SIMPLICIAL DETERMINANT MAP AND THE SECOND TERM OF WEIGHT FILTRATION

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Introduction

The notion of determinant occurs twice in the algebraic K-theory of a scheme X: firstly, we have the map $\det : K_0X \to \mathrm{Pic}X$ which takes a vector bundle on X to its highest exterior power (we assume X is irreducible); secondly, we have the map $\det : K_1X \to \Gamma(X, O_X^*)$ which is induced by the usual determinant map $GL(R) \to R^*$ in the affine case $X = \mathrm{Spec}R$.

Let W^1 be the union of components of rank zero in the G-construction of Gillet and Grayson [GG] associated with the category $\mathcal{P}_X$ of vector bundles on X. We can regard W^1 as the first term of the weight filtration, since

$$\pi_0 W^1 \cong F_\gamma^1 K_0 X = \ker(\mathrm{rank} : K_0X \to \mathbf{Z})$$

and

$$\pi_m W^1 \cong F_\gamma^1 K_m X \cong K_m X \quad \text{for } m \geq 1.$$

In the present paper we define a simplicial set T such that

$$\pi_0 T \cong \mathrm{Pic}X, \pi_1 T \cong \Gamma(X, O_X^*), \text{ and } \pi_m T = 0 \text{ for } m \geq 2$$

and a simplicial map

$$\det : W^1 \to T$$

which yields the above two determinant maps on the homotopy groups:

$$F_\gamma^1 K_0 X = \ker(\mathrm{rank} : K_0X \to \mathbf{Z}) \cong \pi_0 W^1 \overset{\det}{\to} \pi_0 T \cong \mathrm{Pic}X$$

and

$$K_1 X \cong \pi_1 W^1 \overset{\det}{\to} \pi_1 T \cong \Gamma(X, O_X^*).$$

We also describe the homotopy fiber of the map $\det : W^1 \to T$ as a simplicial set W^2. A vertex in W^2 is a triple $(P, P'; \psi)$, where P and P' are vector bundles on X such that $\mathrm{rank}\, P = \mathrm{rank}\, P'$ and $\psi : \det P \tilde{\to} \det P'$ is an isomorphism. An edge in W^2 connecting $(P_0, P_0'; \psi_0)$ to $(P_1, P_1'; \psi_1)$ is a pair of short exact sequences $(P_0 \to P_1 \to P_{1/0}; P_0' \to P_1' \to P_{1/0})$ such that the diagram

$$\begin{array}{ccccc} \det P_1 & \tilde{\to} & \det P_0 & \otimes & \det P_{1/0} \\ \psi_1 \wr \downarrow & & & \wr \downarrow & \psi_0 \otimes 1 \\ \det P_1' & \tilde{\to} & \det P_0' & \otimes & \det P_{1/0} \end{array}$$

commutes, where the horizontal isomorphisms are naturally induced by these short exact sequences. Higher dimensional simplices in W^2 are defined in a similar way.

The long exact sequence associated with $W^2 \to W^1 \to T$ yields

$$\pi_0 W^2 \cong \ker((\mathrm{rank}, \det) : K_0 X \to \mathbf{Z} \oplus \mathrm{Pic}\, X)$$
$$\pi_1 W^2 \cong \ker(\det : K_1 X \to \Gamma(X, O_X^*))$$
$$\pi_m W^2 \cong K_m X \text{ for } m \geq 2.$$

Thus W^2 provides the groups $SK_m X$ as homotopy groups for all $m \geq 0$. The SK-groups can be defined for $m \geq 1$ as the homotopy groups of $BSL^+(R)$ in the affine case

$X = \mathrm{Spec}R$ and by means of the generalized cohomology of the sheafification of BSL^{+} in the general case (cf. [Sou] p.524).

In a future paper we hope to define λ-and γ- operations on W^2 as simplicial maps and prove on the simplicial level that the map $\gamma^1 + \gamma^2 + \cdots$ is contractible. This would imply directly that $F^2_\gamma K_m X = SK_m X$ for each $m \geq 0$.

1. Definitions

Let X be an irreducible scheme.

We denote by $\mathcal{P} = \mathcal{P}_X$ the category of vector bundles on X. Suppose we are given a choice of the tensor product $P_1 \otimes \cdots \otimes P_k$ for each collection of objects $(P_1, \ldots, P_k)$ in $\mathcal{P}$ and a choice of the exterior product $P_1 \wedge \ldots \wedge P_k$ for each admissible filtration $P_1 \rightarrowtail \cdots \rightarrowtail P_k$ (by definition, the latter is isomorphic to the image of $P_1 \otimes P_2 \otimes \cdots \otimes P_k$ in $\bigwedge^K P_k$). These operations satisfy the usual functoriality and compatibility conditions (cf. [Gr, sect 7]).

Let $\mathcal{L} = \mathcal{L}_X$ be the category of linear bundles on X and their isomorphisms. We set $\det P = \bigwedge^{\text{rank P}} P$ *for every P in $\mathcal{P}$, where $\bigwedge^k P$ now stands precisely for the exterior product $P_{\wedge \cdots \wedge} P$ associated with $P \overset{1}{\rightarrowtail} \cdots \overset{1}{\rightarrowtail} P$ (k copies). Thus we obtain the map*

$$\det : Ob\mathcal{P} \to Ob\mathcal{L}.$$

Let $I = O_X$ be the identity linear bundle. We assume that $\det O = I$ *for any zero object O in $\mathcal{P}$. We also assume that an object* $L^{-1} \cong \mathrm{Hom}(L, I)$ *is chosen for each L in $\mathcal{L}$.*

Proposition 1.1. (i) Any exact sequence

$$(1.1) \qquad 0 \to P_0 \to P_1 \to P_{1/0} \to O$$

in $\mathcal{P}$ gives rise to an isomorphism

$$\delta = \delta_{0,1} : \det P_1 \widetilde{\to} \det P_0 \otimes \det P_{1/0}$$

in a natural way;

(ii) Given a commutative diagram of the form

$$(1.2) \qquad \begin{array}{ccccc} & & & & P_{2/1} \\ & & & & \uparrow \\ & & P_{1/0} & \to & P_{2/0} \\ & & \uparrow & & \uparrow \\ P_0 & \to & P_1 & \to & P_2 \end{array}$$

such that the sequences $0 \to P_i \to P_j \to P_{j/i} \to 0$, with $0 \leq i < j \leq 2$, and $0 \to P_{1/0} \to P_{2/0} \to P_{2/1} \to 0$ are exact, the diagram

$$(1.3) \qquad \begin{array}{ccc} \det P_2 & \overset{\delta_{0,2}}{\longrightarrow} & \det P_0 \otimes \det P_{2/0} \\ \delta_{1,2} \downarrow & & \downarrow 1 \otimes \delta_{1/0,2/0} \\ \det P_1 \otimes \det P_{2/1} & \overset{\delta_{0,1} \otimes 1}{\longrightarrow} & \det P_0 \otimes P_{1/0} \otimes \det P_{2/1} \end{array}$$

commutes.

