Phase-Space Weyl Calculus and Global Hypoellipticity of a Class of Degenerate Elliptic Partial Differential Operators

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Abstract

In a recent series of papers M. W. Wong has studied a degenerate elliptic partial differential operator related to the Heisenberg group. It turns out that Wong's example is best understood when replaced in the context of the phase-space Weyl calculus we have developed in previous work; this approach highlights the relationship of Wong's constructions with the quantum mechanics of charged particles in a uniform magnetic field. Using Shubin's classes of pseudodifferential symbols we prove global hypoellipticity results for arbitrary phase-space operators arising from elliptic operators on configuration space.

Key index: degenerate elliptic operators, hypoellipticity, phase-space Weyl calculus, Shubin symbols

Introduction

In a recent series of interesting papers [23, 24, 25] M. W. Wong discusses various properties of the partial differential operator

$$W = -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z) \tag{1}$$

where Z and $\overline{Z}$ are the vector fields on $\mathbb{R}^2$ defined by

$$Z = \frac{\partial}{\partial z} + \frac{1}{2}z \;\; , \;\; \overline{Z} = \frac{\partial}{\partial \overline{z}} + \frac{1}{2}\overline{z}. \tag{2}$$

Writing $z = x + iy$ the operator W has the explicit form

$$W = -\Delta - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{4}(x^2 + y^2) \tag{3}$$

where Δ is the usual Laplace operator in the x, y variables. It turns out that Wong's results are only the "top of an iceberg" because the operator (3) can be viewed as the phase-space version of the Hermite operator $-\Delta + x^2$ obtained by using a quantization procedure we have introduced in previous work. To understand this, let us make the two following independent observations:

- The operator W has a well-known meaning in physics. Consider in fact an electron with mass m and charge e placed in a strong uniform magnetic field $\mathbf{B}$ directed along the z axis: $\mathbf{B} = (0, 0, B_z)$. The Hamiltonian operator is, in a particular choice of gauge (see Section 2),
$$H = -\frac{\hbar^2}{2m}\Delta + i\hbar\omega_L\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \frac{m\omega_L^2}{2}(x^2 + y^2) \tag{4}$$
where ω_L is the "Larmor frequency". This operator reduces to Wong's operator (3) if x and y are swapped and units are chosen so that $\hbar = 1$, $m = 1/2$, and $\omega_L = 1$; the spectrum of the operator (4) is well-known; it consists of the sequence "energy levels" $E_N = (2N + 1)\hbar\omega_L$ (see Subsection 2.1); the spectrum of W is thus given by the sequence of numbers, $E_N = 2N + 1$ for $N = 0, 1, ...$, a fact which Wong rediscovers in [23] using complicated calculations involving the Wigner formalism and special function theory.

- On the other hand, a straightforward calculation shows that we can rewrite the operator W more compactly as
$$W = \left(-i\frac{\partial}{\partial x} - \frac{1}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{1}{2}x\right)^2; \tag{5}$$
setting $p = -y$ this operator becomes
$$W = \left(-i\frac{\partial}{\partial x} + \frac{1}{2}p\right)^2 + \left(i\frac{\partial}{\partial p} + \frac{1}{2}x\right)^2 \tag{6}$$
which makes apparent that W is obtained from the harmonic oscillator Hamiltonian $H(x, p) = \frac{1}{2}(p^2 + x^2)$ using the "quantization rules"
$$x \longrightarrow X = i\frac{\partial}{\partial p} + \frac{1}{2}x \ \ , \ \ p \longrightarrow P = -i\frac{\partial}{\partial x} + \frac{1}{2}p \tag{7}$$

> we have studied and exploited in [9, 10] in connection with the phase space Schrödinger equation of Torres-Vega and Frederick [19, 20]; notice that X and P satisfy the same commutation relation $[X, P] = i$ as satisfied by the operators x and $-i\partial/\partial x$.

This paper consists of two parts. In the first part (Sections 1–2) we will focus on the "phase space Weyl calculus" aspect of Wong's operator and its generalizations; we also briefly review our previous definitions and results from [9, 10]. In the second part of this paper (Section 3) we show that our phase-space Weyl calculus together with Shubin's pseudodifferential calculus (Shubin [18], Chapter IV) allows us not only to recover (in a trivial way) the global hypoellipticity of Wong's operator W, but to prove that every operator

$$\widetilde{A} = a(i\tfrac{\partial}{\partial p} + \tfrac{1}{2}x, -i\tfrac{\partial}{\partial x} + \tfrac{1}{2}p)$$

obtained by replacing formally (x, p) by $(i\frac{\partial}{\partial p} + \frac{1}{2}x, -i\frac{\partial}{\partial x} + \frac{1}{2}p)$ in any positive-definite quadratic form $a = a(x, p)$ is globally hypoelliptic (we will in fact prove a $2n$-dimensional statement, allowing the symbol a to be defined on $\mathbb{R}^{2n}$).

