Spaces of pseudo-Riemannian geodesics and pseudo-Euclidean billiards

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Abstract

Many classical facts in Riemannian geometry have their pseudo-Riemannian analogs. For instance, the spaces of space-like and time-like geodesics on a pseudo-Riemannian manifold have natural symplectic structures (just like in the Riemannian case), while the space of light-like geodesics has a natural contact structure. We discuss the geometry of these structures in detail, as well as introduce and study pseudo-Euclidean billiards. In particular, we prove pseudo-Euclidean analogs of the Jacobi-Chasles theorems and show the integrability of the billiard in the ellipsoid and the geodesic flow on the ellipsoid in a pseudo-Euclidean space.

Contents

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1 Introduction

The space of oriented lines in the Euclidean n-space has a natural symplectic structure. So does the space of geodesic on a Riemannian manifold, at least locally. The structures on the space of geodesics on a *pseudo-Riemannian* manifold are more subtle. It turns out that the spaces of space-like lines and time-like lines in a pseudo-Euclidean space have natural symplectic structures (and so are the corresponding spaces of the geodesics on a pseudo-Riemannian manifold), while the space of light-like (or, null) lines or geodesics has a natural contact structure. Moreover, the corresponding symplectic structures on the manifolds of space- and time-like geodesics blow up as one approaches the border between them, the space of the null geodesics.

Many other familiar facts in Euclidean/Riemannian geometry have their analogs in the pseudo-Riemannian setting, but often with an unexpected twist. For example, assign the oriented normal line to each point of a coori-ented hypersurface in pseudo-Euclidean space; this gives a smooth map from the hypersurface to the space of oriented lines whose image is Lagrangian in the space of of space-like and time-like lines and Legendrian in the space of light-like lines, see Section 2.5. Another example: a convex hypersurface in Euclidean space $\mathbf{R}^n$ has at least n diameters. It turns out that a convex hypersurface in pseudo-Euclidean space V^{k+l} with k space directions and l time directions has at least k space-like diameters and at least l time-like ones, see Section 3.5.

In Section 3 we introduce pseudo-Euclidean billiards. They can be regarded as a particular case of projective billiards introduced in [28]. The corresponding pseudo-Euclidean billiard map has a variational origin and exhibits peculiar properties. For instance, there are special ("singular") points, where the normal to the reflecting surface is tangent to the surface itself (the phenomenon impossible for Euclidean reflectors), at which the billiard map is not defined. These points can be of two different types, and the reflection near them is somewhat similar to the reflection in the two different wedges, with angles $\pi/2$ and $3\pi/2$, in a Euclidean space. We study in detail the case of a circle on a Lorentz plane, see Section 4.

We prove a Lorentz version of the Clairaut theorem on the complete integrability of the geodesic flow on a surface of revolution. Finally, we prove pseudo-Euclidean analogs of the Jacobi-Chasles theorems and show the integrability of the billiard in the ellipsoid and the geodesic flow on the ellipsoid in a pseudo-Euclidean space. Unlike the Euclidean situation, the number of pseudo-confocal conics passing through a point in pseudo-Euclidean space can be different for different points of space, see Section 5.3.

Throughout the paper, we mostly refer to "pseudo-Euclidean spaces" or "pseudo-Riemannian manifolds" to emphasize arbitrariness of the number of space- or time-like directions. "Lorentz" means that the signature is of the form $(k, 1)$ or $(1, l)$. Note also that the contact structure on null geodesics was previously known, at least, for the Lorentz case – see [17, 22], and it had been important in various causality questions in the physics literature. Apparently, pseudo-Euclidean billiards have not been considered before, nor the integrability of pseudo-Riemannian geodesic flows on quadratic surfaces different from pseudospheres.

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2 Symplectic and contact structures on the spaces of oriented geodesics

2.1 General construction

Let M^n be a smooth manifold with a pseudo-Riemannian metric $\langle\, ,\rangle$ of signature $(k,l),\ k+l=n$. Identify the tangent and cotangent spaces via the metric. Let $H: T^*M \to \mathbf{R}$ be the Hamiltonian of the metric: $H(q,p)=\langle p,p\rangle/2$. The geodesic flow in T^*M is the Hamiltonian vector field X_H of H.

A geodesic curve in M is a projection of a trajectory of X_H to M. Let $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0$ be the spaces of oriented non-parameterized space-, time- and light-like geodesics (that is, $H = const > 0, < 0$ or $= 0$, respectively). Let $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- \cup \mathcal{L}_0$ be the space of all oriented geodesic lines. We assume that these spaces are smooth manifolds (locally, this is always the case); then $\mathcal{L}_0$ is the common boundary of $\mathcal{L}_\pm$.

Consider the actions of $\mathbf{R}^*$ on the tangent and cotangent bundles by rescaling (co)vectors. The Hamiltonian H is homogeneous of degree 2 in the variable p. Refer to this action as the dilations. Let E be the Euler field in T^*M that generates the dilations.

Theorem 2.1 *The manifolds $\mathcal{L}_\pm$ carry symplectic structures obtained from T^*M by Hamiltonian reduction on the level hypersurfaces $H=\pm 1$. The manifold $\mathcal{L}_0$ carries a contact structure whose symplectization is the Hamiltonian reduction of the symplectic structure in T^*M on the level hypersurface $H=0$.*

Proof. Consider three level hypersurfaces: $N_{-1} = \{H = -1\}, N_0 = \{H = 0\}$ and $N_1 = \{H = 1\}$. The Hamiltonian reduction on the first and the third yields the symplectic structures in $\mathcal{L}_\pm$. This is the same as in the Riemannian case, see, e.g., [3].

Consider N_0. We have two vector fields on it, X_H and E, satisfying $[E, X_H] = X_H$. Denote the Hamiltonian reduction of N_0 by P, it is the quotient of N_0 by the $\mathbf{R}$-action with the generator X_H (sometimes, P is called the space of *scaled* light-like geodesics). Then $\mathcal{L}_0$ is the quotient of P by the dilations; denote the projection $P \to \mathcal{L}_0$ by π. Note that E descends on P as a vector field $\bar{E}$. Denote by $\bar{\omega}$ the symplectic form on P. Let $\bar{\lambda} = i_{\bar{E}}\bar{\omega}$. We have:

$$d\bar{\lambda} = \bar{\omega},\ L_{\bar{E}}(\bar{\omega}) = \bar{\omega},\ L_{\bar{E}}(\bar{\lambda}) = \bar{\lambda}.$$

Thus $(P, \bar{\omega})$ is a homogeneous symplectic manifold with respect to the Euler field $\bar{E}$. Consider the distribution Ker $\bar{\lambda}$ on P. Since $\bar{E}$ is tangent to this distribution, Ker $\bar{\lambda}$ descends to a distribution on $\mathcal{L}_0$. This is a contact structure whose symplectization in $(P, \bar{\omega})$.

To prove that the distribution on $\mathcal{L}_0$ is indeed contact, let η be a local 1-form defining the distribution. Then $\pi^*(\eta) = \bar{\lambda}$. Hence

$$\pi^*(\eta \wedge d\eta^{n-2}) = \bar{\lambda} \wedge \bar{\omega}^{n-2} = \frac{1}{n-1}\, i_{\bar{E}}\bar{\omega}^{n-1}.$$

Since $\bar{\omega}^{n-1}$ is a volume form, the last form does not vanish. $\Box$

2.2 Examples

Example 2.2 Let us compute the area form on the space of lines in the Lorentz plane with the metric $ds^2 = dxdy$. A vector (a, b) is orthogonal to $(a, -b)$. Let $D(a, b) = (b, a)$ be the linear operator identifying vectors and covectors via the metric.

The light-like lines are horizontal or vertical, the space-like have positive and the time-like negative slopes. Each space $\mathcal{L}_+$ and $\mathcal{L}_-$ has two components. To fix ideas, consider space-like lines having the direction in the first coordinate quadrant. Write the unit directing vector of a line as $(e^{-u}, e^u),\ u \in \mathbf{R}$. Drop the perpendicular $r(e^{-u}, -e^u),\ r \in \mathbf{R}$, to the line from the origin. Then (u, r) are coordinates in $\mathcal{L}_+$. Similarly one introduces coordinates in $\mathcal{L}_-$.

Lemma 2.3 *(cf.[7, 9]) The area form ω on $\mathcal{L}_+$ is equal to $2du \wedge dr$, and to $-2du \wedge dr$ on $\mathcal{L}_-$.*

Proof. Assign to a line with coordinates (u, r) the covector $p = D(e^{-u}, e^u) = (e^u, e^{-u})$ and the point $q = r(e^{-u}, -e^u)$. This gives a section of the bundle $N_1 \to \mathcal{L}_+$, and the symplectic form $\omega = dp \wedge dq$ equals, in the (u, r)-coordinates, $2du \wedge dr$. The computation for $\mathcal{L}_-$ is similar. $\Box$

There exists a Lorentz analog of the Crofton formula which recovers the length of a curve from the area of the set of lines that intersect it (see [7, 9], as well as [27] for Crofton formulas in constant curvature Lorentz spaces).

The difference with the Euclidean (or more generally, Riemannian) case is that the total area of the set of lines intersecting a curve is infinite. Here is a formulation of the Crofton formula in the Lorentz plane [9]. Let γ_1 and γ_2 be space-like curves with common end-points. Let $N_{1,2}(\ell)$ be the number of intersection points of a line ℓ with the curves $\gamma_{1,2}$. Then

$$\text{length } \gamma_2 - \text{length } \gamma_1 = C \int \left(N_2(\ell) - N_1(\ell)\right)\ \omega,$$

integration over the set of space-like lines, oriented in the North-East direction; the constant C depends on the normalization of the area form ω.

Example 2.4 Consider the Lorentz space with the metric $dx^2 + dy^2 - dz^2$; let H^2 be the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$ and $H^{1,1}$ the hyperboloid of one sheet $x^2+y^2-z^2 = 1$. The restriction of the ambient metric to H^2 gives it a Riemannian metric of constant negative curvature and the restriction to $H^{1,1}$ a Lorentz metric of constant curvature. The geodesics of these metrics are the intersections of the surfaces with the planes through the origin; the light-like lines of $H^{1,1}$ are the straight rulings of the hyperboloid. The central projection on a plane induces a (pseudo)-Riemannian metric therein whose geodesics are straight lines (for H^2, this is the Beltrami-Klein model of the hyperbolic plane).

The scalar product in the ambient space determines duality between lines and points by assigning to a vector the orthogonal plane. In particular, to a point of H^2 there corresponds a space-like line in $H^{1,1}$ (which is a closed curve). More precisely, H^2 (which is the upper sheet of the hyperboloid) is identified with the space of positively (or "counterclockwise") oriented space-like lines of $H^{1,1}$, while the lower sheet of the same hyperboloid (that is, H^2 with the opposite orientation) is identified with the space of negatively oriented lines. On the other hand, $H^{1,1}$ is identified with the space of oriented lines in H^2. The space of oriented time-like lines of $H^{1,1}$ (which are not closed) is also identified with $H^{1,1}$ itself. The area forms on the spaces of oriented lines coincide with the area forms on the respective surfaces, induced by the ambient metric.

This construction is analogous to the projective duality between points and oriented great circles of the unit sphere in 3-space.

Problem 2.5 One can use the Crofton formula and the symplectic structure on the space of geodesics to describe all Finsler metrics in a convex domain whose geodesics are straight lines, that is, to solve Hilbert's 4th problem, see [1]. Can one similarly solve an analog of Hilbert's 4th problem for other signatures, that is, to describe pseudo-Finsler metrics whose geodesics are straight lines?

2.3 Pseudo-Euclidean space

Let V^{n+1} be a vector space with an indefinite non-degenerate quadratic form. Decompose V into the orthogonal sum of the positive and negative subspaces; denote by v_1, v_2 the positive and negative components of a vector v, and likewise, for covectors. The scalar product in V is given by the formula $\langle u, v\rangle = u_1 \cdot v_1 - u_2 \cdot v_2$ where $\cdot$ is the Euclidean dot product. Let $S_\pm$ be the unit pseudospheres in V given by the equations $|q_1|^2 - |q_2^2| = \pm 1$.

Proposition 2.6 *$\mathcal{L}_\pm$ is (anti)symplectomorphic to $T^*S_\pm$.*

Proof. Consider the case of $\mathcal{L}_+$. Assign to a space-like line ℓ its unit vector v, so that $|v_1|^2 - |v_2^2| = 1$, and a point $x \in \ell$ whose position vector is orthogonal to v, that is, $\langle x, v\rangle = 0$. Then $v \in S_+$ and $x \in T_v S_+$. Let $\xi \in T_v^* S_+$ be the covector corresponding to the vector x via the metric: $\xi_1 = x_1, \xi_2 = -x_2$. Then the canonical symplectic structure in T^*S_+ is $d\xi \wedge dv = dx_1 \wedge dv_1 - dx_2 \wedge dv_2$.