Proof: for any $m > 0$, we have the Grothendieck filtration

$$P_0 \wedge \cdots \wedge P_0 \rightarrowtail P_0 \wedge \ldots \wedge P_0 \wedge P_1 \rightarrowtail \cdots \rightarrowtail P_0 \wedge P_1 \wedge \cdots \wedge P_1 \rightarrowtail P_1 \wedge \cdots \wedge P_1$$

associated with the left arrow in (1.1) in which all products contain m factors. The successive quotients are the products

$$\underbrace{P_0 \wedge \cdots \wedge P_0}_{r} \otimes \underbrace{P_{1/0} \wedge \cdots \wedge P_{1/0}}_{s}$$

with $r + s = m$, the quotient maps being induced by the right arrow in (1.1). In particular, if $m = \text{rank } P_1$, the only nonvanishing quotient corresponds to the pair $r = \text{rank } P_0, s = \text{rank } P_{1/0}$, and we obtain the isomorphisms

$$\begin{array}{ccccc} \underbrace{P_0 \wedge \cdots \wedge P_0}_{r} & \wedge & \underbrace{P_1 \wedge \cdots \wedge P_1}_{s} & \xrightarrow{\sim} & \underbrace{P_1 \wedge \cdots \wedge P_1}_{m} \\ & \downarrow \wr & & \swarrow_{\delta} & \\ \underbrace{P_0 \wedge \cdots \wedge P_0}_{r} & \otimes & \underbrace{P_{1/0} \wedge \cdots \wedge P_{1/0}}_{s} & & \end{array}$$

The desired isomorphism $\delta_{0,1} : \det P_1 \xrightarrow{\sim} \det P_0 \otimes \det P_{1/0}$ now can be defined from the above diagram.

Let rank $P_i = r_i$ and rank $P_{j/i} = r_{j/i}$ in (1.2). We have the natural commutative diagrams

$$\begin{array}{ccccccccc} \det P_2 & \xleftarrow{\sim} & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} \wedge \underbrace{P_2 \wedge \cdots \wedge P_2}_{r_{2/0}} & \xleftarrow{\sim} & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \wedge & \underbrace{P_1 \wedge \cdots \wedge P_1}_{r_{1/0}} & \wedge & \underbrace{P_2 \wedge \cdots \wedge P_2}_{r_{2/1}} \\ \searrow^{\delta_{0,2}} & & \downarrow & & & & \downarrow & & \\ & & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} \otimes \underbrace{P_{2/0} \wedge \cdots \wedge P_{2/0}}_{r_{2/0}} & \xleftarrow{\sim} & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \otimes & \underbrace{P_{1/0} \wedge \cdots \wedge P_{1/0}}_{r_{1/0}} & \wedge & \underbrace{P_{2/0} \wedge \cdots \wedge P_{2/0}}_{r_{2/1}} \\ & & \searrow^{1 \otimes \delta_{1/0,2/0}} & & & & \downarrow & & \\ & & & & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \otimes & \underbrace{P_{1/0} \wedge \cdots \wedge P_{1/0}}_{r_{1/0}} & \otimes & \underbrace{P_{2/1} \wedge \cdots \wedge P_{2/1}}_{r_{2/1}} \end{array}$$

and

$$\begin{array}{ccccccccc} \det P_2 & \xleftarrow{\sim} & \underbrace{P_1 \wedge \cdots \wedge P_1}_{r_1} \wedge \underbrace{P_2 \wedge \cdots \wedge P_2}_{r_{2/1}} & \xleftarrow{\sim} & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \wedge & \underbrace{P_1 \wedge \cdots \wedge P_1}_{r_{1/0}} & \wedge & \underbrace{P_2 \wedge \cdots \wedge P_2}_{r_{2/1}} \\ \searrow^{\delta_{1,2}} & & \downarrow & & & & \downarrow & & \\ & & \underbrace{P_1 \wedge \cdots \wedge P_1}_{r_1} \otimes \underbrace{P_{2/1} \wedge \cdots \wedge P_{2/1}}_{r_{2/1}} & \xleftarrow{\sim} & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \wedge & \underbrace{P_1 \wedge \cdots \wedge P_1}_{r_{1/0}} & \otimes & \underbrace{P_{2/1} \wedge \cdots \wedge P_{2/1}}_{r_{2/1}} \\ & & \searrow^{\delta_{0,1} \otimes 1} & & & & \downarrow f & & \\ & & & & \underbrace{P_0 \wedge \cdots \wedge P_0}_{r_0} & \otimes & \underbrace{P_{1/0} \wedge \cdots \wedge P_{1/0}}_{r_{1/0}} & \otimes & \underbrace{P_{2/1} \wedge \cdots \wedge P_{2/1}}_{r_{2/1}} \end{array}$$

in which all the arrows are obviously isomorphisms. The desired commutativity in (1.3) is now equivalent to the commutativity of the diagram

$$\begin{array}{ccc} P_0 \wedge \cdots \wedge P_0 \wedge P_1 \wedge \cdots \wedge P_1 \wedge P_2 \wedge \cdots \wedge P_2 & \to & P_0 \wedge \cdots \wedge P_0 \otimes P_{1/0} \wedge \cdots \wedge P_{1/0} \wedge P_{2/0} \wedge \cdots \wedge P_{2/0} \\ \downarrow & & \downarrow \\ P_0 \wedge \cdots \wedge P_0 \wedge P_1 \wedge \cdots \wedge P_1 \otimes P_{2/1} \wedge \cdots \wedge P_{2/1} & \to & P_0 \wedge \cdots \wedge P_0 \otimes P_{1/0} \wedge \cdots \wedge P_{1/0} \otimes P_{2/1} \wedge \cdots \wedge P_{2/1} \end{array}$$

The latter is evident (cf. (E2) in [Gr, sect 7])

