Mellin Operators in a Pseudodifferential Calculus for Boundary Value Problems on Manifolds with Edges

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1. Basic Constructions for Pseudodifferential Boundary Value Problems

Operator-Valued Symbols and Wedge Sobolev Spaces

1.1 Operator-valued symbols. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbf{R}_+\} \subseteq \mathcal{L}(E)$ such that, for $e \in E$, the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous and $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$. In particular, each κ_λ is an isomorphism.

It will be useful to know that there are constants c and M with

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \leq c \max\{\lambda, \lambda^{-1}\}^M. \tag{1.1}$$

This can be easily deduced from the corresponding well-known result on the growth of (additive) strongly continuous semi-groups.

We let $H^s(\mathbf{R})$ be the usual Sobolev space on $\mathbf{R}$, while $H^s(\mathbf{R}_+) = \{u|_{\mathbf{R}_+} : u \in H^s(\mathbf{R})\}$ and $H_0^s(\mathbf{R}_+)$ is the set of all $u \in H^s(\mathbf{R})$ whose support is contained in $\overline{\mathbf{R}}_+$. Furthermore, $H^{s,t}(\mathbf{R}_+) = \{\langle r\rangle^{-t} u : u \in H^s(\mathbf{R}_+)\}$ and $H_0^{s,t}(\mathbf{R}_+) = \{\langle r\rangle^{-t} u : u \in H_0^s(\mathbf{R}_+)\}$. Finally, $\mathcal{S}(\mathbf{R}^q_+) = \{u|_{\mathbf{R}^q_+} : u \in \mathcal{S}(\mathbf{R}^q)\}$.

For all Sobolev spaces on $\mathbf{R}$ and $\mathbf{R}_+$, we will use the group action

$$[\kappa_\lambda f](r) = \lambda^{\frac{1}{2}} f(\lambda r). \tag{1.2}$$

This action extends to distributions by $\kappa_\lambda u(\varphi) = u(\kappa_{\lambda^{-1}}\varphi)$. On $E = \mathbf{C}^l$ use the trivial group action $\kappa_\lambda = id$.

In the above definition, $\langle r\rangle = (1+|r|^2)^{1/2}$ is the function used frequently for estimates in connection with pseudodifferential operators. The definition extends $\langle\eta\rangle$ to $\eta \in \mathbf{R}^q$. It is equivalent, but sometimes more convenient, to estimate in terms of a function $[\eta]$, where $[\eta]$ is strictly positive, and $[\eta] = |\eta|$ for large $|\eta|$. We then have *Peetre's inequality*: For each $s \in \mathbf{R}$ there is a constant C_s with

$$[\eta+\xi]^s \leq C_s [\eta]^s [\xi]^{|s|}.$$

Let E, F be Banach spaces with strongly continuous group actions $\kappa, \tilde\kappa$, let $\Omega \subseteq \mathbf{R}^k$, $a \in C^\infty(\Omega \times \mathbf{R}^n, \mathcal{L}(E,F))$, and $\mu \in \mathbf{R}$. We shall write

$$a \in S^\mu(\Omega, \mathbf{R}^q; E, F),$$

provided that, for every $K \subset\subset \Omega$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde\kappa_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta a(y,\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \leq C \langle\eta\rangle^{\mu-|\alpha|}. \tag{1.3}$$

The space $S^\mu(\Omega, \mathbf{R}^q; E, F)$ is Fréchet topologized by the choice of the best constants C.

The space $S^\mu(\Omega, \mathbf{R}^q; \mathbf{C}^k, \mathbf{C}^l)$ coincides with the ($l \times k$ matrix-valued) elements of Hörmander's class $S^\mu(\Omega, \mathbf{R}^q)$.

Just like in the standard case one has asymptotic summation: Given a sequence $\{a_j\}$ with $a_j \in S^{\mu_j}(\Omega, \mathbf{R}^q; E, F)$ and $\mu_j \to -\infty$, there is an $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$, $\mu = \max\{\mu_j\}$ such that $a \sim \sum a_j$; a is unique modulo $S^{-\infty}(\Omega, \mathbf{R}^q; E, F)$. Note that $S^{-\infty}(\Omega, \mathbf{R}^q; E, F)$ is independent of the choice of κ and $\tilde\kappa$.

A symbol $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ is said to be *classical*, if it has an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ with $a_j \in S^{\mu-j}(\Omega, \mathbf{R}^q; E, F)$ satisfying the homogeneity relation

$$a_j(y, \lambda\eta) = \lambda^{\mu-j} \tilde\kappa_\lambda\, a_j(y,\eta)\, \kappa_{\lambda^{-1}} \tag{1.4}$$

for all $\lambda \geq 1, |\eta| \geq R$ for a suitable constant R. We write $a \in S^\mu_{cl}(\Omega, \mathbf{R}^q; E, F)$. For $E = \mathbf{C}^k$, $F = \mathbf{C}^l$ we recover the standard notion.

There is an extension to projective and inductive limits: Let $\tilde E, \tilde F$ be Banach spaces with group actions. If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ and $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ are sequences of Banach spaces with the same group action, and $F = \text{proj} - \lim F_k$, $E = \text{ind} - \lim E_k$, then let

$$\begin{aligned} S^\mu(\Omega, \mathbf{R}^q; \tilde E, F) &= \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^q; \tilde E, F_k); \\ S^\mu(\Omega, \mathbf{R}^q; E, \tilde F) &= \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^q; E_k, \tilde F); \\ S^\mu(\Omega, \mathbf{R}^q; E, F) &= \text{proj} - \lim_{k,l} S^\mu(\Omega, \mathbf{R}^q; E_k, F_l). \end{aligned}$$

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As an integral part of a pseudodifferential calculus for boundary value problems on manifolds with edges we introduce the algebra of Mellin operators. They represent the typical operators near the edge. In fact we show how to associate an operator-valued Mellin symbol to an arbitrary edge-degenerate pseudodifferential boundary value problem, the so-called 'Mellin quantization' procedure. Furthermore, we introduce a class of adequate Sobolev spaces based on the Mellin transform on which these operators act continuously.

Introduction

The analysis of partial differential operators on manifolds with piecewise smooth geometry, in particular, on manifolds with polyhedral singularities, is of central interest in models in mathematical physics, engineering, and applied sciences.

An important aspect is the understanding of the solvability of differential equations in terms of a Fredholm theory. It is very desirable, for example, to have an appropriate notion of ellipticity implying the Fredholm property and the possibility of constructing parametrices to elliptic elements within a specified calculus, for this allows a precise analysis of the solutions to elliptic equations.

We shall deal with these questions in the context of boundary value problems on a manifold with edges by constructing an algebra of pseudodifferential operators adapted particularly to this situation.

The present paper is a first step in this direction. It focuses on the Mellin type operators, their properties, and the (Mellin) Sobolev spaces they naturally act on. It follows the general strategy of an iterative construction of operator algebras for situations of increasing complexity: Our local model of a manifold with an edge is the wedge $C \times \mathbf{R}^q$, where C is a manifold with boundary and conical singularities. We can therefore rely on the analysis of boundary value problems on manifolds with conical singularities given in [15], [16]. Technically, we regard the operators on the wedge as pseudodifferential operators along the edge of the wedge, taking values in the algebra of boundary value problems on the cone, and we employ the concept of operator-valued symbols on Banach spaces with group actions as presented, e.g., in [20].

The operators we are considering in this article correspond to boundary value problems on a manifold with edges localized to a neighborhood of the edge. They show a typical edge-degeneracy: Denoting the variable in the direction of the cone by t and the variables along the edge by y, derivatives ∂_t or ∂_y will only appear with an additional factor t. This suggests the use of the Mellin transform and associated Mellin Sobolev spaces.

There are two crucial constructions in this context. The first is the Mellin quantization procedure which shows how to pass from an edge-degenerate boundary symbol to a Mellin symbol which induces the same operator up to smoothing errors and vice versa. The second is the so-called kernel cut-off, an analytical procedure that allows to switch to holomorphic Mellin symbols (up to regularizing symbols). While the first step shows that the Mellin calculus is indeed the appropriate tool for this situation, the second one is indispensable for a Fredholm theory within the calculus, for it enables us to work on Sobolev spaces with different weights.

Historically, this paper has several roots. One is Kondrat'ev's article [10], where he analyzed boundary value problems on domains with conical points, another Agranovich&Vishik [1], who employed parameter-dependent operators, furthermore Vishik&Eskin [23], who analyzed boundary value problems without the transmission property, and Boutet de Monvel [3], who constructed a pseudodifferential calculus for boundary value problems. Primarily, however, there is the Mellin calculus for manifolds with conical singularities in the boundaryless case, see, e.g., Schulze [20], as well as the corresponding calculus for manifolds with edges in [7].

1.7 Elementary properties of wedge Sobolev spaces.
(a) $\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}_+^{q+1})$.
(b) $\mathcal{W}^s(\mathbf{R}^q, H_0^s(\mathbf{R}_+)) = H_0^s(\mathbf{R}_+^{q+1})$.
(c) $\mathcal{W}^s(\mathbf{R}^q, \mathbf{C}) = H^s(\mathbf{R}^q)$, using the trivial group action $\kappa_\lambda = id$.

Theorem 1.8. *Let E, F be Banach spaces as in 1.1, $s, \mu \in \mathbf{R}$, and $a \in S^\mu(\mathbf{R}_y^q, \mathbf{R}_\eta^q \times \mathbf{R}_\lambda^l; E, F)$ or $a \in S^\mu(\mathbf{R}_y^q \times \mathbf{R}_{\tilde{y}}^q, \mathbf{R}_\eta^q \times \mathbf{R}_\lambda^l; E, F)$. Then for every $\lambda \in \mathbf{R}^l$*

$$\operatorname{op} a(\lambda) : \mathcal{W}^s_{comp}(\mathbf{R}^q, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\mathbf{R}^q, F)$$

is bounded. If a is independent of y and $\tilde{y}$, then we may omit the subscripts 'comp' and 'loc'.
The mapping op : symbol $\mapsto$ operator is continuous in the corresponding topologies.
A proof may be found in [19, Section 3.2.1].

Boutet de Monvel's Algebra

We start with a review of the relevant spaces and terminology. An central notion in Boutet de Monvel's calculus is the so-called transmission property. It is a condition on the symbols of the pseudodifferential operators that ensures that the operators map functions which are smooth up to the boundary to functions which are smooth up to the boundary.

Definition 1.9. (a) Let $H^+ = \{(e^+ f)^\wedge : f \in \mathcal{S}(\mathbf{R}_+)\}, H_0^- = \{(e^- f)^\wedge : f \in \mathcal{S}(\mathbf{R}_-)\}$, where the hat $\hat{}$ indicates the Fourier transform on $\mathbf{R}$, and $e^\pm$ stands for extension by zero to the opposite half axis. H' denotes the space of all polynomials. Then let

$$H = H^+ \oplus H_0^- \oplus H'.$$

Write $H_d, d \in \mathbf{N}$, for the subspace of all functions $f \in H$ with $f(\rho) = O(\langle\rho\rangle^{d-1})$.
(b) Let $U = U' \times \mathbf{R}, U' \subseteq \mathbf{R}^{n-1}$ open. A symbol $p \in S^\mu(U, \mathbf{R}^q)$ has the *transmission property at* $r = 0$ if for every $k \in \mathbf{N}$

$$D_r^k p(x', r, \xi', \langle\xi'\rangle \rho)|_{r=0} \in S^\mu(U'_{x'}, \mathbf{R}_{\xi'}^{n-1}) \hat{\otimes}_\pi H_{d,\rho}, \tag{1.7}$$

where $d = \text{entier}(\mu) + 1$. Write $p \in S^\mu_{tr}(U, \mathbf{R}^q), p \in S^\mu_{cl,tr}(U, \mathbf{R}^q)$, etc.

Remark 1.10. Recall that

$$\begin{aligned} \mathcal{S}(\mathbf{R}_+) &= \operatorname{proj} - \lim\nolimits_{\sigma,\tau \in \mathbf{N}} H^{\sigma,\tau}(\mathbf{R}_+), \\ \mathcal{S}'(\mathbf{R}_+) &= \operatorname{ind} - \lim\nolimits_{\sigma,\tau \in \mathbf{N}} H_0^{-\sigma,-\tau}(\mathbf{R}_+). \end{aligned}$$

Using the notation of 1.1 we will, in particular, deal with the spaces $S^\mu(U, \mathbf{R}^n; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, $S^\mu(U, \mathbf{R}^n; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$, and $S^\mu(U, \mathbf{R}^n; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$.

