1}\cdot\prod_{i=2}^{m}\frac{1}{(a_i-a_{i-1})!}\cdot\bar{\epsilon}^{(0,2)}*\bar{\epsilon}^{(\psi_2,0)}*\cdots*\bar{\epsilon}^{(\psi_{n-1},0)}\\&+\sum_{1<k\leq n}\frac{1}{(k-1)!}\cdot\sum_{1\leq m\leq n,k=a_2<a_3<\cdots<a_m}(-1)^{m-2}\cdot\prod_{i=3}^{m}\frac{1}{(a_i-a_{i-1})!}\\&\cdot\bar{\epsilon}^{(\psi_1,0)}*\cdots*\bar{\epsilon}^{(\psi_{k-1},0)}*\bar{\epsilon}^{(0,2)}*\bar{\epsilon}^{(\psi_k,0)}*\cdots*\bar{\epsilon}^{(\psi_{n-1},0)}\end{aligned}$$

(7.3)

7.2. **Computation of $\mathbf{E}_2$.** By Equations (7.1) and (6.2), $U_{k,m}$ is given as

$$U_{k,m}=\sum_{1\leq l\leq m\leq n}\frac{(-1)^{l-1}}{l}\cdot\prod_{i=1}^{l}S_{\mathbf{E}_2}(\theta_{b_{i-1}+1},\theta_{b_{i-1}+2},\cdots\theta_{b_i};\tau^{\bullet},\tilde{\tau})\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}$$

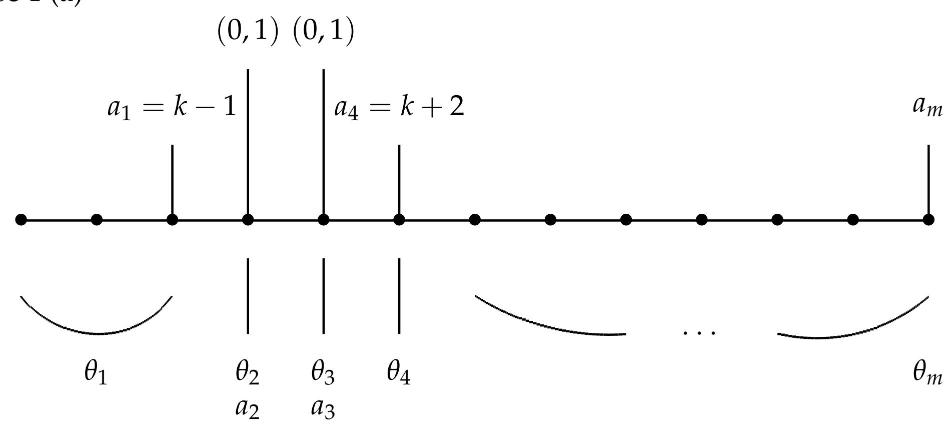
(7.4)

Lemma 7.2. *Consider the notation in Equation* (7.4)*. Then* $U_{k,m} = 0$.

Proof. In order to evaluate $U_{k,m}$ we need to compute the combinatorial coefficients $S_{\mathbf{E}_2}(\theta_{b_{i-1}+1}, \theta_{b_{i-1}+2}, \cdots \theta_{b_i}; \tau^{\bullet}, \tilde{\tau})$ (in short $S_{\mathbf{E}_2}$) appearing on the right hand side of Equation (7.4). To compute $S_{\mathbf{E}_2}$ we divide our analysis into three combinatorial cases (Case 1, Case 2 and Case 3) based on how the $(0,1)$ elements are located in the sequence of (ψ_i, d_i)'s. We denote the contributions to $U_{k,m}$ in each case by $U^1_{k,m}, U^2_{k,m}$ and $U^3_{k,m}$. Moreover, for $i = 1,2,3$ we denote by $S^i_{\mathbf{E}_2}$ the value of the function $S_{\mathbf{E}_2}$ corresponding to $U^i_{k,m}$. By construction $U_{k,m} = U^1_{k,m} + U^2_{k,m} + U^3_{k,m}$.

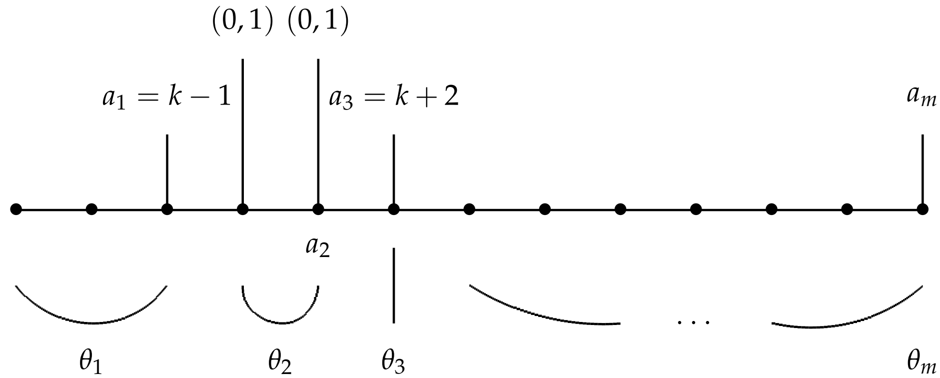
7.2.1. *Computations in Case 1:* Case 1 represents the configurations where, the two $(0,1)$ elements occur adjacent to each other. In this case notationally $U^1_{k,m} = U_{k,k+1}$ (the two $(0,1)$ elements are adjacent). Now we need to choose and distribute a_i in order to obtain equation (6.1). The following diagrams describe the two possible distribution types for a_i, we call them by Case 1 (a) and Case 1 (b). We assume that the first occurrence of a $(0,1)$ element is at k'th location. We denote this by Case 1 (a). In Case 1 (a) case, $a_1 = k-1$ and in Case 1 (b), $a_1 = \theta_1 = 1$.