Notation

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all smooth complex-valued functions on $\mathbb{R}^n$ which decrease, together with their derivatives, faster than the inverse than any polynomial when $|x| \to \infty$. The dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$.

Operators acting on functions (or distributions) defined on $\mathbb{R}^n$ will be denoted by capital letters A, B, C... while operators acting on functions (or distributions) defined on $\mathbb{R}^n$ defined on the symplectic space $(\mathbb{R}^{2n}, \sigma)$ will denoted by covering capital letters with a tilde: $\widetilde{A}, \widetilde{B}, \widetilde{C}$... Functions on $\mathbb{R}^n$ will usually be denoted by lower-case Greek letters ψ, ϕ... while functions on $\mathbb{R}^{2n}$ are denoted by upper-case Greek letters $\Psi, \Phi, ...$ We will use standard multi-index notation: if $\alpha = (\alpha_1, .., \alpha_n)$ is a sequence of non-negative integers we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if $x = (x_1, ..., x_n)$, and $D_x^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We will denote by σ the standard symplectic form on the vector space $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$:

$$\sigma(z, z') = p \cdot x' - p' \cdot x$$

for $z = (x, p)$, $z' = (x', p')$. The symplectic group of $(\mathbb{R}^{2n}, \sigma)$ will be denoted by $\mathrm{Sp}(n)$: it is the group of all linear automorphisms s of $\mathbb{R}^{2n}$ such that $\sigma(sz, sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$.

1 Phase-Space Weyl Calculus

For proofs and a detailed exposition we refer to de Gosson [10] (the phase space calculus was introduced in de Gosson [8, 9] following a suggestion in the paper by Grossmann *et al.* [13]).

1.1 Definitions

Let $A:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator. In view of Schwartz's kernel theorem there exists a distribution $K\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$Af(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy$$

(the integral being interpreted as a partial distribution bracket). The Weyl symbol of A is by definition (see *e.g.* [10], Theorem 6.12) the tempered distribution a given by

$$a(x,p)=\int_{\mathbb{R}^n}e^{-ip\cdot y}K(x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)dy.$$

Defining the twisted Weyl symbol a_σ as being the symplectic Fourier transform of a, that is

$$a_\sigma(z)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^{2n}}e^{-i\sigma(z,z')}a(z)dz$$

the operator A is given by the Bochner integral

$$A=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^{2n}}a_\sigma(z_0)T(z_0)dz_0; \tag{8}$$

here $T(z)$ is the Heisenberg–Weyl operator defined by

$$T(z_0)\psi(x)=e^{i(p_0\cdot x-\frac{1}{2}p_0\cdot x_0)}\psi(x-x_0) \tag{9}$$

if $z_0=(x_0,p_0)$. The operator A can be (at least formally) shown to act on $\psi\in\mathcal{S}(\mathbb{R}^n)$ by the formula

$$A\psi(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{ip\cdot(x-y)}a(\frac{1}{2}(x+y),p)\psi(y)dpdy.$$

We now associate to A an operator $\widetilde{A}:\mathcal{S}(\mathbb{R}^{2n})\longrightarrow\mathcal{S}'(\mathbb{R}^{2n})$ by the formula

$$\widetilde{A}=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^{2n}}a_\sigma(z_0)\widetilde{T}(z_0)dz_0 \tag{10}$$

where $\widetilde{T}(z_0)$ acts on $\mathcal{S}'(\mathbb{R}^{2n})$ via

$$\widetilde{T}(z_0)\Psi(z) = e^{-\frac{i}{2}\sigma(z,z_0)}\Psi(z-z_0). \tag{11}$$

Observe that the operators $\widetilde{T}(z_0)$ satisfy the same canonical commutation relations as the Heisenberg–Weyl operators $T(z_0)$ namely

$$\widetilde{T}(z_1)\widetilde{T}(z_0) = e^{i\sigma(z_1,z_0)}\widetilde{T}(z_0)\widetilde{T}(z_1) \tag{12}$$

hence they correspond as we will see below to some unitary representation of the Heisenberg (not on $L^2(\mathbb{R}^n)$ but on a closed subspace of $L^2(\mathbb{R}^{2n})$).