The correspondence $\ell \mapsto (q, p)$, where $q = x$ and $p = (p_1, p_2) = (v_1, -v_2)$ is the covector corresponding to the vector v via the metric, is a section of the bundle $N_1 \to \mathcal{L}_+$. Thus the symplectic form ω on $\mathcal{L}_+$ is the pull-back of the form $dp \wedge dq$, that is, $\omega = dv_1 \wedge dx_1 - dv_2 \wedge dx_2$. Up to the sign, this is the symplectic structure in T^*S_+. $\Box$

A light-like line is characterized by its point x and a vector v along the line; one has $\langle v, v\rangle = 0$. The same line is determined by the pair $(x + sv, tv)$, $s \in \mathbf{R}, t \in \mathbf{R}_+^*$. The respective vector fields $v\partial x$ and $v\partial v$ are the Hamiltonian and the Euler fields, in this case.

We shall now describe the contact structure in $\mathcal{L}_0$ geometrically.

Assign to a line $\ell \in \mathcal{L}_0$ the set $\Delta(\ell) \subset \mathcal{L}$ consisting of the oriented lines in the affine hyperplane, orthogonal to ℓ. Then $\ell \in \Delta(\ell)$ and $\Delta(\ell)$ is a smooth

$(2n-2)$-dimensional manifold, the space of oriented lines in n-dimensional space. Denote by $\xi(\ell) \subset T_\ell \mathcal{L}$ the tangent hyperplane to $\Delta(\ell)$ at point ℓ.

Denote by S_0 the spherization of the light cone: S_0 consists of equivalence classes of non-zero vectors $v \in V$ with $\langle v, v \rangle = 0$ and $v \sim tv, t > 0$. Let E be the 1-dimensional $\mathbf{R}_+^*$-bundle over S_0 whose sections are functions $f(v)$, homogeneous of degree 1. Denote by $J^1 E$ the space of 1-jets of sections of E; this is a contact manifold.

Proposition 2.7 *1. $\xi(\ell)$ is the contact hyperplane of the contact structure in $\mathcal{L}_0$.*
2. $\mathcal{L}_0$ is contactomorphic to $J^1 E$.[1]

Proof. By construction of Theorem 2.1, the contact hyperplane at ℓ is the projection to $T_\ell \mathcal{L}_0$ of the kernel of the Liouville form vdx (identifying vectors and covectors via the metric). Write an infinitesimal deformation of $\ell = (x, v)$ as $(x + \varepsilon y, v + \varepsilon u)$. This is in Ker vdx if and only if $\langle y, v \rangle = 0$. The deformed line is light-like, hence $\langle v + \varepsilon u, v + \varepsilon u \rangle = 0 \bmod \varepsilon^2$, that is, $\langle u, v \rangle = 0$. Thus both the foot point and the directional vector of the line ℓ move in the hyperplane, orthogonal to ℓ, and therefore the contact hyperplane at ℓ is contained in $\xi(\ell)$. Since the dimensions coincide, $\xi(\ell)$ is this contact hyperplane. In particular, we see that $\Delta(\ell)$ is tangent to $\mathcal{L}_0$ at point ℓ. This proves the first statement.

Assign to $\ell = (x, v)$ the 1-jet of the function $\phi(\ell) = \langle x, \cdot \rangle$ on S_0. This function is homogeneous of degree 1. The function $\phi(\ell)$ is well defined: since v is orthogonal to v and to $T_v S_0$, the function $\phi(\ell)$ does not change if x is replaced by $x + sv$. Thus we obtain a diffeomorphism $\phi : \mathcal{L}_0 \to J^1 E$.

To prove that ϕ preserves the contact structures, let f be a test section of E. By definition of the contact structure in $J^1 E$, the 1-jet extension of f is a Legendrian manifold. Set: $x(v) = \nabla f(v)$ (gradient taken with respect to the pseudo-Euclidean structure). We claim that $\phi(x(v), v) = j^1 f(v)$. Indeed, by the Euler formula,

$$\langle x(v), v \rangle = \langle \nabla f(v), v \rangle = f(v), \tag{1}$$

that is, the value of the function $\langle x(v), \cdot \rangle$ at point v is $f(v)$. Likewise, let $u \in T_v S_0$ be a test vector. Then the value of the differential $d\langle x(v), \cdot \rangle$ on u is $\langle \nabla f(v), u \rangle = df_v(u)$.

[1] In the case of a Lorentz space, $\mathcal{L}_0$ is also contactomorphic to the space of cooriented contact elements of a Cauchy surface, see [17].

It remains to show that the manifold $\phi^{-1}(j^1 f) = \{(x(v), v)\}$ is Legendrian in $\mathcal{L}_0$. Indeed, the contact form is vdx. One has:

$$vd(x(v)) = d\langle x(v), v\rangle - x(v)dv = df - \nabla f dv = 0;$$

the second equality is due to (1). Therefore $\phi^{-1}(j^1 f)$ is a Legendrian submanifold, and the second claim follows. $\Box$

2.4 Symplectic, Poisson and contact structures

The contact manifold $\mathcal{L}_0$ is the boundary between two open symplectic manifolds $\mathcal{L}_\pm$. Suppose that $n \geq 2$, that is, we consider lines in at least three-dimensional space V^{n+1}.

Theorem 2.8 *Neither the symplectic structures of $\mathcal{L}_\pm$, nor their inverse Poisson structures, extend smoothly across the boundary $\mathcal{L}_0$ to the corresponding structure on the total space $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_0 \cup \mathcal{L}_-$.*

Remark 2.9 When $n = 1$ the symplectic strictures go to infinity as we approach the one-dimensional manifold $\mathcal{L}_0$. The corresponding Poisson structures, which are inverses of the symplectic ones, extend smoothly across $\mathcal{L}_0$.

This can be observed already in the explicit computations of Example 2.2. Recall that for the metric $ds^2 = dxdy$ in V^2 and the lines directed by vectors (e^{-u}, e^u), $u \in \mathbf{R}$ the symplectic structure in the corresponding coordinates (u, r) at $\mathcal{L}_+$ has the form $2du \wedge dr$, see Lemma 2.3.

Now consider a neighborhood of a light-like line among all lines, that is, a neighborhood of a point in $\mathcal{L}_0$ regarded as a boundary submanifold between $\mathcal{L}_+$ and $\mathcal{L}_-$. Look at the variation $\xi_\varepsilon = (1, \varepsilon)$ of the horizontal (light-like) direction $\xi_0 = (1, 0)$, and regard (ε, r) as the coordinates in this neighborhood. For $\varepsilon > 0$ the corresponding half-neighborhood lies in $\mathcal{L}_+$, while the coordinates u and ε in this half-neighborhood are related as follows. Equating the slope of $(1, \varepsilon)$ to the slope of (e^{-u}, e^u) we obtain the relation $\varepsilon = e^{2u}$ or $u = \frac{1}{2}\ln\varepsilon$. Then the symplectic structure $\omega = 2du \wedge dr = d\ln\varepsilon \wedge dr = \frac{1}{\varepsilon} d\varepsilon \wedge dr$. One sees that $\omega \to \infty$ as $\varepsilon \to 0$. The Poisson structure, inverse to ω, is given by the bivector field $\varepsilon\frac{\partial}{\partial\varepsilon} \wedge \frac{\partial}{\partial r}$ and it extends smoothly across the border $\varepsilon = 0$.

Example 2.10 Let us compute the symplectic strictures on lines in the 3-dimensional space V^3 with the metric $dxdy - dz^2$. We parametrize the space-like directions by $\xi = (e^{-u}\cosh\phi, e^u\cosh\phi, \sinh\phi)$, where $u \in \mathbf{R}$, $\phi \in \mathbf{R}$. The operator D identifying vectors and covectors has the form $D(a,b,c) = (b/2, a/2, -c)$. Choose the basis of vectors orthogonal to ξ as

$$e_1 = (e^{-u}\sinh\phi, e^u\sinh\phi, \cosh\phi) \quad \text{and} \quad e_2 = (e^{-u}, -e^u, 0)\,.$$

The symplectic structure $\omega = dp \wedge dq$ for $q = r_1e_1 + r_2e_2$ and $p = D\xi = (e^u\cosh\phi/2, e^{-u}\cosh\phi/2, -\sinh\phi)$ has the following explicit expression in coordinates (u, ϕ, r_1, r_2):

$$\omega = -\,d\phi \wedge dr_1 + \cosh\phi\, du \wedge dr_2 - r_2\sinh\phi\, d\phi \wedge du\,.$$

Now we are in a position to prove Theorem 2.8 on non-extendability.

Proof. The impossibility of extensions follows from the fact that the "eigenvalues" of the symplectic structures ω of $\mathcal{L}_\pm$ go to both 0 and ∞, as we approach $\mathcal{L}_0$ from either side. (Of course, according to the Darboux theorem the eigenvalues of the symplectic structures are not well defined, but their zero or infinite limits are.) More precisely, let $\alpha = \sum a_{ij}dx_i \wedge dx_j$ be a meromorphic 2-form written in local coordinates $\{x_i\}$ in a neighborhood of a point P.

Lemma 2.11 *The number of eigenvalues of the matrix $A = (a_{ij})$ which go to 0 or ∞ as $x \to P$ does not dependent on the choice of coordinates $\{x_i\}$.*

Proof. Indeed, under a coordinate change $x = \eta(y)$ the matrix A changes to $(J\eta)^*A(J\eta)$ in coordinates $\{y_j\}$, where $J\eta$ is the Jacobi matrix of the diffeomorphism η. Since $J\eta$ is bounded and non-degenerate, this change preserves (in)finiteness or vanishing of the limits of the eigenvalues of A. $\square$

Now the theorem follows from

Lemma 2.12 *The eigenvalues of the 2-form ω in coordinates (u, ϕ, r_1, r_2) go to both 0 and ∞ as $r_2 \to \infty$ (while keeping other coordinates fixed).*

Proof. Indeed, the matrix of ω has the following (biquadratic) characteristic equation: $\lambda^4+a\lambda^2+b=0$, where $a=1+r_2^2\sinh^2\phi+\cosh^2\phi$, $b=\cosh^2\phi$. As $r_2\to\infty$, so does a, whereas b does not change. Thus the sum of the squares of the roots goes to infinity, whereas their product is constant. Hence the equation has one pair of roots going to 0, while the other goes to infinity. $\square$

The limit $r_2\to\infty$ means that one is approaching the boundary of the space $\mathcal{L}_+$. The infinite limit of the eigenvalues means that the symplectic structure ω does not extend smoothly across $\mathcal{L}_0$, while the zero limit of them means that the inverse to ω Poisson structure is non-extendable as well. The case of higher dimensions n can be treated similarly. $\square$

Remark 2.13 The contact planes in $\mathcal{L}_0$ can be viewed as the subspaces of directions in the tangent spaces $T_*\mathcal{L}_0$, on which the limits of the $\mathcal{L}_\pm$-symplectic structures are finite. One can also see that the existence of extensions of the symplectic or Poisson structures would mean the presence of other intrinsic structures, different from the contact one, on the boundary $\mathcal{L}_0$. Indeed, the existence of a symplectic structure extension would imply the existence of a presymplectic structure (and hence, generically, a characteristic direction field), rather than of a contact distribution, on $\mathcal{L}_0$.

On the other hand, consider the Poisson structures on $\mathcal{L}_\pm$ which are inverses of the corresponding symplectic structures. The assumption of a smooth extension of such Poisson structures would mean the existence of a Poisson structure on $\mathcal{L}_0$ as well. The corresponding foliation of $\mathcal{L}_0$ by symplectic leaves would be integrable, while the contact distribution is not.

One can consider this Poisson structure, up to conformal changes, as a "conformal cosymplectic structure": it does not explode on $\mathcal{L}_0$ but captures many features of the neighboring symplectic structures. This approach is developed in [15].

2.5 Hypersurfaces and submanifolds

Let $M\subset V$ be an oriented smooth hypersurface. Assign to a point $x\in M$ its oriented normal line $\ell(x)$. We obtain a Gauss map $\psi: M\to\mathcal{L}$; denote by ψ_+,ψ_- and ψ_0 its space-, time- and light-like components. Note that if the normal line is light-like then it is tangent to the hypersurface.