Definition Let A be a partially ordered set. We let ArA denote the set $\{j/i|i,j \in A, i \leq j\}$. By multiplicative map on ArA with values in $\mathcal{L}$ we mean a map $D : ArA \to \mathcal{L}$ endowed with a collection of isomorphisms

$$\delta_{i,j,k} : D(k/i) \tilde{\to} D(j/i) \otimes D(k/j) \text{ for every } i \leq j \leq k \text{ in} A$$

such that

$$(1.4) \qquad \begin{array}{c} (i) D(i,i) = I \text{ for every } i \in A; \\ (ii) \text{ for every } i \leq j \text{ in } A, \delta_{i,i,j} \text{ and } \delta_{i,j,j} \end{array}$$

are the natural isomorphisms $D(j/i) \tilde{\to} I \otimes D(j/i)$ and $D(j/i) \tilde{\to} D(j/i) \otimes I$, respectively;

$$\begin{array}{ccc} & (iii) \text{ for every } i \leq j \leq l \text{ in } A, \text{ the diagram} & \\ D(\ell/i) & \overset{\delta_{i,j,\ell}}{\to} & D(j/i) \otimes D(l/j) \\ \delta_{i,k,\ell} \downarrow & & \downarrow 1 \otimes \delta_{j,k,\ell} \\ D(k/i) \otimes D(\ell/k) & \overset{\delta_{i,j,k} \otimes 1}{\to} & D(j/i) \otimes D(k/j) \otimes D(\ell/k) \end{array}$$

commutes. We let $Mult(ArA, L) = \{(D; \delta_{i,j,k})\}$ denote the set of all $\mathcal{L}$-valued multiplicative maps on ArA.

Definition. We define the simplical set $Z = Z.\ \mathcal{L}$ by

$$Z(A) = \text{Mult } (Ar, A, \mathcal{L}),\ A \in \Delta$$

where Δ denotes as usually the category of finite nenempty totally ordered sets and non-decreasing maps.

By definition, there is a unique O-simplex $*$ in Z. A 1–simplex in Z is an object $L = D(1/2)$ of L. A 2–simplex in Z is a tuple $(L_1, L_2, L_{2/1}; \delta)$, where int the above notation $L_1 = D(1/0), L_2 = D(2/0)$, and $L_{2/1} = D(2/1)$ are objects of $\mathcal{L}$ and $\delta = \delta_{0,1,2}$: $L_2 \tilde{\to} L_1 \otimes L_{2/1}$ is an isomorphism. Thus, Z looks in a sense like the classifying space of the Picard group, and it is easy to see that $\pi_1 Z \cong \text{Pic } X$. However, Z is not homotopy equivalent to $B\text{Pic } X$. In fact, we have $\pi_2 Z \cong \text{Aut } I \cong \Gamma(X, O_X^*)$ and $\pi_m Z \cong 0$ for $m \geq 3$ (cf. Proprosition 2.1 and Theorem 3.1).

Given a partially ordered set A, we regard the set ArA as a category in which Mor $(j/i, j'/i')$ consists of a unique mormpism if $i \leq i'$ and $j \leq j'$, otherwise it is empty. Say that a functor $F : ArA \to \mathcal{P}$ is exact if

(i) $F(i/i) = O$ for every $i \in A$, where O denotes a distinguished zero object in $\mathcal{P}$;
(ii) the sequence $O \to F(j/i) \to F(k/i) \to O$ is exact for every $i \leq j \leq k$ in A.

Recall that the S-construction of Waldhausen associated with the category $\mathcal{P}$ is the simplicial set $S = S\mathcal{P}$ given by

$$S(A) = \text{Exact } (ArA, \mathcal{P}),\ A \in \Delta,$$

where $Exact$ refers the set of exact functors.

Proposition 1.1 obviously implies the following

Proposition 1.2. *Let A be a partially ordered set and $F : ArA \to \mathcal{P}$ be an exact functor. Consider the map $D = \det \cdot F : ArA \to \mathcal{L}$.*

(i) *For every $i \leq j \leq k$ in A, the exact sequence $0 \to F(j/i) \to F(k/i) \to F(k/j) \to 0$ gives rise to an isomorphism $\delta_{i,j,k} : D(k/i) \tilde{\to} D(j/i) \otimes D(k/j)$ in a natural way;*

(ii) *For every $i \leq k \leq k \leq \ell$ in A, the diagram (1.4) (iii) commutes, i.e., $(D; \delta_{i,j,k})$ is a multiplicative map ((1.4) (i) and (ii) obviously hold);*

(iii) *This gives rise to a simplicial map*

$$\det : S.\mathcal{P} \to Z.\mathcal{L} \tag{1.5}$$

2. Applying the loop space functor

Let $F : X \to Y$ be a mimplicial map, $A \in \Delta$, and $y_0 \in Y(A)$. Following [GG, sect. 1], we define the right fiber over y_0 to be the simplicial set $y_0|F$ given by

$$(y_0|F)(B) = \overset{\lim}{\leftarrow} \begin{pmatrix} & & X(B) \\ & & \downarrow \\ & \downarrow Y(AB) & \to & Y(B) \\ \{y_0\} \hookrightarrow Y(A) & & \end{pmatrix}, B \in \Delta,$$

where AB denotes the concatenation of A and B, i.e., the disjoint union $A \coprod B$ ordered so $A < B$. By definition, a B-simplex in $y_0|F$ is a pair (y, x), where y is an AB-simplex in Y and x is a B-simplex in X such that the A-face of y is equal to y_0 and the B-face of y is equal to $F(x)$.

If $F = 1 : Y \to Y$, we write $y|F$. Ir is easy to see that $y|Y$ is contractible for any Y and $y \in Y(A)$ (cf. [GG, Lemma 1.4]). A B-simplex in $y|Y$ is an AB-simplex in Y whose A-face coincides with y.

Suppose Y has a distinguished vertex *, i.e., $* \in Y(\{b\})$, where $\{b\} \in \Delta$ is a one-element set. Let $Pr : *|Y \to Y$ denote the natural projection. We define the (simplicial) loop space of Y at $*$ to be the simplicial set

$$\Omega Y = *|Pr$$

or, equivalently, ΩY can be defined from the cartesian square $\begin{matrix} \Omega Y & \to & *|Y \\ | & & \downarrow \\ *|Y & \to & Y \end{matrix}$ (cf. [GG, sect. 2]). By definition, a B-simplex in ΩY is a pair of $\{b\}B$-simplices in Y whose B-faces coincide and whose $\{b\}$-vertices are equal to $*$.

Recall that the G-construction of Gillet and Grayson associated with $\mathcal{P}$ is the simplicial set $G = G.\ \mathcal{P} = \Omega S.\mathcal{P}$. By [GG, Theorem 3.1]; there is a homotopy equivalence $|G| \tilde{\to} \Omega|S|$. It follows that $\pi_m G \cong K_m X$ for $m \geq 0$.