For $\phi \in \mathcal{S}(\mathbb{R}^n)$, $||\phi||_{L^2} = 1$, we define an operator $U_\phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ by the formula

$$U_\phi\psi(z) = \left(\frac{\pi}{2}\right)^{n/2} W(\psi,\phi)(\tfrac{1}{2}z) \tag{13}$$

where $W(\psi,\phi)$ is the cross-Wigner transform of the pair (ψ,ϕ):

$$W(\psi,\phi)(x,p) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{-ip\cdot y}\psi(x+\tfrac{1}{2}y)\overline{\phi(x-\tfrac{1}{2}y)}dy.$$

We will call U_ϕ the *wave-packet transform with window* ϕ; is essentially the short-time Fourier transform V_ϕ used in time-frequency analysis and defined by

$$V_\phi(x,\omega) = \int_{\mathbb{R}^n} e^{-2\pi i\omega\cdot t}\psi(t)\overline{\phi(t-x)}dt.$$

When ϕ is a Gaussian both U_ϕ and V_ϕ are closely related to the Bargmann transform [1].

We have (de Gosson [10], Theorem 10.6 p. 312):

Theorem 1 *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then:*

(i) U_ϕ is an isometry of $L^2(\mathbb{R}^n)$ on a closed subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$;

(ii) We have $U_\phi^*U_\phi = I$ on $L^2(\mathbb{R}^n)$ and the operator $P_\phi = U_\phi U_\phi^*$ is the orthogonal projection in $L^2(\mathbb{R}^{2n})$ onto the space $\mathcal{H}_\phi$;

(iii) The intertwining formulae

$$\widetilde{T}(z_0)U_\phi = U_\phi T(z_0) \quad , \quad \widetilde{A}U_\phi = U_\phi A \tag{14}$$

hold for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

In particular (iii) implies that we have

$$\left(\tfrac{1}{2}x_j + i\frac{\partial}{\partial p_j}\right) U_\phi \psi = U_\phi(x_j \psi) \tag{15a}$$

$$\left(\tfrac{1}{2}p_j - i\frac{\partial}{\partial x_j}\right) U_\phi \psi = U_\phi(-i\frac{\partial}{\partial x_j}\psi) \tag{15b}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. This motivates the following notation: if $\widetilde{A}$ is the phase-space operator with symbol a we will write

$$\widetilde{A} = a(\tfrac{1}{2}x + i\frac{\partial}{\partial p} +, \tfrac{1}{2}p - i\frac{\partial}{\partial x}).$$

1.2 The composition property

We will need the following composition property:

Proposition 2 *(i) Assume that the compose AB of the Weyl operators exists and is a Weyl operator. Then $\widetilde{A}\widetilde{B}$ exists as well and we have $\widetilde{A}\widetilde{B} = \widetilde{AB}$.*

(ii) If the Weyl operator R has kernel $K_R \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ then $\widetilde{R}$ has kernel $K_{\widetilde{R}} \in \mathcal{S}(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$.

Proof. (i) See de Gosson [10], Proposition 10.13, p. 320. (ii) The Weyl symbol r of R is related to the kernel K_R of R by the formula

$$r(x,p) = \int_{\mathbb{R}^n} e^{-ip\cdot y} K(x + \tfrac{1}{2}y, x - \tfrac{1}{2}y) dy$$

hence $r \in \mathcal{S}(\mathbb{R}^{2n})$ if $K_R \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ from which follows that we also have $r_\sigma \in \mathcal{S}(\mathbb{R}^{2n})$ where r_σ is the symplectic Fourier transform of r. We have, using (10),

$$\widetilde{R}\Psi(z) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} r_\sigma(z_0) e^{-\frac{i}{2}\sigma(z,z_0)} \Psi(z - z_0) dz_0$$

that is, setting $u = z - z_0$:

$$\widetilde{R}\Psi(z) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} r_\sigma(z-u) e^{-\frac{i}{2}\sigma(z,z-u)} \Psi(u) du.$$