Proposition 2.14 *The images of $\psi_\pm$ are Lagrangian and the image of ψ_0 is Legendrian.*

Proof. Consider the case of ψ_+ (the time-like case being similar). Denote by $\nu(x)$ the "unit" normal vector to M at point x satisfying $\langle \nu(x), \nu(x) \rangle = 1$. The line $\ell(x)$ is characterized by its vector $\nu(x)$ and its point x; the correspondence $\ell \mapsto (\nu(x), x)$ is section of the bundle $N_1 \to \mathcal{L}_+$ over the image of ψ_+. We need to prove that the form $dx \wedge d\nu(x)$ vanishes on this image. Indeed, the 1-form $\nu(x)dx$ vanishes on M since $\nu(x)$ is a normal vector, hence its differential is zero as well.

In the case of ψ_0, we do not normalize $\nu(x)$. The 1-form $\nu(x)dx$ is still zero on M, and this implies that the image of the Gauss map in $\mathcal{L}_0$ is Legendrian, as in the proof of Proposition 2.7. $\Box$

Remark 2.15 The maps $\psi_\pm$ are immersions but ψ_0 does not have to be one. For example, let M be a hyperplane such that the restriction of the metric to M has a 1-dimensional kernel. This kernel is the normal direction to M at each point. These normal lines foliate M, the leaves of this foliation are the fibers of the Gauss map ψ_0, and its image is an $n-1$-dimensional space.

Remark 2.16 More generally, let $M \subset V$ be a smooth submanifold of any codimension. Assign to a point $x \in M$ the set of all oriented normal lines to M at x. This also gives us a Gauss map $\psi : M \to \mathcal{L}$ with space-, time- and light-like components ψ_+, ψ_- and ψ_0 respectively. In this setting Proposition 2.14 still holds, while the proof requires only cosmetic changes.

Example 2.17 The set of oriented lines through a point provides an example of a submanifold in $\mathcal{L}$ whose intersection with $\mathcal{L}_+ \cup \mathcal{L}_-$ is Lagrangian and with $\mathcal{L}_0$ Legendrian.

Example 2.18 Consider the circle $x^2+y^2=1$ on the Lorentz plane with the metric dx^2-dy^2. Then the caustic, that is, the envelope of the normal lines to the circle, is the astroid $x^{2/3}+y^{2/3}=2^{2/3}$, see figure 1 (note that the caustic of an ellipse in the Euclidean plane is an astroid too). The role of Euclidean circles is played by the pseudocircles, the hyperbolas $(x-a)^2-(y-b)^2=c$: their caustics degenerate to points.

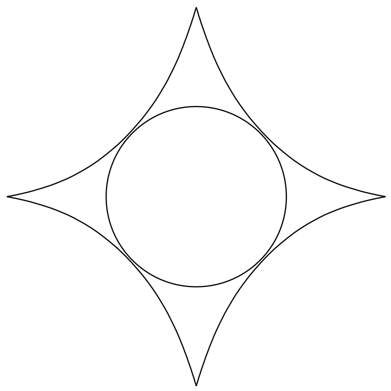

Figure 1: The caustic of a circle in the Lorentz plane

3 Billiard flow and billiard transformation

3.1 Definition of the billiard map

Let M be a pseudo-Riemannian manifold with a smooth boundary $S = \partial M$. The billiard flow in M is a continuous time dynamical system in TM. The motion of tangent vectors in the interior of M is free, that is, coincides with the geodesic flow. Suppose that a vector hits the boundary at point x. Let $\nu(x)$ be the normal to T_xS. If x is a singular point, that is, the restriction of the metric on S is singular or, equivalently, $\langle \nu(x), \nu(x) \rangle = 0$, then the billiard trajectory stops there. Otherwise the billiard reflection occurs.

Since x is a non-singular point, $\nu(x)$ is transverse to T_xS. Let w be the velocity of the incoming point. Decompose it into the the tangential and normal components, $w = t + n$. Define the billiard reflection by setting $w_1 = t - n$ to be the velocity of the outgoing point. Clearly $|w|^2 = |w_1|^2$. In particular, the billiard reflection does not change the type of a geodesic: time-, space- or light-like.

Example 3.1 Let the pseudocircle $x^2 - y^2 = c$ be a billiard curve (or an ideally reflecting mirror) in the Lorentz plane with the metric $dx^2 - dy^2$. Then all normals to this curve pass through the origin, and so every billiard trajectory from the origin reflects back to the origin. The same holds in multi-dimensional pseudo-Euclidean spaces.

Example 3.2 In the framework of Example 2.4, consider two billiards, inner and outer, in the hyperbolic plane H^2. (The latter is an area preserving mapping of the exterior of a strictly convex curve γ defined as follows: given a point x outside of γ, draw a support line to γ and reflect x in the support point; see [32, 33, 10].) The duality between H^2 and $H^{1,1}$ transforms the inner and outer billiard systems in H^2 to the outer and inner billiard systems in $H^{1,1}$. Given a convex closed curve in H^2, the dual curve in $H^{1,1}$ (consisting of the points, dual to the tangent lines of the original curve) is space-like. Thus any outer billiard in H^2 provides an example of a billiard in $H^{1,1}$ whose boundary is a space-like curve.

3.2 Variational approach to the billiard map

The origin of the billiard reflection law is variational. Let $A, B \in M$ be fixed points and let AXB be a billiard trajectory with reflection at point $X \in S$. In Riemannian geometry, the location of point X is determined by requiring the distance $|AX| + |XB|$ to be extremal. In pseudo-Riemannian set-up, consider the functional

$$I_\tau(\gamma_1, \gamma_2) = \int_0^\tau \langle \gamma_1'(t), \gamma_1'(t) \rangle \; dt + \int_\tau^1 \langle \gamma_2'(t), \gamma_2'(t) \rangle \; dt \qquad (2)$$

where $\gamma_1(t),\ 0 \le t \le \tau$ is a path from A to a point X of S and $\gamma_2(t),\ \tau \le t \le 1$ is a path from the point $\gamma_1(\tau)$ to B. Here $\tau \in [0, 1]$ is also a variable.

Proposition 3.3 *Suppose that (X, τ) is a solution of the variational problem (2). Then X is a non-singular point of S, and the curves AX and BX are geodesics related by the billiard reflection in S.*

Proof. Let $u_1(t), u_2(t)$ be variations of γ_1, γ_2. Then $u_1(0) = 0, u_1(\tau) = u_2(\tau)$ are tangent to S at point $X = \gamma_1(\tau) = \gamma_2(\tau)$ and $u_2(1) = 0$. We have:

$$0 = \frac{1}{2} \frac{dI_\tau(\gamma_1 + \varepsilon u_1, \gamma_2 + \varepsilon u_2)}{d\varepsilon} = \int_0^\tau \langle \gamma_1'(t), u_1'(t) \rangle \; dt + \int_\tau^1 \langle \gamma_2'(t), u_2'(t) \rangle \; dt =$$

$$\langle \gamma_1'(t), u_1(t) \rangle |_0^\tau - \int_0^\tau \langle \gamma_1''(t), u_1(t) \rangle \; dt + \langle \gamma_2'(t), u_2(t) \rangle |_\tau^1 - \int_\tau^1 \langle \gamma_2''(t), u_2(t) \rangle \; dt =$$

$$\langle \gamma_1'(\tau) - \gamma_2'(\tau), u_1(\tau) \rangle - \int_0^\tau \langle \gamma_1''(t), u_1(t) \rangle \; dt - \int_\tau^1 \langle \gamma_2''(t), u_2(t) \rangle \; dt$$

for all $u_1(t), u_2(t)$ (here γ'' is the covariant derivative). The vanishing of the integrals implies that AX and XB are geodesics and that $c_1 = \langle \gamma_1'(t), \gamma_1'(t) \rangle$

and $c_2 = \langle \gamma_2'(t), \gamma_2'(t) \rangle$ are constants. We also see that $\gamma_1'(\tau) - \gamma_2'(\tau)$ is normal to $T_X S$. Since $dI_\tau(\gamma_1, \gamma_2)/d\tau = 0$, we conclude that $c_1 = c_2$. It follows that the billiard reflection in S takes $\gamma_1'(\tau)$ to $\gamma_2'(\tau)$. $\Box$

Let $\gamma(t)$ be a parameterized space- or time-like geodesic; its length is $\int |\langle \gamma'(t), \gamma'(t) \rangle|^{1/2}\, dt$. If the end-points of the geodesics are A and X, let $|AX|$ denote its length. One has the following consequence of Proposition 3.3, familiar in the Euclidean set-up.

Corollary 3.4 *The reflection point X extremizes the function $|AX| + |XB|$ (assuming A and B fixed and $X \in S$ variable).*

Example 3.5 It is a classical optical property of an ellipse in the Euclidean plane that a ray of light emanating from one focus reflects to another focus. Consider the Lorentz plane with the metric $dxdy$; fix two points, $A = (-1, 0)$ and $B = (1, 0)$, and let us construct a curve such that every billiard trajectory from A reflects to a trajectory through B. It follows from Corollary 3.4 that the desired curves are the level curves of the function $|AX| + |XB|$ where $X = (x, y)$.

To fix ideas, assume that AX and BX are space-like. Then $|AX| = \sqrt{y(x+1)}$, $|BX| = \sqrt{y(x-1)}$ and the desired curves have the equations

$$\sqrt{y(x+1)} + \sqrt{y(x-1)} = c$$

or, equivalently,

$$\frac{y}{\lambda}\left(2x - \frac{y}{\lambda}\right) = 1 \quad \text{with} \quad \lambda = \frac{c^2}{2}.$$

This is a hyperbola with the asymptotes $y = 0$ and $y = 2\lambda x$.

3.3 Reflection near a singular point

Let us look more carefully at the billiard reflection in a neighborhood of a singular point of a curve in the Lorentz plane. First of all we note that typical singular points can be of two types, according to whether the inner normal is oriented toward or from the singular point as we approach it along the curve, see figure 2.

One can see that the smooth boundary of a strictly convex domain in the Lorentz plane has singular points of the former type only. Indeed, up to a diffeomorphism, there exists a unique germ of normal line field at a singular

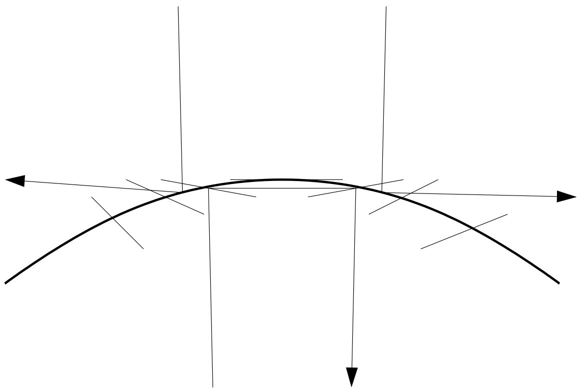

Figure 2: Billiard reflection near a singular point

point of a quadratically non-degenerate curve in the Lorentz plane – the one shown in figure 2. The billiard table may lie either on the convex (lower) or the concave (upper) side of the curve, whence the distinction between he two cases.

Note also that at a singular point the caustic of the curve always touches the curve (cf. Example 2.18). The above two cases differ by the location of the caustic: it can touch the curve from (a) the exterior or (b) the interior of the billiard domain. Billiard inside a circle in the Lorentz plane has singular points of the former type only, cf. figure 1.

The billiard reflections are drastically different in these two cases. In case (b), a generic family of rays gets dispersed in opposite directions on different sides from the singular point. For case (a), the situation is quite different: the scattered trajectories are reflected toward the singular point and hit the curve one more time in its vicinity. Thus one considers the square of the billiard map T^2.

Proposition 3.6 *Assume that a smooth billiard curve γ in the Lorentz plane is quadratically non-degenerate at a singular point O. Consider a parallel beam of lines $\{\ell\}$ reflecting in an arc of γ near point O, on the convex side. Then, as the reflection points tend to O, the lines $T^2(\ell)$ have a limiting direction, and this direction is parallel to ℓ.*

Proof. Let the metric be $dxdy$. In this metric, a vector (a, b) is orthogonal to $(-a, b)$. We may assume that the singular point is the origin, and that γ is the graph $y = f(x)$ where $f(0) = f'(0) = 0$ and $f''(0) > 0$. Consider a downward incoming ray with slope u reflecting in γ at point $(s, f(s))$, then at point $(t, f(t))$, and escaping upward with slope v.

The law of billiard reflection can be formulated as follows: the incoming billiard ray, the outgoing one, the tangent line and the normal to the boundary of the billiard table at the impact point constitute a harmonic quadruple of lines. See [28] for a study of projective billiards. The following criterion is convenient, see [28]. Let the lines be given by vectors a, b, c, d, see figure 3. Then the lines constitute a harmonic quadruple if and only if

$$[a, c][b, d] + [a, d][b, c] = 0 \tag{3}$$

where $[,]$ is the cross product of two vectors.

