Strongly homotopy algebras of a Kähler manifold

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Abstract

It is shown that any compact Kähler manifold M gives canonically rise to two strongly homotopy algebras, the first one being associated with the Hodge theory of the de Rham complex and the second one with the Hodge theory of the Dolbeault complex. In these algebras the product of two harmonic differential forms is again harmonic.

If M happens to be a Calabi-Yau manifold, there exists a third strongly homotopy algebra closely related to the Barannikov-Kontsevich extended moduli space of complex structures.

1 Introduction

Strongly homotopy algebras have been introduced by Stasheff [7] more than 30 years ago in the context of topological H-spaces. Since that time this remarkable structure (and its cousins) has been found at a number of unexpected places, for example in the string field theory, in topological conformal field theories and in Floer's theory (see, e.g., [8, 9, 3, 4]).

In this paper we show that in the heart of the classical Hodge theory of Kähler manifolds (see, e.g., [2]) there lies a strongly homotopy algebra which "explains" the following imperfect behavior of harmonic differential forms: the wedge product, $\alpha \wedge \beta$, of two harmonic forms α and β is *not* harmonic in general. This, of course, can be easily fixed by defining a new product $\circ$,

$$\alpha \circ \beta := [\alpha \wedge \beta]_{\mathrm{Harm}},$$

$[\ ]_{\mathrm{Harm}}$ being the projection to the harmonic constituent, but at a price — the product $\circ$ fails to be associative. Among the main results of the paper is an observation that $\circ$ does satisfy the *higher order associativity conditions* thereby unveiling the structure of strongly homotopy algebra in the Hodge theory of Kähler manifolds.

In fact there are at least two strongly homotopy algebras associated with a compact Kähler manifold M. The first one is real and is associated with the Hodge theory of the de Rham complex $(\Omega^\bullet, d)$; the second one is complex (and is not a complexification of the first) and is associated with the Hodge theory of the Dolbeault complex $(\Omega^{\bullet,\bullet}, \bar{\partial})$.

If M happens to be a Calabi-Yau manifold, there exists a third strongly homotopy algebra closely related to the Barannikov-Kontsevich extended moduli space of complex structures [1].

The paper is organized as follows. In Sect. 2 we remind the definition of a strongly homotopy algebra. In Sect. 3 we consider a specific algebraic data which gives rise to a particular strongly homotopy algebra; a remarkable feature of the construction is that it provides *explicit* formulae for all the higher homotopies. In Sect. 4 we consider a compact Kähler manifold and prove the results mentioned above.

2 Strongly homotopy algebras

A *strongly homotopy algebra*, or shortly A_∞-*algebra*, is by definition a vector superspace V equipped with linear maps,

$$\begin{array}{rccc} \mu_k : & \otimes^k V & \longrightarrow & V \\ & v_1 \otimes \ldots \otimes v_k & \longrightarrow & \mu_k(v_1, \ldots, v_k), \qquad k \geq 1, \end{array}$$

of parity $\tilde{k} := k \bmod 2\mathbb{Z}$ satisfying, for any $n \geq 1$ and any $v_1, \ldots, v_n \in V$, the following *higher order associativity conditions*,

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \mu_k \left(v_1, \ldots, v_j, \mu_l(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n\right) = 0, \tag{1}$$

where $r = \tilde{l}(\tilde{v}_1 + \ldots + \tilde{v}_j) + \tilde{j}(\tilde{l} - 1) + (\tilde{k} - 1)\tilde{l}$ and $\tilde{v}$ denotes the parity of $v \in V$.

Denoting $dv_1 := \mu_1(v_1)$ and $v_1 \circ v_2 := \mu_2(v_1, v_2)$, one may depict explicitly the first four floors of the infinite tower of higher order associativity conditions as follows

$n = 1$**:** $d^2 = 0$,

$n = 2$**:** $d(v_1 \circ v_2) = (dv_1) \circ v_2 + (-1)^{\tilde{v}_1} v_1 \circ (dv_2)$,