For $A \in \Delta$, let $\gamma(A)$ denote the disjoint union $\{L, R\} \coprod A$ ordered so that the symbols L and R are comparable, $L < a$ and $R < a$ for any $a \in A$, and A is an ordered subset in $\gamma(A)$. Let $\Gamma(A) = Ar\gamma(A)$. It is easy to see that the G-constructions can be described as follows:

$$G(A)\text{Exact } (\Gamma(A), \mathcal{P}), A \in \Delta. \tag{2.1}$$

Definition. We define the simolicial set $T = T.\mathcal{L}$ by $T.\mathcal{L} = \Omega Z.\mathcal{L}$. Similarly to (2.1), we can write

(2.2) $$T(A)\text{Mult }(\Gamma(A), \mathcal{L}),\ A \in \Delta.$$

Thus, a p-simplex in T is a collection of objects

$$\begin{bmatrix} & & & L_{p/p-1} \\ & & \cdots & \\ & L_{1/0} & \cdots & L_{p/0} \\ L_0 & L_1 & \cdots & L_p \\ L'_0 & L'_1 & \cdots & L'_p \end{bmatrix}$$

in $\mathcal{L}$ endowed with isomorphisms

$$\delta_{L,i,j} : L_j \tilde{\rightarrow} L_i \otimes L_{j/i} \text{ and } \delta_{R,i,j} L'_j \tilde{\rightarrow} L'_i \otimes L_{j/i}$$

for every $0 \leq i \leq j \leq p$ and

$$\delta_{i,j,k} : L_{k/i} \tilde{\rightarrow} L_{j/i} \otimes L_{k/j}$$

for every $0 \leq i \leq j \leq k \leq p$ satisfying (1.4) (iii) (here we write for short L_i, L'_i, and $L_{j/i}$ for $D(i/L), D(i/R)$, and $D(j/i)$, respectively). In particular, a vertix in T is a pair of objects $\begin{bmatrix} L \\ L' \end{bmatrix}$ in $\mathcal{L}$. And edge connecting $\begin{bmatrix} L_1 \\ L'_1 \end{bmatrix}$ to $\begin{bmatrix} L \\ L' \end{bmatrix}$ is a triple $(L_{1/0}; \delta, \delta')$, where $L_{1/0}$ is an object of $\mathcal{L}$ and $\delta : L_1 \tilde{\rightarrow} L_0 \otimes L_{1/0}, \delta' : L'_1 \tilde{\rightarrow} L'_0 \otimes L_{1/0}$ are isomorphisms.

Proposition 2.1. $|T| \sim \Omega|Z|$.

Proof: By [GG, Lemma 2.1], it suffices to show that the map $*|Z \to Z$ is fibred (see sect. 4 for the definition of a fibred map). In fact, any simplicial map $X \to Z$ is fibred, since Z satisfies the condition (4.1) of Proposition 4.2. The verification of (4.1) for Z is similar to the proof of Proposition 4.3 and we omit it, because we will not use the homotopy equivalence $|T| \sim \Omega|Z|$ in the sequel. qed

Applying the loop space functor to the map (1.5), we obtain a simplicial map $G.\mathcal{P} \to T.\mathcal{L}$ which we will also denote by $\det$. We set $W^0 = G$, and let W^1 be the union of components in G of rank zero, i.e., the components whose vertices $\begin{bmatrix} P \\ Q \end{bmatrix}$ satisfy rank P = rank Q. The restriction of the above map to W^1 yielsd the simplical map

$$\det : W^1 \to T$$

which plays the central role in the paper. By definition, this map takes a simplex $F \in W^1(A) \subset \text{Exact }(\Gamma(A), \mathcal{P})$ to mulitplicative map $D = \det.F : \Gamma(A) \to \mathcal{L}$ (cf. (2.2)) such that for every $i \leq j \leq k$ in $\gamma(A)$ the structural isomorphism $\delta_{i,j,k} : D(k/i) \tilde{\rightarrow} D(j/i) \otimes D(k/j)$ is the isomorphism associated with the exact sequence $0 \to F(j/i) \to F(k/i) \to F(k/j) \to 0$ as in Proposition 1.2.

3. The homotopy groups of the simplicial set T

We make T into a H-space using tensor products in $\mathcal{L}$; i.e., for $D, D' \in T(A)$, we define $D \otimes D' \in T(A)$ by

$$(D \otimes D')(j/i) = D(j/i) \otimes D'(j/i) \text{ for } i \leq j \text{ in } \gamma(\mathrm{A})$$

and let the isomorphism

$$\delta_{i,j,k} : (D \otimes D')(k/i) \xrightarrow{\sim} (D \otimes D')(j/i) \otimes (D \otimes D')(k/j)$$

be the product map

$$D(k/i) \otimes D'(k/i) \xrightarrow{\sim} (D(j/i) \otimes D(k/j)) \otimes (D'(j/i) \otimes D'(k/j)) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} (D(j/i) \otimes D'(j/i)) \otimes (D(k/j) \otimes D'(k/j))$$

where the second arrow denotes the natural permutation map. The verification of (1.4) is trivial (we assume strictly $I \otimes I = I$). This H-space structure on T makes $\pi_0 T$ into a monoid. The vertex $\begin{bmatrix} I \\ I \end{bmatrix}$ is strict identity in T, and therefore its component is the identity element of $\pi_0 T$

Theorem 3.1.

(i) $$\pi_0 T \cong {}_{\mathrm{Pic}}X;$$

(ii) $$\pi_1 T \cong \Gamma(X, O_X^*) \text{ and } \pi_m T \cong 0 \text{ for } m \geq 2.$$

Proof: (i) For any two vertices $\begin{bmatrix} L \\ L' \end{bmatrix}$ and $\begin{bmatrix} M \\ M' \end{bmatrix}$ in T, there exists an edge connecting these vertices if and only if $L \otimes (L')^{-1} \cong M \otimes (M')^{-1}$. Thus the assignment $\begin{bmatrix} L \\ L' \end{bmatrix} \mapsto \{L \otimes (L')^{-1}\}$ gives rise to a bijective map $\pi_0 T \to \mathrm{Pic} X$, and the operation on Pic X obviously agrees with the operation on $\pi_0 T$ induced by the H -space structure.

(ii) It follows from (i) that all the components of T are homotopy equivalent. Nevertheless, we will construct a universal covering for an arbitrary component of T, which will enable us to compute its homotopy groups.