Definition 1.11. Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The exterior direct sum $E \oplus F$ is Fréchet and has the closed subspace $\Delta = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space $E + F := E \oplus F / \Delta$.

1.12 Parameter-dependent operators and symbols in Boutet de Monvel's calculus. Let $U \subseteq \mathbf{R}^{n-1}$ be open. A *parameter-dependent operator of order $\mu \in \mathbf{R}$ and type $d \in \mathbf{N}$ in Boutet de Monvel's calculus* on $U \times \mathbf{R}_+$ is a family of operators

$$A(\lambda) : \begin{matrix} C_0^\infty(U \times \overline{\mathbf{R}}_+)^{n_1} \\ \oplus \\ C_0^\infty(U)^{m_1} \end{matrix} \rightarrow \begin{matrix} C^\infty(U \times \overline{\mathbf{R}}_+)^{n_2} \\ \oplus \\ C^\infty(U)^{m_2} \end{matrix} \tag{1.8}$$

Example 1.2. Let $\gamma_j : \mathcal{S}(\mathbf{R}_+) \to \mathbf{C}$ be defined by

$$\gamma_j f = \lim_{r \to 0^+} \partial_r^j f(r).$$

Then, for all $s > j + 1/2$, we can consider γ_j as a (y, η)-independent symbol in $S^{j+1/2}(\mathbf{R}^q \times \mathbf{R}^q; H^s(\mathbf{R}_+), \mathbf{C})$.

In fact, all we have to check is that $\|\tilde{\kappa}_{[\eta]^{-1}} \gamma_j \kappa_{[\eta]}\| = O([\eta]^{j+1/2})$ for the group actions $\tilde{\kappa}$ on $\mathbf{C}$ and κ on $H^s(\mathbf{R}_+)$. Since the group action on $\mathbf{C}$ is the identity, that on $H^s(\mathbf{R}_+)$ is given by (1.2), everything follows from the observation that

$$\partial_r^j \{[\eta]^{1/2} f([\eta]r)\}|_{r=0} = [\eta]^{j+1/2} \partial_r^j f(0).$$

The following lemma is obvious.

Lemma 1.3. *For $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ and $b \in S^\nu(\Omega, \mathbf{R}^q; F, G)$, the symbol c defined by $c(y, \eta) = b(y, \eta) a(y, \eta)$ (point-wise composition of operators) belongs to $S^{\mu+\nu}(\Omega, \mathbf{R}^q; E, G)$, and $D_\eta^\alpha D_y^\beta a$ belongs to $S^{\mu-|\alpha|}(\Omega, \mathbf{R}^q; E, F)$.*

Lemma 1.4. *Let $a = a(y, \eta) \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E, F))$, and suppose that $a(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda\, a(y, \eta)\, \kappa_{\lambda^{-1}}$ for all $\lambda \geq 1, |\eta| \geq R$. Then $a \in S_{cl}^\mu(\Omega, \mathbf{R}^n; E, F)$, and the symbol semi-norms for a can be estimated in terms of the semi-norms for a in $C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E, F))$.*

Proof. Without loss of generality let $R = 1$. We only have to consider the case of large $|\eta|$. For these, the assumption implies that

$$D_\eta^\alpha D_y^\beta a(y, \eta) = \lambda^{-\mu+|\alpha|} \tilde{\kappa}_{\lambda^{-1}}\, (D_\eta^\alpha D_y^\beta a)(y, \lambda\eta)\, \kappa_\lambda.$$

Letting $\lambda = [\eta]$, we conclude that

$$\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta a(y, \eta) \kappa_{[\eta]} = [\eta]^{\mu-|\alpha|} (D_\eta^\alpha D_y^\beta a)(y, \eta/[\eta]).$$

The norm of the right hand side in $\mathcal{L}(E, F)$ clearly is $O([\eta]^{\mu-|\alpha|})$. Moreover, a is classical, since it is homogeneous of degree μ in the sense of (1.4). □

Definition 1.5. Let $\Omega = \Omega_1 \times \Omega_2 \subseteq \mathbf{R}^q \times \mathbf{R}^q$ be open and $a \in S^\mu(\Omega, \mathbf{R}^q \times \mathbf{R}^l; E, F)$. The parameter-dependent pseudodifferential operator $\mathrm{op}\, a$ is the operator family $\{\mathrm{op}\, a(\lambda) : \lambda \in \mathbf{R}^l\}$ defined by

$$[\mathrm{op}\, a(\lambda) f](y) = \int e^{i(y-\tilde{y})\eta} a(y, \tilde{y}, \eta, \lambda) f(\tilde{y}) d\tilde{y} đ\eta, \tag{1.5}$$

$f \in C_0^\infty(\Omega_2, E), y \in \Omega_1$. This reduces to

$$[\mathrm{op}\, a(\lambda) f](y) = \int e^{iy\eta} a(y, \eta) \hat{f}(\eta) đ\eta \tag{1.6}$$

for symbols that are independent of y'. Here, $\hat{f}(\eta) = \mathcal{F}_{y\to\eta} f(\eta) = \int e^{-iy\eta} f(y) dy$ is the vector-valued Fourier transform of f, and $đ\eta = (2\pi)^{-n} d\eta$.

Definition 1.6. Let E, κ be as in 1.1, $q \in \mathbf{N}, s \in \mathbf{R}$. The *wedge Sobolev space* $\mathcal{W}^s(\mathbf{R}^q, E)$ is the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q) \hat{\otimes}_\pi E$ in the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q, E)} = \left(\int \langle \eta \rangle^{2s} \|\kappa_{\langle \eta \rangle^{-1}} \mathcal{F}_{y\to\eta} u(\eta)\|_E^2 đ\eta \right)^{\frac{1}{2}}.$$

It is a subset of $\mathcal{S}'(\mathbf{R}^q, E)$. There are a few straightforward generalizations: If $\{E_k\}$ is a sequence of Banach spaces, $E_{k+1} \hookrightarrow E_k$, $E = \mathrm{proj} - \lim E_k$, and the group action coincides on all spaces, we let $\mathcal{W}^s(\mathbf{R}^q, E) = \mathrm{proj} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k)$. Similarly we treat inductive limits. For $\Omega \subseteq \mathbf{R}^q$ open we shall write $u \in \mathcal{W}_{comp}^s(\Omega, E)$, if there is a function $\varphi \in C_0^\infty(\Omega)$ such that $u = \varphi u$, and say $u \in \mathcal{W}_{loc}^s(\Omega, E)$, if $u \in \mathcal{D}'(\Omega, E)$ and $\varphi u \in \mathcal{W}^s(\mathbf{R}^q, E)$ for all $\varphi \in C_0^\infty(\mathbf{R}^q)$.

(i) For all $C_0^\infty(G_j)$ functions φ, ψ, supported in the same coordinate neighborhood G_j intersecting the boundary, the push-forward

$$(M_\varphi A(\lambda) M_\psi)_* : \begin{array}{c} C_0^\infty(U_j \times \overline{\mathbf{R}}_+, V_1) \\ \oplus \\ C_0^\infty(U_j, W_1) \end{array} \to \begin{array}{c} C^\infty(U_j \times \overline{\mathbf{R}}_+, V_2) \\ \oplus \\ C^\infty(U_j, W_2) \end{array},$$

induced by $M_\varphi A(\lambda) M_\psi$ and the coordinate maps, is an operator in $\mathcal{B}^{\mu,d}(U_j \times \mathbf{R}_+; \mathbf{R}^l)$.

(ii) If φ, ψ are as before, but the coordinate chart does not intersect the boundary, then all entries in the matrix $(M_\varphi A(\lambda) M_\psi)_*$ – except for the pseudodifferential part – are regularizing.

(iii) If the supports of the functions $\varphi, \psi \in C_0^\infty(G)$ are disjoint, then $M_\varphi A(\lambda) M_\psi$ is a rapidly decreasing function of λ with values in the regularizing operators of type d.

It remains to define the regularizing elements. A *regularizing operator of type* 0 *in Boutet de Monvel's calculus* is an operator R acting on the above spaces with the property that there are continuous extensions

$$R : \begin{array}{c} L^2(\overline{X}, V_1) \\ \oplus \\ L^2(Y, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array} \quad \text{and}$$

$$R^* : \begin{array}{c} L^2(\overline{X}, V_2) \\ \oplus \\ L^2(Y, W_2) \end{array} \to \begin{array}{c} C^\infty(\overline{X}, V_1) \\ \oplus \\ C^\infty(Y, W_1) \end{array}.$$

Here R^* is the formal adjoint with respect to the inner product on the respective spaces. A *regularizing operator of type* d is a sum $R = \sum_{j=0}^d R_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}$ with all R_j regularizing of type zero. We write $\mathcal{B}^{-\infty,d}(X)$ for the regularizing elements of type d and $\mathcal{B}^{-\infty,d}(X; \mathbf{R}^q)$ for the parameter-dependent regularizing elements, i.e., the Schwartz functions on $\mathbf{R}^q$ with values in $\mathcal{B}^{-\infty,d}(X)$.

We topologize $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ as the corresponding non-direct sum of Fréchet spaces.

For each coordinate patch G_j intersecting the boundary, $A(\lambda)$ induces an operator

$$A_j(\lambda) = \begin{bmatrix} P_{j+}(\lambda) + G_j(\lambda) & K_j(\lambda) \\ T_j(\lambda) & S_j(\lambda) \end{bmatrix}$$

on $U_j \times \mathbf{R}_+$. We find a quintuple $a_j(\lambda) = \{p_j(\lambda), g_j(\lambda), k_j(\lambda), t_j(\lambda), s_j(\lambda)\}$ of symbols for $P_j(\lambda)$, $G_j(\lambda)$, $K_j(\lambda)$, $T_j(\lambda)$, $S_j(\lambda)$ in the sense of 1.12.

We shall call A *classical*, if all entries in the quintuples $a_j = \{p_j, g_j, k_j, t_j, s_j\}$ are classical elements in the respective symbol classes, i.e., p_j and s_j are classical pseudodifferential symbols, while g_j, k_j, t_j are classical operator-valued symbols. For an interior patch, we have the pseudodifferential symbol for P_j; all other symbols can be taken to be zero. Write $A \in \mathcal{B}_{cl}^{\mu,d}(X; \mathbf{R}^l)$.

Example 1.14. The Dirichlet problem $\binom{\Delta}{\gamma_0}$ is an operator in Boutet de Monvel's calculus of order 2 and type 1: Clearly, the Laplacian Δ is a differential operator of order 2. As we saw in Example 1.2, the operator of evaluation at the boundary, γ_0, is an operator-valued symbol in $S^{1/2}(\mathbf{R}^q, \mathbf{R}^q; H^s(\mathbf{R}_+), \mathbf{C})$, provided $s > 1/2$. It is not so obvious that this is an operator of type 1: Using the integration by parts formula

$$u(0) = \int_0^\infty [\eta] e^{-r[\eta]} u(r) dr + \int_0^\infty e^{-r[\eta]} \partial_r u(r) dr$$

valid for $u \in \mathcal{S}(\mathbf{R}_+)$, we may we may rewrite γ_0 in the form

$$\gamma_0 = t_0 + t_1 \partial_r.$$

Here, $t_0 \in S^{1/2}(\mathbf{R}^q, \mathbf{R}^q; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$, is given by $t_0 u = \int_0^\infty [\eta] e^{-r[\eta]} u(r) dr$, while the operator-valued symbol $t_1 \in S^{-1/2}(\mathbf{R}^q, \mathbf{R}^q; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$ is defined by integrating $\partial_r u$ against $e^{-r[\eta]}$. Hence γ_0 is of type 1.

The Dirichlet problem is independent of any parameter, but since it is a *differential* boundary value problem, we may also consider it as a parameter-dependent element. Since the order of γ_0 only is $1/2$, of the following form:

$$A(\lambda) = \begin{bmatrix} P_+(\lambda) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum_{j=0}^{d}(\mathrm{op}\, g_j(\lambda) + G_j(\lambda))\partial_r^j & \mathrm{op}\, k(\lambda) + K_0(\lambda) \\ \sum_{j=0}^{d}(\mathrm{op}\, t_j(\lambda) + T_j(\lambda))\partial_r^j & \mathrm{op}\, s(\lambda) + S_0(\lambda) \end{bmatrix}, \tag{1.9}$$

where

(i) $P(\cdot) = \mathrm{op}\, p(\cdot)$ with $p \in S^{\mu}_{tr}(U \times \mathbf{R} \times U \times \mathbf{R}, \mathbf{R}^q; \mathbf{R}^l)$, $P_+ = \mathrm{r}^+ P \mathrm{e}^+$. Here r^+ denotes restriction of functions from $U \times \mathbf{R}$ to $U \times \mathbf{R}_+$, e^+ denotes extension by zero from $U \times \mathbf{R}_+$ to $U \times \mathbf{R}$.

