REAL MULTIPLICATION
AND NONCOMMUTATIVE GEOMETRY

(ein Alterstraum)

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Abstract. Classical theory of Complex Multiplication (CM) shows that all abelian extensions of a complex quadratic field K are generated by the values of appropriate modular functions at the points of finite order of elliptic curves whose endomorphism rings are orders in K. For real quadratic fields, a similar description is not known. However, the relevant (still unproved) case of Stark conjectures ([St1]) strongly suggests that such a description must exist. In this paper we propose to use two–dimensional quantum tori corresponding to real quadratic irrationalities as a replacement of elliptic curves with complex multiplication. We discuss some basic constructions of the theory of quantum tori from the perspective of this Real Multiplication (RM) research project.

ἐπεὶ καὶ τὰ γνώριμα ὀλίγοις γνώριμά ἐστιν

... for even subjects that are known are known only to a few.

Aristotle, Poetics IX, 1451b

Preface

This paper can be read as a union of three largely independent parts.

Section 1 is dedicated to a general problem of noncommutative geometry Connes style: what are morphisms between noncommutative spaces considered as "spectra" of associative rings (perhaps, with an additional structure)? One natural suggestion is to define morphisms as isomorphism classes of biprojective bimodules as in Morita theory. Slightly extending Rieffel's Morita classification of two–dimensional quantum tori, I present a description of the resulting category (Theorem 1.7.1) in terms of what can be called "period pseudolattices" (sec. 1.1), by analogy with period lattices of elliptic curves. (In the context of operator algebras, requiring quite sophisticated modification of basic notions, A. Connes calls such morphisms "correspondences", cf. [Co1], p. 526, and [Jo2]).

Section 2 contains some results on the values and residues of zeta functions of arithmetical progressions in real quadratic fields, in the spirit of earlier work of E. Hecke, continued by G. Herglotz and D. Zagier. Our calculations are strongly motivated by H. M. Stark's conjectures ([St1], [St2]) proposing very special generators of abelian extensions of such fields.

Section 3 is a contribution to the theory of quantum theta functions (see [Ma3]). It gives a partial answer to the question of A. Schwarz ([Sch2]) about the relationship between quantum thetas and representations of quantum tori. The main Theorem 3.7 of this section generalizes a seminal calculation of F. Boca in [Bo2].

I collected these disjoint results under one roof because I feel that they form pieces of a general picture, which could be called Real Multiplication of two–dimensional quantum tori, by analogy with the classical Complex Multiplication of elliptic curves (Kronecker's *Jugendtraum*).

From this perspective, Section 1 outlays basics of the (noncommutative) geometry of Real Multiplication, Section 2 presents certain known or conjectural arithmetical facts in the light of this geometric picture, whereas Section 3 provides elements of function theory.

Unfortunately, the relations between these parts that I can establish are too sparse yet. An important test for such a theory would be a proof of Stark's conjectures for real quadratic fields. If this plan succeeds, a more ambitious project could address Real Multiplication of multidimensional quantum tori, as an analog and an extension of Shimura–Taniyama multidimensional CM theory.

I tried to facilitate reading this paper for potential readers with varied backgrounds by providing many definitions and introductory explanations, so that large parts of this paper can be read as a review. In particular, the introductory Section 0 explains rudiments of the classical Complex Multiplication theory which serves as a guide for the whole enterprise. It discusses as well a possibilty of including this theory in the context of noncommutative geometry. For additional connections, see [Ma4].

Acknowledgement. The crystallization of this project owes much to Matilde Marcolli and our collaboration [MaMar]. Victor Nistor consulted me about the proof of Lemma 1.4.2. Florian Boca's paper [Bo2] and correspondence with him were crucial for recognizing the connection between quantum thetas and modules over quantum tori. Sasha Rosenberg's insights about morphisms between noncommutative spaces developed in [Ro2] helped me to overcome a difficult psychological barrier. Last but not least, I appreciate the proposal of Friedrich Hirzebruch to translate *Alterstraum* in the title as "midlife crisis".

§0. Introduction: Lattices, elliptic curves, and Complex Multiplication

0.0. An overview. Let K be a field of algebraic numbers of one of the three types: $\mathbf{Q}$, a complex quadratic extension of $\mathbf{Q}$, or a real quadratic extension of $\mathbf{Q}$. Consider the following classical problem: describe the maximal abelian extension K^{ab} of K. Of course, the Galois group of such an extension is known for arbitrary agebraic number fields K: it is the idèle class group of K modulo its connected component. However, explicit generators of K^{ab} and the action of the Galois group on them generally remain a mystery, with exception of two classical cases described below.