Case 1 (a)



Now we discuss the second possible distribution of a_i's for Case 1 and we denote this distribution by Case 1 (b). In Case 1 (b) (diagrams below) we set $a_2 = k+1$ and a_3 can be chosen freely (similar to a_4 in Case 1) to have any value as long as $a_3 \geq k+2$:

Case 1 (b)



Let us compute the value of $U_{k,k+1}$. Since Case 1 itself is given by two possible configurations (Case 1 (a) and Case 1 (b)) we denote by $U^{a}_{k,k+1}$ the value of $U^{1}_{k,m}$ when we have the Case 1 (a) configuration, and similarly by $U^{b}_{k,k+1}$ the value of $U^{1}_{k,m}$ when we have the Case 1 (b) configuration. It is trivially seen that $U^{1}_{k,m} = U^{a}_{k,k+1} + U^{b}_{k,k+1}$. We compute the coefficient $S^{1a}_{\mathbf{E}_2}$ and $S^{1b}_{\mathbf{E}_2}$ coming from the fixed distributions of a_i's as shown in Case 1 and Case 1 (b). Consider Diagram Case 1 (a). We set for the variable l in (6.1), $l = 1$ or $l = 2$, (for $l > 2$, $S^{1a}_{\mathbf{E}_2} = 0$). If $l = 1$ then according to formula (6.1) we need to compute $S^{1a}_{\mathbf{E}_2}(\theta_1, \cdots, \theta_m)$. Note that $\tau^{\bullet}(\theta_2) = -1$ and $\tau^{\bullet}(\theta_3) = -1$ then $\tau^{\bullet}(\theta_2) \leq \tau^{\bullet}(\theta_3)$ however $\tilde{\tau}(\theta_1 + \theta_2) \not> \tilde{\tau}(\theta_3 + \cdots + \theta_m)$, hence neither condition (a) nor (b) in Definition 6.3 are satisfied and $S^{1a}_{\mathbf{E}_2}(\theta_1, \cdots, \theta_m) = 0$. Now set $l = 2$. Setting $l = 2$ means that we need to choose $0 = b_0 < b_1 < b_2 = m$ so that b_i, $i = 0, 1, 2$, satisfy the conditions in Definition (6.1). Note that one can choose $b_1 = 1, \cdots, m$. However the only allowed choice for b_1 is to set $b_1 = 2$. We explain this fact further; Set $b_1 = 1$, in that case $\gamma_1 = \theta_1$ and $\gamma_2 = \theta_2 + \cdots + \theta_m$. This configuration is not allowed, since for γ_1, $\tilde{\tau}(\gamma_1) = 0 \neq \tilde{\tau}(\beta, 2) = 1$. One easily observes that using similar arguments, the only allowable choice is to set $b_1 = 2$. Now define:

$$U^{a}_{k,k+1} = \sum_{\Lambda} \frac{-1}{2} S^{1a}_{\mathbf{E}_2}(\theta_1, \theta_2) \cdot S^{1a}_{\mathbf{E}_2}(\theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!},$$

where by similar arguments $S^{1a}_{\mathbf{E}_2}(\theta_1, \theta_2) = (-1)^0 = 1$ and $S^{1a}_{\mathbf{E}_2}(\theta_3, \cdots, \theta_m) = (-1)^{(m-3)}$. Hence

$$U^{a}_{k,k+1} = (-1) \cdot \sum_{\Lambda} \frac{1}{2}(-1)^{(m-3)} \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}$$

$$= (-1) \cdot \sum_{\Lambda} \frac{1}{2}(-1)^{(m-3)} \cdot \frac{1}{(a_3 - a_2)!} \cdot \frac{1}{(a_2 - a_1)!} \cdot \frac{1}{(a_1 - a_0)!} \cdot \prod_{i=4}^{m} \frac{1}{(a_i - a_{i-1})!}.$$

By looking at Diagram Case 1 (a), it is easy to see that $a_0 = 0, a_1 = k-1$, $a_2 = k$ and $a_3 = k+1$. Hence $(a_2 - a_1) = 1$ and $a_1 - a_0 = k-1$. Now we use the result of Lemma 13.9 of [7] and rewrite this equation as follows:

$$U^a_{k,k+1} = (-\frac{1}{2}) \cdot \frac{1}{(a_3-a_2)!} \cdot \frac{1}{(a_2-a_1)!} \cdot \frac{1}{(a_1-a_0)!} \sum_{1\leq m\leq l} (-1)^{(m-3)} \cdot \prod_{i=4}^{m} \frac{1}{(a_i-a_{i-1})!} =$$
$$(-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(1+k))}}{(n-(1+k))!}. \tag{7.5}$$

A similar analysis is carried out for Diagram Case 1 (b). Note that in this case $\theta_2 = (0,1)+(0,1) = (0,2)$. We can set $l = 1$ or $l = 2$. Setting $l = 2$ would result in obtaining a disallowed configuration, since there would always exist at least one γ_i for $i = 1,2$ so that $\tilde{\tau}(\gamma_i) = 0 \neq \tilde{\tau}(\beta, 2) = 1$. Hence we set $l = 1$. Define

$$U^b_{k,k+1} := \sum_{\Lambda} S^{1b}_{\mathbf{E}_2}(\theta_1, \theta_2, \theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i-a_{i-1})!},$$

where by similar arguments, $S^{1b}_{\mathbf{E}_2}(\theta_1, \cdots, \theta_m) = (-1)^{(m-2)}$. Hence

$$\begin{aligned} U^b_{k,k+1} &= \sum_{\Lambda} (-1)^{(m-2)} \cdot \prod_{i=1}^{m} \frac{1}{(a_i-a_{i-1})!} \\ &= \sum_{\Lambda} (-1)^{(m-2)} \cdot \frac{1}{(a_2-a_1)!} \cdot \frac{1}{(a_1-a_0)!} \prod_{i=3}^{m} \frac{1}{(a_i-a_{i-1})!}. \end{aligned}$$