$n = 3$**:** $v_1 \circ (v_2 \circ v_3) - (v_1 \circ v_2) \circ v_3 = d\mu_3(v_1, v_2, v_3) + \mu_3(dv_1, v_2, v_3) + (-1)^{\tilde{v}_1} \mu_3(v_1, dv_2, v_3) + (-1)^{\tilde{v}_1 + \tilde{v}_2} \mu_3(v_1, v_2, dv_3)$,

$n = 4$**:** $\mu_3(v_1, v_2, v_3) \circ v_4 - \mu_3(v_1 \circ v_2, v_3, v_4) + \mu_3(v_1, v_2 \circ v_3, v_4) - \mu_3(v_1, v_2, v_3 \circ v_4) + (-1)^{\tilde{v}_1} v_1 \circ \mu_3(v_2, v_3, v_4) = d\mu_4(v_1, v_2, v_3, v_4) - \mu_4(dv_1, v_2, v_3, v_4) - (-1)^{\tilde{v}_1} \mu_4(v_1, dv_2, v_3, v_4) - (-1)^{\tilde{v}_1 + \tilde{v}_2} \mu_4(v_1, v_2, dv_3, v_4) - (-1)^{\tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3} \mu_4(v_1, v_2, v_2, v_3, dv_4)$.

Therefore A_∞-algebras with $\lambda_k = 0$ for $k \geq 3$ are nothing but the differential associative superalgebras with the differential μ_1 and the associative multiplication defined by μ_2. If, furthermore, $\mu_1 = 0$, one recovers the usual associative superalgebras.

The notion of A_∞-algebra is a very natural extension of the usual concept of associative superalgebra. The following well-known fact [6] may serve as a confirmation of this statement: the moduli space of infinitesimal deformations of an associative superalgebra A

within the class of associative superalgebras is isomorphic to the second Hochschild cohomology group $\mathrm{Hoch}^2(A, A)$, while the moduli space of infinitesimal deformations of the same A within the class of A_∞-algebras can be identified with the full Hochschild cohomology $\mathrm{Hoch}^*(A, A)$. In this sense A_∞-algebra is an "utter" concept.

3 An explicit construction of an A_∞-algebra

Let (V, d) be a differential associative superalgebra, with d denoting the differential, i.e. an odd linear map $d : V \longrightarrow V$ satisfying the Leibnitz identity

$$d(v_1 \cdot v_2) = (dv_1) \cdot v_2 + (-1)^{\tilde{v}_1} v_1 \cdot (dv_2),$$

for any $v_1, v_2 \in V$.

Let W be a linear subspace of V invariant under d. Note that we do *not* assume that W is a subalgebra of V. Instead we make the following

Assumption 3.1 *There exists an odd operator*

$$Q : V \longrightarrow V$$

such that for any $v \in V$ the element $(1 - [d, Q])v$ lies in the subspace W, where $[\,,\,]$ is the supercommutator.

Put another way we assume that the operator $P := \mathrm{Id} - [d, Q]$ defines a (not necessary surjective) map from V to W. It may not be an identity when restricted to $W \subset V$.

Since $[d, P] = 0$, the product

$$v_1 \circ v_2 := (1 - [d, Q])(v_1 \cdot v_2)$$

makes $(W, d, \circ)$ into a differential superalgebra which is *not* associative in general. We shall show, however, that $\circ$ does satisfy the higher associativity conditions; moreover, we will be able to compute all the higher homotopies μ_k *explicitly* in terms of d and Q only.

First we define a series of linear maps

$$\lambda_n : \otimes V^n \longrightarrow V, \qquad n \geq 2,$$

starting with

$$\lambda_2(v_1, v_2) := v_1 \cdot v_2$$

and then recursively, for $n \geq 3$,

$$\begin{aligned} \lambda_n(v_1, \ldots, v_n) \;:=\; & (-1)^{n-1}[Q\lambda_{n-1}(v_1, \ldots, v_{n-1})] \cdot v_n - (-1)^{n\tilde{v}_1} v_1 \cdot [Q\lambda_{n-1}(v_2, \ldots, v_n)] \\ & - \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} (-1)^{k+(l-1)(\tilde{v}_1+\ldots+\tilde{v}_k)} [Q\lambda_k(v_1, \ldots, v_k)] \cdot [Q\lambda_l(v_{k+1}, \ldots, v_n)]. \end{aligned} \tag{2}$$