According to the Kronecker–Weber theorem (KW), $\mathbf{Q}^{ab}$ is generated by roots of unity, i.e. by the points of finite order of the multiplicative group $\mathbf{G}_m$ considered as an algebraic group over $\mathbf{Q}$. For K imaginary quadratic, the multiplicative group should be replaced by the elliptic curve E_K whose $\mathbf{C}$–points are $\mathbf{C}/O_K$, O_K being the ring of integers in K. To get K^{ab}, one must adjoin to K the values of a power of the Weierstrass function at points of finite order of E_K, and the value of the absolute invariant of E_K. To see that points of finite order generate an abelian extension, one observes that the action of the Galois group on them must commute with the action of algebraic endomorphisms furnished by the power maps $x \mapsto x^m$ in the KW case, resp. the complex multiplication (CM) maps written additively on the universal covering of E_K as $x \mapsto ax$, $a \in O_K$. The commutant of this action suitably completed in profinite topology is abelian, and essentially coincides with the completion of the action itself. The universal idèlic description of the Galois group together with reduction modulo p arguments furnish the rest.

Elliptic curves have a rich analytic theory. Curves admitting a complex multiplication form a subfamily of all elliptic curves. The latter can be parametrized by their period lattices Λ i.e. discrete images of the injective homomorphisms $j : \mathbf{Z}^2 \to \mathbf{C}$ modulo a natural equivalence relation. The moduli space of them is $PGL(2,\mathbf{Z}) \setminus (H^+ \cup H^-)$, $H^\pm$ being the upper/lower halfplanes respectively. The curves isogeneous to E_K live over orbits of points $\mathbf{P}^1(K)$. The multiplicative group also appears in this family as the "degenerate elliptic curve" over the cusp, that is the orbit $PGL(2,\mathbf{Z}) \setminus \mathbf{P}^1(\mathbf{Q})$, so that in principle the geometry of the CM and KW cases can be unified.

The cusp corresponds to the very degenerate lattice: j acquires a cyclic kernel. There is an intermediate case of degeneration, invisible in algebraic geometry, where j is still injective, but its image is not discrete. The relevant modular orbit is $PGL(2,\mathbf{Z}) \setminus (\mathbf{P}^1(\mathbf{R}) \setminus \mathbf{P}^1(\mathbf{Q}))$, it contains orbits of $\mathbf{P}^1(K)$ for real quadratic K, but they could not be used in the same way as CM points of the modular curve because of lack of the analog of elliptic curves over this stratum of the moduli space. Hopefully, quantum tori might serve as a substitute.

This introductory section is dedicated to some details of the CM picture and its possible extension to the RM case.

0.1. Category of lattices $\mathcal{L}$. By definition, *a lattice* (of rank 2) is a triple (Λ, V, j), where Λ is a free abelian group of rank two, V is an one–dimensional complex space, and $j : \Lambda \to V$ is an injective homomorphism with discrete image, hence compact quotient.

When no confusion is likely, we will refer to (Λ, V, j) simply as Λ.

A morphism of lattices $(\Lambda', V', j') \to (\Lambda, V, j)$ is a commutative diagram

$$\begin{array}{ccc} \Lambda' & \xrightarrow{j'} & V' \\ \varphi\big\downarrow & & \big\downarrow\psi \\ \Lambda & \xrightarrow[j]{} & V \end{array} \tag{0.1}$$

in which φ is a group homomorphism, and ψ is a $\mathbf{C}$–linear map. Clearly, φ is uniquely determined by ψ, and vice versa. Choosing a basis (λ_1, λ_2) in Λ, taking $j(\lambda_2)$ as the base vector of V we see that in any isomorphism class of lattices one can find a representative given by $j : \mathbf{Z}^2 \to \mathbf{C}$ such that $j(0,1) = 1$, $j(1,0) := \tau$ is a number in $\mathbf{C} \setminus \mathbf{R}$. Changing the sign of λ_1 if needed we can arrange τ to lie in the upper half–plane H.

Let us denote this lattice Λ_τ. Then any non–zero morphism $\Lambda_{\tau'} \to \Lambda_\tau$ is represented by a non–degenerate matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M\,(2, \mathbf{Z})$$

such that

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{0.2}$$

This g is obtained by writing φ in (0.1) as the right multiplication of a row by a matrix; the respective ψ is the multiplication by $(c\tau + d)^{-1}$.

Clearly, (0.2) is an isomorphism, iff $g \in GL(2, \mathbf{Z})$. Thus the moduli space of (isomorphism classes of) lattices is

$$PGL(2, \mathbf{Z}) \setminus (\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R})) = PSL(2, \mathbf{Z}) \setminus H. \tag{0.3}$$

Endomorphisms of a lattice (Λ, V, j) form a ring, with componentwise addition of (ϕ, ψ) and composition as multiplication. It contains $\mathbf{Z}$ and comes together with its embedding in $\mathbf{C}$:

$$\mathrm{End}\,\Lambda = \{a \in \mathbf{C} \,|\, aj(\Lambda) \subset j(\Lambda)\}.$$

0.1.1. Lemma. *(a)* $\mathrm{End}\,\Lambda \neq \mathbf{Z}$ *iff there exists a complex quadratic subfield* K *of* $\mathbf{C}$ *such that* Λ *is isomorphic to a lattice contained in* K.