Since $\sigma(z, z-u) = -\sigma(z,u)$ the kernel of $\widetilde{R}$ is thus given by the formula

$$K_{\widetilde{R}}(z,u) = \left(\frac{1}{2\pi}\right)^n e^{\frac{i}{2}\sigma(z,u)} r_\sigma(z-u). \tag{16}$$

The function $(z,u) \longmapsto r_\sigma(z-u)$ being in $\mathcal{S}(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ so is $K_{\widetilde{R}}$. ■

1.3 Symplectic covariance

Since we are in the business of Weyl operators, let us study the symplectic covariance properties of the corresponding phase space operators. Recall that the symplectic group $\mathrm{Sp}(n)$ has a double covering which can be faithfully represented by a group of unitary operators acting on $L^2(\mathbb{R}^{2n})$; that group is called the metaplectic group and we will denote it by $\mathrm{Mp}(n)$. The standard "metaplectic covariance formula" for Weyl calculus reads as follows: for $s \in \mathrm{Sp}(n)$ let S be any of the two operators in $\mathrm{Mp}(n)$ corresponding to s. Then if A has Weyl symbol a the operator SAS^{-1} has Weyl symbol $a \circ s^{-1}$. In de Gosson [7] we proved that if s has no eigenvalue equal to one, then

$$S = \left(\frac{1}{2\pi}\right)^n \frac{i^{\pm\nu}}{\sqrt{|\det(s-I)|}} \int_{\mathbb{R}^{2n}} \exp\left[\frac{i}{2} M_s z^2\right] T(z) dz \tag{17}$$

where $M_S = M_s^T$ is the symplectic Cayley transform defined by

$$M_S = \tfrac{1}{2} J(s+I)(s-I)^{-1}$$

and ν an integer (the "Conley–Zehnder index") that need not preoccupy us here; in addition we showed that every $S \in \mathrm{Mp}(n)$ can be written as the product of exactly two operators of the type above. In view of formula (10) the operator S determines naturally a phase-space operator

$$\widetilde{S} = \left(\frac{1}{2\pi}\right)^n \frac{i^{\pm\nu}}{\sqrt{|\det(s-I)|}} \int_{\mathbb{R}^{2n}} \exp\left[\frac{i}{2} M_s z^2\right] \widetilde{T}(z) dz \tag{18}$$

satisfying the second intertwining relation (14) in Theorem 1.

Proposition 3 *Let $s \in \mathrm{Sp}(n)$ and*

$$\begin{aligned}\widetilde{A} &= a(\tfrac{1}{2}x + i\tfrac{\partial}{\partial p}, \tfrac{1}{2}p - i\tfrac{\partial}{\partial x}) \\ \widetilde{B} &= (a \circ s^{-1})(\tfrac{1}{2}x + i\tfrac{\partial}{\partial p}, \tfrac{1}{2}p - i\tfrac{\partial}{\partial x}).\end{aligned}$$

We have $\widetilde{B} = \widetilde{S}\widetilde{A}\widetilde{S}^{-1}$ where $\widetilde{S} \in \mathrm{Mp}(n)$ is any of the two metaplectic operators corresponding to s.

Proof. It is an immediate consequence of the usual symplectic covariance formula

$$(a \circ s^{-1})(x, -i\tfrac{\partial}{\partial x}) = Sa(x, -i\tfrac{\partial}{\partial x})S^{-1}$$

for Weyl operators (see de Gosson [10], Chapter 10, §10.3.3). Alternatively, it follows from the metaplectic covariance relation $\widetilde{S}\widetilde{T}(z)\widetilde{S}^{-1} = \widetilde{T}(sz)$. ■

2 Alternative Quantizations

2.1 Some physical considerations

For the physical background we refer the reader to Landau and Lifshitz [16] or Messiah [17]. Consider an electron placed in a strong uniform magnetic field $\mathbf{B}$ directed along the z axis: $\mathbf{B} = (0, 0, B_z)$; if $\mathbf{A}$ is a vector potential defined by $\mathbf{B} = \nabla_{\mathbf{r}} \times \mathbf{A}$ ($\mathbf{r} = (x, y, z)$) the Hamiltonian function is