For example,

$$\begin{aligned}
\lambda_3(v_1,v_2,v_3) &= [Q\lambda_2(v_1,v_2)]\cdot v_3 - (-1)^{\tilde v_1} v_1\cdot[Q\lambda_2(v_2,v_3)],\\
\lambda_4(v_1,v_2,v_3,v_4) &= -[Q\lambda_3(v_1,v_2,v_3)]\cdot v_4 - (-1)^{\tilde v_1+\tilde v_2}[Q\lambda_2(v_1,v_2)]\cdot[Q\lambda_2(v_3,v_4)]\\
&\quad -v_1\cdot[Q\lambda_3(v_2,v_3,v_4)]\\
\lambda_5(v_1,v_2,v_3,v_4,v_5) &= [Q\lambda_4(v_1,v_2,v_3,v_4)]\cdot v_5 + (-1)^{\tilde v_1+\tilde v_2+\tilde v_3}[Q\lambda(v_1,v_2,v_3)]\cdot[Q\lambda_2(v_4,v_5)]\\
&\quad -[Q\lambda_2(v_1,v_2)]\cdot[Q\lambda_3(v_3,v_4,v_5)] - (-1)^{\tilde v_1} v_1\cdot[Q\lambda_4(v_2,v_3,v_4,v_5)].
\end{aligned}$$

Setting formally

$$\lambda_1 := -Q^{-1}$$

(which makes sense because in all our formulae below λ_1 enters in the combination $Q\lambda_1 = -\mathrm{Id}$), one can rewrite the recursion (2) as follows

$$\lambda_n(v_1,\dots,v_n) = -\sum_{\substack{k+l=n+1\\ k,l\geq 1}} (-1)^{k+(l-1)(\tilde v_1+\ldots+\tilde v_k)}[Q\lambda_k(v_1,\dots,v_k)]\cdot[Q\lambda_l(v_{k+1},\dots,v_n)], \tag{3}$$

where now $n\geq 2$.

Lemma 3.2 *The tensors λ_k, $k\geq 2$, satisfy the identities*

$$\Phi_n(v_1,\dots,v_n) \equiv \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{j=0}^{k-1} (-1)^r \lambda_k\left(v_1,\dots,v_j,\lambda_l(v_{j+1},\dots,v_{j+l}),v_{j+l+1},\dots,v_n\right) = 0,$$

$$r = l(\tilde v_1+\ldots\tilde v_j) + j(l-1) + (k-1)l,$$

for any $n\geq 3$ and any $v_1,\dots,v_n\in V$.

Proof. First we split Φ_n,

$$\begin{aligned}
\Phi_n(v_1,\dots,v_n) &= \sum_{\substack{k+l=n+1\\ k,l\geq 2}} (-1)^{(k-1)l}\lambda_k\left(\lambda_l(v_1,\dots,v_l),v_{l+1},\dots,v_n\right) +\\
&\quad \sum_{\substack{k+l=n+1\\ k,l\geq 2}} (-1)^{l(\tilde v_1+\ldots+\tilde v_{k-1})+k-1}\lambda_k\left(v_1,\dots,v_{k-1},\lambda_l(v_k,\dots,v_n)\right) +\\
&\quad \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{j=1}^{k-2} (-1)^r \lambda_k\left(v_1,\dots,v_j,\lambda_l(v_{j+1},\dots,v_{j+l}),v_{j+l+1},\dots,v_n\right) = 0.
\end{aligned} \tag{4}$$

It is an easy calculation using (3) to check that

$$\begin{aligned}
&-\sum_{\substack{k+l=n+1\\ k,l\geq 2}} (-1)^{l+k(\tilde v_1+\ldots+\tilde v_l)}\lambda_l(v_1,\dots,v_l)\cdot[Q\lambda_{k-1}(v_{l+1},\dots,v_n)]\\
&+\sum_{\substack{k+l=n+1\\ k+l\geq 2}} (-1)^{l(\tilde v_1+\ldots+\tilde v_{k-1})}[Q\lambda_{k-1}(v_1,\dots,v_{k-1})]\cdot\lambda_l(v_k,\dots,v_n) \quad = \quad 0.
\end{aligned}$$