(b) If this is the case, denote by O_K *the ring of integers of* K*. There exists a unique integer* $f \geq 1$ *(conductor) such that* $\mathrm{End}\,\Lambda = \mathbf{Z} + fO_K =: R_f$*, and* Λ *is a*

projective module of rank 1 over R_f. Every K, f and a projective module over R_f come from a lattice.

(c) If lattices Λ and Λ' have the same K and f, they are isomorphic if and only if their classes in the Picard group $\mathrm{Pic}\, R_f$ *coincide.*

Automorphisms of a lattice generally form a group $\mathbf{Z_2}$ (ψ is multiplication by ± 1.) However, integers of two imaginary quadratic fields obtained by adjoining to $\mathbf{Q}$ a primitive root of unity of degree 4 (resp. 6) furnish examples of lattices with automorphism group of order 4 (resp. 6). Only these two fields produce lattices with such extra symmetries.

0.2. Category of elliptic curves $\mathcal{E}$. For any lattice (Λ, V, j) the quotient space $V/j(\Lambda)$ is an one–dimensional complex torus which has a canonical structure of (the set of complex points of) an algebraic curve E_Λ of genus 1 with base point 0. Such curves form a category $\mathcal{E}$ (morphisms should respect base points).

0.3. The functor $P : \mathcal{E} \to \mathcal{L}$. Let E be an elliptic curve. The functor "period lattice" P is defined on objects by the following prescription: $P(E) = (\Lambda_E, V_E, j_E)$ where $V_E =$ the tangent space to E at the base point, considered as its Lie algebra, Λ_E the kernel of the of the exponential map $V_E \to E(\mathbf{C})$, and j_E its canonical embedding. On morphisms, ψ is the induced tangent map and φ its restriction to the period lattices.

0.3.1. Theorem. *P is an equivalence of categories.*

This simple result is crucial for the theory of complex multiplication.

0.4. Abelian extensions of complex quadratic fields. Let now K be a complex quadratic extension of $\mathbf{Q}$. Choose and fix an embedding $K \to \mathbf{C}$. Denote by O_K the ring of integers of K.

There are three related but somewhat different ways to describe the maximal abelian extension K^{ab} of K.

(A) *Approach via elliptic curves.*

Here one starts with a single elliptic curve E_K associated to the lattice $O_K \subset \mathbf{C}$. It turns out that its minimal definition field containing K is generated by the value of its absolute invariant $J(E_K)$, and is the maximal unramified extension of K. One can also give a beautiful description of the total set of conjugates of $J(E_K)$ and the action of the Galois group on this set. Namely, any lattice whose endomorphism ring is precisely O_K, is represented by an ideal in O_K, and two lattices are isomorphic iff they lie in the same class. Absolute invariants of the respective elliptic curves are conjugate to each other, and the action of the Galois group is induced by a geometric twisting operation producing from a curve an isogenous curve.

The remaining part of the K^{ab} is generated by the values at points of finite order of E of a special function t. In Weierstrass notation, it is $t=\wp(z,O_K)^u$ where u is the order of the automorphism group of E, so that our algebraic numbers can be described as the values of a transcendental function

$$\left[\frac{1}{z^2}+\sum_{\lambda\in O_K\setminus\{0\}}\left(\frac{1}{(z+\lambda)^2}-\frac{1}{\lambda^2}\right)\right]^u, \tag{0.4}$$

at $z\in K$. In geometric terms, t is an appropriate coordinate on the projective line

$$\mathbf{P}^1=E_K/O_K^*, \tag{0.5}$$

to which the points of finite order are mapped.

(B) *Approach via modular curves.*

In another approach, one considers an extension of K generated by roots of unity and absolute invariants of all elliptic curves admitting complex multiplication by an order in O_K. Not all of K^{ab} is generated in this way, it remains to produce an additional infinite extension with a Galois group of period two, but in a sense the most essential part of K^{ab} is obtained in this way.

This approach stresses the geometry and arithmetic of *the moduli space (stack)* of elliptic curves rather than that of elliptic curves themselves. This space has special points which can be characterized as fixed points of certain correspondences, and fields of definition of these points are of primary interest.

For a brief introduction to both approaches, see [Se] and [Ste].

(C) *Approach via Stark's numbers.*

The general conjectures due to H. M. Stark provide (hypothetical) generators of abelian extensions which are values of zetas (or their derivatives, or the Taylor coefficients next to the residue) similar to (0.4). In the CM case these conjectures are proved in [St2], by reducing them to the more classical and geometrical forms of the theory sketched in (A), (B). No independent arguments are known.

To provide the basis for comparison with the RM case, we will briefly describe these numbers.