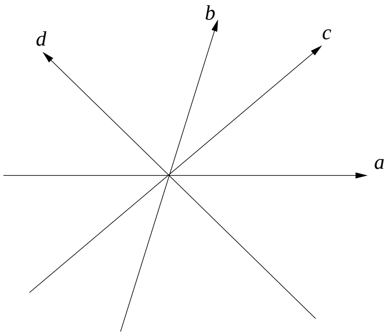

Figure 3: Harmonic quadruple of lines given by four vectors

For the first reflection, we have

$$a = (1, f'(s)),\ b = (-1, f'(s)),\ c = (t - s, f(t) - f(s)),\ d = (u, 1).$$

Substitute to (3) and compute the determinants to obtain:

$$u(f'(s))^2(t - s) = f(t) - f(s).$$

Similarly, for the second reflection, we have:

$$v(f'(t))^2(t - s) = f(t) - f(s),$$

and hence

$$u(f'(s))^2 = v(f'(t))^2. \tag{4}$$

Write $f(x) = ax^2 + O(x^3)$, then $f'(x) = 2ax + O(x^2)$, and

$$\frac{f(t) - f(s)}{t - s} = a(s + t) + O(s^2, st, t^2).$$

The above quantity equals $u(f'(s))^2$ which is $O(s^2)$, hence $t = -s + O(s^2)$. It follows that $(f'(t))^2 = (f'(s))^2 = 4a^2s^2 + O(s^3)$ and, by (4), that $v = u$. $\square$

Thus a ray meeting a curve near a singular point emerges, after two reflections, in the opposite direction. This resembles the billiard reflection in a right angle in the Euclidean plane, see figure 4. In contrast, the reflection of a parallel beam on the concave side of a Lorentz billiard near a singular point resembles the Euclidean billiard reflection from the angle $3\pi/2$ (cf. figures 2 and 4). Of course, this behavior excludes the existence of smooth caustics in Lorentz billiards, cf., e.g, [32, 33] for the Euclidean case.

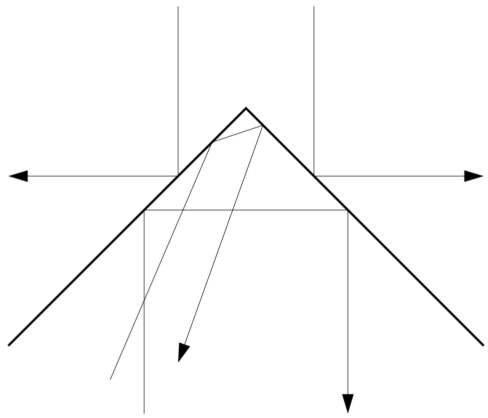

Figure 4: Euclidean billiard reflection in a right angle

3.4 Symplectic and contact properties of the billiard map

Now we discuss symplectic properties of the billiard transformation. To fix ideas, let the billiard table be geodesically convex.[2] Denote by $\mathcal{L}^0$ the set of oriented lines that meet S at non-singular points. Then we have billiard

[2] Alternatively, one may consider the situation locally, in a neighborhood of an oriented line transversally intersecting S at a non-singular point.

transformations T_+, T_- and T_0 acting on $\mathcal{L}_+^0, \mathcal{L}_-^0$ and $\mathcal{L}_0^0$, respectively. The (open dense) subsets $\mathcal{L}_\pm^0 \subset \mathcal{L}_\pm$ and $\mathcal{L}_0^0 \subset \mathcal{L}_0$ carry the same symplectic or contact structures as the ambient spaces.

Theorem 3.7 *The transformations T_+ and T_- are symplectic and T_0 is a contact transformation.*

Proof. We adopt the approach of R. Melrose [20, 21]; see also [3, 2]. Identify tangent vectors and covectors via the metric. We denote vectors by v and covectors by p.

Consider first the case of space-like geodesics (the case of time-like ones is similar). Let $\Sigma \subset T^*M$ be the hypersurface consisting of vectors with foot-point on S. Let $Z = N_1 \cap \Sigma$ and let $\Delta \subset Z$ consist of the vectors tangent to S.

Denote by $\nu(q) \in T_q^*M$ a conormal vector to S at point $q \in S$. Consider the characteristics of the canonical symplectic form ω in T^*M restricted to Σ. We claim that these are the lines $(q, p + t\nu(q)), t \in \mathbf{R}$.

Indeed, in local Darboux coordinates $\omega = dp \wedge dq$. The line $(q, p + t\nu(q))$ lies in the fiber of the cotangent bundle T^*M over the point q and the vector $\xi = \nu(q)\partial/\partial p$ is tangent to this line. Then $i_\xi \omega = \nu(q)dq$. This 1-form vanishes on Σ since $\nu(q)$ is a conormal vector to S at q. Thus ξ has the characteristic direction. Note that the quotient space by the characteristic foliation is T^*S.

Next we claim that the restriction of ω to $Z - \Delta$ is a symplectic form. Indeed, $Z - \Delta \subset N_1$ is transverse to the trajectories of the geodesic flow, that is, the leaves of the characteristic foliation of $N_1 \subset T^*M$.

The intersections of Z with the leaves of the characteristic foliation on N_1 determine an involution, τ, which is free on $Z - \Delta \subset N_1$. If $(q, v) \in Z$ is a vector, let $q_1 \in S$ be the other intersection point of the geodesic generated by (q, v) with S and v_1 the vector translated to point q_1 along the geodesic. Then $\tau(q, v) = (q_1, v_1)$.

Consider the intersections of Z with the leaves of the characteristic foliation on Σ. We claim that this also determine an involution, σ, which is free on $Z - \Delta \subset \Sigma$. Indeed, let $(q, v) \in Z$, i.e., $q \in S, \langle v, v\rangle = 1$. The characteristic line is $(q, v + t\nu(q))$, where $\nu(q)$ is a normal vector, and its intersection with Z is given by the equation $\langle p + t\nu(q), p + t\nu(q)\rangle = 1$. Since $\langle \nu(q), \nu(q)\rangle \neq 0$, this equation has two roots and we have an involution. One root is $t = 0$, the other is different from 0 if $\langle v, \nu(q)\rangle \neq 0$, that is, v is not tangent to S.

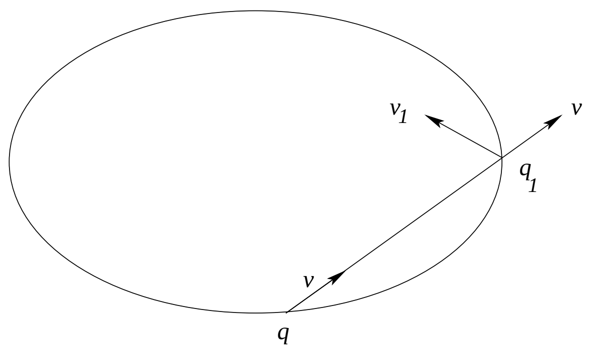

Figure 5: The billiard map $F = \sigma \circ \tau$ is a composition of two involutions: $\tau(q, v) = (q_1, v), \sigma(q_1, v) = (q_1, v_1)$

Let $F = \sigma \circ \tau$; this is the billiard map on Z, see figure 5. Since both involutions are defined by intersections with the leaves of the characteristic foliations, they preserve the symplectic structure $\omega|_Z$. Thus F is a symplectic transformation of $Z - \Delta$. Let $P : Z - \Delta \to \mathcal{L}_+^0$ be the projection. Then P is a symplectic 2-to-1 map and $P \circ F = T_+ \circ P$. It follows that T_+ preserves the symplectic structure in $\mathcal{L}_+^0$.

In the case of T_0, we have the same picture with N_0 replacing N_1 and its symplectic reduction P in place of $\mathcal{L}_+^0$. We obtain a symplectic transformation of P that commutes with the action of $\mathbf{R}_+^*$ by dilations. Therefore the map T_0 preserves the contact structure of $\mathcal{L}_0^0$. $\square$

Remark 3.8 Consider a convex domain D in the Lorentz plane with the metric $dxdy$. The light-like lines are either horizontal or vertical. The billiard system in D, restricted to light-like lines, coincides with the system described in [4] in the context of Hilbert's 13th problem (namely, see figure 6 copied from Figure 3 on p. 8 of [4]).

3.5 Diameters

A convex hypersurface in $\mathbf{R}^n$ has at least n diameters, which are 2-periodic billiard trajectories in this hypersurface. In a pseudo-Euclidean space with signature (k, l) the result is as follows.

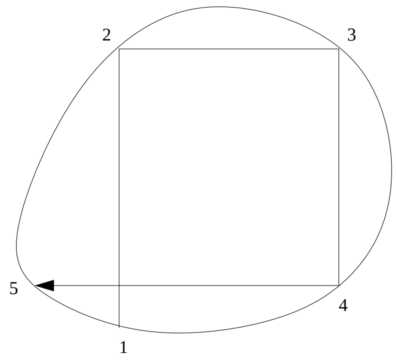

Figure 6: A dynamical system on an oval

Theorem 3.9 *A smooth strictly convex closed hypersurface has at least k space-like and l time-like diameters.*

Proof. Denote the hypersurface by Q. Consider the space of chords $Q \times Q$ and set $f(x, y) = \langle x - y, x - y\rangle/2$. Then f is a smooth function on $Q \times Q$. The group $\mathbf{Z}_2$ acts on $Q \times Q$ by interchanging points, and this action is free off the diagonal $x = y$. The function f is $\mathbf{Z}_2$-equivariant.

First we claim that a critical point of f with non-zero critical value corresponds to a diameter (just as in the Euclidean case). Indeed, let $u \in T_xQ, v \in T_yQ$ be test vectors. Then $d_xf(u) = \langle x-y, u\rangle$ and $d_yf(v) = \langle x-y, v\rangle$. Since these are zeros for all u, v, the (non-degenerate) chord $x - y$ is orthogonal to Q at both end-points. Note that such a critical chord is not light-like, due to convexity of Q.

Fix a sufficiently small generic $\varepsilon > 0$. Let $M \subset Q \times Q$ be a submanifold with boundary given by $f(x, y) \geq \varepsilon$. Since the boundary of M is a level hypersurface of f, the gradient of f (with respect to an auxiliary metric) has inward direction along the boundary, and the inequalities of Morse theory apply to M. Since $\mathbf{Z}_2$ acts freely on M and f is $\mathbf{Z}_2$-equivariant, the number of critical $\mathbf{Z}_2$-orbits of f in M is not less than the sum of $\mathbf{Z}_2$ Betti numbers of $M/\mathbf{Z}_2$.

We claim that M is homotopically equivalent to S^{k-1} and $M/\mathbf{Z}_2$ to $\mathbf{RP}^{k-1}$. Indeed, M is homotopically equivalent to the set of space-like oriented lines intersecting Q. Retract this set to the set of space-like oriented lines through the origin. The latter is the spherization of the cone $|q_1|^2 > |q_2|^2$, and the projection $(q_1, q_2) \mapsto q_1$ retracts it to the sphere S^{k-1}.

Since the sum of $\mathbf{Z}_2$ Betti numbers of $\mathbf{RP}^{k-1}$ is k, we obtain at least k space-like diameters. Replacing M by the manifold $\{f(x,y) \le -\varepsilon\}$ yields l time-like diameters. $\Box$

Remark 3.10 Another way to prove Theorem 3.9 is to consider the "width function." Namely, for any space/time-like direction ℓ consider the two planes normal to ℓ and tangent to the hypersurface Q. We define the (square) width function $W(\ell)$ by assigning to ℓ the square of the pseudo-Euclidean length of the segment cut out by these planes from ℓ. Similarly to the Euclidean case, this function $W(\ell)$ is smooth on space- or time-like directions and its critical points correspond to space- or time-like diameters, that is, 2-periodic billiard trajectories in Q.

By definition, the function $W(\ell)$ is positive on space-like directions and negative on time-like ones. It assumes the same values on the lines of opposite orientation. Define it to be equal to 0 on light-like directions. One can show that this function extends continuously (and even smoothly) across the hypersurface of light-like directions.

Now choose sufficiently small $\varepsilon > 0$ and we consider the submanifold $N \subset \mathcal{L}_+$ with boundary given by $W(\ell) \ge \varepsilon$. Finally, the same application of $\mathbf{Z}_2$-equivariant Morse theory to the function $W(\ell)$, as described above, provides an estimate on the number of its critical points and completes the proof.

Remark 3.11 Any closed immersed curve in the Lorentz plane has at least two diameters: these are chords xy with the maximal and the minimal values of $\langle x-y, x-y\rangle$; one of them is time-like and another space-like.