Then the first two sums in (4) reduce, again with the help of (3), to the following expression

$$-\sum_{\substack{k+l=n+1\\k,l\geq 2}}\ \sum_{\substack{s+t=k\\s\geq 2\\t\geq 1}}(-1)^p\left[Q\lambda_s\left(\lambda_l(v_1,\ldots,v_l),v_{l+1},\ldots,v_{s+l-1}\right)\right]\cdot\left[Q\lambda_t(v_{l+s},\ldots,v_n)\right]$$

$$-\sum_{\substack{k+l=n+1\\k,l\geq 2}}\ \sum_{\substack{s+t=k\\s\geq 1\\t\geq 2}}(-1)^q\left[Q\lambda_s\left(v_1,\ldots,v_s\right)\right]\cdot\left[Q\lambda_t\left(v_{s+1},\ldots,v_{k-1},\lambda_l(v_k,\ldots,v_n)\right)\right],$$

where $p=k(l-1)+s+(t-1)(\tilde{v}_1+\ldots+\tilde{v}_{s+l-1}+l)$ and $q=l(\tilde{v}_1+\ldots+\tilde{v}_{k-1})+k-1+s+(t-1)(\tilde{v}_1+\ldots+\tilde{v}_s)$. The third sum in (4) splits into the following two sums

$$\sum_{\substack{k+l=n+1\\k,l\geq 2}}\ \sum_{j=1}^{k-2}\ \sum_{\substack{s+t=k\\1\leq s\leq j\\t\geq 1}}(-1)^a\left[Q\lambda_s(v_1,\ldots,v_s)\right]\cdot\left[Q\lambda_t\left(v_{s+1},\ldots,v_j,\lambda_l(v_{j+1},\ldots,v_{j+l}),v_{j+l-1},\ldots,v_n\right)\right]$$

$$-\sum_{\substack{k+l=n+1\\k,l\geq 2}}\ \sum_{j=1}^{k-2}\ \sum_{\substack{s+t=k\\s\geq j+1\\t\geq 1}}(-1)^b\left[Q\lambda_s\left(v_1,\ldots,v_j,\lambda_l(v_{j+1},\ldots,v_{j+l}),v_{j+l+1},\ldots,v_{s+l-1}\right)\right]\cdot\left[Q\lambda_t(v_{s+l},\ldots,v_n)\right],$$

where $a=r+s-1+(t-1)(\tilde{v}_1+\ldots+\tilde{v}_s)$ and $b=r+s+(t-1)(\tilde{v}_1+\ldots+\tilde{v}_{s+l-1}+l)$.

Substituting these two expressions back into (4) one easily gets the following recursive formula

$$\begin{aligned}\Phi_n(v_1,\ldots,v_n) &= \sum_{\substack{k+l=n\\k\geq 3\\l\geq 1}}(-1)^{(l-1)(\tilde{v}_1+\ldots+\tilde{v}_k)+k}\left[Q\Phi_k(v_1,\ldots,v_k)\right]\cdot\left[Q\lambda_l(v_{k+1},\ldots,v_n)\right]\\ &\quad-\sum_{\substack{k+l=n\\k\geq 1\\l\geq 3}}(-1)^{l(\tilde{v}_1+\ldots+\tilde{v}_k)}\left[Q\lambda_k(v_1,\ldots,v_k)\right]\cdot\left[Q\Phi_l(v_{k+1},\ldots,v_n)\right],\end{aligned}$$

where $n\geq 4$.

Finally we compute

$$\Phi_3(v_1,v_2,v_3)=(v_1\cdot v_2)\cdot v_3-v_1\cdot(v_2\cdot v_3)=0.$$

Thus $\Phi_n=0$ for all $n\geq 3$. $\square$

Lemma 3.3 *The tensors λ_k, $k\geq 2$, satisfy, for any $n\geq 2$ and any $v_1,\ldots,v_n\in V$, the identities*