Let $(\Lambda,\mathbf{C},j)$ be a lattice in $\mathbf{C}$, $\lambda_0\in\Lambda\otimes\mathbf{Q}$. Put

$$\zeta(\Lambda,\lambda_0,s):=\sum_{\lambda\in\Lambda}\frac{1}{|j(\lambda_0+\lambda)|^{2s}} \tag{0.6}$$

where j is extended by $\mathbf{Q}$–linearity. These series admit meromorphic continuation and may have a pole of the first order at $s=1$ and zero of the first order at $s=0$.

Similar behavior is exhibited in the real case. Zeta-functions for two isomorphic lattices differ by a factor A^s where A is a positive real number. In the CM case we will restrict the choice of Λ in the isomorphism class by considering only lattices with $j(\Lambda) \subset K$. Then A can be modulus squared of any number in K.

We have the following simple lemma.

0.4.1. Lemma. *(a) Assume that $F(s)$ vanishes at $s=0$. Then for any $A>0$*

$$\frac{d}{ds}F(s)\Big|_{s=0} = \frac{d}{ds}(A^sF(s))\Big|_{s=0}.$$

In particular,

$$S_0(\Lambda,\lambda_0) := e^{\zeta'(\Lambda,\lambda_0,0)} \tag{0.7}$$

is an invariant of the isomorphism class of (Λ,λ_0).

(b) Assume that

$$F(s) = \frac{r}{s-1} + v + O(s-1),\ \ r\neq 0$$

near $s=1$. Then the similar formula holds for $A^sF(s)$, with the ratio v/r replaced by $v/r+\log A$. In particular, the following coset

$$S_1(\Lambda,\lambda_0) := e^{v/r} \bmod N_{K/\mathbf{Q}}(K^*) \in \mathbf{C}^*/N_{K/\mathbf{Q}}(K^*) \tag{0.8}$$

is an invariant of the isomorphism class of (Λ,λ_0). Here v, r are calculated via $\zeta(\Lambda,\lambda_0,s)$ for any representative of this class satisfying $j(\Lambda)\subset K$,

The essence of Stark's conjectures consists in the prediction that invariants of the type $S_0(\Lambda,\lambda_0)$ are algebraic units in appropriate abelian extensions of K, and that the action of the Frobenius elements of the Galois group upon them can be explicitly described.

Stark's proof in the CM case is based upon a direct calculation of these invariants, which in turn reduces to the second Kronecker limit formula. A version involving $S_1(\Lambda,\lambda_0)$ might be more feasible from the computational viewpoint, the two versions being essentially equivalent thanks to the classical functional equations. These calculations show that Stark's numbers a priori defined as values of some transcendental functions admit an algebraic geometric interpretation demonstrating their arithmetical nature.

0.5. Real quadratic fields. In this paper we propose some constructions parallel to (A) – (C) above, for the case of real quadratic fields.

(A) *Geometry of real multiplication.*

Replacing lattices by pseudolattices and elliptic curves by quantum tori, we develop in in §1 the geometric framework parallel to that of 0.1–0.3 above.

(B) *Geometry of noncommutative modular curves.*

The space $PGL(2,\mathbf{Z})\setminus(\mathbf{P}^1(\mathbf{R})\setminus\mathbf{P}^1(\mathbf{Q}))$ as an invisible stratum of the classical modular curve was studied in [MaMar]. In particular, it was shown that its $K-$theory can be written in terms of modular symbols, and that classical modular forms of weight two and their Mellin transforms are represented by interesting densities on this stratum.

For the purposes of real multiplication, however, more relevant might be noncommutative spaces which represent the orbits $PGL(2,\mathbf{Z})\setminus\mathbf{P}^1(K)$ corresponding to the individual real quadratic K. It seems that the remarkable paper by Bost and Connes ([BoCo]) and its extensions [ArLR], [HaL], furnish the right language to describe the arithmetic phenomena that interest us. Provisionally, [BoCo] appears to describe the KW case (cusp) from the noncommutative viewpoint. However, a satisfactory generalization of [BoCo] to more general number fields is not developed as yet (cf. however [HaL], [ArLR], [Coh1]).

We expect that other noncommutative spaces, besides quantum tori and modular curves, must play an essential role in the future theory. In particular, the projective line (0.5) might be replaced by the crossed product of the algebra of functions on K and its automorphism group of the type $x\mapsto ax+b$ where $a\in O_K^*$, $b\in O_K$. This looks even closer to the spaces studied in [BoCo] and [HaL].

(C) *Stark's numbers.*

For real quadratic fields K, one should consider series of the type (0.6), in which $|a|^{-2s}=(a\overline{a})^{-s}$ is replaced by $N_{K/\mathbf{Q}}(a)^{-2s}$ and furnished with a slight additional twist: the typical term of (0.6) is multiplied by the sign of the conjugate of $j(l_0+l)$, cf. §2 below for more details. More important is the following complication: the action of an infinite cyclic group of units makes each term in (0.6) to repeat infinitely often, so to make sense of the whole expression one should only sum over cosets modulo the relevant group.

