Max-Planck-Institut für Mathematik
Bonn

A Fubini rule for ∞-coends

by

Fosco Loregian

Max-Planck-Institut für Mathematik
Preprint Series 2018 (68)

A Fubini rule for ∞-coends

Fosco Loregian

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 18-68

A FUBINI RULE FOR ∞-COENDS

FOSCO LOREGIAN

Abstract. We prove a Fubini rule for ∞-co/ends of ∞-functors $F : \mathcal{C}^o \times \mathcal{C} \to \mathcal{D}$. This allows to lay down "integration rules", similar to those in classical co/end calculus, also in the setting of ∞-categories.

Contents

1. Introduction

In [Lur17, §5.2.1] (we shorten the reference to this source to "HA" from now on, and similarly we call simply "T" the reference [Lur09]) the author introduces the definition of *twisted arrow* ∞*-category* of an ∞-category; in [GHN15] this paves the way to the definition of co/end for a ∞-functor $F : \mathcal{C}^o \times \mathcal{C} \to \mathcal{D}$. Here we briefly recall how this construction works.

Definition 1.1. Let $\epsilon\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be the functor $[n] \mapsto [n] \star [n]^o$. The *edgewise subdivision* $\mathrm{esd}(X)$ of a simplicial set S is defined to be the composite $\epsilon^* S$. If $\mathcal{C}$ is an ∞-category, we define $\mathrm{Tw}(\mathcal{C})$ to be the simplicial set $\epsilon^*\mathcal{C}$. The n-simplices of $\mathrm{Tw}(\mathcal{C})$ are, in particular, determined as

$$\mathrm{Tw}(\mathcal{C})_n \cong \mathbf{sSet}([n], \mathrm{Tw}(\mathcal{C})) = \mathbf{sSet}([n] \star [n]^o, \mathcal{C}).$$

Date: Tuesday 15$^{\text{th}}$ January, 2019.

Remark 1.2. In dimension 0 and 1, the n-simplices of $\mathrm{Tw}(\mathcal{C})$ correspond respectively to edges f of $\mathcal{C}$ and to commutative squares

$$\begin{array}{ccc} \cdot & \xleftarrow{\;s\;} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[\;t\;]{} & \cdot \end{array}$$

The canonical natural transformations given by the embedding of $[n], [n]^{\mathrm{o}}$ in the join entail that there is a well-defined *projection map* $\Sigma_{\mathcal{C}} : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{o}} \times \mathcal{C}$. Note that from HA.5.2.1.11 we deduce that $\Sigma_{\mathcal{C}}$ is the right fibration HA.5.2.1.3 (this entails that if $\mathcal{C}$ is an ∞-category, then $\mathrm{Tw}(\mathcal{C})$ is also an ∞-category) classified by $\mathrm{Map} : \mathcal{C}^{\mathrm{o}} \times \mathcal{C} \to \mathcal{S}$.

If the ∞-category $\mathcal{C}$ is of the form $\mathrm{N}(A)$ for some category A, $\mathrm{Tw}(\mathcal{C})$ corresponds to the nerve of the classical twisted arrow category of A, as defined in [ML98, IX.6.3].

Definition 1.3. Let $F : \mathcal{C}^{\mathrm{o}} \times \mathcal{C} \to \mathcal{D}$ be a functor; when it exists, the *end* of F is the limit

$$\int_C F := \lim_{\mathrm{Tw}(\mathcal{C})} (F \cdot \Sigma)$$

Dually, when it exists, the *coend* of F is the colimit

$$\int^C F := \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} (F \cdot \Sigma)$$

It is clear that a sufficient condition for $\int^C F$ to exists is that $\mathcal{D}$ is cocomplete, and dually a sufficient condition for $\int_C F$ to exist is that $\mathcal{D}$ is complete.

[GHN15] employs this notation to prove [*ibi*, Thm. 1.1] that

Theorem 1.4. *Suppose $F\colon \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ is a functor of ∞-categories,*

- LC1*) if $\mathcal{E} \to \mathcal{C}$ is a cocartesian fibration associated to F. Then $\mathcal{E}$ is the* lax colimit[1] *of the functor F.*
- LC2*) if $\mathcal{E} \to \mathcal{C}$ is a cartesian fibration associated to F. Then $\mathcal{E}$ is the* oplax colimit *of the functor F.*

[1]The lax colimit of $F : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ is defined by the coend

$$\int^C \mathcal{C}_{C/} \times F(C).$$

Dually, the oplax colimit of F is defined by the coend

$$\int^C \mathcal{C}_{/C} \times F(C),$$

where in both cases the weights are the *slice* ∞-categories of T.1.2.9.2 and T.1.2.9.5.