By Diagram Case 1 (b), it is easy to see that $a_0 = 0, a_1 = k-1$ and $a_2 = k+1$, hence $(a_2 - a_1) = 2$ and $a_1 - a_0 = k-1$. Now use the result of Lemma 13.9 of [7] and rewrite this equation as follows:

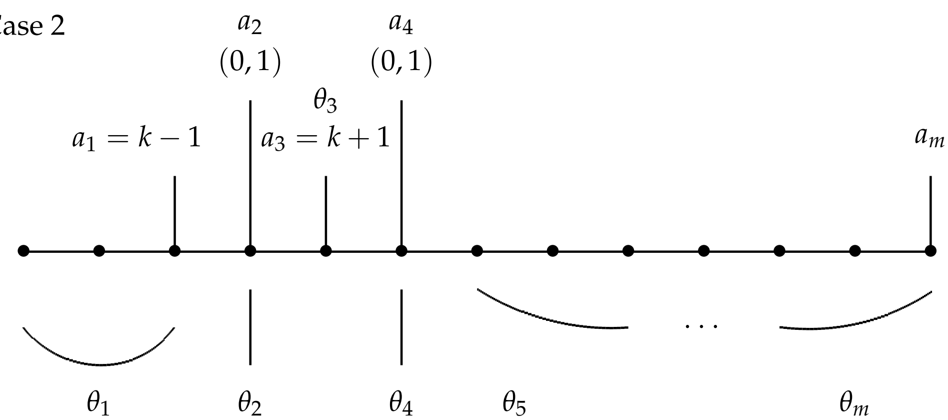
$$\begin{aligned} U^b_{k,k+1} &= \frac{1}{(a_1-a_0)!} \cdot \frac{1}{(a_2-a_1)!} \cdot \sum_{1\leq m\leq l} (-1)^{(m-2)} \cdot \prod_{i=3}^{m} \frac{1}{(a_i-a_{i-1})!} \\ &= \frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(1+k))}}{(n-(1+k))!}. \end{aligned} \tag{7.6}$$

By definition $U^1_{k,m} = U^a_{k,k+1} + U^b_{k,k+1}$. Therefore, adding the values of the function U_i, $i = 1,2$ obtained from the two distributions in Case 1 and Case 1 (b), we obtain:

$$U^1_{k,m} = \frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-1-k)}}{(n-1-k)!} + (-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-1-k)}}{(n-1-k)!} = 0 \tag{7.7}$$

7.2.2. *Computations in Case 2:* Case 2 represents the configurations where there exists some $1 \leq k \leq n$ for which there exists only one element of type $(\beta_k, 0)$ between the two elements of type $(0,1)$ such that $\beta_k \neq 0$. Here notationally we use $U^2_{k,m} := U_{k,k+2}$ (since there exists one $(\beta_k, 0)$ element in between the two $(0,1)$ elements). Moreover we denote the S functions in this case by $S^2_{\mathbf{E}_2}$. The set of allowable distributions for a_i's is given as:

Case 2



Consider the Diagram Case 2. Here we can argue that the only possible value for l in both diagrams is $l = 2$. For $l = 1$ consider θ_2 and θ_3 in the first diagram. Note that $\tau^\bullet(\theta_2) \leq \tau^\bullet(\theta_3)$ but $\tilde{\tau}(\theta_1 + \theta_2) \not> \tilde{\tau}(\theta_3 + \cdots + \theta_m)$ hence $S^2_{\mathbf{E}_2}(\theta_1, \cdots, \theta_m) = 0$. Setting $l = 2$ means that we need to choose $0 = b_0 < b_1 < b_2 = m$ so that b_i, $i = 0,1,2$, satisfy the conditions in Definition (6.1). Note that one can choose $b_1 = 2$ or $b_1 = 3$. We denote these values by choice (a) and (b) respectively. Set $b_1 = 2$ and define:

$$U^a_{k,k+2} := \sum_{\Lambda} \frac{-1}{2} S^2_{\mathbf{E}_2}(\theta_1, \theta_2) \cdot S^2_{\mathbf{E}_2}(\theta_3, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!},$$

Following similar computations and via the result of Lemma 13.9 of [7] we obtain the following identity:

$$U^a_{k,k+2} = (-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} \tag{7.8}$$