We show in §2 that an adaptation of Hecke's calculations leads to formulas for Stark's numbers which are compatible with the general picture of "passing to the quantum limit": cf. 1.8 below.

Our hope, based upon this calculation, is that an appropriate algebraic geometric refinement of (A) and (B) will lead to a proof of Stark's conjectures, in the same way as it worked in the CM case.

Again, a natural question arises: can one see in noncommutative geometry Stark's numbers of the cyclotomic (KW) case ? In fact, closely related numbers appear in [BoCo] in their description of the arithmetical symmetry breaking, and in [Jo1], as indices of subfactors of the hyperfinite factor of type II_1. This suggests interesting questions in the framework of our program.

0.6. Elliptic curves as non–commutative spaces. $\mathbf{C}$–points of the elliptic curve E_τ associated with the lattice $\mathbf{Z}+\mathbf{Z}\tau$ can be identified via the exponential map with

$$\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)\cong \mathbf{C}^*/(q^{\mathbf{Z}}),\ q:=e^{2\pi i\tau}.$$

To treat E_τ as a non–commutative space means to study the appropriate crossed product of an algebra of functions on $\mathbf{C}^*$ with its automorphism group generated by the shift $z\mapsto qz$. The simplest crossed product of this type is $A_q^{alg}:=\mathbf{C}[z,z^{-1}][v,v^{-1}]$ where $vz=qzv$, but more sophisticated versions (various completions and their subalgebras) are really interesting.

There is a lot of results in the theory of lattice models and q–deformations that can be interpreted in the light of non–commutative geometry and function theory of elliptic curves. It would be worthwhile to review them systematically for two reasons: first, to get an environment in which elliptic curves and quantum tori could be treated more or less uniformly, and second, because the non–commutative setting considerably enriches even the classical picture.

I will restrict myself by two examples illustrating these points.

0.6.1. Semistable bundles and regular modules. Here we explain the basic reslut of [BG], generalized in [BEG]. The relevant crossed product is $A_q^{form}:=\mathbf{C}((z,z^{-1}))[v,v^{-1}]$ where $\mathbf{C}((z,z^{-1}))$ denotes the field of formal Laurent series finite in negative degrees. Consider the following categories.

Category I. Its objects are left A_q^{form}–modules M, which are finite–dimensional as $\mathbf{C}((z,z^{-1}))$–spaces and which satisfy the following *regularity* condition: there exists a free $\mathbf{C}[[z]]$–submodule $M_0\subset M$ of maximal rank such that $v^{\pm 1}(M_0)\subset M_0$.

Morphisms are usual module homomorphisms. The tensor product over $\mathbf{C}((z,z^{-1}))$ extends to a structure of rigid tensor category.

Category II. Its objects are semistable degree 0 holomorphic vector bundles over the elliptic curve E_τ. Semistability means that the associated principal bundle admits a global holomorphic connection, which is automatically flat. Morphisms and tensor structure are evident ones.

One of the main results of [BG] consists in the construction of an equivalence between the categories I and II. This equivalence is compatible with tensor products.

An interesting comment made in [BG] connects this result with a problem in the theory of finite difference equations.

Consider first differential equations of the form

$$z\,\frac{d\chi}{dz}=m(z)\,\chi(z) \tag{0.9}$$

where $\chi(z)$ is a column of formal series from $\mathbf{C}((z, z^{-1}))$ and $m(z)$ a matrix of such series. We can try to classify such equations by identifying those which can be obtained from each other by a linear transformation of $\chi(z)$. This is equivalent to the gauge transformation of $m(z)$:

$$m(z) \mapsto g(z)\, m(z)\, g(z)^{-1} + z\, \frac{dg}{dz}\, g(z)^{-1}. \tag{0.10}$$

It is known that if (0.9) has a regular (Fuchsian) singularity at $z = 0$, then the formal classification coincides with the analytic one, and the latter is furnished by monodromy around 0.

Now, a finite difference, or q–version of (0.9) is

$$\chi(qz) = m(z)\, \chi(z) \tag{0.11}$$

and the gauge equivalence (0.10) is replaced by

$$m(z) \mapsto g(qz)\, m(z)\, g(z)^{-1}. \tag{0.12}$$

We can identify the problem of classification of the equations (0.11) up to gauge equivalence with the problem of classification up to isomorphism of A_q^{form}–modules finite–dimensional over $\mathbf{C}((z, z^{-1}))$. To this end, given $m(z)$, treat it as the matrix of the operator v in a basis. The Baranovski–Ginzburg theorem then implies that for regular modules this classification coincides with the classification of semistable vector bundles over E_τ. But the regularity condition for modules is the standard q–version of Fuchsian regularity. Hence semistable vector bundles over E_τ should be regarded as a q–version of the monodromy data.

It would be important to reconstruct the complete category of coherent sheaves and/or its derived category in terms of an appropriate crossed product. See an interesting discussion in [So], especially 3.3.