(ii) The symbols g_j, t_j, k, and s belong to the following spaces:

$g_j \in S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+)^{n_1}, \mathcal{S}(\mathbf{R}_+)^{n_2})$,
$t_j \in S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+)^{n_1}, \mathbf{C}^{m_2})$,
$k \in S^{\mu}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}^{m_1}, \mathcal{S}(\mathbf{R}_+)^{n_2})$, and
$s \in S^{\mu}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}^{m_1}, \mathbf{C}^{m_2})$;

(iii) for $j = 0, \ldots, d$, the operators G_j, T_j, K_0, and S_0 are rapidly decreasing families of integral operators with smooth kernels:

G_j has an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty((U \times \overline{\mathbf{R}}_+) \times (U \times \overline{\mathbf{R}}_+)))$,
T_j has an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty((U \times \overline{\mathbf{R}}_+) \times U))$,
K_0 has an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times (U \times \overline{\mathbf{R}}_+)))$, and
S_0 has an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times U))$.

Of course, all these integral kernels take values in matrices of the corresponding sizes.

(iv) ∂_r is the normal derivative, i.e., the derivative with respect to the variable in $\mathbf{R}_+$ on $U \times \mathbf{R}_+$.

We call an operator $A_0(\lambda) = \begin{bmatrix} \sum_{j=0}^{d} G_j(\lambda)\partial_r^j & K_0(\lambda) \\ \sum_{j=0}^{d} T_j(\lambda)\partial_r^j & S_0(\lambda) \end{bmatrix}$, $\lambda \in \mathbf{R}^l$, with the above choice of G_j, T_j, K_0, and S_0 a *regularizing parameter-dependent operator of type d in Boutet de Monvel's calulus.* It is a consequence of Theorem 1.8 that the operators in (1.9) indeed have the desired mapping property.

We shall write $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ for a parameter-dependent operator of order μ and type d, and $A \in \mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ for a regularizing parameter-dependent operator of type d.

The decomposition $P_+ + G$ is not unique; certain regularizing pseudodifferential operators provide examples for operators that belong to both classes. The topology on $\mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ and $\mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ is that of a non-direct sum of Fréchet spaces.

Given an operator $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ in the notation of (1.9) we let $g = \sum_{j=0}^{d} g_j\partial_r^j$, and $t = \sum_{j=0}^{d} t_j\partial_r^j$. We then have a quintuple $a = \{p, g, k, t, s\}$ of symbols for A. It is not unique, but any other choice differs only by a quintuple inducing a regularizing element.

1.13 Boutet de Monvel's algebra on a manifold. Symbol levels. Let X be an n-dimensional C^∞ manifold with boundary Y, embedded in an n-dimensional manifold G without boundary, all not necessarily compact. In the following we shall denote by X the open interior of X, while $\overline{X}$ denotes the closure. Let V_1, V_2 be vector bundles over G and let W_1, W_2 be vector bundles over Y.

By $\{G_j\}$ denote a locally finite open covering of G, and suppose that the coordinate charts map $X \cap G_j$ to $U_j \times \mathbf{R}_+ \subset \mathbf{R}^n_+$ and $Y \cap G_j$ to $U_j \times \{0\}$ for a suitable open set $U_j \subseteq \mathbf{R}^{n-1}$, unless $G_j \cap Y = \emptyset$.

For a smooth function φ on G write M_φ for the operator of multiplication with the diagonal matrix $\mathrm{diag}\{\varphi, \varphi|_Y\}$. We will say that $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$, if

$$A(\lambda) : \begin{matrix} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{matrix} \to \begin{matrix} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{matrix}, \tag{1.10}$$

is an operator with the following properties:

for all $k \le l$ and all differential operators D of order $\le l-k$ on G, cf. [19, Section 2.1.1, Proposition 2].

(c) We let $\mathcal{H}^{s,\gamma}(X^\wedge) = \{f|_{X^\wedge} : f \in \mathcal{H}^{s,\gamma}(G^\wedge)\}$, endowed with the quotient norm:

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \inf\{\|f\|_{\mathcal{H}^{s,\gamma}(G^\wedge)} : f \in \mathcal{H}^{s,\gamma}(G^\wedge), f|_{X^\wedge} = u\}.$$

(d) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$, where the subscript '$loc$' refers to the t-variable only. Moreover, $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

(e) $\mathcal{H}^{0,0}(X^\wedge)$ has a natural inner product

$$(u,v)_{\mathcal{H}^{0,0}(X^\wedge)} = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}\, dz.$$

(f) If φ is the restriction to $X^\wedge$ of a function in $C_0^\infty(G \times \mathbf{R})$, then the operator M_φ of multiplication by φ,

$$M_\varphi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge),$$

is bounded for all $s, \gamma \in \mathbf{R}$, and the mapping $\varphi \mapsto M_\varphi$ is continuous in the corresponding topology.

Definition 2.4. Let $\mathcal{F}$ be a subspace of $\mathcal{D}'(X^\wedge)$ or $\mathcal{D}'(G^\wedge)$ with a stronger topology. Suppose that φ is a smooth function on $G \times \overline{\mathbf{R}}_+$ and that multiplication by φ is continuous on $\mathcal{F}$. Then $[\varphi]\mathcal{F}$ denotes the closure of the space $\{\varphi u : u \in \mathcal{F}\}$ in $\mathcal{F}$.

2.5 The spaces H^s_{cone}. Let $\{G_j\}_{j=1}^J$ be a finite covering of G by open sets, $\kappa_j : G_j \to U_j$ the coordinate maps onto bounded open sets in $\mathbf{R}^n$, and $\{\varphi_j\}_{j=1}^J$ a subordinate partition of unity. The maps κ_j induce a push-forward of functions and distributions: For a function u on G_j

$$(\kappa_{j*}u)(x) = u(\kappa_j^{-1}(x)), \quad x \in U_j; \tag{2.3}$$

for a distribution u ask that $(\kappa_{j*}u)(\varphi) = u(\varphi \circ \kappa_j)$, $\varphi \in C_0^\infty(U_j)$. For $j = 1, \ldots, J$, consider the diffeomorphism

$$\chi_j : U_j \times \mathbf{R} \to \{(x[t], t) : x \in U_j, t \in \mathbf{R}\} =: C_j \subset \mathbf{R}^{n+1}.$$

given by $\chi_j(x,t) = (x[t], t)$. Its inverse is $\chi_j^{-1}(y,t) = (y/[t], t)$. For $s \in \mathbf{R}$ we define $H^s_{cone}(G \times \mathbf{R})$ as the set of all $u \in H^s_{loc}(G \times \mathbf{R})$ such that, for $j = 1, \ldots, J$, the push-forward $(\chi_j\kappa_j)_*(\varphi_j u)$, which may be regarded as a distribution on $\mathbf{R}^{n+1}$ after extension by zero, is an element of $H^s(\mathbf{R}^{n+1})$. The space $H^s_{cone}(G \times \mathbf{R})$ is endowed with the corresponding Hilbert space topology. We let

$$H^s_{cone}(X^\wedge) \;=\; \{u|_{X \times \mathbf{R}_+} : u \in H^s_{cone}(G \times \mathbf{R})\}.$$

For more details see Schrohe&Schulze [16, Section 4.2]. The subscript "cone" is motivated by the fact that, away from zero, these are the Sobolev spaces for an infinite cone with center at the origin and cross-section X. In particular, the space $H^s_{cone}(S^n \times \mathbf{R}_+)$ coincides with $H^s(\mathbf{R}^{n+1} \setminus \{0\})$.

Definition 2.6. For $s, \gamma \in \mathbf{R}$ and $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(r) \equiv 1$ near $r = 0$, let

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{u \in \mathcal{D}'(X^\wedge) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)\, u \in H^s_{cone}(X^\wedge)\}. \tag{2.4}$$

The definition is independent of the choice of ω by 2.3(f). In the notation of 2.4,

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]\, H^s_{cone}(X^\wedge). \tag{2.5}$$

We endow it with the Banach topology

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^\wedge)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega)u\|_{H^s_{cone}(X^\wedge)}.$$

In fact, this is a Hilbert topology with the inner product inherited from $\mathcal{H}^{s,\gamma}$ and H^s_{cone}.

Theorem 2.7. *For $s > 1/2$ and $\gamma \in \mathbf{R}$ the restriction $\gamma_0 u = u|_{Y^\wedge}$ of u to $Y^\wedge$ induces a continuous operator*

$$\gamma_0 : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge).$$

we may even replace γ_0 by $\Lambda\gamma_0$, where Λ is a (parameter-dependent) order reduction of order $3/2$, and still have order 2.

Here, the vector bundle W_1 is zero, while V_1, V_2, W_2 can be taken trivial one-dimensional.

Proposition 1.15. *Let $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^l)$, $B \in \mathcal{B}^{\mu',d'}(X;\mathbf{R}^l)$, and $\alpha,\beta \in \mathbf{C}$. Then*
(a) $\alpha A + \beta B \in \mathcal{B}^{\mu'',d''}(X;\mathbf{R}^l)$ *for* $\mu'' = \max\{\mu,\mu'\}$, $d'' = \max\{d',d'\}$.
(b) $A \circ B \in \mathcal{B}^{\mu'',d''}(X;\mathbf{R}^l)$ *for* $\mu'' = \max\{\mu+\mu'\}$, $d'' = \max\{\mu'+d,d'\}$.
We assume in here that the vector bundles A and B act on are such that the addition and composition make sense.

For a proof see Rempel&Schulze [13, Section 2.3.3.2].

2. Wedge Sobolev Spaces

In the following, we let G be a closed compact manifold of dimension n, and let X be an embedded n-dimensional submanifold with boundary, Y.

2.1 Parameter-dependent order reductions on G. For each $\mu \in \mathbf{R}$ there is a pseudodifferential operator Λ^μ with local parameter-dependent elliptic symbols of order μ, depending on the parameter $\tau \in \mathbf{R}$, such that

$$\Lambda^\mu(\tau) : H^s(G,V) \to H^{s-\mu}(G,V)$$

is an isomorphism for all τ.

In order to construct such an operator one can e.g. start with symbols of the form $\langle\xi,(\tau,C)\rangle^\mu \in S^\mu(\mathbf{R}^n,\mathbf{R}^n_\xi;\mathbf{R}_\tau)$ with a large constant $C > 0$ and patch them together to an operator on the manifold G with the help of a partition of unity and cut-off functions.

Alternatively, one can choose a Hermitean connection on V and consider the operator $(C+\tau^2-\Delta)^{\frac{\mu}{2}}$, where Δ denotes the connection Laplacian and C is a large positive constant.

Definition 2.2. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$. We recall that the classical Mellin transform Mu of a complex-valued $C_0^\infty(\mathbf{R}_+)$-function u is given by

$$(Mu)(z) = \int_0^\infty t^{z-1}u(t)\,dt. \tag{2.1}$$

M extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{1/2})$. Of course, (1) also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ for $u \in C_0^\infty(\mathbf{R}_+)$ motivates the following definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+,E).$$

The inverse of M_γ is given by

$$[M_\gamma^{-1}h](z) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz.$$

2.3 Totally characteristic Sobolev spaces. (a) Let $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ be a family of parameter-dependent order reductions as in 2.1. For $s,\gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(G^\wedge)$ is the closure of $C_0^\infty(G^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(G^\wedge)} = \left\{\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z)Mu(z)\|^2_{L^2(G)}\,|dz|\right\}^{1/2}. \tag{2.2}$$

Recall that n is the dimension of X and G.

(a) The space $\mathcal{H}^{s,\gamma}(G^\wedge)$ is independent of the particular choice of the order reducing family.

(b) For $s = l \in \mathbf{N}$ we obtain the alternative description

$$u \in \mathcal{H}^{l,\gamma}(G^\wedge) \quad \text{iff} \quad t^{n/2-\gamma}(t\partial_t)^k Du(x,t) \in L^2(G^\wedge)$$

$\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge))$ of the operator $\tilde{\kappa}_{[\eta]^{-1}}a(y,\eta)\kappa_{[\eta]}$. This in turn simply is multiplication by $\sigma([\eta]^{-1}\cdot)$, which is uniformly bounded by another application of 2.3(b).