Similarly set $b_1 = 3$ and define:

$$U^b_{k,k+2} = \sum_{\Lambda} \frac{-1}{2} S^2_{\mathbf{E}_2}(\theta_1, \theta_2, \theta_3) \cdot S^2_{\mathbf{E}_2}(\theta_4, \cdots, \theta_m) \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!},$$

and similarly, we obtain

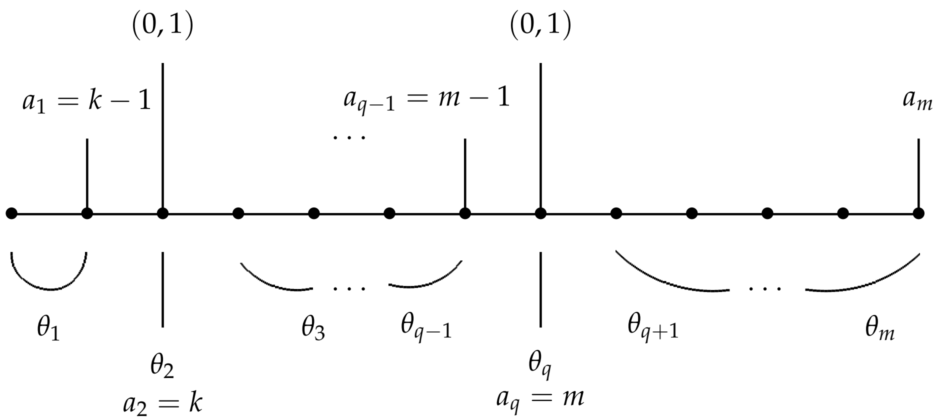
$$U^b_{k,k+2} = \sum_{\Lambda} \frac{1}{2} (-1)^{(m-4)} \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!}.$$

By adding the contributions due to the two choices of $b_1 = 2$ and $b_1 = 3$, we obtain

$$U^2_{k,m} = U^a_{k,k+2} + U^b_{k,k+2} =$$
$$(-\frac{1}{2}) \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} + \frac{1}{2} \cdot \frac{1}{(k-1)!} \cdot \frac{(-1)^{(n-(k+2))}}{(n-(k+2))!} = 0 \tag{7.9}$$

7.2.3. *Computations in Case 3:* Case 3 represents the configurations where for some $1 \leq k < m \leq n$, there exists at least 2 elements $(\beta_k, 0)$ and $(\beta_m, 0)$ between the two elements of type $(0,1)$ such that $\beta_k \neq 0$ and $\beta_m \neq 0$.

Case 3



Following similar analysis it turns out that the contributions for case 3 vanish, i.e:

$$U^3_{k,m} = \sum_{0<a_0<\cdots<a_m} \frac{(-1)}{2} \cdot \left[(-1)^{(q-3)} \cdot (-1)^{(m-q)} + (-1)^{(q-2)} \cdot (-1)^{(m-q)} \right] \cdot \prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!} = 0. \tag{7.10}$$

We conclude that the contributions in Cases 1, 2 and 3 are all equal to zero. Hence

$$U_{k,m} = U^1_{k,m} + U^2_{k,m} + U^3_{k,m} = 0.$$

This finishes the proof of Lemma 7.2. □

Recall that for $\mathbf{E}_1$ in Equation (7.3), the $(n-1)$'th $\mathcal{K}$-theory class, β_{n-1} was placed in the n'th spot, hence by change of variable n to $l-1$, the equation for $\mathbf{E}_1$ is given as:

$$\mathbf{E}_1 = \frac{(-1)^l}{(l)!} \cdot \bar{\epsilon}^{(0,2)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^\bullet) + \sum_{1 \leq k \leq l} \frac{(-1)^{l-k}}{(k-1)!(l-k)!}$$
$$\cdot \bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet) * \cdots\cdots * \bar{\epsilon}^{(\beta_k,0)}(\tau^\bullet) * \bar{\epsilon}^{(0,2)}(\tau^\bullet) * \bar{\epsilon}^{(\beta_{k+1},0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^\bullet)$$
$$= \sum_{0 \leq k \leq l} \frac{(-1)^{l-k}}{(k-1)!(l-k)!} \cdot \bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet) * \cdots * \bar{\epsilon}^{(\beta_k,0)}(\tau^\bullet) * \bar{\epsilon}^{(0,2)}(\tau^\bullet) * \bar{\epsilon}^{(\beta_{k+1},0)}(\tau^\bullet)$$
$$* \cdots * \bar{\epsilon}^{(\beta_l,0)}(\tau^\bullet)$$
(7.11)

The coefficients in (7.11) are precisely equal to those appearing on the right hand side of Equation (292) in [7]. By rewriting the product of stack functions in terms of a nested brackets we obtain an equation analogous to the computation of Joyce and Song in [7] (Proposition 15.10). Simply replace $\bar{\epsilon}^{(0,1)}(\tau^\bullet)$ in Equation (292) in [7] with $\bar{\epsilon}^{(0,2)}(\tau^\bullet)$ and obtain:

$$\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}) = \sum_{1 \leq l, \beta_1 + \cdots + \beta_l = \beta} \frac{(-1)^l}{l!} [[\cdots [[\bar{\epsilon}^{(0,2)}(\tau^\bullet), \bar{\epsilon}^{(\beta_1,0)}(\tau^\bullet)], \bar{\epsilon}^{(\beta_2,0)}(\tau^\bullet)],$$
$$\cdots], \bar{\epsilon}^{(\beta_l,0)}(\tau^\bullet)]$$
(7.12)

7.3. **Wallcrossing for numerical invariants.** In this section we use Equation (7.12) to compute the wallcrossing identity between invariants of $\tilde{\tau}$-semistable objects in $\mathcal{B}_p$ and the generalized Donaldson-Thomas invariants.

Proposition 7.3. *(a). Let $\nu^{(\beta,0)}_{\mathfrak{M}_{\mathcal{B}_p}}$ and $\nu^{\beta}_{\mathfrak{M}}$ denote Behrend's constructible functions* [1] *(Section 1.3) on the moduli stack of objects in $\mathcal{B}_p$ (with fixed class $(\beta, 0)$) and the moduli stack of sheaves with Chern character β respectively. The following identity holds true:*

$$\nu^{(\beta,0)}_{\mathfrak{M}_{\mathcal{B}_p}} \equiv \pi_0^*(\nu^{\beta}_{\mathfrak{M}})$$

where π_0 is the map $\pi_0 : \mathfrak{M}^{(\beta,0)}_{\mathcal{B}_p} \to \mathfrak{M}^{\beta}$ which sends $(F, 0, 0)$ with $[(F, 0, 0)] = (\beta, 0)$ to F with Chern character β.