0.6.2. Quantum pentagon identity. As above, let $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$. This time we will consider the elliptic curve $E_{2\tau}$ represented by an appropriate completion of the algebra $\mathbf{C}[u, u^{-1}, v, v^{-1}]$ with $uv = q^2 vu$. Put

$$e_q(t) := \prod_{n\geq 0} (1 + q^{2n+1} t). \tag{0.13}$$

If t here is understood as a complex number, we get one of the standard classical expressions, for example, occuring in the product formula for the elliptic theta function

$$\theta_q(t) := \sum_n q^{n^2} t^n = e_q(t)\, e_q(t^{-1}).$$

The following noncommutative identities (with $uv = q^2vu$) are however nonclassical:

$$e_q(u)\, e_q(v) = e_q(u+v), \tag{0.14}$$

$$e_q(v)\, e_q(u) = e_q(u)\, e_q(vu)\, e_q(v). \tag{0.15}$$

In view of (0.14), $e_q(t)$ is sometimes called the q–exponential function.

The second identity (0.15) was proved by Faddeev and Kashaev in [FK] and called there a quantum version of the Rogers pentagon identity for the dilogarithm. To explain this, I remind here the classical version of the Rogers identity:

$$L(x) + L(y) - L(xy) = L\left(\frac{x-xy}{1-xy}\right) + L\left(\frac{y-xy}{1-xy}\right), \tag{0.16}$$

where

$$L(x) := L_2(x) + \frac{1}{2}\log(1-x)\log x$$

and

$$L_2(x) := -\int_0^x \log(1-z)\,\frac{dz}{z} = \sum_{n\geq 1}\frac{x^n}{n^2}.$$

As q tends to the cusp 1, we have a classical asymptotic expansion in τ for the logarithm of $e_q(t)$ for which we write a few first terms in the exponentiated form

$$e_q(t) = \frac{1}{\sqrt{1+qt}}\exp\left(L_2(-t)/4\pi i\tau\right)(1+O(\tau)). \tag{0.17}$$

It remains to combine (0.15) and (0.17). This is not quite straightforward. Faddeev and Kashaev argue that an appropriate infinite–dimensional representation of the commutation relations $uv = q^2vu$ and the corresponding notion of the symbol of an operator in this representation produce (0.16).

It is remarkable and promising that (0.15) looks much neater than (0.16) and shows that (0.16) is a boundary reflection of a phenomenon which is both more global and essentially noncommutative.

§1. Pseudolattices, quantum tori, and Real Multiplication

1.1. Category of pseudolattices $\mathcal{PL}$. By definition, *a pseudolattice* (of rank 2) is a quadruple (L, V, j, s), where L is a free abelian group of rank two, V is an one–dimensional complex space, $j : L \to V$ is an injective homomorphism whose image lies on a real line, and finally s is an orientation of this line. Since this line contains 0, a choice of s defines the notion of positive and negative halves of it. Clearly, this line is the topological closure of $j(L)$.

A strict morphism of pseudolattices $(L', V', j', s') \to (L, V, j, s)$ is a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{j'} & V' \\ \varphi \downarrow & & \downarrow \psi \\ L & \xrightarrow[j]{} & V \end{array} \tag{1.1}$$

in which φ is a group homomorphism, and ψ is a $\mathbf{C}$–linear map, which transforms the orientation s' to s. Clearly, ϕ and ψ uniquely determine each other. Moreover, such a strict morphism is a strict isomorphism iff both ϕ and ψ are isomorphisms.

Omitting the condition that ψ respects orientations, we get the notion of *weak morphism.*

As with lattices, several simple observations will help us to clarify the structure of this category.

(i) The orientation s makes L a totally ordered group: by definition, $l > m$ iff $j(l-m)$ lies in the s–positive half–line. Choosing a basis (l_1, l_2) in L and taking $j(l_2)$ as the base vector of V, we see that in any strict isomorphism class of pseudolattices one can find a representative given by $j : \mathbf{Z}^2 \to \mathbf{C}$ such that $j(0,1) = 1$, $j(1,0) := \theta$ is an irrational real number. The remaining piece of data is the sign $\varepsilon = \pm 1$ such that $l > 0$ iff $\varepsilon j(l) > 0$.

Let us denote this pseudolattice (L_θ, ε). Then any non–zero strict morphism $(L_{\theta'}, \varepsilon') \to (L_\theta, \varepsilon)$ is represented by a non–degenerate matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M\,(2, \mathbf{Z})$$

such that

$$\theta' = \frac{a\theta + b}{c\theta + d}, \quad \operatorname{sgn}(c\theta + d) = \varepsilon\varepsilon'. \tag{1.2}$$

This g is obtained by writing φ in (1.1) as the right multiplication of a row by a matrix.

If $g \in GL(2, \mathbf{Z})$, (1.2) is an isomorphism. Since we can replace g by $-g$ without violating the first condition in (1.2), two lattices are strictly isomorphic iff they are weakly isomorphic.