$$\begin{aligned}\Theta_n(v_1,\ldots,v_n) &\equiv d\lambda_n(v_1,\ldots,v_n)+\sum_{j=0}^{n-1}(-1)^{n-1+\tilde{v}_1+\ldots+\tilde{v}_j}\lambda_n(v_1,\ldots,v_j,dv_{j+1},v_{j+2},\ldots,v_n)\\ &\quad-\sum_{\substack{k+l=n+1\\k,l\geq 2}}\ \sum_{j=0}^{k-1}(-1)^r\lambda_k\left(v_1,\ldots,v_j,[d,Q]\lambda_l(v_{j+1},\ldots,v_{j+l}),v_{j+l+1},\ldots,v_n\right)\\ &= 0\end{aligned}$$

where $r=l(\tilde{v}_1+\ldots\tilde{v}_j)+j(l-1)+(k-1)l$.

A scetch of the proof. Following the same scenario as the one used in the proof of Lemma 3.2 (i.e. studing separately the group of terms with extreme values of the index j, where some cancellations occur, and applying (3) throughout), one gets the following recursion formula

$$\begin{aligned}
\Theta_n(v_1,\dots,v_n) \;=\; & \sum_{\substack{k+l=n\\ k\geq 2\\ l\geq 1}} (-1)^{(l-1)(\tilde v_1+\ldots+\tilde v_k)+k}\,[Q\Theta_k(v_1,\dots,v_k)]\cdot[Q\lambda_l(v_{k+1},\dots,v_n)] \\
& - \sum_{\substack{k+l=n\\ k\geq 1\\ l\geq 2}} (-1)^{l(\tilde v_1+\ldots+\tilde v_k)}\,[Q\lambda_k(v_1,\dots,v_k)]\cdot[Q\Theta_l(v_{k+1},\dots,v_n)]\,,
\end{aligned}$$

where $n\geq 3$.

Next we compute

$$\Theta_2(v_1,v_2) = d(v_1\cdot v_2) - (dv_1)\cdot v_2 - (-1)^{\tilde v_1} v_1\cdot(dv_2) = 0.$$

Thus $\Phi_n = 0$ for all $n\geq 1$. $\square$

Theorem 3.4 *Let (V,d) be a differential associative superalgebra and let $(W,d)\subset(V,d)$ be a subcomplex satisfying the Assumption 3.1. Then the linear maps*

$$\mu_k : \otimes^k W \longrightarrow W, \quad k\geq 1,$$

defined by

$$\begin{aligned}
\mu_1 &:= d,\\
\mu_k &:= (1-[d,Q])\lambda_k, \quad \textit{for } k\geq 2,
\end{aligned}\tag{5}$$

with λ_k being given by (3), satisfy the higher order associativity identitites (1) for all $n\geq 1$ and thus define on W the structure of A_∞-algebra.

Proof. Denote the l.h.s. of the equation (1) by Ψ_n. Then Ψ_1 and Ψ_2 vanish because $(W.d)$ is a subcomplex of (V,d). For $n\geq 3$ one has

$$\Psi_n = (1-[d,Q])\,(\Phi_n+\Theta_n)\,.$$

By Lemmas 3.2 and 3.3, the tensors Φ_n and Θ_n vanish for all $n\geq 3$. This completes the proof. $\square$

Note that the higher homotopies μ_n, $n\geq 2$, can be written as follows

$$\mu_n(v_1,\dots,v_n) = -\sum_{\substack{k+l=n+1\\ k,l\geq 1}} (-1)^{k+(l-1)(\tilde v_1+\ldots+\tilde v_k)}[Q\lambda_k(v_1,\dots,v_k)]\circ[Q\lambda_l(v_{k+1},\dots,v_n)].$$

4 A_∞-algebras of a Kähler manifold

4.1. Hodge theory: a brief overview. Let M be a compact Kähler manifold, $\Omega^\bullet M$ the algebra of real smooth differential forms on M, $\Omega^{\bullet,\bullet}M$ the algebra of complex differential forms with Hodge gradation, and

$$\partial : \Omega^{\bullet,\bullet}M \longrightarrow \Omega^{\bullet+1,\bullet}M, \qquad \bar\partial : \Omega^{\bullet,\bullet}M \longrightarrow \Omega^{\bullet,\bullet+1}M,$$

the standard derivations. We also consider two real derivations

$$d = \partial + \bar\partial : \Omega^\bullet M \longrightarrow \Omega^{\bullet+1}M, \qquad d_c = i(\partial - \bar\partial) : \Omega^\bullet M \longrightarrow \Omega^{\bullet+1}M.$$