Problem 3.12 In Euclidean geometry, the fact that a smooth closed convex hypersurface in $\mathbf{R}^n$ has at least n diameters has a far-reaching generalization due to Pushkar' [25], see also [34] and [11]: a generic immersed closed manifold $M^k \to \mathbf{R}^n$ has at least $(B^2 - B + kB)/2$ diameters, that is, chords that are perpendicular to M at both end-points; here B is the sum of the $\mathbf{Z}_2$-Betti numbers of M. It is interesting to find a pseudo-Euclidean analog of this result.

Problem 3.13 Another generalization, in Euclidean geometry, concerns the least number of periodic billiard trajectories inside a closed smooth strictly convex hypersurface. In dimension 2, the classical Birkhoff theorem asserts that, for every n and every rotation number k, coprime with n, there exist at least two n-periodic billiard trajectories with rotation number k, see, e.g., [32, 33]. In higher dimensions, a similar result was obtained recently [12, 13]. It is interesting to find analogs for billiards in pseudo-Euclidean space. A possible difficulty is that variational problems in this set-up may have no solutions: for example, not every two points on the hyperboloid of one sheet in Example 2.4 are connected by a geodesic!

4 Case study: billiard inside a circle in the Lorentz plane

Consider the plane with the metric $ds^2 = dxdy$. Then a vector (a,b) is orthogonal to $(a,-b)$. Let $D(a,b) = (b,a)$ be the linear operator identifying vectors and covectors via the metric.

Consider the circle $x^2 + y^2 = 1$ and the billiard system inside it. There are four singular points: $(\pm 1, 0), (0, \pm 1)$. The phase space consists of the oriented lines intersecting the circle and such that the impact point is not singular. The billiard map on light-like lines is 4-periodic. One also has two 2-periodic orbits, the diameters having slopes ± 1.

Let t be the cyclic coordinate on the circle. Let us characterize a line by the coordinates of its first and second intersection points with the circle, (t_1, t_2). The billiard map T sends (t_1, t_2) to (t_2, t_3).

Theorem 4.1 *1) The map T is given by the equation*

$$\cot\left(\frac{t_2 - t_1}{2}\right) + \cot\left(\frac{t_2 - t_3}{2}\right) = 2\cot 2t_2. \tag{5}$$

2) The area form is given by the formula

$$\omega = \frac{\sin\left((t_2 - t_1)/2\right)}{|\sin(t_1 + t_2)|^{3/2}}\, dt_1 \wedge dt_2. \tag{6}$$

3) The map is integrable: it has an invariant function

$$I = \frac{\sin\left((t_2 - t_1)/2\right)}{|\sin(t_1 + t_2)|^{1/2}}. \tag{7}$$

4) The lines containing the billiard segments, corresponding to a fixed value λ of the (squared) integral I^2, are tangent to the conic

$$x^2 + y^2 + 2\lambda xy = 1 - \lambda^2, \quad \lambda \in \mathbf{R}. \tag{8}$$

These conics for different λ are all tangent to the four lines – two horizontal and two vertical – tangent to the unit circle.

In principal axes (rotated 45°), the family (8) writes as

$$\frac{x^2}{1-\lambda} + \frac{y^2}{1+\lambda} = 1.$$

Before going into the proof of Theorem 4.1 let us make some comments and illustrate the theorem by figures.

In the familiar case of the billiard inside an ellipse in the Euclidean plane, the billiard trajectories are tangent to the family of conics, confocal with the given ellipse, see, e.g., [32, 33]. These conics are either the confocal ellipses inside the elliptic billiard table or the confocal hyperbolas. The billiard map, restricted to an invariant curve of the integral, is described as follows: take a point A on the boundary of an elliptic billiard table, draw a tangent line to the fixed confocal ellipse (or hyperbola – for other values of the integral) until the intersection with the boundary ellipse at point A_1; take A_1 as the next point of the billiard orbit, etc.

In our case, for a fixed billiard trajectory inside the unit circle, there is a quadric inscribed into a 2×2 square to which it is tangent, see the family of ellipses in figure 7. For instance, two 4-periodic trajectories in figure 8 are tangent to one and the same inscribed ellipse.

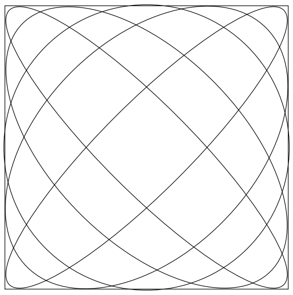

Figure 7: Ellipses inscribed into a square

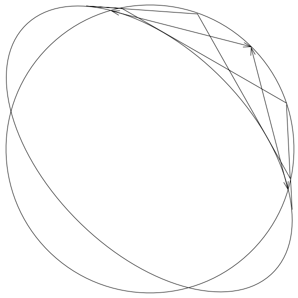

Figure 8: Two 4-periodic trajectories on the invariant curve with the rotation number 1/4: one orbit is a self-intersecting quadrilateral, and the other one consists of two segments, traversed back and forth

Figures 9*a* and 9*b* depict two billiard orbits in the configuration space consisting of 100 time-like billiard segments. It is easy to recognize the inscribed ellipse as an envelope of the segments of the billiard orbit in figure 9*a*: all the reflections occur on one of the two arcs of the circle outside of the ellipse, and the tangent line to the ellipse at its intersection point with the circle is the Lorentz normal to the circle at this point.

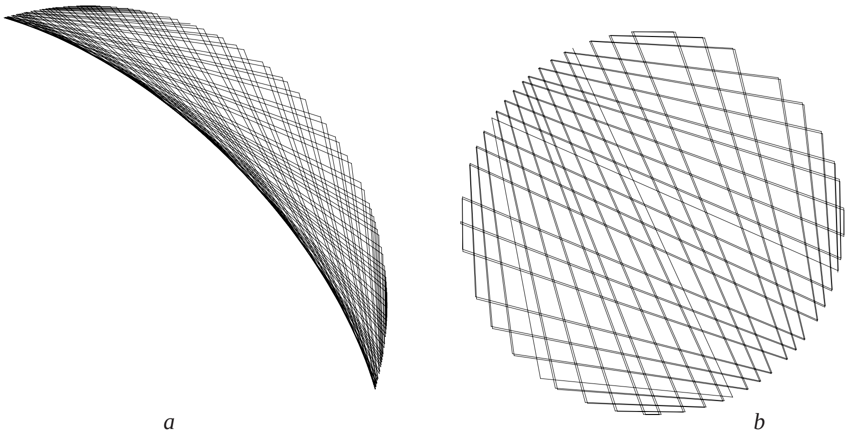

Figure 9: Two orbits consisting of 100 billiard segments: the lines containing the billiard segments on the left are tangent to an ellipse, and those on the right to a hyperbola

The corresponding envelope is less evident for the orbit in figure 9*b*: extensions of the billiard chords have a hyperbola as their envelope, see figure

10.

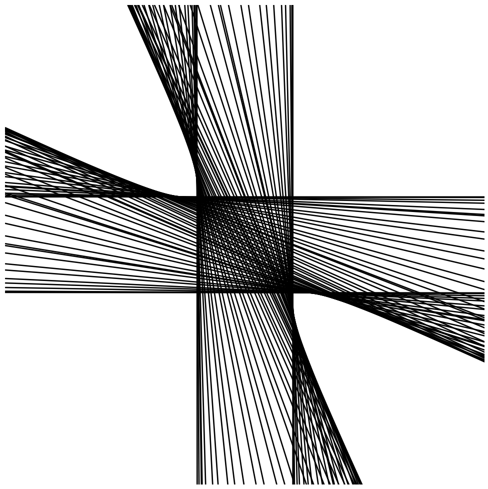

Figure 10: Extensions of billiard chords tangent to a hyperbola

The level curves of the integral I are shown in figure 11 depicting a $[-\pi,\pi]\times[-\pi,\pi]$ torus with coordinates (t_1,t_2). The four hyperbolic singularities of the foliation I=const at points

$$(3\pi/4,-\pi/4),(\pi/4,-3\pi/4),(-\pi/4,3\pi/4),(-3\pi/4,\pi/4)$$

correspond to two 2-periodic orbits of the billiard map; these orbits are hyperbolically unstable[3] (unlike the case of an ellipse in the Euclidean plane where the minor axis is a stable 2-periodic orbit). The white spindle-like regions surround the lines $t_1+t_2=\pi n,\ n\in\mathbf{Z}$; these lines correspond to the light-like rays. The four points $(0,0),(\pi/2,\pi/2),(-\pi/2,-\pi/2),(\pi,\pi)$ are singular: every level curve of the integral I pass through them.

Figures 12a and 12b show two topologically different invariant curves, and figures 9a and 9b discussed above depict two billiard orbits in the configuration space corresponding respectively to those two invariant curves.

Note also that it follows from the Poncelet porism (see, e.g., [8]) that the dynamics on each invariant curve is conjugated to a rotation of the circle. In particular, if some point of an invariant circle is periodic then all points of this invariant circle are periodic with the same period, cf. figure 8.

Now we shall prove Theorem 4.1.

[3] As indicated by the hyperbolic crosses made by the level curves at these points.

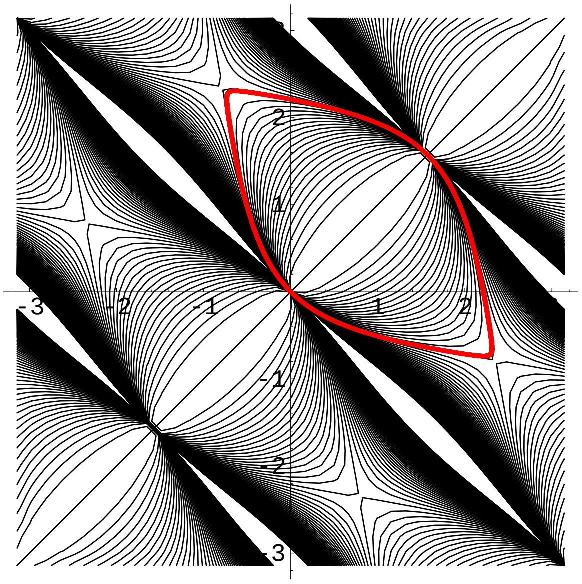

Figure 11: Level curves of the integral I in the (t_1, t_2) coordinates

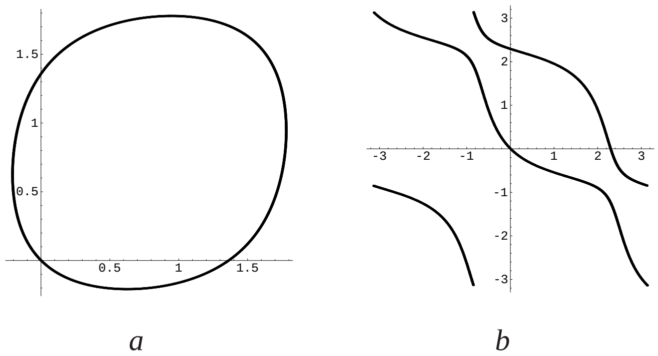

Figure 12: Two invariant curves (the right one consists of two components, the phase points "‘jump" from one component to the other)

Proof. We use another criterion for harmonicity of a quadruple of lines from [28], similar to (3). Consider four concurrent lines, and let α, ϕ, β be the angles made by three of them with the fourth, see figure 13. Then the lines are harmonic if and only if

$$\cot \alpha + \cot \beta = 2 \cot \phi. \tag{9}$$

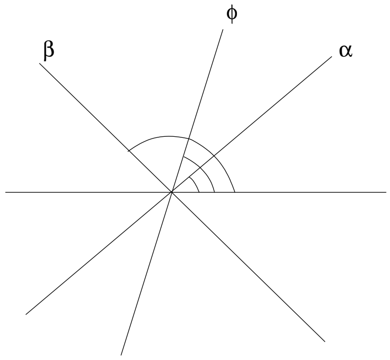

Figure 13: Harmonic quadruple of lines given by angles

In our situation, the billiard curve is $\gamma(t) = (\cos t, \sin t)$. The tangent vector is $\gamma'(t) = (-\sin t, \cos t)$ and the normal is $(\sin t, \cos t)$. Consider the impact point t_2. By elementary geometry, the rays (t_2, t_1) and (t_2, t_3) make the angles $(t_1 - t_2)/2$ and $(t_3 - t_2)/2$ with the tangent line at $\gamma(t_2)$, and the normal makes the angle $\pi - 2t_2$ with this tangent line. Then equation (9) becomes (5).

It is straightforward to compute the area form from Lemma 2.3 in the (t_1, t_2)-coordinates; the result (up to a constant factor) is (6).