(ii) We can choose a positive basis in L. This shows that any pseudolattice is isomorphic to some $(L_\theta, \varepsilon = 1)$ with irrational real θ which can be even taken in (0,1). We will denote it simply L_θ. Two such pseudolattices are isomorphic iff their invariants θ lie in the same $PGL(2,\mathbf{Z})$–orbit, that is, their continued fraction expansions coincide starting from some place. Thus set–theoretically, the moduli space of the isomorphism classes of pseudolattices is

$$(PGL(2,\mathbf{Z}) \setminus \mathbf{P}^1(\mathbf{R})) \setminus \{\text{cusp}\} \tag{1.3}$$

where the cusp is the orbit of rational numbers.

(iii) Weak endomorphisms of a pseudolattice L (we omit other structures in notation if there is no danger of confusion) form a ring w-$\mathrm{End}\,L$ (w stands for weak), with componentwise addition of (ϕ,ψ) and composition as multiplication. It contains $\mathbf{Z}$ and comes together with its embedding in $\mathbf{R}$ as $\{a\in\mathbf{R}\,|\,aj(L)\subset j(L)\}$. The non–negative part of this ring is the semiring $\mathrm{End}\,L$.

1.1.1. Lemma. *(a) w-$\mathrm{End}\,L\ne\mathbf{Z}$ iff there exists a real quadratic subfield K of $\mathbf{R}$ such that L is isomorphic to a pseudolattice contained in K.*

(b) If this is the case, we will say that L is an RM pseudolattice. Denote by O_K the ring of integers of K. There exists a unique integer $f\ge 1$ (conductor) such that w-$\mathrm{End}\,L=\mathbf{Z}+fO_K=:R_f$, and L is a projective module of rank 1 over R_f.

The module L is endowed with a total ordering respected by $\mathrm{End}\,L$.

Every K, f and a ordered projective module over R_f come from a lattice.

(c) If pseudolattices L and L_1 have the same K and f, they are isomorphic if and only if their classes in the Picard group $\mathrm{Pic}\,R_f$ *coincide.*

Unlike the case of lattices, the automorphism group of a pseudolattice is always infinite, it is isomorphic to $\mathbf{Z}\times\mathbf{Z}_2$.

For RM pseudolattices embedded in one and the same real quadratic field K, we will say that an isomorphism $L\to L_1 : l\mapsto al_1$, $a\in K$, is an isomorphism *in a narrow sense*, if $N_{k/\mathbf{Q}}(a)>0$.

1.2. Two–dimensional quantum tori. We now want to define analogs of elliptic curves for pseudolattices, that is, some geometric objects representing quotients $V/j(L)$ where (L,V,j,s) is a pseudolattice.

Choosing L_θ as a representative of the respective isomorphism class, we can naively replace $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\theta)$ by $\mathbf{C}^*/(e^{2\pi i\theta})$ ("Jacobi uniformization"), and then interpret the last quotient as an "irrational rotation algebra", or two–dimensional quantum torus T_θ. We recall that this torus is (represented by) the universal C^*–algebra A_θ generated by two unitaries U,V with the commutation rule $UV=e^{2\pi i\theta}VU$. A choice of such generating unitaries is called *a frame*; it is not unique.

The next task is to define morphisms between these quantum tori, with properties that would allow us to imitate the framework of 0.3. Already isomorphisms present a problem: we want fractional linear transforms (1.2) to produce isomorphic quantum tori. M. Rieffel's seminal discovery was that to this end we should consider Morita equivalences between appropriate categories of modules as isomorphisms between the tori themselves. Morita equivalences are essentially given by tensor multiplication by a bimodule. Taking this lead, we will formally introduce the general Morita morphisms of associative rings, stressing those traits of the formalism that play a central role in the structure theory of quantum tori (of arbitrary dimension). Our presentation also prepares ground for introducing versions of quantum tori with more algebraic geometric flavor.

1.3. Morita category. Let A, B be two associative rings. A *Morita morphism* $A \to B$ by definition, is the isomorphism class of a bimodule ${}_AM_B$, which is projective and finitely generated separately as module over A and B.

The composition of morphisms is given by the tensor product ${}_AM_B \otimes {}_BM'_C$, or ${}_AM \otimes {}_BM'_C$ for short.

If we associate to ${}_AM_B$ the functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_B : \ N_A \mapsto N \otimes_A M_B,$$

the composition of functors will be given by the tensor product, and isomorphisms of functors will correspond to the isomorphisms of bimodules.

We imagine an object A of the (opposite) Morita category as a noncommutative space, right A–modules as sheaves on this space, and the tensor multiplication by ${}_AM_B$ as the pull–back functor, in the spirit of A. Rosenberg's program [Ro2]. We have chosen to work with right modules, but passing to the opposite rings allows one to reverse left and right in all our statements.