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 \tag{19}$$

with $\mathbf{p} = (p_x, p_y, p_z)$. Choosing the vector potential $\mathbf{A}$ such that $\mathbf{A} = \frac{1}{2}(\mathbf{r} \times \mathbf{B})$ ("symmetric gauge") and disregarding the unessential component p_z, the Hamiltonian H takes the particular form

$$\mathcal{H}_{\text{sym}} = \frac{1}{2m}(p_x^2 + p_y^2) - \omega_L L_z + \frac{m\omega_L^2}{2}(x^2 + y^2)$$

with $\omega_L = eB_z/2mc$ ("Larmor frequency") and $L_z = xp_y - yp_x$ is the angular moment in the z direction. The corresponding quantum operator is given by

$$H_{\text{sym}} = -\frac{\hbar^2}{2m}\Delta + i\hbar\omega_L\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \frac{m\omega_L^2}{2}(x^2 + y^2) \tag{20}$$

where Δ is the Laplacian in the x, y variables. As already observed in the Introduction this operator reduces to Wong's operator (3) if units are chosen appropriately.

Suppose now we choose the vector potential as $\mathbf{A} = (-B_z y, 0, 0)$ (this is called the "Landau gauge" in Physics); then the Hamiltonian function takes the simple form

$$\mathcal{H}_{\text{Lan}} = \frac{1}{2m}\left[\left(p_x + \frac{eB_z}{c}y\right)^2 + p_y^2\right]$$

and the corresponding the quantum operator is

$$H_{\text{Lan}} = -\frac{\hbar^2}{2m}\Delta - i\hbar\omega y\frac{\partial}{\partial x} + \frac{1}{2}m\omega^2 y^2 \tag{21}$$

where $\omega = eB_z/2mc$ is the "cyclotron frequency". It is easy to determine the spectrum of H_{Lan} (which is the same as that of H_{sym} since a change of gauge does not affect the spectrum). Noticing that x does not appear explicitly in the function H_{Lan} the momentum p_x is thus a conserved quantity; setting

$\psi(x,y)=e^{\frac{i}{\hbar}p_x x}\phi(y)$ it is easy to check that the eigenvalue problem $H_{\mathrm{Lan}}\psi = E\psi$ reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dy^2}+\frac{1}{2}m\omega^2(y-y_0)^2=E\psi$$

where $y_0=-p_x c/eB_z$; but this is just the eigenvalue problem for a translated harmonic oscillator with mass m and frequency ω, whose spectrum consists of the sequence

$$E_N=(N+\tfrac{1}{2})\hbar\omega \quad , \quad N=0,1,2,...$$

Choosing appropriate units, we recover the spectrum of Wong's operator W as announced in the Introduction.

It turns out that the harmonic oscillator operator ("Hermite operator")

$$H_{\mathrm{Her}}=\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2 \tag{22}$$

as well as the operators H_{sym} and H_{Lan} are all obtained from the Hamiltonian function

$$\mathcal{H}=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2x^2$$

by applying different quantizations rules:

- H_{Her} corresponds to the standard prescription $(x,p)\longrightarrow(x,-i\hbar\frac{\partial}{\partial x})$;
- H_{sym} corresponds to the rule $(x,p)\longrightarrow(i\hbar\frac{\partial}{\partial p}+\frac{1}{2}x,-i\hbar\frac{\partial}{\partial x}+\frac{1}{2}p)$;
- H_{Lan} corresponds to the rule $(x,p)\longrightarrow(i\hbar\frac{\partial}{\partial p}+\frac{1}{2}x,-i\hbar\frac{\partial}{\partial x})$.

Let us generalize this discussion to more general operators.

2.2 Extension; Bopp quantization

As seen above, there is a certain arbitrariness in the definition of our phase-space Weyl operator $\widetilde{A}$ associated with A. If we replace the "quantum translation operator" $\widetilde{T}(z_0)$ defined by (11) by any operator satisfying the canonical commutation relations (12) we will obtain another operator having similar properties as $\widetilde{A}$. Assume for instance that we define (choosing units in which $\hbar=1$),

$$\widetilde{T}'(z_0)\Psi(z)=e^{i(p_0\cdot x-\frac{1}{2}p_0\cdot x_0)}\Psi(z-z_0)$$

which amounts to extending the Heisenberg–Weyl operator (9) in a trivial way by allowing them to act on phase-space functions. The corresponding phase-space Weyl operator is obtained by replacing definition (10) by the expression

$$\widetilde{A'} = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\widetilde{T}'(z_0)dz_0.$$

This choice corresponds to the quantization rules

$$x_j \longrightarrow X = i\frac{\partial}{\partial p_j} + \frac{1}{2}x_j \ , \ p_j \longrightarrow -i\frac{\partial}{\partial x_j} \tag{23}$$

studied in de Gosson [8]; setting $y = -p$ this quantizes the harmonic oscillator Hamiltonian into the operator

$$W' = -\Delta + x^2 - 2ix\frac{\partial}{\partial y} \tag{24}$$

in place of Wong's operator (3) (see de Gosson [9] for a discussion of other possible quantizations compatible with the canonical commutation relations, and their interpretation in terms of classical phases).