The Kähler metric g on M gives rise to a Hermitian metric on $\Omega^{\bullet,\bullet}$ defined, say for (p,q)-forms

$$\alpha = \sum \alpha_{i_1\ldots i_p\bar j_1\ldots\bar j_q} dz^{i_1}\wedge\ldots\wedge dz^{i_p} d\bar z^{\bar j_1}\wedge\ldots\wedge d\bar z^{\bar j_q},$$

and

$$\beta = \sum \beta_{s_1\ldots s_p\bar t_1\ldots\bar t_q} dz^{s_1}\wedge\ldots\wedge dz^{s_p} d\bar z^{\bar t_1}\wedge\ldots\wedge d\bar z^{\bar t_q},$$

by the integral

$$(\alpha,\beta) := \int_M \left(\sum g^{i_1\bar s_1}\ldots g^{i_p\bar s_p} g^{t_1\bar j_1}\ldots g^{t_q\bar j_q}\, \alpha_{i_1\ldots i_p\bar j_1\ldots\bar j_q}\, \overline{\beta_{s_1\ldots s_p\bar t_1\ldots\bar t_q}}\right) \mathrm{vol}_g,$$

where $g_{i\bar j}$ are components of g in a local holomorphic coordinate chart $\{z^i\}$. The scalar product (α,β) of differential forms with different Hodge type is set to be zero.

This scalar product is used to define conjugate operators

$$\partial^* : \Omega^{\bullet,\bullet}M \longrightarrow \Omega^{\bullet-1,\bullet}M, \quad \bar\partial^* : \Omega^{\bullet,\bullet}M \longrightarrow \Omega^{\bullet,\bullet-1}M, \quad \Lambda : \Omega^{\bullet,\bullet}M \longrightarrow \Omega^{\bullet-1,\bullet-1}M,$$

by the formulae

$$(\partial\alpha,\beta) = (\alpha,\partial^*\alpha), \quad (\bar\partial\alpha,\beta) = (\alpha,\bar\partial^*\beta), \quad (\alpha\wedge\omega,\beta) = (\alpha,\Lambda\beta),$$

where ω denotes the Kähler form on M and $\alpha,\beta\in\Omega^{\bullet,\bullet}M$ are arbitrary. We shall also need the following real operators

$$d^* = \partial^* + \bar\partial^*, \qquad d_c^* = -i(\partial^* - \bar\partial^*).$$

There are four Laplacians,

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial,$$

$$\Delta_d = dd^* + d^*d, \qquad \Delta_{d_c} = d_c d_c^* + d_c^* d_c.$$

One of the central results of the Hodge theory says that

$$\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d = \frac{1}{2}\Delta_{d_c}.$$

A differential form $\alpha \in \Omega^{\bullet,\bullet}$ is called *harmonic* if $\Delta_d \alpha = 0$. The vector space Harm of all real (or complex) harmonic forms on M is finite dimensional and is isomorphic to its de Rham (or Dolbeault) cohomology. There is a natural orthogonal projection

$$\begin{array}{rccc} [\ ]_{\mathrm{Harm}}: & \Omega^{\bullet,\bullet} & \longrightarrow & \mathrm{Harm} \\ & \alpha & \longrightarrow & [\alpha]_{\mathrm{Harm}}. \end{array}$$

Note that all the Laplacians Δ become isomorphisms when restricted to the orthogonal complement, $\mathrm{Harm}^{\perp}$, of the space of harmonic forms.