Next let us treat multiplication by $\psi = \psi(y)$ and show that it furnishes a bounded operator on $\mathcal{W}^s(\mathbf{R}^q, E)$ for every Banach space E with group action κ. In fact, since the wedge Sobolev spaces are defined as the completion of $\mathcal{S}(\mathbf{R}^q, E)$ in the corresponding norm, it is sufficient to show that, for a pure tensor $u = u_0 \otimes e$ in $\mathcal{S}(\mathbf{R}^q) \otimes E$ we have

$$\|\psi u\|_{\mathcal{W}^s(\mathbf{R}^q,E)} \le C\|u\|_{\mathcal{W}^s(\mathbf{R}^q,E)}$$

with a constant independent of u. Choose an integer $l > q/2$. With the help of Peetre's inequality and (1.1), in particular the fact that

$$\begin{aligned} \|\kappa_{[\eta+\xi]^{-1}}\kappa_{[\eta]}\|_E &= \|\kappa_{[\eta+\xi]^{-1}[\eta]}\|_E \\ &\le C\max\{[\eta+\xi]^{-1}[\eta], [\eta+\xi][\eta]^{-1}\}^M \le C'[\xi]^M, \end{aligned}$$

we get the following estimate

$$\begin{aligned} &\|\psi u\|^2_{\mathcal{W}^s(\mathbf{R}^q,E)} \\ =\ & \int [\eta]^{2s}|\mathcal{F}(\psi u_0)(\eta)|^2\|\kappa_{[\eta]^{-1}}e\|^2_E d\eta \\ =\ & (2\pi)^{q/2}\int[\eta]^{2s}\left|\int \mathcal{F}u_0(\eta-\xi)\mathcal{F}\psi(\xi)\,d\xi\right|^2 \|\kappa_{[\eta]^{-1}}e\|^2_E d\eta \\ \le\ & (2\pi)^{q/2}\int[\eta]^{2s}\int|\mathcal{F}u_0(\eta-\xi)\mathcal{F}\psi(\xi)[\xi]^l|^2\,d\xi\ \|\kappa_{[\eta]^{-1}}e\|^2_E d\eta\ \int[\xi]^{-2l}d\xi \\ =\ & C\iint[\eta+\xi]^{2s}|\mathcal{F}u_0(\eta)|^2|\mathcal{F}\psi(\xi)|^2[\xi]^{2l}\|\kappa_{[\eta+\xi]^{-1}}e\|^2_E d\eta d\xi \\ \le\ & C'\iint[\eta]^{2s}[\xi]^{2|s|+2l+2M}|\mathcal{F}u_0(\eta)|^2|\mathcal{F}\psi(\xi)|^2\|\kappa_{[\eta]^{-1}}e\|^2_E d\eta d\xi \\ \le\ & C''\|\psi\|_{H^{|s|+l+M}(\mathbf{R}^q)}\,\|u\|^2_{\mathcal{W}^s(\mathbf{R}^q,E)}. \end{aligned}$$

Here the first inequality is Cauchy-Schwarz'. □

3. Operator-Valued Mellin Symbols

As before, we let G be a closed compact manifold of dimension n, and let X be an embedded n-dimensional submanifold with boundary, Y.

Definition 3.1. (a) For $\mu \in \mathbf{R}, d \in \mathbf{N}$, we define $M_O^{\mu,d}(X;\mathbf{R}^q)$ as the space of all functions

$$a \in \mathcal{A}\left(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q)\right)$$

with the following property: Given $c_1 < c_2$ in $\mathbf{R}$

$$a(\beta + i\tau) \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^q\times\mathbf{R}_\tau), \tag{3.1}$$

uniformly for all $\beta \in [c_1, c_2]$.

We call the elements of $M_O^{\mu,d}(X;\mathbf{R}^q)$ holomorphic Mellin symbols of order μ and type d. We are assuming that the vector bundles $a(z)$ is acting on are independent of z.

The topology of $M_O^{\mu,d}(X)$ is given by the semi-norm systems for the topology of $\mathcal{A}\left(\mathbf{C},\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)\right)$ and, for families $\{a_\beta : \beta \in \mathbf{R}\}$, the topology of uniform convergence on compact subsets of $\mathbf{R}_\beta$ in $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q\times\mathbf{R}_\tau)$. Clearly, $M_O^{\mu,d}(X;\mathbf{R}^q)$ is a Fréchet space with this topology.

(b) $M_{O,cl}^{\mu,d}(X;\mathbf{R}^q)$ is the corresponding space with $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)$ replaced by $\mathcal{B}_{cl}^{\mu,d}(X;\mathbf{R}^q)$.

By r denote the normal coordinate in a neighborhood of Y. Then the operators $\gamma_j : u \mapsto \partial_r^j u|_{Y^\wedge}$ define continuous mappings

$$\gamma_j \quad : \quad \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-j-1/2,\gamma-1/2}(Y^\wedge).$$

Proof. For one thing this can be deduced from the trace theorem for the usual Sobolev spaces. Note that the shift in the weight $\gamma \mapsto \gamma - 1/2$ is due to the fact that $\dim Y = n-1$. We shall give an independent proof in 3.4, below. □

The following lemma is obvious after 2.3(d):

Lemma 2.8. $\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

Lemma 2.9. *A strongly continuous group action κ_λ can be defined on $\mathcal{K}^{s,\gamma}(X^\wedge)$ by*

$$(\kappa_\lambda f)(r) = \lambda^{\frac{n+1}{2}} f(\lambda r), \quad f \in \mathcal{K}^{s,\gamma}(X^\wedge),\ s \geq 0.$$

This action is unitary on $\mathcal{K}^{0,0}(X^\wedge)$ and extends to distributions by $(\kappa_\lambda u)(\varphi) = u(\kappa_{\lambda^{-1}}\varphi)$ for $u \in \mathcal{D}'(X^\wedge)$, $\varphi \in C_0^\infty(X^\wedge)$.

Proof. It is lengthy but straightforward to see that κ is strongly continuous; it is unitary on $\mathcal{K}^{0,0}(X^\wedge)$ in view of Lemma 2.8. □

Remark 2.10. The definitions of the spaces $\mathcal{H}^{s,\gamma}$ and $\mathcal{K}^{s,\gamma}$ also make sense for functions and distributions taking values in a vector bundle V. We shall then write $\mathcal{H}^{s,\gamma}(X^\wedge, V)$ and $\mathcal{K}^{s,\gamma}(X^\wedge, V)$, respectively. In later constructions we will often have to deal with direct sums

$$\mathcal{K}^{s,\gamma}(X^\wedge, V) \oplus \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge, W)$$

for vector bundles V and W over X and Y, respectively. On these spaces we use the natural group action

$$\kappa_\lambda(u,v) = (\lambda^{\frac{n+1}{2}} u(\lambda\cdot), \lambda^{\frac{n}{2}} v(\lambda\cdot)).$$

Definition 2.11. For real s and γ we let $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbf{R}^q) = \mathcal{W}^s(\mathbf{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$.

Theorem 2.12. *The restriction operator γ_0 induces a continuous map*

$$\gamma_0 : \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbf{R}^q) \to \mathcal{W}^{s-1/2,\gamma-1/2}(Y^\wedge \times \mathbf{R}^q).$$

Proof. We know from Theorem 2.7 that $\gamma_0 : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)$ is a bounded operator. So we may consider it once more an operator-valued symbol, independent of y, η. Just as in Example 1.2 one checks that $\gamma_0 \in S^{1/2}(\mathbf{R}^q \times \mathbf{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge))$. Now Theorem 1.8 gives the assertion. □

Proposition 2.13. *Let $\varphi \in \mathcal{S}(\overline{X}^\wedge \times \mathbf{R}^q)$. Then the operator of multiplication by φ furnishes a bounded operator on $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbf{R}^q)$ for all $s, \gamma \in \mathbf{R}$. Its norm depends continuously on the semi-norms for φ in $\mathcal{S}(\overline{X}^\wedge \times \mathbf{R}^q)$.*

Proof. We shall use a tensor product argument based on the identity $\mathcal{S}(\overline{X}^\wedge \times \mathbf{R}^q) = \mathcal{S}(\overline{X}^\wedge)\hat{\otimes}_\pi \mathcal{S}(\mathbf{R}^q)$. Let $\varphi = \sigma \otimes \psi$ with $\sigma \in \mathcal{S}(\overline{X}^\wedge)$ and $\psi \in \mathcal{S}(\mathbf{R}^q)$ be a pure tensor. We shall show the separate continuity of the multiplications. Since both $\mathcal{S}(\overline{X}^\wedge)$ and $\mathcal{S}(\mathbf{R}^q)$ are Fréchet spaces this will imply the joint continuity and establish the proof.

Let us first deal with multiplication by σ, denoted for the moment by M_σ. We may consider this multiplication as the application of a pseudodifferential operator with the (y,η)-independent operator-valued symbol $a(y,\eta) = M_\sigma$. Let us check that a is an element of $S^0(\mathbf{R}^q \times \mathbf{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for all s. First of all, an application of 2.3(b) together with interpolation shows that M_σ is bounded on $\mathcal{K}^{s,\gamma}(X^\wedge)$. In view of the independence of y and η we now only have to estimate the norm in *given by the point-wise composition in Boutet de Monvel's calculus:* $(a,b) \mapsto c$ *with* $c(z,\eta) = a(z,\eta) \circ b(z,\eta)$.

The proof is straightforward from the definition and Proposition 1.15.

3.6 Operator-valued Mellin symbols. Let $\gamma, \mu \in \mathbf{R}$, $\Omega \subseteq \mathbf{R}^q$, and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q))$. Recall that $[\cdot]$ is a smooth positive function on $\mathbf{R}^q$ coinciding with $|\cdot|$ outside a neighborhood of zero.

Given $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ define

$$a(y,\eta) = \omega_1(t[\eta])t^{-\mu}\{\mathrm{op}_M^\gamma f(t,y,z,t\eta)\}\omega_2(t[\eta]).$$

According to (3.2) this furnishes a function a on $\Omega \times \mathbf{R}^q$ with values in $\mathcal{L}(\mathcal{K}_1^{s,\gamma}(X^\wedge), \mathcal{K}_2^{s,\gamma}(X^\wedge))$ for all $s > d - 1/2$. We will show that a in fact is an element of $S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma})$. The proof is based on Proposition 3.8, below, and a tensor product argument given in Corollary 3.9. We shall keep the notation $\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma}, \gamma, \mu, f, a, \omega_1, \omega_2$ fixed.

First let us note the following:

Lemma 3.7. *If f is independent of t, then there is a $C > 0$ such that*

$$a(y,\lambda\eta) = \lambda^\mu \kappa_\lambda a(y,\eta)\kappa_{\lambda^{-1}}$$

for all $\lambda \geq 1$, and $|\eta| \geq C$.

Proof. We have

$$\kappa_\lambda\{\mathrm{op}_M^\gamma f(y,z,t\eta)\} = \{\mathrm{op}_M^\gamma f(y,z,\lambda t\eta)\}\kappa_\lambda.$$

Next choose C so large that $[\eta] = |\eta|$ for $|\eta| \geq C$. For $u \in C_0^\infty(\mathbf{R}_+)$ and $\lambda \geq 1$ this implies that

$$\begin{aligned} &\kappa_\lambda\{\omega_1(t[\eta])t^{-\mu}\{\mathrm{op}_M^\gamma f(y,z,t\eta)\}\omega_2(t[\eta])\kappa_{\lambda^{-1}}u\} \\ =\ &\omega_1(\lambda t[\eta])(\lambda t)^{-\mu}\{\mathrm{op}_M^\gamma f(y,z,t\lambda\eta)\}\omega_2(\lambda t[\eta])u. \end{aligned}$$

Since $\omega_j(\lambda t[\eta]) = \omega_j(t[\lambda\eta])$, $j = 1, 2$, this gives the desired result. □

The proposition, below, shows the assertion for the case where the symbol f is independent of t.

Proposition 3.8. *Let $g = g(y,z,\eta) \in C^\infty(\Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$ be independent of t. Then the function b defined by*

$$b(y,\eta) = \omega_1(t[\eta])t^{-\mu}\{\mathrm{op}_M^\gamma g(y,z,t\eta)\}\omega_2(t[\eta])$$

is an element of $S_{cl}^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma})$, and the symbol semi-norms for b can be estimated in terms of those for g.