We shall give two proofs that I is an integral. First, our Lorentz billiard is a particular case of a projective billiard in a circle. It is proved in [28] that every such billiard map has an invariant area form

$$\Omega = \frac{1}{\sin^2((t_2 - t_1)/2)}\ dt_1 \wedge dt_2. \tag{10}$$

(This form is the symplectic structure on the space of oriented lines for the projective – or Klein-Beltrami – model of hyperbolic geometry inside the unit disc.) Thus T has two invariant area forms, and (the cube root of) their ratio is an invariant function.

The second proof imitates a proof that the billiard inside an ellipse in the Euclidean plane is integrable, see [33]. Let us restrict attention to space-like lines. Assign to a line its first intersection point with the circle, q, and the unit vector along the line, v. Then $\langle D(q), q\rangle = 1$ and $\langle v, v\rangle = 1$. We claim that $I = \langle D(q), v\rangle$ is invariant under the billiard map.

The billiard map is the composition of the involutions τ and σ, see proof of Theorem 3.7. It turns out that each involution changes the sign of I.

Indeed, $\langle D(q) + D(q_1), q_1 - q\rangle = 0$ since D is self-adjoint. Since v is collinear with $q_1 - q$, we have: $\langle D(q) + D(q_1), v\rangle = 0$, and hence I is odd with respect to τ.

Since the circle is given by the equation $\langle D(q), q\rangle = 1$, the normal at point q_1 is $D(q_1)$. By definition of the billiard reflection, the vector $v + v_1$ is collinear with the normal at q_1, hence $\langle D(q_1), v\rangle = -\langle D(q_1), v_1\rangle$. Thus I is odd with respect to σ as well. The invariance of $I = \langle D(q), v\rangle$ follows, and it is straightforward to check that, in the (t_1, t_2)-coordinates, this integral is (7).

To find the equation of the envelopes and prove *4)* we first rewrite the integral I in the standard, Euclidean, coordinates (p, α) in the space of lines: p is the signed length of the perpendicular from the origin to the line and α the direction of this perpendicular, see [26]. One has

$$\alpha = \frac{t_1 + t_2}{2},\ p = \cos\left(\frac{t_2 - t_1}{2}\right). \tag{11}$$

Fix a value of the integral I by setting

$$\frac{\sin^2((t_2 - t_1)/2)}{\sin(t_1 + t_2)} = \lambda.$$

It follows from (11) that $1 - p^2 = \lambda \sin 2\alpha$, and hence $p = \sqrt{1 - \lambda \sin 2\alpha}$. (See figure 14 which shows the level curves of the (squared) integral $I^2 = (1 - p^2)/\sin 2\alpha$ in the (α, p)-coordinates.) We use α as a coordinate on the level curve corresponding to a fixed value of λ, and p as a function of α (of course, this function depends on λ as a parameter).

The envelope of a 1-parameter family of lines given by a function $p(\alpha)$ is the curve

$$(x(\alpha), y(\alpha)) = p(\alpha)(\cos\alpha, \sin\alpha) + p'(\alpha)(-\sin\alpha, \cos\alpha),$$

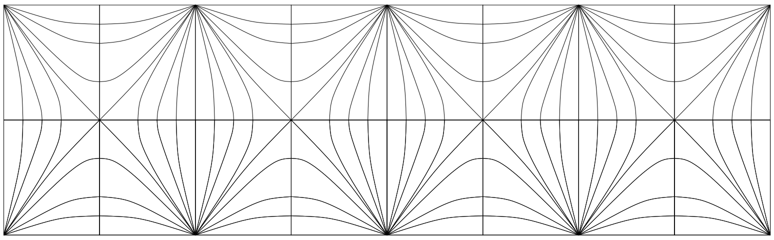

Figure 14: Level curves of the integral I in the (α, p)-coordinates

see [26]. In our case, we obtain the curve

$$(x(\alpha), y(\alpha)) = (1 - \lambda \sin 2\alpha)^{-1/2}(\cos\alpha - \lambda \sin\alpha, \sin\alpha - \lambda\cos\alpha).$$

It is straightforward to check that this curve satisfies equation (8).

It is also clear that the conics (8) are tangent to the lines $x = \pm 1$ and $y = \pm 1$. Indeed, if, for example, $y = 1$ then the left hand side of (8) becomes $(x + \lambda)^2 + 1 - \lambda^2$, and equation (8) has a multiple root $x_{1,2} = -\lambda$. $\Box$

Remark 4.2 Yet another proof of the integrability of the Lorentz billiard inside a circle can be deduced from the duality (the skew hodograph transformation) between Minkowski billiards discovered in [14]. This duality trades the shape of the billiard table for that of the unit (co)sphere of the metric. In our case, the billiard curve is a circle and the unit sphere of the metric is a hyperbola; the dual system is the usual, Euclidean billiard "inside" a hyperbola.

5 The billiard inside a quadric and the geodesic flow on a quadric

5.1 Geodesics and characteristics

Let us start with a general description of the geodesics on a hypersurface in a pseudo-Riemannian manifold.

Let M^n be a pseudo-Riemannian manifold and $S^{n-1} \subset M$ a smooth hypersurface. The geodesic flow on S is a limiting case of the billiard flow inside S when the billiard trajectories become tangent to the reflecting hypersurface. Assume that S is free of singular points, that is, S is a pseudo-Riemannian submanifold: the restriction of the metric in M to S is non-degenerate. The infinitesimal version of the billiard reflection law gives the following characterization of geodesics: a geodesic on S is a curve $\gamma(t)$ such that $\langle \gamma'(t), \gamma'(t) \rangle = \text{const}$ and the acceleration $\gamma''(t)$ is orthogonal to S at point $\gamma(t)$ (the acceleration is understood in terms of the covariant derivative) – see [23]. Note that the type (space-, time-, or light-like) of a geodesic curve remains the same for all t.

Let $Q \subset \mathcal{L}$ be the set of oriented geodesics tangent to S. Write $Q = Q_+ \cup Q_- \cup Q_0$ according to the type of the geodesics. Then $Q_\pm$ are hypersurfaces in the symplectic manifolds $\mathcal{L}_\pm$ and Q_0 in the contact manifold $\mathcal{L}_0$.

Recall the definition of characteristics on a hypersurface X in a contact manifold Y, see [3]. Assume that the contact hyperplane C at point $x \in X$ is not tangent to X; we say that x is a non-singular point. Then $C \cap T_xX$ is a hyperplane in C. Let λ be a contact form. Then $\omega = d\lambda$ is a symplectic form on C; a different choice of the contact form, $f\lambda$, gives a proportional symplectic form $f(x)\omega$ on C. The characteristic line at x is the skew-orthogonal complement of the hyperplane $C \cap T_xX$ in C.

Theorem 5.1 *1) The characteristics of the hypersurfaces $Q_\pm \subset \mathcal{L}_\pm$ consist of oriented geodesics in M tangent to a fixed space- or time-like geodesic on S.*
2) The hypersurface $Q_0 \subset \mathcal{L}_0$ consists of non-singular points and its characteristics consist of oriented geodesics in M tangent to a fixed light-like geodesic on S.

Proof. The argument is a variation on that given in the proof of Theorem 3.7, cf. [2, 3], and we use the notation from this proof. In particular, we identify vectors and covectors using the metric.

Consider Q_+ (the case of Q_- being similar). We have the submanifold $\Delta \subset N_1$ consisting of the unit space-like vectors tangent to S; the projection $N_1 \to \mathcal{L}_+$ identifies Δ with Q_+. Likewise, the projection $\Sigma \to T^*S$ makes it possible to consider Δ as a hypersurface in T^*S. The characteristics of $\Delta \subset T^*S$ are the geodesics on S.

We need to show that the two characteristic directions on Δ, induced by its inclusions into $\mathcal{L}_+$ and into T^*S, coincide. We claim that the restriction of the canonical symplectic structure ω in T^*M on its codimension 3 submanifold Δ has 1-dimensional kernel at every point. If this holds then both characteristic directions on Δ coincide with these kernels and therefore with each other.

The kernel of the restriction of ω on Δ is odd-dimensional. Assume its dimension is 3; then $\Delta \subset T^*M$ is a coisotropic submanifold. We will show that this is not the case.

Let $\nu(q)$ be the normal vector to S at point $q \in S$. Since q is not singular, $\nu(q)$ is transverse to T_qS. Thus the vector $u = \nu(q)\ \partial/\partial q$ is transverse to $\Delta \subset T^*M$. So is the vector $v = \nu(q)\ \partial/\partial p$. Let w be another transverse vector such that u, v, w span a transverse space to Δ. Note that

$$\omega(v, u) = (dp \wedge dq)(\nu(q)\ \partial/\partial p, \nu(q)\ \partial/\partial q) = \langle \nu(q), \nu(q) \rangle \neq 0.$$

Since ω is a symplectic form, $0 \neq i_{u \wedge v}\ \omega^n = C\ \omega^{n-1}$ with $C \neq 0$, and the $2n-3$ form $i_w \omega^{n-1}$ is a volume form on Δ. This contradicts to the fact that $T_{(q,p)}\Delta$ contains a 3-dimensional subspace skew-orthogonal to $T_{(q,p)}\Delta$, and the first statement of the theorem follows.

For the second statement, we replace N_1 by N_0 and $\mathcal{L}_+$ by the space of scaled light-like geodesics P. Then P and Δ are acted upon by the dilations. Using the notation from Theorem 2.1, $\pi(\Delta) = Q_0$. The characteristics of $\Delta \subset P$ consist of scaled oriented geodesics tangent to a fixed light-like geodesic on S.

To show that the points of $Q_0 \subset \mathcal{L}_0$ are non-singular, it suffices to prove that the hypersurface $\Delta \subset P$ is not tangent to the kernel of the 1-form $\bar\lambda$. We claim that this kernel contains the vector v transverse to Δ. Indeed,

$$\bar\lambda(v) = \bar\omega(\bar E, v) = (dp \wedge dq)(p\ \partial/\partial p, \nu(q)\ \partial/\partial p) = 0,$$

and hence $\ker\ \bar\lambda \neq T_{(q,p)}\Delta$.

Finally, the characteristics of the conical hypersurface $\Delta = \pi^{-1}(Q_0)$ in the symplectization P of the contact manifold $\mathcal{L}_0$ project to the characteristics of $Q_0 \subset \mathcal{L}_0$, see [3], and the last claim of the theorem follows. $\Box$

5.2 Geodesics on a Lorentz surface of revolution

Geodesics on a surface of revolution in the Euclidean space have the following Clairaut first integral: $r \sin\alpha = const$, where r is the distance from a given point on the surface to the axis of revolution, and α is the angle of the geodesic at this point with the projection of the axis to the surface.

Here we describe an analog of the Clairaut integral for Lorentz surfaces of revolution. Let S be a surface in the Lorentz space V^3 with the metric $ds^2 = dx^2 + dy^2 - dz^2$ obtained by a revolution of the graph of a function $f(z)$ about the z-axis: it is given by the equation $r = f(z)$ for $r^2 = x^2 + y^2$.

Consider the tangent plane $T_P S$ to the surface S at a point P on a given geodesic γ. Define the following 4 lines in this tangent plane: the axis projection l_z (meridian), the revolution direction l_ϕ (parallel), the tangent to the geodesic l_γ, and one of the two null directions l_{null} on the surface at the point P, see figure 15. We denote the corresponding cross-ratio of this quadruple of lines as $\mathrm{cr} = \mathrm{cr}(l_z, l_\phi, l_\gamma, l_{null})$.

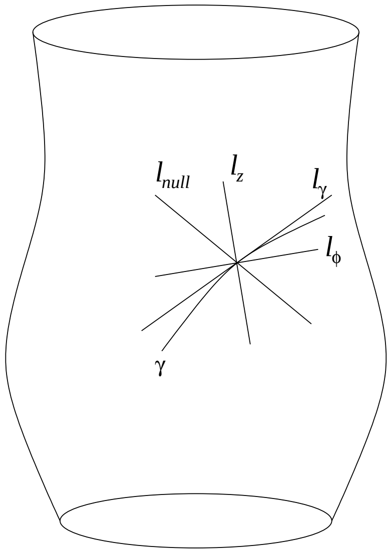

Figure 15: A quadruple of tangent lines on a surface of revolution

Theorem 5.2 *The function $r^2/(1-\mathrm{cr}^2)$ is constant along any geodesic γ on the Lorentz surface of revolution.*

Proof. The Clairaut integral in either Euclidean or Lorentz setting is a specification of the Noether theorem, which gives the invariance of the angular momenta $m = r \cdot v_\phi$ with respect to the axis of revolution. (Here (r, ϕ) are the polar coordinates in the (x, y)-plane.)

In the Euclidean case, we have $r\phi' = |v| \sin\alpha$ and combining the invariance of the magnitude of v along the geodesics with preservation of $m = r^2\phi'$ we immediately obtain the Clairaut integral $r \sin\alpha = const$.