Lemma 1.5. Let $\mathcal{C}$ be a small ∞-category, and $\mathcal{D}$ be a presentable ∞-category; then $\mathcal{D}$ is tensored and cotensored over $\mathcal{S} = \mathrm{N}(\mathcal{K}\mathrm{an})$. This entails that there is a two-variable adjunction

$$\mathcal{D}^{\mathrm{o}} \times \mathcal{D} \xrightarrow{\mathrm{Map}} \mathcal{S} \qquad \mathcal{S} \times \mathcal{D} \xrightarrow{\odot} \mathcal{D} \qquad \mathcal{S}^{\mathrm{o}} \times \mathcal{D} \xrightarrow{\pitchfork} \mathcal{D}$$

such that

$$\mathcal{D}(X \odot D, D') \cong \mathcal{S}(X, \mathcal{D}(D, D')) \cong \mathcal{D}(D, X \pitchfork D')$$

From the existence of these isomorphisms it is clear that

$$V \odot (W \odot D) \cong W \odot (V \odot D) \cong (V \times W) \odot D \tag{1.1}$$

$$V \pitchfork (W \pitchfork D) \cong W \pitchfork (V \pitchfork D) \cong (V \times W) \pitchfork D \tag{1.2}$$

2. The Fubini rule

Lemma 2.1. Let $F : \mathcal{C}^{\mathrm{o}} \times \mathcal{C} \to \mathcal{D}$ be a ∞-functor and $\mathcal{C}, \mathcal{D}$ ∞-categories as in the assumptions of Lemma 1.5. Then

- $F \mapsto \int^C F$ is functorial, and it is a left adjoint;
- $F \mapsto \int_C F$ is functorial, and it is a right adjoint.

Proof. We only prove the first statement for coends; the other one is dual.

Since $\int^C F = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C})}(F \cdot \Sigma)$ acts on F as a composition of ∞-functors, it is clearly functorial; then in the diagram

$$\int^C : \ [\mathcal{C}^{\mathrm{o}} \times \mathcal{C}, \mathcal{D}] \underset{\mathrm{Ran}_\Sigma}{\overset{\Sigma^*}{\rightleftarrows}} [\mathrm{Tw}(\mathcal{C}), \mathcal{D}] \underset{c}{\overset{\mathrm{colim}_{\mathrm{Tw}(\mathcal{C})}}{\rightleftarrows}} \mathcal{D}$$

the composition $\int^C = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C})} \cdot \Sigma^*$ is a left adjoint because it is a composition of left adjoints ($c = t^*$ is the constant functor inverse image of the terminal map $\mathrm{Tw}(\mathcal{C}) \to *$). Dually, the left adjoint to the end functor $\int_C$ is given by $\mathrm{Lan}_\Sigma \cdot c(D)$. □

Loosely speaking, the Fubini rule for co/ends asserts that when the domain of a functor $F : \mathcal{A}^{\mathrm{o}} \times \mathcal{A} \to \mathcal{D}$ results as a product $(\mathcal{C} \times \mathcal{E})^{\mathrm{o}} \times (\mathcal{C} \times \mathcal{E})$, then the co/ends of F can be computed as "iterated integrals"

$$\int^{(C,E)} F \cong \iint^{CE} F \cong \iint^{EC} F \tag{2.1}$$

$$\int_{(C,E)} F \cong \iint_{CE} F \cong \iint_{EC} F \tag{2.2}$$

These identifications hide a slight abuse of notation, that is worth to make explicit in order to avoid confusion: thanks to Lemma 2.1 the three objects of (2.1) can be thought as images of F along certain functors, and the Fubini rule asserts that they are linked by canonical isomorphisms; we can easily turn these functors into having the same type by means of the cartesian closed structure of **sSet**:

$$\iint^{CE} := [\mathcal{C}^o \times \mathcal{C} \times \mathcal{E}^o \times \mathcal{E}, \mathcal{D}] \cong [\mathcal{C}^o \times \mathcal{C}, [\mathcal{E}^o \times \mathcal{E}, \mathcal{D}]] \xrightarrow{[\mathcal{C}^o \times \mathcal{C}, \int^E]} [\mathcal{C}^o \times \mathcal{C}, \mathcal{D}] \xrightarrow{\int^C} \mathcal{D} \tag{2.3}$$

$$\iint^{EC} := [\mathcal{C}^o \times \mathcal{C} \times \mathcal{E}^o \times \mathcal{E}, \mathcal{D}] \cong [\mathcal{E}^o \times \mathcal{E}, [\mathcal{C}^o \times \mathcal{C}, \mathcal{D}]] \xrightarrow{[\mathcal{E}^o \times \mathcal{E}, \int^C]} [\mathcal{E}^o \times \mathcal{E}, \mathcal{D}] \xrightarrow{\int^E} \mathcal{D} \tag{2.4}$$

$$\int^{(C,E)} := [\mathcal{C}^o \times \mathcal{C} \times \mathcal{E}^o \times \mathcal{E}, \mathcal{D}] \cong [(\mathcal{C} \times \mathcal{E})^o \times (\mathcal{C} \times \mathcal{E}), \mathcal{D}] \to \mathcal{D}. \tag{2.5}$$

(of course, we can provide similar definitions for the iterated end functor).