Proof. This is proven in [7] (Proposition 13.12). ☐

Definition 7.4. (Joyce-Song) [7] (Definition 13.11). Define $\mathcal{S}$ to be the subset of (β, d) in $C(\mathcal{B}_p) \subset K(\mathcal{B}_p)$ such that $P_\beta(t) = \frac{k}{d!} p(t)$ for $k = 0, \cdots, N$ and $d = 0$ or 1. Then $\mathcal{S}$ is a finite set [5] (Therem 3.37). Define a Lie algebra

$\tilde{L}(\mathcal{B}_p)$ to be the $\mathbb{Q}$-vector space with the basis of symbols $\lambda^{(\beta,d)}$ with $(\beta,d)\in\mathcal{S}$ with the Lie bracket

$$[\tilde{\lambda}^{(\beta,d)},\tilde{\lambda}^{(\gamma,e)}]=(-1)^{\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))}\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))\tilde{\lambda}^{(\beta+\gamma,d+e)} \tag{7.13}$$

for $(\beta+\gamma,d+e)\in\mathcal{S}$ and $[\tilde{\lambda}^{(\beta,d)},\tilde{\lambda}^{(\gamma,e)}]=0$ otherwise. It can be seen that $\bar{\chi}_{\mathcal{B}_p}$ is antisymmetric and hence, equation (7.13) satisfies the Jacobi-identity and that makes $\tilde{L}(\mathcal{B}_p)$ into a finite-dimensional nilpotent Lie algebra over $\mathbb{Q}$.

In order to define the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}:\mathbf{SF}^{ind}_{al}\mathfrak{M}_{\mathcal{B}_p}\to\tilde{L}(\mathcal{B}_p)$ apply [7] (Definition 5.3) to the moduli stack $\mathfrak{M}_{\mathcal{B}_p}$ and $\tilde{L}(\mathcal{B}_p)$. Now we study the image of $\bar{\epsilon}^{(\beta,2)}(\tilde{\tau})$, $\bar{\epsilon}^{(0,2)}(\tau^{\bullet})$, $\bar{\epsilon}^{(\beta_i,0)}(\tau^{\bullet})$ and $\bar{\epsilon}^{(0,1)}(\tau^{\bullet})$ under the morphism $\tilde{\Psi}^{\mathcal{B}_p}$:

Definition 7.5. Define the invariant $\mathbf{B}^{ss}_p(X,\beta,2,\tilde{\tau})$ associated to $\tilde{\tau}$-semistable objects of type $(\beta,2)$ in $\mathcal{B}_p$ by

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\beta,2)}(\tilde{\tau}))=\mathbf{B}^{ss}_p(X,\beta,2,\tilde{\tau})\cdot\tilde{\lambda}^{(\beta,2)},$$

where $\tilde{\Psi}^{\mathcal{B}_p}$ is given by the Lie algebra morphism defined in [7] (Section 13.4).

According to result of part (b) of Proposition 5.12 and the fact that $[\mathrm{Spec}(\mathbb{C})/\mathrm{GL}_2(\mathbb{C})]$ has dimension -4 we obtain the following:

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\delta}^{(0,2)}(\tau^{\bullet}))=\tilde{\lambda}^{(0,2)} \tag{7.14}$$

The next two identities are proved by Joyce and Song in [7] (13.5):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,1)}(\tau^{\bullet}))=-\tilde{\lambda}^{(0,1)}. \tag{7.15}$$

Now suppose that $\beta=\sum_i\beta_i$ and β_i is indecomposable or (equivalently) there exist no strictly semistable sheaves with class β_i, then by [7] (13.5):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(\beta_i,0)}(\tau^{\bullet}))=-\overline{DT}^{\beta_i}(\tau)\tilde{\lambda}^{(\beta_i,0)} \tag{7.16}$$

where $\overline{DT}^{\beta_i}(\tau)$ is the generalized Donaldson-Thomas invariant defined by Joyce and Song in Definition (5.15) in [7]. Now apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ to both sides of Equation (7.12) and use the results obtained in (7.14), (7.15) and (7.16). We obtain the following equation:

$$\begin{aligned}&\mathbf{B}^{ss}_p(X,\beta,2,\tilde{\tau})\cdot\tilde{\lambda}^{(\beta,2)}=\\&\sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta}\frac{(-1)^l}{l!}\cdot[[\cdots[[\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet})),-\overline{DT}^{\beta_1}(\tau)\tilde{\lambda}^{(\beta_1,0)}],-\\&\overline{DT}^{\beta_2}(\tau)\tilde{\lambda}^{(\beta_2,0)}],\cdots],-\overline{DT}^{\beta_l}(\tau)\tilde{\lambda}^{(\beta_l,0)}]\end{aligned} \tag{7.17}$$

7.3.1. *Computation of* $\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^\bullet))$. By part (b) of Proposition 5.12 the characteristic stack function of moduli stack of strictly $\tau^\bullet$-semistable objects in class $(0,2)$ is given by:

$$\bar{\delta}^{(0,2)}(\tau^\bullet)=\bar{\delta}(\mathfrak{M}^{(0,2),s}_{\mathcal{B}_p}(\tau^\bullet))=\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_2(\mathbb{C})}\right].$$