In the Lorentz setting we first find the cross-ratio discussed above. Let v be the velocity vector along a geodesic, and v_r, v_ϕ, v_z be its radial, angle and axis projections respectively. Suppose the point P has coordinates $(0, y_0, z_0)$; choose (z, x) as the coordinates in the corresponding tangent plane T_PS. The line elements l_z, l_ϕ, and l_γ have the directions $(1, 0)$, $(0, 1)$, and (v_z, v_ϕ), respectively. The direction of null vectors in this tangent plane is the intersection of the cone of null vectors $x^2+y^2-z^2 = 0$ (in the coordinates centered at P) with the plane $y = f'(z_0)z$ tangent to S at P. Thus the corresponding null directions are $x = \pm\sqrt{1-(f'(z_0))^2}z$. We choose the "plus" direction for l_{null} and find the cross-ratio

$$\mathrm{cr} = \mathrm{cr}(l_z, l_\phi, l_\gamma, l_{null}) = \frac{[l_z, l_{null}][l_\gamma, l_\phi]}{[l_z, l_\gamma][l_{null}, l_\phi]} = \frac{v_z\sqrt{1-(f'(z_0))^2}}{v_\phi},$$

that is $v_z^2(1-(f'(z_0))^2) = v_\phi^2 \cdot \mathrm{cr}$.

Now recall that the Lorentz length of v is preserved: $v_r^2 + v_\phi^2 - v_z^2 = 1$. Taking into account that $v_r = f'(z_0)v_z$ in the tangent plane T_PS we exclude from this relation both v_r and v_z and express v_ϕ via the cross-ratio: $v_\phi^2 = 1/(1-\mathrm{cr}^2)$. Thus the preservation of the angular momenta $m = r \cdot v_\phi$ yields the Lorentz analog of the Clairaut theorem: $m^2 = r^2/(1-\mathrm{cr}^2) = const$ along any given geodesic on S. □

Note that choosing the other sign for the null direction l_{null} or another order of arguments in the cross-ratio $\mathrm{cr}(l_z, l_\phi, l_\gamma, l_{null})$ gives an equivalent conserved quantity.

Corollary 5.3 *At the singular points of a surface of revolution the geodesics become tangent to the direction of the axis projection l_z.*

Proof. Indeed, when a geodesic hits a singular point the cross-ratio has a finite limit (obtained from the Clairaut integral along the geodesic). On the other hand, the null direction l_{null} tends to the axis projection l_z. This forces the tangent element l_γ to approach l_z as well. □

A surface of revolution is an example of a warped product. A nice alternative description of the geodesics via Maupertuis' principle in the general context of warped products is given in [35].

5.3 Analogs of the Jacobi and the Chasles theorems

An ellipsoid with distinct axes in Euclidean space

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1 \qquad (12)$$

gives rise to the confocal family of quadrics

$$\frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \cdots + \frac{x_n^2}{a_n^2+\lambda} = 1.$$

The Euclidean theory of confocal quadrics comprises the following theorems: through a generic point in space there pass n confocal quadrics, and they are pairwise orthogonal at this point (Jacobi);[4] a generic line is tangent to $n-1$ confocal quadrics whose tangent hyperplanes at the points of tangency with the line are pairwise orthogonal (Chasles); and the tangent lines to a geodesic on an ellipsoid are tangent to fixed $n-2$ confocal quadrics (Jacobi-Chasles) – see [3, 2].

We shall construct a pseudo-Euclidean analog of this theory and adjust the proofs accordingly.

Consider pseudo-Euclidean space V^n with signature (k,l), $k+l=n$ and let $E: V \to V^*$ be the self-adjoint operator such that $\langle x,x\rangle = E(x)\cdot x$ where dot denotes the pairing between vectors and covectors. Let $A: V \to V^*$ be a positive-definite self-adjoint operator defining an ellipsoid $A(x)\cdot x = 1$. Since A is positive-definite, both forms can be simultaneously reduced to principle axes.[5] We assume that $A = \mathrm{diag}(a_1^2,\ldots,a_n^2)$ and $E = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$. An analog of the confocal family is the family of quadrics Q_λ

$$\frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \cdots + \frac{x_k^2}{a_k^2+\lambda} + \frac{x_{k+1}^2}{a_{k+1}^2-\lambda} + \cdots + \frac{x_n^2}{a_n^2-\lambda} = 1 \qquad (13)$$

where λ is a real parameter or, in short, $(A+\lambda E)^{-1}(x)\cdot x = 1$. We assume that $a_1 < \cdots < a_k < 0 < a_{k+1} < \cdots < a_n$.

The following result is a pseudo-Euclidean version of the Jacobi theorem.

[4] The respective values of λ are called the elliptic coordinates of the point.

[5] In general, it is not true that a pair of self-adjoint maps can be simultaneously reduced to principle axes. The simplest example in the plane is $ds_1^2 = x^2 - y^2,\ ds_2^2 = xy$.

Theorem 5.4 *Through every generic point $x \in V$ there pass either n or $n-2$ quadrics from the family (13). In the latter case, all quadrics have different topological types and in the former two of them have the same type. The quadrics are pairwise orthogonal at point x.*

Proof. Given a point x, we want to find λ satisfying equation (13), which reduces to a polynomial in λ of degree n. Denote by $f(\lambda)$ the function on the left-hand-side of (13). This function has poles at $\lambda = -a_1^2, \ldots, -a_k^2, a_{k+1}^2, \ldots, a_n^2$. At every negative pole $f(\lambda)$ changes sign from negative to positive, and at every positive pole from positive to negative. Let us analyze the behavior of $f(\lambda)$ as $\lambda \to \pm\infty$. One has:

$$f(\lambda) = \frac{1}{\lambda}\langle x, x\rangle - \frac{1}{\lambda^2}\sum_{i=1}^{n} a_i^2 x_i^2 + O\left(\frac{1}{\lambda^3}\right),$$

hence if x is not light-like then the sign of $f(\lambda)$ at $+\infty$ is equal, and at $-\infty$ opposite, to that of $\langle x, x\rangle$, whereas if x is light-like then $f(\lambda)$ at $\pm\infty$ is negative. The graph of the function $f(\lambda)$ in the case $\langle x, x\rangle < 0$ is shown in Figure 16. Thus $f(\lambda)$ assumes value 1 at least $k-1$ times for negative λ and at least $l-1$ times for positive ones. Being a polynomial of degree n, the number of roots is not greater than n.

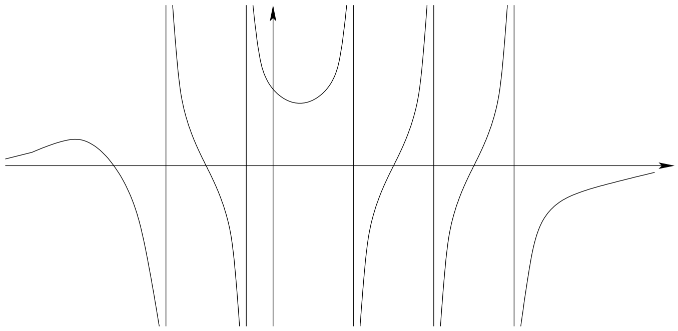

Figure 16: The graph of the function $f(\lambda)$ for a time-like point x

Note that the topological type of the quadric changes each time that λ passes through a pole of $f(\lambda)$. It follows that if there are $n-2$ quadrics passing through x then they all have different topological types, and the ellipsoid (corresponding to $\lambda = 0$) is missing. On the other hand, if there are n quadrics passing through x then two of them have the same topological type and there are $n-1$ different types altogether. Note, in particular, that if x lies on the original ellipsoid then there are n quadrics passing through it.

To prove that Q_λ and Q_μ are orthogonal to each other at x, consider their normal vectors (half the gradients of the left-hand-sides of (13) with respect to the pseudo-Euclidean metric)

$$N_\lambda = \left(\frac{x_1}{a_1^2+\lambda}, \frac{x_2}{a_2^2+\lambda}, \ldots, \frac{x_k}{a_k^2+\lambda}, -\frac{x_{k+1}}{a_{k+1}^2-\lambda}, \ldots, -\frac{x_n}{a_n^2-\lambda} \right),$$

and likewise for N_μ. Then

$$\langle N_\lambda, N_\mu \rangle = \sum_{i=1}^{k} \frac{x_i^2}{(a_i^2+\lambda)(a_i^2+\mu)} - \sum_{i=k+1}^{n} \frac{x_i^2}{(a_i^2-\lambda)(a_i^2-\mu)}. \tag{14}$$

The difference of the left-hand-sides of equations (13), taken for λ and μ, is equal to the right-hand-side of (14) times $(\mu-\lambda)$, whereas the right-hand-side is zero. Thus $\langle N_\lambda, N_\mu \rangle = 0$.

More conceptually, we want to find λ such that equation (13) holds. Set $y = (A+\lambda E)^{-1}(x)$, then we are looking for λ such that $(A+\lambda E)(y) = x$ and $x \cdot y = 1$. Then y is a point of the dual space V^* that belongs to the polar dual hypersurface Q_λ^*. (Recall that the dual hypersurface consists of the covectors that vanish on the tangent hyperplanes to the original one and take value 1 at the respective points). The dual hypersurfaces Q_λ^* constitute a pencil of quadrics $(A+\lambda E)(y) \cdot y = 1$. Polar duality interchanges points and hyperplanes. We are given a hyperplane $x \cdot y = 1$ and want to find hypersurfaces from the pencil tangent to it. This can be interpreted as a Lagrange multipliers problem.

Consider the function $g(x) = A(y) \cdot y - (x \cdot y)^2$ subject to the constraint $h(y) := E(y) \cdot y = c$ where $c \in \{-1, 0, 1\}$. If a critical point exists then $dg + \lambda dh = 0$, that is,

$$A(y) - (x \cdot y)x + \lambda E(y) = 0 \quad \text{or} \quad (A+\lambda E)(y) = (x \cdot y)x.$$

This is an equation of a line in V^*, and the intersection of this line with the hyperplane $x \cdot y = 1$ yields the desired equations.

If E were positive then the hypersurface $h(y) = 1$ would be a sphere and an even function on S^{n-1} would have at least n critical points. However in the pseudo-Euclidean case there may be fewer critical points (but not fewer than $n-2$, as we have seen). The pairwise orthogonality of the corresponding quadrics at the intersection points is reduced to the orthogonality of the principal axes of the quadrics of the dual family, similarly to the Euclidean case, cf. [3]. $\square$

Example 5.5 Consider the simplest example motivated by Section 4 : $A = \mathrm{diag}(1,1), E = \mathrm{diag}(1,-1)$. Figure 17 depicts the partition of the Lorentz plane according to the number of conics from a pseudo-confocal family passing through a point: the boundary consists of the lines $|x \pm y| = \sqrt{2}$.

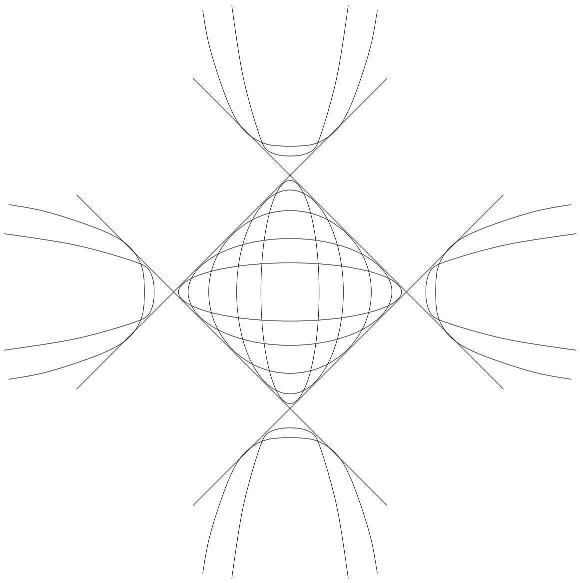

Figure 17: A pseudo-confocal family of conics

Problem 5.6 It is interesting to describe the topology of the partition of V according to the number of quadrics from the family (13) passing through a point. In particular, how many connected components are there?

Next, consider a pseudo-Euclidean version of the Chasles theorem.

Theorem 5.7 *A generic space- or time-like line ℓ is tangent to either $n-1$ or $n-3$, and a generic light-like line to either $n-2$ or $n-4$, quadrics from the family (13). The tangent hyperplanes to these quadrics at the tangency points with ℓ are pairwise orthogonal.*

Proof. Let v be a vector spanning ℓ. Suppose first that v is space- or time-like. Project V along ℓ on the orthogonal complement U to v. A quadric determines a hypersurface in this $(n-1)$-dimensional space, the set of critical values of its projection (the apparent contour). If one knows that these hypersurfaces also constitute a family (13) of quadrics, the statement will follow from Theorem 5.4.