It is noteworthy that the quantization rules $(x,p) \longrightarrow (X,P)$ listed above obey the canonical commutation rules $[X,P] = i\hbar$ and thus correspond to different (but of course isomorphic) representations of the Heisenberg group (the rule $(x,p) \longrightarrow (x, -i\hbar\frac{\partial}{\partial x})$ corresponds to the usual Schrödinger representation). There are of course other choices. One easily verifies that for any quadruple of real numbers $(\alpha,\beta,\gamma,\delta)$ such that $\alpha\delta - \beta\gamma = 1$ the operators

$$X = \alpha x + i\beta\hbar\frac{\partial}{\partial p} \ \ , \ P = \ \gamma p + i\delta\hbar\frac{\partial}{\partial x} \tag{25}$$

satisfy $[X,P] = i\hbar$ and therefore define a *bona fide* quantization rule for which the statements in Theorem 1 remain true for an adequate redefinition $U_\phi^{(\alpha,\beta,\gamma,\delta)}$ of the transform U_ϕ for which the intertwining relations (14) should be replaced by

$$\left(\alpha x + i\beta\hbar\frac{\partial}{\partial p}\right) U_\phi^{(\alpha,\beta,\gamma,\delta)}\psi = U_\phi^{(\alpha,\beta,\gamma,\delta)}(x_j\psi)$$
$$\left(\ \gamma p + i\delta\hbar\frac{\partial}{\partial x}\right) U_\phi^{(\alpha,\beta,\gamma,\delta)}\psi = U_\phi^{(\alpha,\beta,\gamma,\delta)}(-i\hbar\tfrac{\partial}{\partial x_j}\psi).$$

Applying the rules (25) to the harmonic oscillator Hamiltonian one obtains a whole class of degenerate elliptic operators:

$$\begin{aligned}\widetilde{A}^{(\alpha,\beta,\gamma,\delta)} = -\frac{1}{2}&\left(\delta^2\frac{\partial^2}{\partial x^2} + \beta^2\frac{\partial^2}{\partial p^2}\right) \\ &- i\left(\alpha\beta x\frac{\partial}{\partial p} - \delta\gamma p\frac{\partial}{\partial x}\right) + \frac{1}{4}(\alpha^2x^2 + \gamma^2p^2).\end{aligned}$$

We notice that the choice $\alpha = \gamma = 1$, $\beta = -\delta = 1/2$ yields the so-called "Bopp quantization" (Bopp [3]) rules

$$X_{\text{Bopp}} = x + i\frac{\hbar}{2}\frac{\partial}{\partial p} \quad , \; P_{\text{Bopp}} = p - i\frac{\hbar}{2}\frac{\partial}{\partial x}$$

which play an important role in deformation quantization. Still, physically, the phase-space quantization choice

$$\widetilde{X} = \frac{1}{2}x + i\hbar\frac{\partial}{\partial p} \quad , \quad \widetilde{P} = \frac{1}{2}p - i\hbar\frac{\partial}{\partial x} \tag{26}$$

has particular symmetry properties which makes it more attractive; it seems to play a role in the study of quantum gravity (Isidro and de Gosson [14, 15]).

3 Hypoellipticity in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$

Let us now study the question of global hypoellipticity for a class of phase-space operators generalizing those of Wong.

3.1 Global hypoellipticity and Shubin symbols

Let A be a partial differential operator (or more generally, a pseudodifferential operator). One says that A is *hypoelliptic* (in the usual sense) if $A\psi \in C^\infty(\mathbb{R}^n)$ implies that $\psi \in C^\infty(\mathbb{R}^n)$. Recall that the partial differential operator

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha \quad , \quad a_\alpha \in C^\infty(\mathbb{R}^n)$$