Joyce in [6] (Section 6.2) has shown that given a stack function $\left[\left(\left[\frac{\mathcal{U}}{\mathrm{GL}_2(\mathbb{C})}\right],\nu\right)\right]$, where $\mathcal{U}$ is a quasi-projective variety, one has the following identity:

$$\begin{aligned}\left[\left(\left[\frac{\mathcal{U}}{\mathrm{GL}_2(\mathbb{C})}\right],a\right)\right]&=F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m^2)\left[\left(\left[\frac{\mathcal{U}}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right]\\&+F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m)\left[\left(\left[\frac{\mathcal{U}}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right],\end{aligned}\tag{7.18}$$

where and $\mu\circ i_1$ and $\mu\circ i_2$ are the obvious embeddings and:

$$F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m^2)=\frac{1}{2}\quad,\ F(\mathrm{GL}_2(\mathbb{C}),\mathbb{G}_m^2,\mathbb{G}_m)=-\frac{3}{4}.\tag{7.19}$$

Now substitute (7.19) in (7.18) and obtain:

$$\bar{\delta}^{(0,2)}(\tau^\bullet)=\frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right]-\frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right].\tag{7.20}$$

In order to compute $\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^\bullet))$ one uses the definition of $\bar{\epsilon}^{(0,2)}(\tau^\bullet)$ in [7] (Definition 3.10):

$$\bar{\epsilon}^{(0,2)}(\tau^\bullet)=\bar{\delta}^{(0,2)}(\tau^\bullet)-\frac{1}{2}\cdot\bar{\delta}^{(0,1)}(\tau^\bullet)*\bar{\delta}^{(0,1)}(\tau^\bullet).\tag{7.21}$$

Substitute the right hand side of (7.20) in (7.21) and obtain:

$$\begin{aligned}\bar{\epsilon}^{(0,2)}(\tau^\bullet)&=\frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right],\mu\circ i_1\right)\right]\\&-\frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right],\mu\circ i_2\right)\right]-\frac{1}{2}\cdot\bar{\delta}^{(0,1)}(\tau^\bullet)*\bar{\delta}^{(0,1)}(\tau^\bullet).\end{aligned}\tag{7.22}$$

Next we compute $\bar{\delta}^{(0,1)}(\tau^\bullet)*\bar{\delta}^{(0,1)}(\tau^\bullet)$ (which is equal to $\bar{\epsilon}^{(0,1)}(\tau^\bullet)*\bar{\epsilon}^{(0,1)}(\tau^\bullet)$ since there exist no strictly $\tau^\bullet$-semistable objects in $\mathcal{B}_p$ with class $(0,1)$).

7.3.2. *Computation of* $\bar{\epsilon}^{(0,1)}(\tau^\bullet)*\bar{\epsilon}^{(0,1)}(\tau^\bullet)$. By Proposition 5.12, $\bar{\epsilon}^{(0,1)}(\tau^\bullet)=[\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m]$. In order to compute $\bar{\epsilon}^{(0,1)}(\tau^\bullet)*\bar{\epsilon}^{(0,1)}(\tau^\bullet)$ we need to follow the definition of multiplication in the Ringel Hall algebra of stack functions given in [7] (Definition 2.7); Consider objects (F_i,V_i,ϕ_i) in $\mathcal{B}_p$ of type (β_i,d_i)

for $i = 1, \cdots, 3$. Let $\mathfrak{Exact}_{\mathcal{B}_p}$ denote the moduli stack of exact sequences of objects in $\mathcal{B}_p$ of the form:

$$0 \to (F_1, V_1, \phi_1) \to (F_2, V_2, \phi_2) \to (F_3, V_3, \phi_3) \to 0 \tag{7.23}$$

Let $\pi_i : \mathfrak{Exact}_{\mathcal{B}_p} \to \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet)$ for $i = 1, 2, 3$ be the projection map that sends the exact sequence in (7.23) to the left, middle and right terms respectively. Denote by $\mathcal{Z}$ the fibered product

$$([\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \times [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m]) \times_{\rho_1\times\rho_2, \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet)\times\mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet), \pi_1\times\pi_3} \mathfrak{Exact}_{\mathcal{B}_p}.$$

According to [7] (Definition 2.7) $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = (\pi_2 \circ \Phi)_* \mathcal{Z}$ where the map Φ is given by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\Phi"] \arrow[d] & \mathfrak{Exact}_{\mathcal{B}_p} \arrow[r, "\pi_2"] \arrow[d, "\pi_1\times\pi_3"] & \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet) \\
{[\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m] \times [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m]} \arrow[r, "\rho_1\times\rho_2"] & \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet) \times \mathfrak{M}_{\mathcal{B}_p}(\tau^\bullet) &
\end{tikzcd}$$

Lemma 7.6. *The product $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet)$ is given as*

$$\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{A}^1 \rtimes \mathbb{G}_m^2}, \iota\right)\right]$$

where ι is defined to be the corresponding embedding.

Proof. This is essentially proved in [14] (Lemma 5.3). $\square$

by a computation of Joyce and Song in [7] (page 158) and Lemma 7.6:

$$\begin{aligned}\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) &= \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{A}^1 \rtimes \mathbb{G}_m^2}, \iota\right)\right] = \\ &- \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}, e_2\right)\right] + \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}, e_1\right)\right],\end{aligned} \tag{7.24}$$

where $e_1 = \mu \circ i_1$ and $e_2 = \mu \circ i_2$ denote the corresponding embedding maps. Since $\bar{\epsilon}^{(0,1)}(\tau^\bullet) * \bar{\epsilon}^{(0,1)}(\tau^\bullet) = \bar{\delta}^{(0,1)}(\tau^\bullet) * \bar{\delta}^{(0,1)}(\tau^\bullet)$, by substituting the right hand side of (7.24) in (7.22) one obtains:

$$\begin{aligned}\bar{\epsilon}^{(0,2)}(\tau^\bullet) &= \frac{1}{2}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}\right], \mu \circ i_1\right)\right] - \frac{3}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \mu \circ i_2\right)\right] \\ &- \frac{1}{2}\left(-\left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}, \mu \circ i_2\right)\right] + \left[\left(\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m^2}, \mu \circ i_1\right)\right]\right) \\ &= -\frac{1}{4}\left[\left(\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], \mu \circ i_2\right)\right].\end{aligned} \tag{7.25}$$