For $\{L\} \in$ Pic X, let T_L denote the component of the vertex $\begin{bmatrix} L \\ I \end{bmatrix}$ in T. We define the simplicial set $\widetilde{T}_L$ as follows. An A-simplex x in $\widetilde{T}_L$ is a tuple $x = (D; E_i, i \in A)$, where $D \in T_L(A) \subset$ Mult $(\Gamma(A), \mathcal{L})$ and $\mathcal{E}_i D(i/L) \xrightarrow{\sim} L \otimes D(i/R)$, $i \in A$, are isomorphisms such that the diagram

(3.1)
$$\begin{array}{ccc} D(j/L) & \overset{\delta_{L,i,j}}{\longrightarrow} & D(i/L) \otimes D(j/i) \\ \mathcal{E}_j \downarrow & & \downarrow \mathcal{E}_i \otimes 1 \\ L \otimes D(j/R) & \overset{1 \otimes \delta_{R,i,j}}{\longrightarrow} & L \otimes D(i/R) \otimes D(j/i) \end{array}$$

commutes for every $i < j$ and A. Thus, a vertex in $\widetilde{T}_L$ is a pair of objects $\begin{bmatrix} L_0 \\ L'_0 \end{bmatrix}$ in $\mathcal{L}$ endowed with an isomorphism $\mathcal{E}_o : L_0 \xrightarrow{\sim} L \otimes L'_0$. We have an obvious simplicial map $\widetilde{T}_L \to T_L$ which forgets the choice of $\mathcal{E}_i$.

Lemma 3.2. *Let* $D \in T_L(A)$ *and* $k \in A$. *Then for any isomorphism* $\mathcal{E}_k : D(k/L) \tilde{\rightarrow} L \otimes D(k/R)$, *there exist uniquely determined isomorphisms* $\mathcal{E}_i : D(i/L) \tilde{\rightarrow} L \otimes D(i/R)$, *with* $i \in A, i \neq k$, *such that* $x = (D; \mathcal{E}_i, i \in A) \in \widetilde{T}_L(A)$.

Proof: The uniqueness of $\mathcal{E}_j$ for $j > k$ follows directly from diagram (3.1). For $i < k$ it follows from (3.1) that the isomorphism $\mathcal{E}_i \otimes 1 : D(i/L) \otimes D(k/i) \tilde{\rightarrow} L \otimes D(i/R) \otimes D(k/i)$ is uniquely determined. But for any linear bundles L, L', and L'', with $\{L'\} = \{L''\}$ in Pic X, the map $Iso(L'/L'') \to Iso(L' \otimes L, L'' \otimes L)$ given by $\mathcal{E} \mapsto \mathcal{E} \otimes 1$ is a bijection, since $\mathcal{E}$ can be restored from the diagram

$$\begin{array}{ccc} L' & \xrightarrow{\mathcal{E}} & L'' \\ \uparrow \wr & & \uparrow \wr \\ L' \otimes L \otimes L^{-1} & \xrightarrow{\mathcal{E}\otimes 1 \otimes 1} & L'' \otimes L \otimes L^{-1} \end{array}$$

in which the vertical arrows are induced by the natural map $L \otimes L^{-1} \to I$ (in particular, Aut$L$ is naturally isomorphic to Aut$I \cong \Gamma(X, O_X^*)$ for any $L \in \mathcal{L}$). Hence the isomorphisms $\mathcal{E}_i$, with $i < k$, are also uniquely determined. The commutativity of (3.1) for an arbitrary pair $i < j$ can be deduced from the commutativity for (i, k) and for j, k and the properties (1.4) of the isomorphisms δ. ☺

Given a simplex $x = (D; \mathcal{E}_i) \in \widetilde{T}_L(A)$ and an element $\mathcal{E} \in$ Aut L, we define $\mathcal{E}(x)$ to be the simplex

$$\mathcal{E}(x) = \left(D; \mathcal{E} \otimes 1_{D(i/R)}\right) \cdot \mathcal{E}_i, i \in A \in \widetilde{T}_L(A).$$

This is really a simplex in $\widetilde{T}_L$, because the diagram

$$\begin{array}{ccccc} D(j/L) & \xrightarrow{\delta_{L,i,j}} & D(i/L) \otimes D(j/i) & \xrightarrow{\mathcal{E}_i \otimes 1} & L \otimes D(i/R) \otimes D(j/i) \\ \mathcal{E}_j \downarrow & & & & \downarrow \mathcal{E} \otimes 1 \otimes 1 \\ L \otimes D(j/R) & \xrightarrow{\mathcal{E}\otimes 1} & L \otimes D(j/R) & \xrightarrow{1 \otimes \delta_{R,i,j}} & L \otimes D(i/R) \otimes D(j/i) \end{array}$$

obviously commutes for every $i < j$ in A. Thus we obtain a free left action of the group Aut L on the simplicial set $\widetilde{T}_L$, and it follows from Lemma 3.2 that the forgetful map $\widetilde{T}_L \to T_L$ is the quotient map associated with this action. Hence $\left|\widetilde{T}_L\right| \to |T_L|$ is a covering, and to complete the proof of theorem 3.1, it now remains to show the following

Proposition 3.3. $\widetilde{T}_L$ *is contractible.*

Proof: We will define simplicial maps $f : *|Z \to \widetilde{T}_L$ and $g : \widetilde{T}_L \to *|Z$ such that $g \cdot f = 1$, and $f \cdot g$ admits a simplicial homotopy to the identity map of $\widetilde{T}_L$. This will be enough, since $*|Z$ is contractible (cf. [GG, Lemma 1.4]).

We can describe the simplicial set $*|Z$ as follows. For $A \in \Delta$, let $\sigma(A)$ denote the concatenation $\{b\}A$, where b is a symnol ("base element"), and let $\sum(A) = Ar\sigma(A)$. Then we can identify $(*|Z)(A)$ with the set Mult $(\sum(A), \mathcal{L})$. Given $D \in$ Mult $(\sum(A), \mathcal{L})$, we define $f(D) : \Gamma(A) \to \mathcal{L}$ by

$$\begin{array}{lll} f(D)(j/i) = D(j/i) & \text{for } i < j & \text{in } A; \\ f(D)(j/L) = L \otimes D(j/b) & \text{for } j \in A; & \\ f(D)(j/R) = D(j/b) & \text{for } j \in A. & \end{array}$$

The isomorphisms δ for $f(D)$ are naturally induced by those for D, and we also define the map $\mathcal{E}_i : f(D)(i/L) \tilde{\rightarrow} L \otimes f(D)(i/R)$ to be the identity map for every $i \in A$. This makes

$f(D)$ an A-simplex in $\widetilde{T}_L$. The definition obviously agrees with the face and degeneracy maps, and we obtain the simplicial map $f : *|Z \to \widetilde{T}_L$.

Given a simplex of $\widetilde{T}_L(A)$, i.e., a multiplicative map $D : \Gamma(A) \to \mathcal{L}$ endowed with isomorphisms $\mathcal{E}_i$ satisfying (3.1), we let $g(D)$ be the composite map $\sum(A) \hookrightarrow \Gamma(A) \xrightarrow{D} \mathcal{L}$, where the inclusion $\sum(A) \hookrightarrow \Gamma(A)$ is the identity on ArA and sends b to R. Then $g : \widetilde{T}_L \to *|Z$ is a simplicial map, and obviously we have $g \cdot f = 1_{*|Z}$.