Proof. For fixed y and η, the operator $b(y,\eta)$ is an element of $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma})$ by 3.3. Moreover, it is a smooth function of y and η, and its semi-norms in $C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma}))$ depend continuously on those for g. According to the lemma above it is homogeneous of degree μ for large $|\eta|$. The assertion therefore follows from Lemma 1.4. □

Corollary 3.9. It is now easy to see that a is an element of $S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s,\gamma})$ for all $s > d - 1/2$. Indeed, we use the fact that

$$C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q)) = C^\infty(\overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q)).$$

Employing the continuity of the mapping $g \mapsto b$ in Proposition 3.9 it is therefore sufficient to consider the case where

$$f(t,y,z,\eta) = \varphi(t)g(y,z,\eta)$$

with $\varphi \in C^\infty(\overline{\mathbf{R}}_+)$ and $g \in C^\infty(\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q))$ independent of t. Choose a function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(t)\omega_1(t[\eta]) = \omega_1(t[\eta])$. This is possible, since $[\eta]$ is bounded away from zero. We have

$$a(y,\eta) = M_\varphi \omega_1(t[\eta])\{\mathrm{op}_M^\gamma g(y,z,t\eta)\}\omega_2(t[\eta]).$$

Example 3.2. Let $\mu \in \mathbf{N}$ and let $A_k \in \mathcal{B}^{\mu-k,d}(X)$, $k = 0, \dots, \mu$, be *differential* boundary value problems. Then

$$a(z) = \sum_{k=0}^{\mu} A_k z^k \in M_O^{\mu,d}(X).$$

3.3 Mellin symbols and operators. Let $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$. For each fixed $(t, t', z) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \Gamma_{1/2-\gamma}$, we have a boundary value problem

$$f(t,t',z): \begin{array}{c} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{array} \rightarrow \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array}$$

in Boutet de Monvel's calculus.

For $u \in C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(\overline{Y}^\wedge, W_1) = C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$ we define the Mellin operator $\mathrm{op}_M^\gamma f$ by

$$\{\mathrm{op}_M^\gamma f\} u(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-z} f(t,t',z) u(t') dt'/t' dz.$$

If f is independent of t', this reduces to

$$\{\mathrm{op}_M^\gamma f\} u(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} t^{-z} f(t,z) M u(z) dz.$$

It is easy to see the continuity of

$$\mathrm{op}_M^\gamma f: \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \rightarrow \begin{array}{c} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array}.$$

For $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$ and $\omega_1, \omega_2 \in C_0^\infty(\mathbf{R}_+)$ we obtain a bounded extension

$$\omega_1 \{\mathrm{op}_M^\gamma f\} \omega_2: \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \rightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array} \tag{3.2}$$

provided $s > d - 1/2$. A proof is given in [16, Proposition 2.1.5].

In the following we shall use the abbreviations

$$\begin{aligned} \mathcal{K}_1^{s,\gamma} &= \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \quad \text{and} \\ \mathcal{K}_2^{s,\gamma} &= \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}-\mu}(X^\wedge, V_2) \oplus \mathcal{K}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(Y, W_2). \end{aligned}$$

3.4 Alternative proof of Theorem 2.7. We consider the operator of evaluation at the boundary γ_0. As we saw in Example 1.14, it is a parameter-dependent operator in Boutet de Monvel's calculus of order 1/2 and type 1. We may therefore regard it as a Mellin operator with a Mellin symbol independent of t, t', and z. The mapping properties (3.2), applied with the choice V_1, W_2 = trivial one-dimensional, W_1, V_2 = zero, show that for every choice of cut-off functions ω_1, ω_2 near zero and $s > 1/2$,

$$\omega_1 \gamma_0 \omega_2 : \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)$$

is bounded. Away from zero, the spaces $\mathcal{K}^{\cdot,\cdot}$ coincide with usual Sobolev spaces on the cone, hence the result there follows from the usual trace theorem.

Proposition 3.5. *Given $\mu, \mu' \in \mathbf{Z}$ and $d, d' \in \mathbf{N}$, let $\mu'' = \mu + \mu'$ and $d'' = \max\{\mu' + d, d'\}$. Then there is a continuous multiplication*

$$M_O^{\mu,d}(X; \mathbf{R}^q) \times M_O^{\mu',d'}(X; \mathbf{R}^q) \rightarrow M_O^{\mu'',d''}(X; \mathbf{R}^q)$$

Note that T cannot be continued to a function in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)$.

Proof. (a) is trivial. For (b) write $\varphi(x) = (x-1)/\ln x$; this is a smooth function on $\mathbf{R}_+$, $\varphi(1) = 1$. The observation that $(t'\partial_{t'})^k[t'^{-1}T(t,t')]|_{t=t'} = (-x\partial_x)^k\varphi(x)|_{x=1}$ shows the first claim. For the second note that $t^{-1}T(t,t') = \varphi(x)/x$, while the third and fourth follow by replacing φ by $1/\varphi$. □

Proposition 3.14. *For $p \in C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R}\times\mathbf{R}^q_\eta))$ define $g \in C^\infty(\mathbf{R}_+\times\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q_\eta))$ by*

$$g(t,t',y,i\tau,\eta) = p(t,y,-T(t,t')^{-1}\tau,\eta)t'T(t,t')^{-1}. \tag{3.3}$$

Then $\operatorname{op}_t p(t,y,\tau,\eta) = \operatorname{op}_M^{1/2} g(t,t',y,i\tau,\eta)$.

Conversely let $f \in C^\infty(\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q_\eta))$ and define $q \in C^\infty(\mathbf{R}_+\times\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau\times\mathbf{R}^q_\eta))$ by

$$q(t,t',y,\tau,\eta) = f(t,y,-iT(t,t')\tau,\eta)T(t,t')/t'. \tag{3.4}$$

Then $\operatorname{op}_t q(t,t',y,\tau,\eta) = \operatorname{op}_M^{1/2} f(t,y,i\tau,\eta)$.

The subscript t with op indicates that the pseudodifferential action is with respect to t and the covariable τ only.

Proof. The proof is a straightforward computation. For completeness let us sketch (3.3), omitting for better legibility the variables x and y. The proof of the second identity is analogous.

$$\begin{aligned} &\{\operatorname{op}_t p(t,\tau,\eta)\}u(t,\eta) \\ =\ & (2\pi)^{-1}\iint e^{i(t-t')\tau}p(t,\tau,\eta)u(t')dt'd\tau \\ =\ & (2\pi)^{-1}\int\int_0^\infty (t/t')^{iT(t,t')\tau}p(t,\tau,\eta)u(t')dt'd\tau \\ =\ & (2\pi)^{-1}\int\int_0^\infty (t/t')^{i\tau}p(t,T(t,t')^{-1}\tau,\eta)t'T(t,t')^{-1}u(t')dt'/t'd\tau. \end{aligned}$$

□

As a preparation for the proof of Theorem 3.17, below, we need the following well-known facts. For a proof see e.g. Schrohe&Schulze [16, 2.1.12, 2.3.3].

Lemma 3.15. *Given a sequence $f_j \in C^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu_j,d}(X;\Gamma_0\times\mathbf{R}^q))$ with $\mu_j \to -\infty$, there is a symbol $f \in C^\infty(\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$, $\mu = \max\{\mu_j\}$ such that $f \sim \sum f_j$; the symbol f is unique modulo $C^\infty(\mathbf{R}_+\times\Omega, \mathcal{B}^{-\infty,d}(X;\Gamma_0\times\mathbf{R}^q))$. If the symbols f_j even belong to $C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu_j,d}(X;\Gamma_0\times\mathbf{R}^q))$, then we find $f \in C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$; it is unique modulo $C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{-\infty,d}(X;\Gamma_0\times\mathbf{R}^q))$.*

Lemma 3.16. *Given $f \in C^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$ there is a symbol $g \in C^\infty(\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$ with*

$$\operatorname{op}_M^{1/2} f \equiv \operatorname{op}_M^{1/2} g \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q)); \tag{3.5}$$

it has the asymptotic expansion

$$g(t,y,z,\eta) \sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k\partial_z^k f(t,t',y,z,\eta)|_{t'=t}. \tag{3.6}$$

Conversely, every symbol with this asymptotic expansion satisfies relation (3.5).

Theorem 3.17. *Let $p \in C^\infty(\mathbf{R}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}\times\mathbf{R}^q_\eta))$ be edge-degenerate. Then there is a symbol $f \in C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q_\eta))$ with*

$$\operatorname{op}_t p(t,y,\tau,\eta) = \operatorname{op}_M^{1/2} f(t,y,i\tau,t\eta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q)). \tag{3.7}$$

Here, M_φ denotes the operator of multiplication by $\omega\varphi$. We note that

$$\kappa_{[\eta]^{-1}} M_\varphi \kappa_{[\eta]} = M_{\tilde\varphi},$$

where $\tilde\varphi(t) = \omega([\eta]^{-1}t)\varphi([\eta]^{-1}t)$. The norm of this operator on $\mathcal{K}_2^{s,\gamma}$ is uniformly bounded in η; it can be estimated in terms of the semi-norms for φ. Therefore M_φ furnishes an element in $S^0(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s,\gamma}, \mathcal{K}_2^{s,\gamma})$, and we get the statement from Lemma 1.3.

Theorem 3.10. *Let $\gamma, \mu \in \mathbf{R}$, $\Omega \subseteq \mathbf{R}^q$, and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma} \times \mathbf{R}^q))$. Then the operator*

$$\mathrm{op}\,\{\omega_1(t[\eta])t^{-\mu}\{\mathrm{op}_M^\gamma f(t,y,z,t\eta)\}\omega_2(t[\eta])\} : \mathcal{W}^s_{comp}(\mathbf{R}^q, \mathcal{K}_1^{s,\gamma}) \to \mathcal{W}^{s-\mu}_{loc}(\mathbf{R}^q, \mathcal{K}_2^{s,\gamma})$$

is continuous.

Proof. This now is immediate from Theorem 1.8. □

Lemma 3.11. *We use the notation of Theorem 3.10 and let $\beta \in \mathbf{R}$. Then*

$$\omega_1(t[\eta])\{\mathrm{op}_M^\gamma f(t,y,z,t\eta)\}\omega_2(t[\eta])t^\beta = \omega_1(t[\eta])t^\beta\{\mathrm{op}_M^{\gamma-\beta} T^{-\beta} f(t,y,z,t\eta)\}\omega_2(t[\eta]).$$

In case f even is an element in $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_{\mathcal{O}}^{\mu,d}(X; \mathbf{R}^q))$ we additionally have

$$\omega_1(t[\eta])\{\mathrm{op}_M^\gamma f(t,y,z,t\eta)\}\omega_2(t[\eta])t^\beta = \omega_1(t[\eta])t^\beta\{\mathrm{op}_M^\gamma T^{-\beta} f(t,y,z,t\eta)\}\omega_2(t[\eta]).$$

Here we consider both sides as operators on $C_0^\infty(\mathbf{R}_+, C^\infty(X))$; $T^{-\beta}$ is the translation operator defined by $T^{-\beta}f(t,y,z,t\eta) = f(t,y,z-\beta,t\eta)$.

Proof. Using a tensor product argument, it is sufficient to treat the case where f is independent of t and y, i.e., $f \in \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma} \times \mathbf{R}^q)$. But then

$$\begin{aligned}
\{\mathrm{op}_M^\gamma f(z,t\eta)\}t^\beta u(t) &= \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-z} f(z,t\eta) t'^\beta u(t')dt'/t'dz \\
&= t^\beta \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-(z+\beta)} f(z,t\eta) u(t')dt'/t'dz \\
&= t^\beta \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma+\beta}} \int_0^\infty (t/t')^{-z} T^{-\beta} f(z,t\eta) u(t')dt'/t'dz,
\end{aligned}$$

so the first assertion is obvious. In case f is holomorphic, Cauchy's theorem allows us to shift the contour of integration, and we obtain the second statement. □

Mellin Quantization

Definition 3.12. A symbol $p = p(t,y,\tau,\eta)$ in $C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R}_\tau \times \mathbf{R}^q_\eta))$ is called *edge degenerate*, if there is a symbol $\tilde p$ in $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R}_\tau \times \mathbf{R}^q_\eta))$ with $p(t,y,\tau,\eta) = \tilde p(t,y,t\tau,t\eta)$.

We shall now show that given an edge degenerate symbol we can find a Mellin symbol which induces the same operator up to a smoothing perturbation and vice versa. We start with an analysis of the following simple function.