Now apply the Lie algebra morphism $\tilde{\Psi}^{\mathcal{B}_p}$ to $\bar{\epsilon}^{(0,2)}(\tau^{\bullet})$. By Equation (7.25):

$$\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet})) = \chi^{na}(\frac{-1}{4}\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], (\mu\circ i_2)^*\nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}})\tilde{\lambda}^{(0,2)}.$$

Note that by Proposition 5.12 $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^{\bullet}) \cong [\mathrm{Spec}(\mathbb{C})/\,\mathrm{GL}_2(\mathbb{C})]$ and hence $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ has relative dimension 3 over $\mathfrak{M}^{(0,2)}_{ss,\mathcal{B}_p}(\tau^{\bullet})$. Moreover, $\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]$ is given by a single point with Behrend's multiplicity -1 and

$$(\mu\circ i_2)^*\nu_{\mathfrak{M}^{(0,2)}_{\mathcal{B}_p}})\tilde{\lambda}^{(0,2)} = (-1)^3\cdot\nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]},$$

therefore:

$$\begin{aligned}\tilde{\Psi}^{\mathcal{B}_p}(\bar{\epsilon}^{(0,2)}(\tau^{\bullet})) &= \chi^{na}\left(\frac{-1}{4}\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right], (-1)^3\cdot\nu_{\left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathbb{G}_m}\right]}\right)\tilde{\lambda}^{(0,2)}\\ &= (-1)^1\cdot(-1)^3\cdot\frac{-1}{4}\tilde{\lambda}^{(0,2)} = \frac{-1}{4}\tilde{\lambda}^{(0,2)}.\end{aligned}$$

The wall-crossing identity (7.17) is given as follows:

$$\begin{aligned}&\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau})\cdot\tilde{\lambda}^{(\beta,2)} =\\ &\sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta}\frac{-1}{4}\cdot\frac{(1)}{l!}\prod_{i=1}^{l}\overline{DT}^{\beta_i}(\tau)\cdot[[\cdots[[\tilde{\lambda}^{(0,2)},\tilde{\lambda}^{(\beta_1,0)}],\tilde{\lambda}^{(\beta_2,0)}],\cdots],\tilde{\lambda}^{(\beta_l,0)}].\end{aligned} \tag{7.26}$$

Now we use the fact that by [7] (Definition 5.13) the generators $\tilde{\lambda}^{(\beta,d)}$ satisfy the following property:

$$[\tilde{\lambda}^{(\beta,d)},\tilde{\lambda}^{(\gamma,e)}] = (-1)^{\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))}\bar{\chi}_{\mathcal{B}_p}((\beta,d),(\gamma,e))\tilde{\lambda}^{(\beta+\gamma,d+e)} \tag{7.27}$$

this enables us to simplify (7.26) as follows:

$$\begin{aligned}&\mathbf{B}_p^{ss}(X,\beta,2,\tilde{\tau})\cdot\tilde{\lambda}^{(\beta,2)} =\\ &\sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta}\frac{-1}{4}\cdot\frac{(1)}{l!}\cdot\prod_{i=1}^{l}\Big(\overline{DT}^{\beta_i}(\tau)\cdot\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2),(\beta_i,0))\Big)\\ &\cdot(-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))}\cdot\tilde{\lambda}^{(\beta,2)}\end{aligned} \tag{7.28}$$

by canceling $\tilde{\lambda}^{(\beta,2)}$ from both sides we obtain the wallcrossing equation and this finishes our computation:

$$\mathbf{B}^{ss}_{p}(X,\beta,2,\tilde{\tau}) = \sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4}\cdot\left[\frac{(1)}{l!}\cdot\prod_{i=1}^{l}\left(\overline{DT}^{\beta_i}(\tau)\cdot\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2)\right.\right.$$

$$\left.\left.,(\beta_i,0))\cdot(-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))}\right)\right].$$

(7.29)

8. $\tilde{\tau}$-SEMISTABLE OBJECTS IN $\mathcal{B}_p$ VERSUS JOYCE-SONG STABLE ($\hat{\tau}$-STABLE) HIGHLY FROZEN TRIPLES

Definition 8.1. Given a highly frozen triple (E,F,ϕ,ψ) as in Definition 3.3 fix the Chern character of F to be equal to β. By the Grothendieck-Riemann-Roch theorem, fixing the Chern character of F would induce a fixed unique Hilbert polynomial for F , say $P_F = P$. Define $\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ to be the moduli stack of $\hat{\tau}$-stable (Definition 3.4) highly frozen triples of type (P,r).