There are very important for our purposes Hodge identities

$$[\Lambda, \bar{\partial}] = -i\partial^*, \qquad [\Lambda, \partial] = i\partial^*, \qquad [\Lambda, d] = d_c^*, \tag{6}$$

and the Hodge decompositions, for an arbitrary $\alpha \in \Omega^{\bullet,\bullet}$,

$$\begin{aligned} \alpha &= [\alpha]_{\mathrm{Harm}} + \partial G_{\partial} \partial^*(\alpha) + \partial^* G_{\partial} \partial(\alpha) \\ \alpha &= [\alpha]_{\mathrm{Harm}} + \bar{\partial} G_{\bar{\partial}} \bar{\partial}^*(\alpha) + \bar{\partial}^* G_{\bar{\partial}} \bar{\partial}(\alpha) \\ \alpha &= [\alpha]_{\mathrm{Harm}} + d G_d d^*(\alpha) + d^* G_d d(\alpha) \\ \alpha &= [\alpha]_{\mathrm{Harm}} + d_c G_{d_c} d_c^*(\alpha) + d_c^* G_{d_c} d_c(\alpha), \end{aligned}$$

where $G : \Omega^{\bullet,\bullet} \longrightarrow \Omega^{\bullet,\bullet}$ are the Green operators defined by

$$G_{\partial} \mid_{\mathrm{Harm}} = 0, \qquad G_{\partial} \mid_{\mathrm{Harm}^{\perp}} = \Delta_p^{-1},$$

and analogously for all others. A classical fact:

$$G_{\partial} = G_{\bar{\partial}} = 2G_d = 2G_{d_c}.$$

Note that $\partial G_{\partial} = G_{\partial}\partial$, $\partial^* G_{\partial} = G_{\partial}\partial^*$, etc.

For further details about Hodge theory on Kähler manifolds we refer to [2].

4.2. Real A_∞-algebra of a Kähler manifold. Consider the subcomplex $(W := \mathrm{Ker} d_c^*, d)$ in the de Rham differential superalgebra $(\Omega^\bullet M, d)$. It does satisfy the Assumption 3.1 with

$$Q = d_c G_d \Lambda.$$

Indeed,

$$1 - [d, Q] = 1 - d_c G_d [\Lambda, d] = 1 - d_c G_d d_c^*$$

which, according to the Hodge decomposition, is precisely the projector from $\Omega^\bullet M$ to its subspace $\mathrm{Ker} d_c^*$.

By Theorem 3.4, the space $\mathrm{Ker} d_c^*$ has canonically the structure of an A_∞-algebra with the higher homotopeis given explicitly by the formulae (5) and (2). For example,

$$\begin{aligned} \mu_1 &= d, \\ \mu_2(v_1, v_1) &= (1 - d_c G_d d_c^*)(v_1 \wedge v_2) \equiv v_1 \circ v_2, \\ \mu_3(v_1, v_2, v_3) &= [d_c G_d \Lambda(v_1 \wedge v_2)] \circ v_3 - (-1)^{\tilde{v}_1} v_1 \circ [d_c G_d \Lambda(v_2 \wedge v_3)], \\ &\text{etc.} \end{aligned}$$

In this algebra the product of two harmonic forms is obviously harmonic as well. Another interesting observation is that the complex $(\mathrm{Ker} d_c^*, d)$ is precisely the one which gives rise to a Frobenius manifold structure on the de Rham cohomology of M [5]. The same comments will apply to our next example.

4.3. Complex A_∞-algebra of a Kähler manifold. Consider the subcomplex ($W :=$ Ker$\partial^*, \bar\partial$) in the Dolbeault differential superalgebra $(\Omega^{\bullet,\bullet}M, \bar\partial)$. It satisfies the Assumption 3.1 with

$$Q = i\partial G_\partial \Lambda.$$

Indeed,

$$1 - [d, Q] = 1 - i\partial G_\partial[\Lambda, \partial] = 1 - \partial G \partial^*$$

which, according to the Hodge decomposition, is precisely the projector from $\Omega^{\bullet,\bullet}$ to the subspace Ker∂^*.

Therefore, by Theorem 3.4, the space Ker∂^* has canonically the structure of an A_∞-algebra and one may explicitly write all the higher homotopies μ_k in terms of operators ∂, ∂^*, G_∂ and Λ.

4.4. A_∞-algebra of a Calabi-Yau manifold. A compact Kähler manifold M is called Calabi-Yau if $c_1(M) = 0$ or, equivalently, if it admits a nowhere vanishing holomorphic section Ω of the canonical line bundle of M.