Lemma 3.13. *For $t, t' > 0$ let $T(t,t') = \frac{t-t'}{\ln t - \ln t'}$. Then T is a smooth positive function on $\mathbf{R}_+ \times \mathbf{R}_+$, $T(t,t) = t$. Moreover:*

(a) *Write $x = t/t'$. We have $t'\partial_{t'} = -x\partial_x$ and $t'^{-1}T(t,t') = \frac{x-1}{\ln x}$.*

(b) *For each $k \in \mathbf{N}$ the functions*

$$(t'\partial_{t'})^k[t'^{-1}T(t,t')]|_{t'=t}, \quad (t'\partial_{t'})^k[t^{-1}T(t,t')]|_{t'=t},$$

$$(t'\partial_{t'})^k[t'T(t,t')^{-1}]|_{t'=t}, \quad \text{and} \quad (t'\partial_{t'})^k[t'T(t,t')^{-1}]|_{t'=t}$$

are constant in t.

Proof. The Mellin symbol f_γ can be computed in terms of the function $f = f_{1/2}$ in Theorem 3.17. The definition of op_M^γ shows that

$$\mathrm{op}_t\, p(t,y,\tau,\eta) \equiv \mathrm{op}_M^{1/2} f_{1/2}(t,y,i\tau,t\eta) = \mathrm{op}_M^\gamma g_\gamma(t,t',y,1/2-\gamma+i\tau,t\eta),$$

where $g_\gamma(t,t',y,1/2-\gamma+i\tau,\eta) = (t/t')^{1/2-\gamma} f_{1/2}(t,y,i\tau,\eta)$. We can convert g_γ to a t'-independent symbol f_γ with

$$\begin{aligned} f_\gamma(t,y,1/2-\gamma+i\tau,\eta) &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k D_\tau^k g_\gamma(t,t',y,1/2-\gamma+i\tau,\eta)|_{t'=t} \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k (\frac{t}{t'})^{1/2-\gamma}|_{t'=t}\ D_\tau^k f_{1/2}(t,y,i\tau,\eta) \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(1/2-\gamma)^k D_\tau^k f_{1/2}(t,y,i\tau,\eta). \end{aligned} \tag{3.9}$$

Here we used that $(-t'\partial_{t'})^k(t/t')^{1/2-\gamma}|_{t'=t} = (x\partial_x)^k x^{1/2-\gamma}|_{x=1} = (1/2-\gamma)^k$. Since $f_{1/2}$ is smooth up to $t=0$, the asymptotic summation can be carried out in $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}\times\mathbf{R}^q_\eta))$, and this is all we need.

If p is classical, then so is $f_{1/2}$ by Theorem 3.17, hence f_γ will be classical. □

Kernel Cut-Off

We shall now analyse the behavior of symbols $f \in C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}\times\mathbf{R}^q))$ under operations of the type

$$f \mapsto M_{\rho\to z}\varphi(\rho)M^{-1}_{\gamma,\zeta\to\rho} f(t,y,\zeta,\eta).$$

Here, φ is either a function in $C_0^\infty(\mathbf{R}_+)$ or of the form $1-\psi$ with $\psi \in C_0^\infty(\mathbf{R}_+)$. For the proof, the specific choice of γ is of little importance. We therefore let $\gamma = 1/2$, so we can work conveniently on the imaginary axis $i\mathbf{R} = \Gamma_0$.

Theorem 3.20. *Let $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho)\equiv 1$ near $\rho = 1$. Let $f \in C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$. Then the operator-valued function $f_{1-\psi}$ defined by*

$$f_{1-\psi}(t,y,z,\eta) = M_{\rho\to z}(1-\psi(\rho))M^{-1}_{1/2,\zeta\to\rho} f(t,y,\zeta,\eta)$$

is an element of $C^\infty(\overline{\mathbf{R}}_+\times\mathbf{R}^q; \mathcal{B}^{-\infty,d}(X;\Gamma_0))$. Moreover, the mapping $(\psi, f)\mapsto f_{1-\psi}$ is separately continuous from $C_0^\infty(\mathbf{R}_+)\times C^\infty(\overline{\mathbf{R}}_+\times\Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q))$ to $C^\infty(\overline{\mathbf{R}}_+\times\Omega; \mathcal{B}^{-\infty,d}(X;\Gamma_0\times\mathbf{R}^q))$.

Proof. Using a tensor product argument as above it is sufficient to treat the case where f is independent of t and y, i.e., $f = f(z,\eta) \in \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q_\eta)$. First note that $\mathcal{B}^{-\infty,d}(X;\Gamma_0\times\mathbf{R}^q)) = \mathcal{S}(\Gamma_0, \mathcal{B}^{-\infty,d}(X;\mathbf{R}^q))$. In view of the identity

$$M_{\rho\to z}(\ln^M\rho\,(-\rho\partial_\rho)^N h) = \left(\frac{d}{dz}\right)^M z^N(Mh)(z)$$

valid for, say, $h \in C_0^\infty(\mathbf{R}_+)$, we only have to check that, for all $M, N \in \mathbf{N}$, and each semi-norm p_j on $\mathcal{B}^{\mu-j,d}(X;\mathbf{R}^q)$, the semi-norms

$$\|p_j(\ln^M\rho(\rho\partial_\rho)^N\{(1-\psi(\rho))(M^{-1}_{1/2}f)(\rho,\eta)\})\|_{L^2(\mathbf{R}_\rho)} \tag{3.10}$$

are finite and depend continuously on the semi-norms for f and ψ, respectively. For fixed ρ,

$$\begin{aligned} 2\pi(1-\psi(\rho))(M^{-1}_{1/2}f)(\rho,\eta) &= \int(1-\psi(\rho))\rho^{-i\tau}f(i\tau,\eta)d\tau \\ &= (1-\psi(\rho))\ \ln^{-L}\rho\int(i\partial_\tau)^L\rho^{-i\tau}f(i\tau,\eta)d\tau \\ &= (1-\psi(\rho))\ \ln^{-L}\rho\int\rho^{-i\tau}(-i\partial_\tau)^L f(i\tau,\eta)d\tau \end{aligned}$$

Conversely, given $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_0 \times \mathbf{R}^q_\eta))$ *there is an edge-degenerate boundary value problem* p *such that relation* (3.7) *holds. The corresponding statement holds for classical symbols, i.e., for* $\mathcal{B}^{\mu,d}$ *replaced by* $\mathcal{B}^{\mu,d}_{cl}$.

Proof. Let $p(t,y,\tau,\eta) = \tilde{p}(t,y,t\tau,t\eta)$ with $\tilde{p} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R} \times \mathbf{R}^q))$. We know from Proposition 3.14 and Lemma 3.16 that

$$\mathrm{op}_t p(t,y,\tau,\eta) \equiv \mathrm{op}_M^{1/2} g(t,t',y,i\tau,\eta) \equiv \mathrm{op}_M^{1/2} \tilde{g}(t,y,i\tau,\eta),$$

where

$$\begin{aligned}
\tilde{g}(t,y,i\tau,\eta) &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k D_\tau^k g(t,t',i\tau)|_{t'=t} \\
&\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k D_\tau^k \{p(t,y,-T(t,t')^{-1}\tau,\eta)t'T(t,t')^{-1}\}|_{t'=t} \\
&\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k D_\tau^k \{\tilde{p}(t,y,-T(t,t')^{-1}t\tau,t\eta)t'T(t,t')^{-1}\}|_{t'=t}.
\end{aligned}$$

Next we prove that, for each k, the function f_k defined by

$$f_k(t,y,\tau,\eta) = (-t'\partial_{t'})^k D_\tau^k \{\tilde{p}(t,y,-T(t,t')t\tau,\eta)t'T(t,t')^{-1}\}|_{t'=t}$$

is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R} \times \mathbf{R}^q))$. In fact, Leibniz' formula implies that

$$\begin{aligned}
&(-t'\partial_{t'})^k D_\tau^k \{\tilde{p}(t,y,-T(t,t')^{-1}t\tau,\eta)t'T(t,t')^{-1}\} \\
= &\sum_{k_1+k_2+k_3=k} c_{k_1,k_2,k_3}(-t'\partial_{t'})^{k_1}\{D_\tau^k \tilde{p}(t,y,-T(t,t')^{-1}t\tau,\eta)\} \\
&\times(-t'\partial_{t'})^{k_2}\{t'T(t,t')^{-1}\}(-t'\partial_{t'})^{k_3}\{-T(t,t')^{-1}t\}^k,
\end{aligned}$$

hence Lemma 3.13 shows that we only have to check the derivatives

$$(-t'\partial_{t'})^{k_1}\{D_\tau^k \tilde{p}(t,y,-T(t,t')^{-1}t\tau,\eta)\}.$$

For $k_1 = 1$, this is just $D_\tau^{k+1}\tilde{p}(t,y,-T(t,t')^{-1}t\tau,\eta)\tau t'\partial_{t'}T(t,t')^{-1}t$. Together with iteration, Lemma 3.13 again yields the smoothness.

According to Lemma 3.15 we can find an $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_0 \times \mathbf{R}^q))$ with $f \sim \sum_{k=0}^\infty f_k$. Then $f(t,y,\tau,t\eta) \sim \sum f_k(t,y,\tau,t\eta)$ in $C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_0 \times \mathbf{R}^q))$, and hence

$$\mathrm{op}_M^{1/2} f(t,y,i\tau,t\eta) \equiv \mathrm{op}_t p(t,y,\tau,\eta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{R}^q)).$$

Clearly, the same argument applies with $\mathcal{B}^{\mu,d}$ replaced by $\mathcal{B}^{\mu,d}_{cl}$.

The converse statement follows in the same way, using the second part of Proposition 3.14 and the asymptotic expansion formula for pseudodifferential double symbols. □

3.18 Mellin quantization for arbitrary weights. We have solved the question how to associate to an edge-degenerate boundary value problem $p \in C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R} \times \mathbf{R}^q_\eta))$ a Mellin symbol $f_{1/2} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_0 \times \mathbf{R}^q_\eta))$ with $\mathrm{op}_t\, p(t,y,\tau,\eta) \equiv \mathrm{op}_M^{1/2} f_{1/2}(t,y,i\tau,t\eta) \bmod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{R}^q_\eta))$. This allows us to treat the case of arbitrary weights.

Theorem 3.19. *For every edge-degenerate* $p \in C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R} \times \mathbf{R}^q_\eta))$ *and every* $\gamma \in \mathbf{R}$ *there is an* $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma} \times \mathbf{R}^q_\eta))$ *such that*

$$\mathrm{op}_M^\gamma f_\gamma(t,y,1/2-\gamma+i\tau,t\eta) \equiv \mathrm{op}_t p(t,y,\tau,\eta) \tag{3.8}$$

modulo $C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{R}^q))$. *The corresponding statement holds for classical symbols, i.e., for* $\mathcal{B}^{\mu,d}$ *replaced by* $\mathcal{B}^{\mu,d}_{cl}$.

Proof. By definition, $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q)) = \mathcal{L}(C^\infty(\mathbf{R}_+), \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$ with the topology of bounded convergence. Let us start by showing that $\varphi M_{1/2}^{-1} f \in \mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$ and that the mapping is separately continuous in φ and f.

Let $\psi \in C^\infty(\mathbf{R}_+)$ and denote by $\langle\ ,\ \rangle$ the evident $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)$-valued pairing which extends the $L^2(\mathbf{R}_+, \frac{d\rho}{\rho})$ bilinear form. Then

$$\begin{aligned}
\langle \varphi M_{1/2,\tau\to\rho}^{-1} f, \psi\rangle &= \langle \varphi \int_{-\infty}^{\infty} \rho^{-i\tau} f(i\tau,\eta)d\tau, \psi\rangle \\
&= \langle \int_{-\infty}^{\infty} \rho^{-1-i\tau} f(i\tau,\eta)d\tau, \rho\varphi\psi\rangle \\
&= \langle (-\rho\partial_\rho)^N \int_{-\infty}^{\infty} \rho^{-1-i\tau}(1+i\tau)^{-N} f(i\tau,\eta)d\tau, \rho\varphi\psi\rangle \\
&= \int_0^\infty \int_{-\infty}^{\infty} \rho^{-i\tau}(1+i\tau)^{-N} f(i\tau,\eta)d\tau\, \rho^{-1}(\rho\partial_\rho)^N(\rho\varphi(\rho)\psi(\rho))\frac{d\rho}{\rho}. \qquad (3.12)
\end{aligned}$$

The last integral is an L^1-integral with values in $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)$, provided N is sufficiently large. This follows from the fact that, for every semi-norm q on $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)$, we have $q(f(i\tau,\eta)) = O(\langle\tau,\eta\rangle^\mu)$.