Theorem 8.2. *The moduli stack $\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ is a $\mathrm{GL}_r(\mathbb{C})$-torsor over $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})$. In particular locally in the flat topology $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})\cong\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})\times[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}]$.*

Proof. First prove that there exists a map $\pi^{\hat{\tau}}_{\tilde{\tau}}:\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})\to\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})$:

Let $p\in\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})(\mathrm{Spec}(\mathbb{C}))$ be a point represented by (E,F,ϕ,ψ) as in Definition 3.3. Now forget the choice of isomorphism $\psi: E\xrightarrow{\cong}\mathcal{O}_X^{\oplus r}(-n)$ and obtain (E,F,ϕ) which itself is represented by a complex $I^\bullet := [V\otimes\mathcal{O}_X(-n)\to F]$ such that $E\cong V\otimes\mathcal{O}_X(-n)$ for V a $\mathbb{C}^{\oplus r}$-vector space. Now use Remark 5.1 and identify the complex $I^\bullet$ with an object (F,V,ϕ_V) of type (β,r) in $\mathcal{B}_p$. Now use the following lemma:

Lemma 8.3. *The highly frozen triple (E,F,ϕ,ψ) is $\hat{\tau}$-stable if and only if the associated (F,V,ϕ_V) of type (β,r) is $\tilde{\tau}$-semistable.*

Proof. 1. $\hat{\tau}$-stability$\Rightarrow$ $\tilde{\tau}$-semistability:

One proves the claim by contradiction. Suppose (F,V,ϕ_V) is not $\tilde{\tau}$-semistable. Then there exists a subobject (F',V,ϕ'_V), a quotient object $(Q,0,0)$ and an exact sequence

$$0\to(F',V,\phi'_V)\to(F,V,\phi_V)\to(Q,0,0)\to 0,$$

such that $\tilde{\tau}(F',V,\phi'_V)=1$ and $\tilde{\tau}(Q,0,0)=0$. Now use the identification of (F,V,ϕ_V) and (F',V,ϕ'_V) with the complexes $I^\bullet := V\otimes\mathcal{O}_X(-n)\to F$

and $I'^{\bullet} := V \otimes \mathcal{O}_X(-n) \to F'$ respectively [7] (Page 185) and consider the following commutative diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_X^{\oplus r}(-n) & \longrightarrow & F' \\
\cong\downarrow & & \downarrow \\
\mathcal{O}_X^{\oplus r}(-n) & \longrightarrow & F \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Q \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{8.1}
$$

From the right vertical short exact sequence in diagram (8.1) it is seen that since F and Q are both objects in $\mathcal{A}_p$ and since $\mathcal{A}_p$ is an abelian category it contains kernels and hence $p(F') = p$. Hence we obtain a contradiction with $\hat{\tau}$-stability of (E, F, ϕ, ψ).

2. $\tilde{\tau}$-semistability$\Rightarrow$ $\hat{\tau}$-stability: Similarly suppose (E, F, ϕ, ψ) is not $\hat{\tau}$-stable. Then there exists a proper nonzero subsheaf $F' \subset F$ such that ϕ factors through F' and $p(F') = p(F) = p$. Now obtain the diagram in (8.1) and consider the right vertical short exact sequence. By the same reasoning as above $p(Q) = p$. Hence $Q \in \mathcal{A}_p$ and the complex $0 \to Q$ represents an object in $\mathcal{B}_p$ given by $(Q, 0, 0)$ with $\tilde{\tau}(Q, 0, 0) = 0$. Hence (F, V, ϕ_V) is not $\tilde{\tau}$-semistable which contradicts the assumption. Now in order to show that $\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})$ is a principal $\mathrm{GL}_r(\mathbb{C})$ bundle over $\mathfrak{M}^{(\beta,r)}_{ss,\mathcal{B}_p}(\tilde{\tau})$ replace $\mathfrak{M}^{(\beta,r)}_{\mathcal{B}_p^{\mathbf{R}}}(\tilde{\tau})$ in Proposition 5.11 with $\mathfrak{M}^{(P,r)}_{s,\mathrm{HFT}}(\hat{\tau})$. This finishes the proof of Lemma 8.3 as well as Theorem 8.2. □

Now fix the rank $r = 2$. Theorem 8.3 enables us to define the invariants of highly frozen triples of type $(P, 2)$ to be equal to invariants of $\tilde{\tau}$-semistable objects of type $(\beta, 2)$ in the category $\mathcal{B}_p$. These invariants themselves, via the identity in (7.29), are computed with respect to the generalized Donaldson-Thomas invariants:

Definition 8.4. Define the invariant of $\hat{\tau}$-stable highly frozen triples of type $(P, 2)$ as follows:

$$\mathrm{HFT}(X, P, 2, \hat{\tau}) = \mathbf{B}^{ss}_p(X, \beta, 2, \tilde{\tau}), \tag{8.2}$$

where $\mathbf{B}^{ss}_p(X, \beta, 2, \tilde{\tau})$ denotes the invariant of $\tilde{\tau}$-semistable objects of type $(\beta, 2)$ in $\mathcal{B}_p$.

Corollary 8.5. *Let $r = 2$ and the Hilbert polynomial of the sheaf F appearing in the highly frozen triples be given by P. By Equation* (7.29) *the invariants of rank 2*

$\hat{\tau}$-stable highly frozen triples can be expressed in terms of generalized Donaldson-Thomas invariants:

$$\mathrm{HFT}(X,P,2,\hat{\tau}) = \sum_{1\leq l,\beta_1+\cdots+\beta_l=\beta} \frac{-1}{4}\cdot\Bigg[\frac{(1)}{l!}\cdot\prod_{i=1}^{l}\Bigg(\overline{DT}^{\beta_i}(\tau)\cdot\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots+\beta_{i-1},2)$$

$$,(\beta_i,0))\cdot(-1)^{\bar{\chi}_{\mathcal{B}_p}((0,2),(\beta_1,0))+\sum_{i=1}^{l}\bar{\chi}_{\mathcal{B}_p}((\beta_1+\cdots\beta_{i-1},2),(\beta_i,0))}\Bigg)\Bigg].$$

(8.3)

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