We now proceed to show that there is a simplicial homotopy connecting $f \cdot g$ and $1_{\widetilde{T}_L}$. A p-simplex x in $\widetilde{T}_L$ is a collection of objects in $\mathcal{L}$ of the form

$$x = \begin{bmatrix} & & & L_{p/p-1} \\ & & \cdots & \\ & L_{1/0} & \cdots & L_{p/0} \\ L_0 & L_1 & \cdots & L_p \\ L'_0 & L'_1 & \cdots & L'_p \end{bmatrix}$$

endowed with isomorphisms δ (cf.(2.3)) and isomorphisms $\mathcal{E}_i : L_i \to L \otimes L'_i$ for $0 \leq i \leq p$. By definition,

$$(f \cdot g)(x) = \begin{bmatrix} & & & L_{p/p-1} \\ & & \cdots & \\ & L_{1/0} & \cdots & L_{p/0} \\ L \otimes L'_0 & L \otimes L'_1 & \cdots & L \otimes L'_p \\ L'_0 & L'_1 & \cdots & L'_p \end{bmatrix}$$

where the isomorphisms $\mathcal{E}_i : L \otimes L'_i \widetilde{\to} L \otimes L'_i$ are the identity maps.

A p-simplex in $\Delta[1]$ can be thought of as a representation of the set $[p] = \{0, 1, \ldots, p\}$ in the form of a concatenation $[p] = \{0, \ldots, n\}\{n+1, \ldots, p\}$, where $n \in \{-1, 0, \ldots, p\}$. For short, we will denote this simplex by n. We define homotopy $H : \widetilde{T}_L \times \Delta[1] \to \widetilde{T}_L$ by

$$H(x; n) = \begin{bmatrix} & & & L_{p/p-1} \\ & & \cdots & \\ \cdots L_{n/0} & L_{n+1/0} & \cdots & L_{p/0} \\ L_0 \cdots L_n & L \otimes L'_{n+1} & \cdots & L \otimes L'_p \\ L'_0 \cdots L'_n & L'_{n+1} & \cdots & L'_p \end{bmatrix}$$

where $\mathcal{E}_i$ is as in x for $0 \leq i \leq n$ and $\mathcal{E}_i = 1_{L \otimes L'_i}$ for $n+1 \leq i \leq p$.

To make $H(x; n)$ into a simplex of $\widetilde{T}_L$ it remains to define the isomorphisms δ. They will be the same as in x except the case $\delta_{L,i,j} : L \otimes L'_j \widetilde{\to} L_i \otimes L_{j/i}$, where $i \leq n$ and $j \geq n+1$. In this case we define δ to be the composite isomorphism in any of the two possible ways in the diagram

$$\begin{array}{ccc} L \otimes L'_j & \overset{1 \otimes \delta_{R,i,j}}{\longrightarrow} & L \otimes L'_i \otimes L_{j/i} \\ \mathcal{E}_j \uparrow & \searrow & \uparrow \mathcal{E}_i \otimes 1 \\ L_j & \overset{\delta_{L,i,j} \text{ of } x}{\longrightarrow} & L_i \otimes L_{j/i} \end{array}$$

which is commutative by virtue of (3.1). One checks directly the required compatibility conditions (1.4) (iii) and (3.1) for $H(x; n)$, whence H is the desired simplicial homotopy. This completes the proof of Proposition 3.3 and Theorem 3.1. ☺

4. The map $\det : W^1 \to T$ is fibred

Let $F : X \to Y$ be a map of simplicial sets, $A \in \Delta$, and $y_0 \in Y(A)$. Any map $f : A' \to A$ in Δ gives rise to the base change map $y_0|F \to (f^*y_0)|F$ which takes a B-simplex (y, x) to (f^*y, x) where we write simply f^*y to denote the inverse image of y under the map $A'B \overset{f\coprod 1}{\to} AB$ (cf. sect. 2). We say that F is fibred if $y|F \to (f^*y)|F$ is a homotopy equivalence for any $f : A' \to A$ in Δ and any $y \in Y(A)$.

Theorem B'[GG, p. 580]. *If $F : X \to Y$ is a fibred simplicial map, then for any $A \in \Delta$ and $y \in Y(A)$ the square*

$$\begin{array}{ccc} y|F & \to & X \\ \downarrow & & \downarrow \\ y|Y & \to & Y \end{array}$$

is homotopy cartesian, and therefore $|y|F|$ can be regarded as homotopy fiber of the map $|F| : |X| \to |Y|$.

Theorem 4.1. *The map $\det : W^1 \to T$ defined in sect. 2 is fibred.*

We claim that in fact any simplicial map $X \to T$ is fibred. The latter follows from Propositions 4.2 and 4.3 below.

Proposition 4.2. *Suppose that Y is a simplicial set such that*

(4.1) for any map $f : \{a\} \to A$ in Δ and any simplex $y \in Y(A)$ there exists a simplicial map $\varphi : (f^*y)|Y \to y|Y$ such that the diagram

$$\begin{array}{ccc} (f^*y)|Y & \overset{\varphi}{\to} & y|Y \\ \downarrow & & \downarrow \\ Y & \overset{1}{\to} & Y \end{array}$$

commutes, where the vertical arrows take $\{a\}B$ (resp. AB)-simplices to their B-faces, and

(4.1) (i) $\qquad (f^*y)\Big|Y \overset{\varphi}{\to} y\Big|Y \overset{f^*}{\to} (f^*y)|Y$ is the identity map;

(4.1) (ii) there exists a simplicial homotopy $h : (y|Y)\mathrm{x}\Delta[1] \to y|Y$ which connects the map $y|Y \overset{f^*}{\to} (f^*y)|Y \overset{\varphi}{\to} y|Y$ with the identity map and which is constant on the B-part (see the definition of $y|Y$ in sect. 2), i.e., the diagram

$$\begin{array}{ccc} (y|Y)\mathrm{x}\Delta[1] & \overset{h}{\to} & y|Y \\ \downarrow & & \downarrow \\ Y & \overset{1}{\to} & Y \end{array}$$

commutes.

Then any simplicial map $F : X \to Y$ is fibred.

Proof: It suffices to prove that for any map $f : \{a\} \to A$ in Δ and any $y_0 \in Y(A)$, the map $y_0|F \to (f^*y_0)|F$ is a homotopy equivalence, for given a map $g : A' \to A$, we see that the base change maps $y_0|F \to (g_1^*g^*y_0)|F$ and $(g^*y_0)|F \to (g_1^*g^*y_0)|F$ are homotopy equivalences for any map $g_1 : \{a\} \to A'$, the assertion for $y_0|F \to (g^*y_0)|F$ follows.