Once that this has been clarified, we can deduce the isomorphisms (2.1) and (2.2) from the fact that the three functors $\iint^{CE}, \iint^{EC}, \int^{(C,E)}$ have right adjoints isomorphic to each other, and hence they must be isomorphic themselves.

Theorem 2.2 (Fubini rule for co/ends)**.** *Let $F : \mathcal{C}^o \times \mathcal{C} \times \mathcal{E}^o \times \mathcal{E} \to \mathcal{D}$ be a ∞-functor. Then the ∞-functors $\iint^{CE}, \iint^{EC}, \int^{(C,E)}$ of* (2.3), (2.4), (2.5) *are naturally isomorphic.*

In order to prove 2.2 we need a preliminary observation characterizing the right adjoint to $\int^C : [\mathcal{C}^o \times \mathcal{C}, \mathcal{D}] \to \mathcal{D}$.

Lemma 2.3. The functor $R = \mathrm{Ran}_\Sigma(c(\_))$ acts "cotensoring with mapping space": more precisely, the functor $RD : \mathcal{C}^o \times \mathcal{C} \to \mathcal{D}$ is isomorphic to the functor

$$(C, C') \mapsto \mathrm{Map}_\mathcal{C}(C, C') \pitchfork D$$

Dually, the functor $L = \mathrm{Lan}_\Sigma(c(\_))$ acts "tensoring with mapping space": more precisely, the functor $LD : \mathcal{C}^o \times \mathcal{C} \to \mathcal{D}$ is isomorphic to the functor

$$(C, C') \mapsto \mathrm{Map}_\mathcal{C}(C, C') \odot D$$

Proof. We only prove the first statement for coends; the other one is dual.

Being $c(D)$ the constant functor on $D \in \mathcal{D}$, the pointwise formula for right Kan extensions (see [Cis, 6.4.9] for the fact that "Ran are limits") yields that the desired limit consists of cotensoring with the slice category $(C, C')/\Sigma$ regarded as a simplicial set (in the Kan-Quillen model structure); now, if we consider the diagram

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{C}}(C,C') & \xrightarrow{\sim} & (C,C')/\Sigma & \longrightarrow & \mathrm{Tw}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^0 & \xrightarrow[\sim]{} & (\mathcal{C}^o \times \mathcal{C})_{(C,C')/} & \longrightarrow & \mathcal{C}^o \times \mathcal{C}
\end{array}$$

expressing the fiber of Σ, i.e. the mapping spaces $\mathrm{Map}_{\mathcal{C}}(C,C')$, as suitable pullbacks, we can easily see that $(\mathcal{C}^o \times \mathcal{C})_{(C,C')/}$ is contractible in Kan-Quillen (it has an initial object), hence, in the ∞-category of spaces, the object $(C,C')/\Sigma$ exhibits the same universal property of $\mathrm{Map}_{\mathcal{C}}(C,C')$. Since the functor $\_ \pitchfork D$ preserves Kan-Quillen weak equivalences, it turns out that

$$\mathrm{Ran}_{\Sigma}(c(D)) \cong (C,C')/\Sigma \pitchfork D \cong \mathrm{Map}_{\mathcal{C}}(C,C') \pitchfork D,$$

and this concludes the proof. □

Proof of 2.2. The Fubini rule now follows from uniqueness of adjoints (T.5.2.1.3, T.5.2.1.4): in diagram

$$\begin{array}{ccc}
\lambda F. \int^C\int^E F & \longmapsto & \lambda D.\lambda CC'.\lambda EE'.\mathrm{Map}_{\mathcal{C}}(C,C') \pitchfork \Big(\mathrm{Map}_{\mathcal{E}}(E,E') \pitchfork D\Big) \\
 & & \| \wr \\
\lambda F. \int^E\int^C F & \longmapsto & \lambda D.\lambda EE'.\lambda CC'.\mathrm{Map}_{\mathcal{E}}(E,E') \pitchfork \Big(\mathrm{Map}_{\mathcal{C}}(C,C') \pitchfork D\Big) \\
 & & \| \wr \\
 & & \lambda D.\lambda CEC'E'\Big(\mathrm{Map}_{\mathcal{C}}(C,C') \times \mathrm{Map}_{\mathcal{E}}(E,E')\Big) \pitchfork D \\
 & & \| \wr \\
\lambda F. \int^{(C,E)} F & \longmapsto & \mathrm{Map}_{\mathcal{C}\times\mathcal{E}}\big((C,E),(C',E')\big) \pitchfork D
\end{array}$$

the vertical isomorphisms on the right are justified by (1.2). A completely analogous argument, using (1.1) instead, and the left adjoints given by tensoring with the derived mapping space, gives the Fubini rule for (2.2). □

Acknowledgements. This document was partially written during the author's stay at MPIM.

References

[Cis] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf.

[GHN15] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, arXiv preprint arXiv:1501.02161 (2015).

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.
[Lur17] ———, *Higher algebra*, online version September 18, 2017.
[ML98] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

Fosco Loregian
Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn — Germany
flore@mpim-bonn.mpg.de