(or, more generally, a classical pseudodifferential operator) is said to be *elliptic* if its principal symbol

$$a_m(x, p) = \sum_{|\alpha| = m} a_\alpha(x) p^\alpha$$

has the property that $a_m(z) = 0$ if and only if $z = 0$. An elliptic operator is hypoelliptic (in the usual sense), as is easily seen by constructing an approximate inverse, or *parametrix* (see for instance Shubin [18] or Trèves [21]). More precisely B (resp. B') is called a left (resp. right) parametrix if

$$BA = I + R \;\; \text{(resp. } AB' = I + R')$$

where R and R' are smoothing operators, that is $R, R' : \mathcal{S}'(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ (equivalently, R and R' have smooth kernels). The hypoellipticity of an elliptic operator easily follows using the existence of a left parametrix; assume in fact that $A\psi = \phi$ is in $C^\infty(\mathbb{R}^n)$. Then $\psi = B\phi - R\psi$ is also in $C^\infty(\mathbb{R}^n)$: we have $B\phi \in C^\infty(\mathbb{R}^n)$ because $\phi \in C^\infty(\mathbb{R}^n)$ and, on the other hand $R\psi \in C^\infty(\mathbb{R}^n)$ because R is smoothing.

For our purposes the notion of ordinary hypoellipticity as just described is rather useless, because it gives no possibility of controlling the behaviour at infinity. It is preferable to use the notion of global hypoellipticity as introduced by Shubin [18], Corollary 25.1, p. 186 (also Boggiatto *et al.* [2], p.70). We will say that a linear operator $A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is *globally hypoelliptic* if

$$\psi \in \mathcal{S}'(\mathbb{R}^n) \text{ and } A\psi \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \psi \in \mathcal{S}(\mathbb{R}^n). \tag{27}$$

Shubin [18] (Chapter IV, §23) has introduced very convenient classes of globally hypoelliptic operators. Let $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ ($m_0, m_1 \in \mathbb{R}$ and $0 < \rho \leq 1$) be the complex vector space of all functions $a \in C^\infty(\mathbb{R}^n)$ for which there exists $R \geq 0$ such that for $|z| \geq R$ the following estimates hold:

$$C_0|z|^{m_0} \leq |a(z)| \leq C_1|z|^{m_1} \tag{28a}$$

$$|D_z^\alpha a(z)| \leq C_\alpha |a(z)||z|^{-\rho|\alpha|} \tag{28b}$$

with $C_0, C_1, C_\alpha \geq 0$. The main properties we will need are summarized in the following Theorem:

Theorem 4 (Shubin) *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ and A the Weyl operator with symbol a.*

(i) There exists a Weyl operator B with symbol $b \in H\Gamma_\rho^{-m_1,-m_0}(\mathbb{R}^{2n})$ such that $BA = I + R_1$ and $AB = I + R_2$ where R_1, R_2 have kernels in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$

(ii) The operator A is globally hypoelliptic.

(Note that (ii) immediately follows from (i)). We will call the operator B a *Shubin parametrix* of$A$.

In the context of Wong's operator W the following example is crucial:

Example 5 *The Hermite operator $-\Delta + |x|^2$ is globally hypoelliptic: it suffices to note that the Weyl symbol of $-\Delta + |x|^2$ is $a(z) = |z|^2$ and thus trivially satisfies the estimates (28) with $m_0 = m_1 = 2$, $\rho = 1$).*

In next subsection we generalize this example.

3.2 Main result

We claim that:

Theorem 6 *Assume that the Weyl symbol a of A is in $H\Gamma_{\rho}^{m_1,m_0}(\mathbb{R}^{2n})$ (hence A is a globally hypoelliptic pseudodifferential operator). Then the phase-space operator $\widetilde{A}$ is globally hypoelliptic:*

$$\Psi \in \mathcal{S}'(\mathbb{R}^{2n}) \ , \ \widetilde{A}\Psi \in \mathcal{S}(\mathbb{R}^{2n}) \Longrightarrow \Psi \in \mathcal{S}(\mathbb{R}^{2n}).$$

Proof. Let B be a Shubin parametrix of A: $BA = I + R$ where R is an operator with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Writing $\widetilde{A}\Psi = \Psi'$ we have $\Psi = \widetilde{B}\Psi' - \widetilde{R}\Psi$. Clearly $B\Psi' \in \mathcal{S}(\mathbb{R}^{2n})$ hence $\widetilde{B}\Psi' \in \mathcal{S}(\mathbb{R}^{2n})$ so it suffices to show that $\widetilde{R}\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ for all $\Psi \in \mathcal{S}'(\mathbb{R}^{2n})$; for this it suffices to show that the kernel of $\widetilde{R}$ is in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, but this follows from Proposition 2, (ii). ∎