Consider a differential associative superalgebra

$$V := \Lambda^\bullet TM \otimes \Lambda^\bullet \overline{\Omega^1 M}$$

with the differential $\bar\partial$ and the multiplication $\cdot$ being given by the ordinary wedge products on $\Lambda^p TM$ and $\Lambda^q \overline{\Omega^1 M}$. It was first considered by Barannikov and Konstevich [1] in the context of mirror symmetry.

The holomorphic volume form defines an isomorphism

$$\begin{array}{rccc} \Omega: & \Lambda^\bullet TM \otimes \Lambda^\bullet \overline{\Omega^1 M} & \longrightarrow & \Lambda^{\dim M - \bullet}\Omega^1 M \otimes \Lambda^\bullet \overline{\Omega^1 M} \\ & \alpha \otimes \beta & \longrightarrow & (\alpha \lrcorner \Omega) \otimes \beta. \end{array}$$

Then for any linear operator

$$s : \Omega^{\bullet,\bullet} \longrightarrow \Omega^{\bullet,\bullet}$$

there is associated a linear operator

$$\hat s : V \longrightarrow V$$

defined as the composition:

$$\hat s : V \stackrel{\Omega}{\longrightarrow} \Omega^{\bullet,\bullet} \stackrel{s}{\longrightarrow} \Omega^{\bullet,\bullet} \stackrel{\Omega^{-1}}{\longrightarrow} V.$$

In this way one obtains the operators $\hat\partial$, $\hat\partial^*$, $\hat G_\partial$, etc. Note that $\hat{\bar\partial} = \bar\partial$. Note also that $\hat\partial$ is not a derivation of $(V, \cdot)$.

Consider the following subcomplex

$$(W := \mathrm{Ker}\hat\partial, \bar\partial)$$

in the Barannikov-Kontsevich differential superalgebra $(V, \bar\partial)$. It does satisfy the Assumption 3.1 with

$$Q = -i\hat\Lambda \hat G_\partial \hat\partial.$$

Indeed,

$$1-[d,Q]=1-i\widehat{[\Lambda,\bar{\partial}]}\hat{G}_{\partial}\hat{\partial}=1-\hat{\partial}^{*}\hat{G}_{\partial}\hat{\partial},$$

which, according to the Hodge decomposition, is precisely the projector from V to the subspace $\mathrm{Ker}\hat{\partial}$.

Therefore, by Theorem 3.4, the space $\mathrm{Ker}\hat{\partial}$ has canonically the structure of an A_∞-algebra and one may explicitly write down all the higher homotopies μ_k as explained in Sect.3. It is again a remarkable fact that the same complex $(\mathrm{Ker}\hat{\partial},\bar{\partial})$ admits not only an A_∞ but also a very special Gerstenhaber-Batalin-Vilkoviski structure which makes the extended moduli space of complex structures on M into a Frobenius manifold [1].

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References

[1] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices, no.4 (1998), 201-215.

[2] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 1978.

[3] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992) (Bürkhäuser, Basel), 1994, pp. 97-121.

[4] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol.1 (Zürich, 1994) (Bürkhäuser, Basel), 1995, pp. 120-139.

[5] S.A. Merkulov, *Formality of canonical symplectic complexes and Frobenius manifolds*, Internat. Math. Res. Notices, no. 14 (1998), 727-733.

[6] M. Penkava and A. Schwarz, *A_∞ algebras and the cohomology of moduli spaces*, Lie Groups and Lie algebras: E.B. Dynkin's seminar, AMS Providence, 1995, pp. 91-107.

[7] J.D. Stasheff, *On the homotopy associativity of H-spaces, II*, Trans. Amer. Math. Soc. **108** (1963), 293-312.

[8] J.D. Stasheff, *Higher homotopy algebras: string field theory and Drinfeld's quasi-Hopf algebras*, in Proceedings of the XXth International Conference on Differential Geometric methods in Theoretical Physics (New York, 1991)(River Edge, NJ), vol.1, 1992, pp.408-425.

[9] J.D. Stasheff, *Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli space*, Perspectives on Mathematics and Physics (R.C. Penner and S.Y. Yau, eds.), International Press, 1994, hep-th/930461, pp. 265-288.

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