Moreover, if the semi-norms for ψ in $C^\infty(\mathbf{R}_+)$ tend to zero, then the last integral tends to zero in all semi-norms of $\mathcal{B}^{\mu,d}(X;\mathbf{R}^q)$. So it indeed defines an element of $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$.

Now let us show separate continuity. As ψ varies over a bounded set in $C^\infty(\mathbf{R}_+)$, the integral in (3.12) can be estimated in terms of finitely many semi-norms for $f \in \mathcal{B}^{\mu,d}(X;\Gamma_0\times\mathbf{R}^q)$ and finitely many semi-norms for $\varphi \in \mathcal{D}_K, K \subset \mathbf{R}_+$ compact. Finally note that the Mellin transform yields a continuous map from $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$ to $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$. Indeed, this follows from the relations

$$\begin{aligned}
\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q)) &= \mathcal{E}'(\mathbf{R}_+)\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\mathbf{R}^q) \quad \text{and} \\
\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q)) &= \mathcal{A}(\mathbf{C})\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\mathbf{R}^q),
\end{aligned}$$

together with the well-known fact that the Mellin transform maps $\mathcal{E}'(\mathbf{R}_+)$ to $\mathcal{A}(\mathbf{C})$ continuously. □

The next proposition settles Step (i) of the Outline 3.22.

Lemma 3.24. *Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{Y}$ be Fréchet spaces, and assume that $\mathcal{E}$ and $\mathcal{F}$ are embedded in a common vector space $\mathcal{X}$. Suppose $T : \mathcal{E}+\mathcal{F} \to \mathcal{Y}$ is a linear map, and the restrictions*

$$T : \mathcal{E} \to \mathcal{Y}, \quad T : \mathcal{F} \to \mathcal{Y}$$

are continuous in the topologies of $\mathcal{E}$ and $\mathcal{F}$. Then

$$T : \mathcal{E}+\mathcal{F} \to \mathcal{Y}$$

is continuous in the topology of the non-direct sum.

Proof. Let $\{p_1, p_2, ...\}$, $\{q_1, q_2, ...\}$, be increasing systems of semi-norms for $\mathcal{E}$ and $\mathcal{F}$ respectively. Denote the translation invariant metric in $\mathcal{Y}$ by d. Then a system of semi-norms for $\mathcal{E}+\mathcal{F}$ is given by $r_j(x) = \inf\{p_j(e) + q_j(f) : e + f = x\}$. So suppose $x_0 \in \mathcal{E}+\mathcal{F}$ and $V \subseteq \mathcal{Y}$ is an ε-ball about Tx_0. Then there is a $j \in \mathbf{N}$ and a $\delta > 0$ such that $d(Te, 0) < \frac{\varepsilon}{2}$ and $d(Tf, 0) < \frac{\varepsilon}{2}$, provided that $e \in \mathcal{E}$, $f \in \mathcal{F}$, $p_j(e) < \delta$ and $q_j(f) < \delta$. This implies that $Tx \in V$ for all x with $r_j(x - x_0) < \delta$: In this case we can find $e_1 \in \mathcal{E}$, $f_1 \in \mathcal{F}$ such that $e_1 + f_1 = x - x_0$ and $p_j(e_1) + q_j(f_1) < \delta$. Hence $d(Tx, Tx_0) = d(T(x - x_0), 0) \le d(T(e_1), 0) + d(T(f_1), 0) < \varepsilon$. □

In order to settle Step (ii) of the Outline 3.22, we first note that an operator in $\mathcal{B}^{-\infty,d}(X;\Gamma_0\times\mathbf{R}^q)$ can also be viewed as an element of $\mathcal{S}(\Gamma_0\times\mathbf{R}^q, C^\infty(X\times X))$, where $C^\infty(X\times X)$ is the Fréchet space of smooth kernel sections.

Lemma 3.25. *Let $h \in \mathcal{S}(\Gamma_0)$, $s \in \mathcal{S}(\Gamma_0, C^\infty(X\times X))$. Then* after integration by parts. Since $f \in \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q)$ we conclude that $(1-\psi(\rho))(M_{1/2}^{-1}f)(\rho,\eta) \in \mathcal{B}^{\mu-L,d}(X;\mathbf{R}^q)$ for arbitrary L, so it belongs to $\mathcal{B}^{-\infty,d}(X;\mathbf{R}^q)$. Next write for large L

$$\begin{aligned} & \ln^M \rho\,(\rho\partial_\rho)^N[(1-\psi(\rho))(M_{1/2}^{-1}f)(\rho,\eta)] \\ = \; & \frac{1}{2\pi}\int \ln^M \rho\,(\rho\partial_\rho)^N[\rho^{-i\tau}(1-\psi(\rho))\ln^{-L}\rho]\,(\partial_\tau^L f)(i\tau,\eta)d\tau. \end{aligned}$$

Denoting $\psi_j(\rho) := (\rho\partial_\rho)^j(1-\psi(\rho))$, we conclude from Leibniz' rule that the integral is a linear combination of terms of the form

$$\ln^{M-L-j_3}\rho\;\psi_{j_2}(\rho)\int_{-\infty}^{\infty}\rho^{-i\tau}\tau^{j_1}(\partial_\tau^L f)(i\tau,\eta)d\tau, \tag{3.11}$$

where $j_1+j_2+j_3 = N$. For a semi-norm p_j on $\mathcal{B}^{\mu-j,d}(X;\mathbf{R}^q)$ and fixed M,N, choose $L > M+N+j+2$. Then $M-L-j_3 < 0$; moreover $(1+\tau^2)\tau^{j_1}(\partial_\tau^L f)(i\tau,\eta) \in \mathcal{B}^{\mu-j,d}(X;\mathbf{R}_\tau \times \mathbf{R}^q_\eta)$, so that

$$p_j\left(\int_{-\infty}^{\infty}\rho^{-i\tau}(\tau^{j_1}\partial_\tau^L f(i\tau,\eta))\,d\tau\right) \le C$$

with a constant $C = C(L,j_1,j)$ independent of ρ. We conclude that the semi-norm in (3.10) can be estimated by finitely many expressions

$$\text{const.}\left\{\int_0^\infty \left|\ln^{M-L-j_3}\rho\;\psi_{j_2}(\rho)\right|^2\frac{d\rho}{\rho}\right\}^{1/2} < \infty.$$

Thus all the semi-norms in (3.10) are finite; they continuously depend on the semi-norms for f in $\mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q)$ and on those for ψ in $\mathcal{D}_K, K \subset \mathbf{R}_+$ compact. Here, $\mathcal{D}_K$ denotes those elements in $C_0^\infty(\mathbf{R}_+)$ that have support in K. This completes the proof. □

Theorem 3.21. *Let $\varphi \in C_0^\infty(\mathbf{R}_+)$ and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q))$. Then the operator-valued function f_φ defined by*

$$f_\varphi(t,y,z,\eta) = M_{\rho\to z}\varphi(\rho)M^{-1}_{1/2,\zeta\to\rho}f(t,y,\zeta,\eta)$$

is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \mathbf{R}^q; M_O^{\mu,d}(X;\mathbf{R}^q))$. Moreover, the mapping $(\varphi,f) \mapsto f_\varphi$ is separately continuous from $C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q))$ to $C^\infty(\overline{\mathbf{R}}_+ \times \Omega; M_O^{\mu,d}(X;\mathbf{R}^q))$.

3.22 Outline of the proof. Using a tensor product argument as in Corollary 3.9, it is no restriction to assume that f is independent of t and y, i.e., $f \in \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q)$. We shall first see quite easily in Lemma 3.23 that f_φ is an operator-valued function in $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$. It is more difficult to show that it also defines a family of parameter-dependent operators along each line Γ_β, uniformly for β in compact intervals, in the sense of Definition 3.1. To this end we proceed in the following steps, keeping the notation f, φ, f_φ fixed:

(i) For non-direct sums, it is sufficient to consider each summand separately.

(ii) Show the assertion for regularizing elements.

(iii) Reduce to the case $X = \mathbf{R}^n_+$.

(iv) We then only have to deal with operator-valued symbols of 5 types, with

(v) an additional consideration concerning the transmission property.

Lemma 3.23. *The function f_φ is an element of $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$, and the mapping*

$$C_0^\infty(\mathbf{R}_+) \times \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q) \to \mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X;\mathbf{R}^q))$$

given by $(\varphi,f) \mapsto f_\varphi$ is separately continuous.

$$= \|\int_{-\infty}^{\infty} \tilde\kappa_{[\xi,\sigma,\eta][v]^{-1}} \hat\psi(\tau-\sigma)\tilde\kappa_{[\xi,\sigma,\eta]^{-1}} D_\xi^{\beta_1} D_\sigma^{\beta_2} D_\eta^{\beta_3} p(\xi,\sigma,\eta)\kappa_{[\xi,\sigma,\eta]}\kappa_{[v][\xi,\sigma,\eta]^{-1}} d\sigma\|_{\mathcal{L}(E,F)}$$

$$\leq \int_{-\infty}^{\infty} \|\tilde\kappa_{[\xi,\sigma,\eta][v]^{-1}}\|_{\mathcal{L}(F)} |\hat\psi(\tau-\sigma)| \|\tilde\kappa_{[\xi,\sigma,\eta]^{-1}} D_\xi^{\beta_1} D_\sigma^{\beta_2} D_\eta^{\beta_3} p(\xi,\sigma,\eta)\kappa_{[\xi,\sigma,\eta]}\|_{\mathcal{L}(E,F)}$$

$$\cdot \|\kappa_{[v][\xi,\sigma,\eta]^{-1}}\|_{\mathcal{L}(E)} d\sigma.$$

Here we have used the fact that (scalar) multiplication by $\hat\psi(\tau-\sigma)$ commutes with the action of $\tilde\kappa$. According to (1.1) there are constants c and M such that

$$\|\kappa_{[\xi,\sigma,\eta]^{-1}[v]}\|_{\mathcal{L}(E)},\ \|\tilde\kappa_{[\xi,\sigma,\tau][v]^{-1}}\|_{\mathcal{L}(F)} \quad \leq \quad cL(\xi,\sigma,\eta,v)^M,$$

where $L(\xi,\sigma,\eta,v) = \max\{[\xi,\sigma,\eta]^{-1}[v], [\xi,\sigma,\eta][v]^{-1}\}$. Peetre's inequality implies that

$$[\xi,\sigma,\eta]^{-1}[v] \leq C[(\xi,\sigma,\eta)-(\xi,\tau,\eta)] = C[\sigma-\tau]$$

and, by symmetry, $[\xi,\sigma,\eta][v]^{-1} \leq C[\sigma-\tau]$ for a suitable constant C. Together with the facts that

$$\|\tilde\kappa_{[\xi,\sigma,\eta]^{-1}}(D_\xi^{\beta_1} D_\sigma^{\beta_2} D_\eta^{\beta_3} p)(\xi,\sigma,\eta)\kappa_{[\xi,\sigma,\eta]}\|_{\mathcal{L}(E,F)} = O([\xi,\sigma,\eta]^{\mu-|\beta|})$$

and that $\hat\psi$ is rapidly decreasing we conclude with Peetre's inequality that the final integral above is $O([v]^{\mu-|\beta|})$.

This shows our claim. Clearly, all estimates depend continuously on φ and p, thus they depend continuously on a, and the corresponding mapping is separately continuous. □

We now complete the proof of Theorem 3.21 with Step (v) of the outline, i.e., the observation that the transmission property is preserved under the construction. This is the contents of the following lemma.

Lemma 3.27. *Let* $p \in S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0 \times \mathbf{R}^q)$. *Then*

$$q = M(\varphi M^{-1}_{1/2} p) \in \mathcal{A}(\mathbf{C}, S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^{n+q})). \tag{3.14}$$

Moreover, for every $\beta \in \mathbf{R}$,

$$q|_{\Gamma_\beta} \in S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_\beta \times \mathbf{R}^q). \tag{3.15}$$

The corresponding estimates are satisfied uniformly for β *in compact intervals. The mapping* $(\varphi, p) \mapsto q$ *is separately continuous as a map from*

$$C_0^\infty(\mathbf{R}_+) \times S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0 \times \mathbf{R}^q)$$

to this Fréchet subspace of $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^n, \mathbf{R}^{n+q}))$.

Proof. If it were not for the subscript "*tr*", (3.14) and (3.15) would follow from the Lemma above, because the usual symbol classes correspond to the operator-valued symbols with $E = F = \mathbf{C}$ and trivial group action.