Quadratic forms on $\mathbb{R}^{2n}$ are very interesting objects: they can be viewed as the Hamiltonians functions generalizing the harmonic oscillator. We recall the following very useful symplectic diagonalization result, which goes back to Williamson [22]: for every real positive-definite symmetric matrix there exists $s \in \mathrm{Sp}(n)$ such that $s^T M s = D$ where D is a diagonal matrix of the type $\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ the diagonal entries of Λ consisting of the moduli $\omega_1, ..., \omega_n$ of the eigenvalues of JM (these are precisely of the type $\pm i\omega_j$ since JM is equivalent to the antisymmetric matrix $M^{1/2}JM^{1/2}$).

Theorem 6 has the following interesting consequence:

Corollary 7 *Let a be a positive-definite quadratic form on $\mathbb{R}^{2n}$: $a(z) = \frac{1}{2}Mz \cdot z$ with $M = M^T > 0$.*

(i) The associated phase space operator

$$\widetilde{A} = a(\tfrac{1}{2}x + i\tfrac{\partial}{\partial p}, \tfrac{1}{2}p - i\tfrac{\partial}{\partial x})$$

obtained by the quantization rule (26) is globally hypoelliptic.

(ii) There exists $s \in \mathrm{Sp}(n)$ such that the operator $\widetilde{B} = \widetilde{S}\widetilde{A}\widetilde{S}^{-1}$ with symbol $b = a \circ s^{-1}$ is given by the formula

$$\widetilde{B} = \sum_{j=1}^{n} \frac{\omega_j}{2} \left[\left(\tfrac{1}{2}x + i\tfrac{\partial}{\partial p} \right)^2 + \left(\tfrac{1}{2}p - i\tfrac{\partial}{\partial x} \right)^2 \right].$$

Proof. (i) It suffices to show that $a \in H\Gamma_{\rho}^{m_1,m_0}(\mathbb{R}^{2n})$ for some m_1, m_0, ρ. Writing $S^T M S = D$ with S and D as above, and ordering the entries of D so that $\omega_1 \leq \cdots \leq \omega_n$ we have

$$\frac{\omega_1}{2}|z|^2 \leq a(z) \leq \frac{\omega_n}{2}|z|^2.$$

We have $|sz| \leq ||s|| \cdot |z|$ and $|s^{-1}z| \leq ||s^{-1}|| \cdot |z|$ hence there exist constants C_0, C_1 such that $C_0|z|^2 \leq a(z) \leq C_1|z|^2$ which is condition (28a) with $m_0 = m_1 = 2$. Let α be a multi-index; if $|\alpha| > 2$ then $|D_z^\alpha a(z)| = 0$. Suppose $|\alpha| \leq 2$; in view of the homogeneity of a there exists $C_\alpha > 0$ such that

$$|D_z^\alpha a(z)| \leq C_\alpha |z|^2 |z|^{-|\alpha|} = C'_\alpha a(z)|z|^{-|\alpha|}$$

so that condition (28b) holds with $\rho = 1$. The statement (ii) is an obvious consequence of the fact that if $a(z) = \frac{1}{2}Mz \cdot z$ then

$$b(z) = a(s^{-1}s) = \sum_{j=1}^{n} \frac{\omega_j}{2}(x_j^2 + p_j^2).$$

■

(Of course, in the first part of the proof of Corollary 7 we could have used standard diagonalization of the symbol a by orthogonal matrices).

4 Concluding Remarks

A question which poses itself is whether the global hypoellipticity of Section 3 can be generalized to other functional spaces than the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$. As has been pointed out to me by Franz Luef (NuHAG, Vienna), this might very well be the case if one considers Feichtinger's [5, 6] weighted modulation spaces $M_v^{p,q}$ (also see Gröchenig [12], Chapter 11 for a study of these spaces). The main interest of modulation spaces comes from the fact that they allow a simultaneous control of both local regularity and decay at infinity. Besides their intrinsic interest in Functional Analysis, they play an important role not only in time-frequency analysis (for which they were originally designed), but also in the study of the regularity of the solutions of Schrödinger's equation as we have shown in [11]. We will come back to this important question in a forthcoming paper.

Another fact which is certainly worth to be scrutinized is the following. As we pointed out several times in this paper, a change of phase-space quantization seems to correspond to a change of gauge. Is there any "universal rule" behind this property which we only checked for physical operators associated with a magnetic field?

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