Let $\varphi : (f^*y_0)|Y \to y_0|Y$ be the map of (4.1). We define a map $\Phi : (f^*y_0)|F \to y_0|F$ by $(y, x) \rightarrowtail (\varphi(y), x)$. Then, by virtue of (4.1) (i), the composite map $(f^*y_0)|F \overset{\Phi}{\to}$

$y_0|F \xrightarrow{f^\bullet} (f^*y_0)|F$ is the identity map. We define a homotopy $H : (y_0|F) \times \Delta[1] \to y_0|F$ which connects the map $y_0|F \xrightarrow{f^\bullet} (f^*y_0)|F \xrightarrow{\Phi} y_0|F$ with the identity map, by letting $H(y, x; n) = (h(y; n), x)$ for $(y, x) \in (y_0|F)(B)$ and $n \in \Delta[1](B)$ ☺

Proposition 4.3. T *satisfies (4.1).*

Proof: Let $f : \{a\} \to A$ be the inclusion $\{t\} \hookrightarrow [p] = \{0, 1, \ldots, p\}$. We assume y_0 is a p-simplex in T given by (2.3). Then f^*y_0 is the vertex $\begin{bmatrix} L_t \\ L'_t \end{bmatrix}$. A q-simplex x in $(f^*y_0)|T$ is a collection of objects of $\mathcal{L}$ of the form

$$\begin{bmatrix} & & & M_{q/q-1} \\ & & \cdots & \\ & M_{1/0} & \cdots & M_{q/0} \\ & M_{0,t} & \cdots & M_{q,t} \\ L_t & M_0 & \cdots & M_q \\ L'_t & M'_0 & \cdots & M'_q \end{bmatrix}$$

together with isomorphisms δ satisfying (1.4). We set

$$\varphi(x) = \left[\begin{array}{cccccccccccc|lll} & & & & & & & & & & & M_{q/q-1} & & & \\ & & & & & & & & & & \cdots & & & & \\ & & & & & & & & & M_{1/0} & \cdots & M_{q/0} & & & \\ & & & & & & & & & L^{-1}_{j/t} \otimes M_{0,t} & \cdots & L^{-1}_{j/t} \otimes M_{q,t} & j-th & \text{row} & t+1 \leq j \leq p \\ & & & & & \cdots & L_{j/t} & \cdots & & M_{0,t} & \cdots & M_{q,t} & t-th & \text{row} & \\ & & & \cdots & L_{t/i} & \cdots & L_{j/i} & \cdots & & L_{t/i} \otimes M_{0,t} & \cdots & L_{t/i} \otimes M_{q,t} & i-th & \text{row} & 0 \leq i \leq t-1 \\ L_0 & \cdots & L_i & \cdots & L_t & \cdots & L_j & \cdots & L_p & M_0 & \cdots & M_q & & & \\ L'_0 & \cdots & L'_i & \cdots & L'_t & \cdots & L'_j & \cdots & L'_p & M'_0 & \cdots & M'_q & & & \end{array}\right]$$

(recall that an inverse object L^{-1} is chosen for every L in $\mathcal{L}$). To make $\varphi(x)$ a q-simplex in $y_0|T$, we have to define the isomorphisms δ and verify (1.4). This amounts to the study of various locations of three (resp. six) objects in the above picture. In each case δ is naturally induced by the corresponding isomorphisms for x and y_0, and (1.4) for $\varphi(x)$ follows easily from the same properties of x and y_0.

Thus we obtain a simplicial map $\varphi : (f^*y_0)|T \to y_0|T$, and obviously $f^* \cdot \varphi$ is the identity map of $(f^*y_0)|T$. It remains to define a homotopy $h : (y_0|T) \times \Delta[1] \to y_0|T$ satisfying (4.1) (ii) which connects the map $\varphi \cdot f^*$ with $1_{y_0|T}$.

A q-simplex y in $y_0|T$ is a collection of objects of $\mathcal{L}$

$$y = \begin{bmatrix} & & & & & & M_{q/q-1} \\ & & & & & \cdots & \\ & & & & M_{1/0} & \cdots & M_{q/0} \\ & & & & M_{o,p} & \cdots & M_{q,p} \\ & & & L_{p/p-1} & M_{0,p-1} & \cdots & M_{q,p-1} \\ & & & & & \cdots & \\ & L_{1/0} & \cdots & L_{p/0} & M_{0,0} & \cdots & M_{q,0} \\ L_0 & L_1 & \cdots & L_p & M_0 & \cdots & M_q \\ L'_0 & L'_1 & \cdots & L'_p & M'_0 & \cdots & M'_q \end{bmatrix}$$

together with isomorphisms δ satisfying (1.4). Let $n \in \{-1, 0, \ldots, q\}$ denote a q-simplex in $\Delta[1]$. We set

$$h(y;n) = \begin{bmatrix} & & & & & & & M_{q/q-1} \\ & & & & & & \cdots & \\ & & & & M_{1/0} & \cdots & & M_{q/0} \\ & & & & & \cdots & & \\ & & & M_{0,j} & \cdots\ M_{n,j} & L_{j/t}^{-1} \otimes M_{n+1,t} & \cdots & L_{j/t}^{-1} \otimes M_{q,t} \\ & \cdots & L_{p/t} & M_{0,t} & \cdots\ M_{n,t} & M_{n+1,t} & \cdots & M_{q,t} \\ & \cdots & L_{p/i} & M_{0,i} & \cdots\ M_{n,i} & L_{t/i} \otimes M_{n+1,t} & \cdots & L_{t/i} \otimes M_{q,t} \\ L_0 & \cdots & L_p & M_0 & \cdots\ M_n & M_{n+1} & \cdots & M_q \\ L_0' & \cdots & L_p' & M_0' & \cdots\ M_n' & M_{n+1}' & \cdots & M_q' \end{bmatrix}$$

Again we have to consider various locations of objects in order to define the isomorphisms δ for $h(y;n)$ and check (1.4). We omit this trivial verification. This completes the proof of Proposition 4.3 and Theorem 4.1.

5. The second term of the weight filtration

We define the simplicial set W^2 as follows. For $A \in \Delta$,, an A-simplex in W^2 is a tuple $(F; \psi_i, i \in A)$, where $F : \Gamma(A) \to \mathcal{P}$ is an exact functor such that rank $F(i/L)$ = rank $F(i/R)$ for every $i \in A$ (i.e., $F \in W^1(A)$; cf. sect. 2) and $\psi_i : \det F(i/L) \xrightarrow{\sim} \det F(i/R)$ are isomorphisms compatible with the isomorphisms δ in $\det \cdot F$ (cf. Proposition 1.2), i.e., for every $i < j$ in A the diagram

$$\begin{array}{ccc} \det F(j/L) & \overset{\delta}{\to} & \det F(i/L) \otimes \det F(j/i) \\ \psi_j \downarrow & & \downarrow \psi_i \otimes 1 \\ \det F(j/R) & \underset{\sim}{\overset{\delta}{\to}} & \det F(i/R) \otimes \det F(j/i) \end{array} \tag{5.1}$$

commutes.