So we have to show that the transmission property is preserved under the operation $f \mapsto f_\varphi$. This, however, is simple: a symbol $a \in S^\mu(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0 \times \mathbf{R}^q)$ has the transmission property iff

$$\partial^k_{x_n} a(x', 0, \xi', \langle\xi'\rangle\xi_n, z, \eta) \in S^\mu(\mathbf{R}^{n-1}_{x'}, \mathbf{R}^{n-1}_{\xi'} \times \Gamma_{0,z} \times \mathbf{R}^q_\eta)\hat\otimes_\pi H_{\xi_n}.$$

In the present situation

$$\begin{aligned}\partial^k_{x_n} q(x',0,\xi',\langle\xi'\rangle\xi_n,z,\eta) &= M_{\rho\to z}\varphi(\rho)M^{-1}_{1/2,\zeta\to\rho}\partial^k_{x_n} p(x',0,\xi',\langle\xi'\rangle\xi_n,\zeta,\eta)\\ &\in \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1+q}))\hat\otimes_\pi H_{\xi_n}\end{aligned}$$

by a tensored version of the argument in 3.26. The last space coincides with $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1+q})\hat\otimes_\pi H_{\xi_n})$ and (3.14) is proven. For (3.15) we can argue in the same way: Restriction to Γ_β furnishes an element in $S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times \Gamma_\beta \times \mathbf{R}^q)\hat\otimes_\pi H_{\xi_n}$.

(a) $H := M(\varphi M_{1/2}^{-1}h) \in \mathcal{A}(\mathbf{C})$, *and* $H|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ *for every* β, *with estimates uniformly in* β *for* β *in compact intervals. The corresponding induced mapping* $(\varphi, h) \mapsto H$ *from* $C_0^\infty(\mathbf{R}_+) \times \mathcal{S}(\Gamma_0)$ *into this subspace of* $\mathcal{A}(\mathbf{C})$ *is separately continuous.*

(b) $F := M(\varphi M_{1/2}^{-1}s) \in \mathcal{A}(\mathbf{C}, C^\infty(X \times X))$, $F|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, C^\infty(X \times X))$ *for every* β, *with estimates uniformly in* β *for* β *in compact intervals. The mapping* $(\varphi, s) \mapsto F$ *is separately continuous from* $C_0^\infty(\mathbf{R}_+) \times \mathcal{S}(\Gamma_0, C^\infty(X \times X))$ *to this subspace of* $\mathcal{A}(\mathbf{C}, C^\infty(X \times X))$.

Proof. (a) By the Mellin inversion formula in 2.2, $M_{1/2}^{-1}h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-is}h(is)ds$. The integral converges; we can differentiate under the integral sign for the derivatives to see that it furnishes a smooth function.

Hence $\varphi M_{1/2}^{-1}h \in C_0^\infty(\mathbf{R}_+)$. It is easy to check that its Mellin transform therefore is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals. Clearly, the mapping $(\varphi, g) \mapsto \varphi g$ is separately continuous from $C_0^\infty(\mathbf{R}_+) \times C^\infty(\mathbf{R}_+)$ to $C_0^\infty(\mathbf{R}_+)$, and the Mellin transform is continuous from $C_0^\infty(\mathbf{R}_+)$ to the subspace of $\mathcal{A}(\mathbf{C})$ consisting of functions that restrict to $\mathcal{S}(\Gamma_\beta)$, uniformly for β in compact intervals, i.e., the space $M_O^{-\infty}$ for $\dim X = 0$. So the separate continuity follows.

(b) follows from (a), noting that $\mathcal{S}(\Gamma_\beta, C^\infty(X \times X)) = \mathcal{S}(\Gamma_\beta)\hat{\otimes}_\pi C^\infty(X \times X)$ and $\mathcal{A}(\mathbf{C}, C^\infty(X \times X)) = \mathcal{A}(\mathbf{C})\hat{\otimes}_\pi C^\infty(X \times X)$. For the continuity assertion we use the continuity of the Mellin transform from $C_0^\infty(\mathbf{R}_+, C^\infty(X \times X))$ to the corresponding subspace of $\mathcal{A}(\mathbf{C}, C^\infty(X \times X))$. □

Since the topology of $\mathcal{B}^{\mu,d}(X; \mathbf{R}^q)$ is precisely that of a non-direct sum of the regularizing terms and the local terms, Step (iii) of the Outline 3.22 is immediate. We next attack Step (iv). Notice that all entries of the symbol quintuple in 1.12 are operator-valued symbols.

Lemma 3.26. *Let* E, F *be Banach spaces with strongly continuous group actions* κ, $\tilde{\kappa}$. *Let* $\mu \in \mathbf{R}$, $m, k, q \in \mathbf{N}$, *and*

$$a = a(x, \xi, z, \eta) \in S^\mu(\mathbf{R}_x^m, \mathbf{R}_\xi^k \times \Gamma_{0,z} \times \mathbf{R}_\eta^q; E, F).$$

Then the function

$$A = A(z) = M_{\rho \to z}(\varphi(\rho) M_{1/2, z \to \rho}^{-1} a)$$

is analytic on $\mathbf{C}$ *with values in* $S^\mu(\mathbf{R}^m, \mathbf{R}^{k+q}; E, F)$. *Moreover, for all* $\beta \in \mathbf{R}$,

$$A|_{\Gamma_\beta} \in S^\mu(\mathbf{R}^m, \mathbf{R}^k \times \Gamma_\beta \times \mathbf{R}^q; E, F), \tag{3.13}$$

uniformly for β *in compact intervals. The mapping* $(\varphi, a) \mapsto A$ *from* $C_0^\infty(\mathbf{R}_+) \times S^\mu(\mathbf{R}^m, \mathbf{R}^k \times \Gamma_0 \times \mathbf{R}^q; E, F)$ *to this Fréchet subspace of* $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^m, \mathbf{R}^{k+q}; E, F))$ *is separately continuous.*

Proof. In view of the identities

$$\begin{aligned} S^\mu(\mathbf{R}^m, \mathbf{R}^k \times \Gamma_0 \times \mathbf{R}^q; E, F) &= C^\infty(\mathbf{R}^m)\hat{\otimes}_\pi S^\mu(\mathbf{R}^0, \mathbf{R}^k \times \Gamma_0 \times \mathbf{R}^q; E, F). \\ \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^m, \mathbf{R}^{k+q}; E, F)) &= C^\infty(\mathbf{R}^m)\hat{\otimes}_\pi \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^0, \mathbf{R}^{k+q}; E, F)) \end{aligned}$$

we may assume that $m = 0$, i.e., $a \in S^\mu(\mathbf{R}^k \times \Gamma_0 \times \mathbf{R}^q; E, F)$ is independent of x. From Lemma 3.23 we know that $A = M(\varphi M_{1/2}^{-1}a) \in \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{k+q}; E, F))$. This proves the first part of the statement.

Next consider $A|_{\Gamma_\beta}$. We may assume $\beta = 0$ due to the relation $(Mf)(z + \beta) = M_{t \to z}(t^\beta f)(z)$: Replacing $A|_{\Gamma_\beta}$ by $A|_{\Gamma_0}$ corresponds to replacing $\varphi(t)$ by $t^{-\beta}\varphi(t) \in C_0^\infty(\mathbf{R}_+)$. For the analysis of $A|_{\Gamma_0}$ it is more convenient to switch from the Mellin to the Fourier transform. We write the variable on Γ_0 in the form $z = i\tau$, $\tau \in \mathbf{R}$, and let $p(\tau, \eta) = a(i\tau, \eta)$. A simple computation gives

$$(M_{1/2}\varphi M_{1/2}^{-1}a)(i\tau, \eta) \quad = \quad (\mathcal{F}_{\tilde{\tau} \to r}\varphi(e^{-r})\mathcal{F}_{r \to \tau}^{-1}p)(\tau, \eta).$$

The symbol p is an element of $S^\mu(\mathbf{R}^{k+1+q}; E, F)$, and $r \mapsto \varphi(e^{-r}) = \psi(r)$ is a function in $C_0^\infty(\mathbf{R})$. So our task is reduced to showing that $q = \mathcal{F}\psi(r)\mathcal{F}^{-1}p \in S^\mu(\mathbf{R}^{k+1+q}; E, F)$. We abbreviate $v = (\xi, \tau, \eta)$ and consider a derivative $D_v^\beta q = D_\xi^{\beta_1} D_\tau^{\beta_2} D_\eta^{\beta_3} q$. We then estimate

$$\begin{aligned} &\|\tilde{\kappa}_{[v]^{-1}}\{D_v^\beta[\mathcal{F}_{r \to \tau}\psi(r)\mathcal{F}^{-1}p](v)\}\kappa_{[v]}\|_{\mathcal{L}(E,F)} \\ =\ &\|\tilde{\kappa}_{[v]^{-1}} D_v^\beta(\hat{\psi} * p)(v)\kappa_{[v]}\|_{\mathcal{L}(E,F)} \end{aligned}$$

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Finally the separate continuity of the mapping follows from the closed graph theorem and the continuity properties established in Lemma 3.26, since the topology of the space with the transmission property is finer than the original one. The closed graph theorem indeed can be applied: a mapping $A : C_0^\infty(\mathbf{R}_+) \to \mathcal{Y}$, $\mathcal{Y}$ a locally convex space, is continuous if and only if its restriction to the Fréchet spaces $\mathcal{D}_K$ are continuous. □

Remark 3.28. The Mellin quantization procedure in connection with the kernel cut-off allows us to associate to an edge degenerate boundary symbol a *holomorphic* Mellin symbol.

We have now seen that, starting from an arbitrary Mellin symbol, the operation $f \mapsto f_\psi$ furnishes a holomorphic Mellin symbol. Assuming additionally that $\psi(\rho) \equiv 1$ near $\rho = 1$, the symbols f and f_ψ will differ only by a regularizing symbol along the line Γ_0, where we started. An interesting question is the following: Suppose we initially have a holomorphic Mellin symbol h. What can we say about the difference $h - h_\psi$? Will it also be small along other lines? The theorem below shows that the difference is as good as we can expect it to be.

Theorem 3.29. *Given $h \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$ and $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ near $\rho = 1$, the difference $h - h_\psi$ is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{-\infty,d}(X;\mathbf{R}^q))$.*

Proof. Choose $\beta \in \mathbf{R}$ and a nonnegative integer $M > \mu + |\beta| + 1$. Then $D_z^M h(t,\cdot,\eta)$ is integrable along Γ_β. Moreover, the analyticity of the function $z \mapsto \rho^{-z} D_z^M h(t,z,\eta)$ together with Cauchy's theorem implies that

$$\int_{-\infty}^{\infty} \rho^{-i\tau}(D_z^M h)(t,i\tau,\eta)d\tau = \int_{-\infty}^{\infty} \rho^{-(\beta+i\tau)}(D_z^M h)(t,\beta+i\tau,\eta)d\tau\,,$$

so that $M_{1/2}^{-1}(D_z^M h)(t,\rho,\eta) = \rho^{-\beta} M_{1/2,\zeta\to\rho}^{-1}(D_z^M h(t,\zeta+\beta,\eta))$. Hence, for $z = \beta + i\tau$,

$$\begin{aligned}
&(h - h_\psi)(t,z,\eta)\\
=\;& \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))(M_{1/2}^{-1}h)\,(t,\rho,\eta)\; d\rho\\
=\;& \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))\ln^{-M}\rho\;(M_{1/2}^{-1}(D_z^M h))(t,\rho,\eta)\; d\rho\\
=\;& \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))\ln^{-M}\rho\;\rho^{-\beta}M_{1/2,\zeta\to\rho}^{-1}(D_z^M h)(t,\zeta+\beta,\eta)\; d\rho\\
=\;& \int_0^\infty \rho^{i\tau-1}(1-\psi(\rho))M_{1/2,\zeta\to\rho}^{-1}h(t,\zeta+\beta,\eta)\; d\rho\\
=\;& \left[M_{1/2,\rho\to z}(1-\psi(\rho))M_{1/2,\zeta\to\rho}^{-1}h(t,\zeta+\beta,\eta)\right](z-\beta).
\end{aligned}$$

On the other hand, the function $(t,z,\eta) \mapsto h(t,z+\beta,\eta)$ is an element of $C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X;\mathbf{R}^q))$; the corresponding symbol estimates hold uniformly for β in compact intervals. Applying Theorem 3.20, $h - h_\psi|_{\Gamma_\beta} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_\beta \times \mathbf{R}^q))$, uniformly for β in compact intervals. □

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