Let $Q \subset V$ be a smooth star-shaped hypersurface and let $W \subset V^*$ be the annihilator of v. Suppose that a line parallel to v is tangent to Q at point x. Then the tangent hyperplane $T_x Q$ contains v. Hence the respective covector $y \in V^*$ from the polar dual hypersurface Q^* lies in W. Thus polar duality takes the points of tangency of Q with the lines parallel to v to the intersection of the dual hypersurface Q^* with the hyperplane W.

On the other hand, $U = V/(v)$ and $W = (V/(v))^*$. Therefore the apparent contour of Q in U is polar dual to $Q^* \cap W$. If Q belongs to the family of quadrics (13) then Q^* belongs to the pencil $(A + \lambda E)y \cdot y = 1$. The intersection of a pencil with a hyperplane is a pencil of the same type (with the new A positive definite and the new E having signature $(k-1, l)$ or $(k, l-1)$, depending on whether v is space- or time-like). It follows that the polar dual family of quadrics, consisting of the apparent contours, is of the type (13) again, as needed.

Note that, similarly to the proof of Theorem 5.4, if ℓ is tangent to the original ellipsoid then it is tangent to $n-1$ quadrics from the family (13).

If v is light-like then we argue similarly. We choose as the "screen" $U = V/(v)$ any hyperplane transverse to v. The restriction of E to W is degenerate: it has 1-dimensional kernel and its signature is $(k-1, l-1, 1)$. The family of quadrics, dual to the restriction of the pencil to W, is now given by the formula

$$\frac{x_1^2}{b_1^2+\lambda} + \frac{x_2^2}{b_2^2+\lambda} + \cdots + \frac{x_{k-1}^2}{b_{k-1}^2+\lambda} + \frac{x_{k+1}^2}{b_{k+1}^2-\lambda} + \cdots + \frac{x_{k+l-1}^2}{b_{k+l-1}^2-\lambda} = 1 - \frac{x_k^2}{b_k^2}$$

which is now covered by the $(n-1)$-dimensional case of Theorem 5.4. $\Box$

Note that in Example 5.5 a generic light-like line is tangent to no conic, whereas the four exceptional light-like lines $|x \pm y| = \sqrt{2}$ are tangent to infinitely many ones.

5.4 Complete integrability

The following theorem is a pseudo-Euclidean analog of the Jacobi-Chasles theorem.

Theorem 5.8 *1) The tangent lines to a fixed space- or time-like (respectively, light-like) geodesic on a quadric in pseudo-Euclidean space V^n are tangent to $n-2$ (respectively, $n-3$) other fixed quadrics from the family (13).*
2) A space- or time-like (respectively, light-like) billiard trajectory in a quadric in pseudo-Euclidean space V^n remains tangent to $n-1$ (respectively, $n-2$) fixed quadrics from the family (13).
3) The sets of space- or time-like oriented lines in pseudo-Euclidean space V^n, tangent to $n-1$ fixed quadrics from the family (13), are Lagrangian submanifolds in the spaces $\mathcal{L}_\pm$. The set of light-like oriented lines, tangent to $n-2$ fixed quadrics from the family (13), is a co-Legendrian submanifold in $\mathcal{L}_0$ foliated by codimension one Legendrian submanifolds which are quotients of $\mathbf{R}^{n-2}$ by a discrete subgroup.

Proof. Let ℓ be a tangent line at point x to a geodesic on the quadric Q_0 from the family (13). By Theorem 5.7, ℓ is tangent to $n-2$ (or $n-3$, in the light-like case) quadrics from this family. Denote these quadrics by $Q_{\lambda_j},\ j=1,\ldots,n-2$.

Let N be a normal vector to Q_0 at point x. Consider an infinitesimal rotation of the tangent line ℓ along the geodesic. Modulo infinitesimals of the second order, this line rotates in the 2-plane generated by ℓ and N. By Theorem 5.7, the tangent hyperplane to each Q_{λ_j} at its tangency point with ℓ contains the vector N. Hence, modulo infinitesimals of the second order, the line ℓ remains tangent to every Q_{λ_j}, and therefore remains tangent to each one of them.

The billiard flow inside an ellipsoid in n-dimensional space is the limit case of the geodesic flow on an ellipsoid in $(n+1)$-dimensional space, whose minor axis goes to zero. Thus the second statement follows from the first one.

Now we prove the third statement. Consider first the case of space- or time-like lines $\mathcal{L}_\pm$. Let ℓ be a generic oriented line tangent to quadrics $Q_{\lambda_j},\ j=1,\ldots,n-1$, from the family (13). Choose smooth functions f_j defined in neighborhoods of the tangency points of ℓ with Q_{λ_j} in V^n whose level hypersurfaces are the quadrics from the family (13). Any line ℓ' close to ℓ is tangent to a close quadric $Q_{\lambda'_j}$. Define the function F_j on the space of oriented lines whose value at ℓ' is the (constant) value of f_j on $Q_{\lambda'_j}$.

We want to show that $\{F_j, F_k\} = 0$ where the Poisson bracket is taken with respect to the symplectic structure defined in Section 2. Consider the value $dF_k(\text{sgrad } F_j)$ at ℓ. The vector field sgrad F_j is tangent to the characteristics of the hypersurface $F_j = \text{const}$, that is, the hypersurface consisting of the lines, tangent to Q_{λ_j}. According to Theorem 5.1, these characteristics consist of the lines, tangent to a fixed geodesic on Q_{λ_j}. According to statement 1 of the present theorem, these lines are tangent to Q_{λ_k}, hence F_k does not change along the flow of sgrad F_j. Thus $dF_k(\text{sgrad } F_j) = 0$, as claimed.

Finally, in the light-like case, consider the homogeneous symplectic manifold P^{2n-2} of scaled light-like lines whose quotient is $\mathcal{L}_0$. Then, as before, we have homogeneous of degree zero, Poisson-commuting functions $F_j,\ j = 1, \dots, n-2$, on P. Therefore a level submanifold $M^n = \{F_1 = c_1, \dots, F_{n-2} = c_{n-2}\}$ is isotropic: the symplectic orthogonal complement to TM in TP is contained in TM. The commuting vector fields sgrad F_j define an action of the Abelian group $\mathbf{R}^{n-2}$ on M whose orbits are isotropic submanifolds. Furthermore, M is invariant under the Euler vector field E, and since E commutes with each sgrad F_j, the Euler vector field preserves the foliation on isotropic submanifolds. Hence the quotient by E is a co-Legendrian submanifold in $\mathcal{L}_0$ foliated by Legendrian submanifolds which are quotients of $\mathbf{R}^{n-2}$ by a discrete subgroup. $\Box$

Example 5.9 Let γ be a geodesic on a generic ellipsoid Q_0 in 3-dimensional Lorentz space and let x be a point of γ. Then, upon each return to point x, the curve γ has one of at most two possible directions (a well known property in the Euclidean case).

Indeed, if γ is not light-like then the tangent lines to γ are tangent to a fixed quadratic surface, say Q_1, from the family (13). The intersection of the tangent plane T_xQ_0 with Q_1 is a conic, and there are at most two tangent lines from x to this conic. If γ is light-like then its direction at point x is in the kernel of the restriction of the metric to T_xQ_0 which consists of at most two lines.

Remark 5.10 The functions $F_1, \dots, F_{n-1}$ from the proof of Theorem 5.8 can be considered as functions on the tangent bundle TM. The proof of Theorem

5.8 implies that these functions and the energy function $F_0(x, v) = \langle v, v\rangle$ pairwise Poisson commute with respect to the canonical symplectic structure on T^*M (as usual, identified with TM via the metric).

Note also that the geodesic flow on the contact manifold of light-like geodesics on an ellipsoid seems to give an example of a contact integrable system, considered, e.g. in [5, 16]. Indeed, it is obtained from an integrable system commuting with dilations on the corresponding symplectization of the contact manifold. (Note however, that this is not a torus contact manifold as the $\mathbf{R}^{n-2}$-action of the Hamiltonians F_j is defined only locally.) After quotienting out the dilations we obtain here a co-Legendrian submanifold N^{n-1} in a contact $2n-3$-dimensional manifold, and the projections of symplectic gradients of F_j are $n-2$ commuting vector fields on N. Thus N is foliated by Legendrian tori of codimension 1. The simplest example is the geodesic flow on an ellipsoid Q in 3-D Lorentz space: for light vectors we have only the energy integral (equal to zero). The space of light-like lines tangent to Q is 2-dimensional; the leaves of the foliation by Legendrian curves consists of the lines tangent to a geodesic.

5.5 Alternative approach to integrability

Another approach to complete integrability of the geodesic flow on the ellipsoid in Euclidean space, also applicable to the billiard in the ellipsoid, is described in [29, 30]; see also [31, 24] for a survey. In a nutshell, in the case of billiards one constructs another symplectic form on the space of oriented lines invariant under the billiard map, and for the geodesic flow one constructs another metric on the ellipsoid, projectively equivalent to the Euclidean one: this means that their non-parameterized geodesics coincide (for geodesic flows, this integrability mechanism was independently discovered by Matveev and Topalov [18, 19]). We refer to the cited papers for the problem of obtaining integrals of the billiard map or the geodesic flow in this set-up.

We shall outline a similar approach in the pseudo-Euclidean setting.

Let Q be an ellipsoid in pseudo-Euclidean space V^n given by the equation $A(x) \cdot x = 1$ and let the scalar product be $\langle u, v\rangle = E(u) \cdot v$ where A and E are self-adjoint operators $V \to V^*$. Assume that $A = \mathrm{diag}(a_1^2, \dots, a_n^2)$ and $E = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$. Denote the billiard map in Q by T. Consider the interior of Q as the projective, or Cayley-Klein, model of hyperbolic geometry and let Ω be the respective symplectic structure on the space of oriented lines (obtained by the standard symplectic reduction).

Theorem 5.11 *1) The symplectic structure Ω is invariant under T.*
2) The restrictions of the metrics

$$\langle dx, dx \rangle \quad \text{and} \quad \frac{A(dx)\cdot dx}{\langle A(x), A(x)\rangle} \tag{15}$$

on the ellipsoid Q are projectively equivalent.

Proof. We already mentioned in the proof of Theorem 4.1 that the billiard in pseudo-Euclidean space is a particular case of the projective billiard. Let Q be a smooth hypersurface in vector space V and $v(x)$ a smooth transverse vector field along Q. The projective billiard reflection law is as follows: the velocity of the incoming mass-point is decomposed at the impact point x on the transversal, proportional to $v(x)$, and the tangential components; the former instantaneously changes sign and whereas latter remains intact. In the case of pseudo-Euclidean billiards, $v(x)$ is the normal to Q at point x.

It is proved in [29] that if Q is a Euclidean sphere and $v(x)$ is an *exact* vector field then the projective billiard map preserves the symplectic structure Ω. A transverse field $v(x)$ on the unit sphere is exact if $v(x) = x + \text{grad } f(x)$ where f is a smooth function on the sphere. Exactness is a projectively-invariant property.

We have: $Q = \langle EA(x), x\rangle$, and hence the vector $EA(x)$ is normal to Q at point x. Consider those (non-singular) points x for which $EA(x)$ is transverse to T_xQ, that is, $\langle EA(x), EA(x)\rangle \neq 0$. The linear map $A^{1/2}$ takes Q to the unit sphere S and the normal field $EA(x)$ to the field $A^{1/2}EA^{1/2}(y)$. Set $B = A^{1/2}EA^{1/2}$ and consider the normalized field $v(y) = B(y)/(B(y)\cdot y)$. Then

$$v(y) = y + \frac{1}{2}\,\text{grad }\,\log(B(y)\cdot y),$$

and therefore $v(y)$ is exact. Thus $T^*(\Omega) = \Omega$.

To prove the second statement, let $v(y)$ be a transverse vector field along a hypersurface Q in a vector space. A curve on Q is called a v-geodesic if its osculating plane at every point x is spanned by its tangent vector and the vector $v(x)$. It is proved in [30] that if $v(y) = y + \text{grad}\, g(y)$ is an exact vector field on the unit Euclidean sphere then the v-geodesics are the geodesics of the metric, conformally equivalent to the Euclidean metric with the conformal factor $\exp(-2g(y))$.

In our situation, this factor is $1/(B(y)\cdot y)$. Applying the linear map $A^{-1/2}$ and returning back to the original ellipsoid, we obtain the metric given by the second formula (15). Note that the denominator of this formula (15) vanishes precisely at the singular points of the ellipsoid where $\langle EA(x), EA(x)\rangle = \langle A(x), A(x)\rangle = 0$. $\square$

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