For short, let $P_i = F(i/L)$, $P_i' = F(i/R)$, and $P_{j/i} = F(j/i)$. We see, in particular, that a vertex in W^2 is a triple $(P, P'; \psi)$ where P and P' are objects of $\mathcal{P}$ such that rank P = rank P' and $\psi : \det P \xrightarrow{\sim} \det P'$ is an isomorphism. An edge in W^2 connecting $(P_0, P_0'; \psi_0)$ to $(P_1, P_1'; \psi_1)$ is a pair of short exact sequences $\left(0 \to P_0 \to P_1 \to P_{1/0} \to 0, 0 \to P_0' \to P_1' \to P_{1/0} \to 0\right)$ such that the diagram

$$\begin{array}{ccc} \det P_1 & \overset{\overset{\delta}{\sim}}{\to} & \det P_0 \otimes \det P_{1/0} \\ \psi_1 \downarrow & & \downarrow \psi_0 \otimes 1 \\ \det P_1' & \overset{\overset{\delta}{\sim}}{\to} & \det P_0' \otimes \det P_{1/0} \end{array}$$

commutes.

There is an obvious simplicial map $W^2 \to W^1$ which forgets the choice of the isomorphisms ψ_i.

Theorem 5.1. *$W^2 \to W^1 \overset{\det}{\to} T$ is a homotopy fibration sequence.*

This assertion together with Theorem 3.1 yield a long exact sequence

$$\ldots \to 0 \to \pi_2 W^2 \xrightarrow{\sim} \pi_2 W^1 \to 0 \to \pi_1 W^2 \to K_1 X \overset{\text{epi}}{\to} \Gamma(X, O_X^*) \overset{0}{\to}$$
$$\to \pi_0 W^2 \to \ker(\text{rank} : K_0 X \to \mathbf{Z}) \to \text{ Pic } X \to 0$$

Corollary 5.2.

(i) $$\pi_0 W^2 \cong \ker\left((\text{rank}, \det) : K_0 X \to \mathbf{Z} \oplus \mathit{Pic}\ X\right)$$

(ii) $$\pi_1 W^2 \cong \ker\left(\det : K_1 X \to \Gamma(X, O_X^*)\right)$$

(iii) $$\pi_m W^2 \cong K_m X \text{ for } m \geq 2$$

☺

Proof of the theorem. Let * denote the vertex $\begin{bmatrix} I \\ I \end{bmatrix}$ of T regarded as a $\{b\}$-simplex. By Theorem B' and Theorem 4.1, it suffices to construct homotopy inverse maps $f : *|\det \to W^2$ and $g : W^2 \to *|\det$.

A p-simplex in $*|\det$ is a pair (x, F), where

(5.2) $$x = \begin{bmatrix} & & & & L_{p/p-1} \\ & & & \cdots & \\ & & L_{1/0} & \cdots & L_{p/0} \\ & L_{0/b} & L_{1/b} & \cdots & L_{p/b} \\ L_b = I & L_0 & L_1 & \cdots & L_p \\ L'_b = I & L'_0 & L'_1 & \cdots & L'_p \end{bmatrix}$$

is a collection of objects of $\mathcal{L}$ endowed with isomorphisms δ (i.e., x is a $\{b\}[p]$-simplex in T whose $\{b\}$-vertex is $*$) and F is a p-simplex in W^1 such that $\det F$ is equal to the p -face of x.

We set

$$\psi_i = \delta_{R,b,i}^{-1} \cdot \delta_{L,b,i} : L_i \tilde{\rightarrow} L'_i$$

where $\delta_{L,b,i} : L_i \tilde{\rightarrow} I \otimes L_{i/b}$ and $\delta_{R,b,i} : L'_i \tilde{\rightarrow} I \otimes L_{i/b}$, and claim that $(F; \psi_i, 0 \leq i \leq p)$ is a p-simplex in W^2. For it suffices to verify (5.1) for every $i < j$ in $[p]$. This follows from the diagram

$$\begin{array}{ccc} L_j & \overset{\delta_{L,i,j}}{\underset{\sim}{\rightarrow}} & L_i \otimes L_{j/i} \\ \delta_{L,b,j} \downarrow \wr & & \downarrow \wr \delta_{L,b,i} \otimes 1 \\ I \otimes L_{j/b} & \overset{1 \otimes \delta_{b,i,j}}{\underset{\sim}{\rightarrow}} & I \otimes L_{i/b} \otimes L_{j/i} \\ \uparrow & & \uparrow \wr\, \delta_{R,b,i} \otimes 1 \\ \delta_{R,b,j} \quad L'_j & \overset{\delta_{R,i,j}}{\underset{\sim}{\rightarrow}} & L'_i \otimes L_{j/i} \end{array}$$

in which both parts are commutative by virtue of (1.4) (iii) for x.

Thus we obtain a simplicial map $f : *|\det \to W^2$. We define a homotopy inverse map $g : W^2 \to *|\det$ as follows. Given a p-simplex $(F; \psi_i, 0 \leq i \leq p)$ in W^2, we set $L_i = \det F(i/L), L'_i = L_{i/b} = \det F(i/R)$, and $L_{j/i} = \det F(j/i)$ for $0 \leq i < j \leq p$.

We define a $\{b\}[p]$-simplex x by (5.2), where the isomorphisms δ in the p-face are the same as in $\det F$ (cf. Proposition 1.2). Further, we set

$$\delta_{L,b,i} : L_i = \det F(i/L) \xrightarrow{\psi_i} \det F(i/R) = L_{i/b} \to I \otimes L_{i/b};$$
$$\delta_{R,b,i} : L_i' = \det F(i/R) \xrightarrow{1} \det F(i/R) = L_{i/b} \to I \otimes L_{i/b},$$

where $L_{i/b} \to I \otimes L_{i/b}$ is the natural map (recall that $I = O_X$), and

$$\delta_{b,i,j} : L_{j/b} = \det F(j/R) \overset{\delta_{R,i,j} \text{of} \det F}{\longrightarrow} \det F(i/R) \otimes \det F(j/i) = L_{i/b} \otimes L_{j/i};$$

the compatibility condition follows trivially. Thus (x, F) is a p-simplex in $*|\det$, which gives rise to a simplicial map g. Clearly, $f.g = 1_{W^2}$, and it is easy to define a simplicial homotopy which connects $g \cdot f$ with $1_{*|\det}$. Theorem 5.1 is proved. ☺

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