Two bimodules ${}_AM_B$ and ${}_BN_A$ supplied with two bimodule isomorphisms ${}_AM \otimes_B N_A \to {}_AA_A$ and ${}_BN \otimes_A M_B \to {}_BB_B$ define mutually inverse Morita isomorphisms (equivalences) between A and B. The basic example of this kind is furnished by $B = \mathrm{Mat}\,(n, A)$, $M = {}_AA^n{}_B$ and $N = {}_BA^n{}_A$.

We will now briefly summarize Morita's theory.

(A) *Characterization of functors* $S : \mathrm{Mod}_A \to \mathrm{Mod}_B$ *of the form* $N_A \mapsto N \otimes_A M_B$. They are precisely functors satisfying any of the two equivalent conditions:

(i) *S is right exact and preserves direct sums.*

(ii) *S admits a right adjoint functor $T : \mathrm{Mod}_B \to \mathrm{Mod}_A$ (which is then naturally isomorphic to $\mathrm{Hom}_B(M_B, *)$).*

We will call such functors *continuous.*

(B) *Characterization of continuous functors S such that T is also continuous and $ST \cong 1$.* Let S be given by ${}_AM_B$ and T by ${}_BN_A$. Then $M \otimes_B N \cong {}_AA_A$. Moreover, in this case

(iii) *M_B and $_BN$ are projective.*

(iv) *$_AM$ and N_A are generators.*

In particular, equivalences $\mathrm{Mod}_A \to \mathrm{Mod}_B$ are automatically continuous. Hence any pair of mutually quasi–inverse equivalences must be given by a couple of biprojective bigenerators as above.

(C) *Finite generation and balance.* Any right module M_B can be considered as a bimodule $_AM_B$ where $A = B' := \mathrm{End}_B(M_B)$. We can then similarly produce the ring $B'' = A' := \mathrm{End}_A(_AM)$. Module M_B is called *balanced* if $B'' = B$. Similarly, one can start with a left module. With this notation, we have:

(v) M_B is a generator iff $_{B'}M$ is balanced and finitely generated projective.

Properties (i)–(v) can serve as a motivation for our definition of the Morita category above.

1.4. Projective modules, idempotents, traces, and K_0. Projective right A–modules up to isomorphism are exactly ranges of idempotents in various matrix rings Mat (n, A) acting from the left upon (column) vector modules A^n. Morphisms between such modules are also conveniently described in terms of these idempotents. The following (well known) Proposition summarizes the relevant information in the form convenient for us.

We prefer to work with all n simultaneously. So we will denote by $\mathcal{M}A$ the ring of infinite matrices $(a_{ij})\, i, j \geq 1$, $a_{ij} \in A$, $a_{ij} = 0$ for $i + j$ big enough (depending on the matrix in question). Notice that $\mathcal{M}A$ is not unital even if A is. Similarly, denote by A^∞ the left $\mathcal{M}A$–module of infinite columns (a_i), $i \geq 1$, with coordinates in A and such that $a_i = 0$ for large i.

Denote by Pr_A the category of finitely generated right A–modules. Denote by pr_A the category whose objects are projectors (idempotents) $p \in \mathcal{M}A$, $p^2 = p$, whereas morphisms are defined by

$$\mathrm{Hom}\,(p, q) := q\,\mathcal{M}A\,p$$

with the composition induced by the multiplication in $\mathcal{M}A$.

There is a natural functor $pr_A \to Pr_A$ defined on objects by $p \mapsto pA^\infty$. In order to define it on morphisms, we remark that morphisms $pA^\infty \to qA^\infty$ can be naturally described by matrices in the following way. Clearly, pA^∞ contains the columns p_k of p which generate pA^∞ as a right A–module. We can then apply any $\varphi : pA^\infty \to qA^\infty$ to all p_k and arrange the resulting vectors into a matrix $\Phi \in \mathcal{M}A$ with k–th column $\varphi(p_k)$. One checks that $\Phi p = \Phi$, and since also $q\Phi = \Phi$, we have $\Phi \in q\mathcal{M}Ap$. Conversely, any such matrix determines a unique morphism $pA^\infty \to qA^\infty$.

1.4.1. Proposition. *(a) The functor $pr_A \to Pr_A$ described above is an equivalence of categories.*

(b) pA^∞ is isomorphic to qA^∞ iff there exist $X, Y \in \mathcal{M}A$ such that $p = XY, q = YX$ (von Neumann's equivalence of idempotents). Hence in this case $p - q \in [\mathcal{M}A, \mathcal{M}A]$.

We have already checked most of the statements implicit in (a). As for (b), consider two mutually inverse isomorphisms $pA^\infty \to qA^\infty$ and $qA^\infty \to pA^\infty$. Assume that the first one sends columns of p to the columns of qBp whereas the second one sends columns of q to the columns of pCq. Writing that their compositions send p to p and q to q, we see that one can take $X = pCq$, $Y = qBp$. Conversely, if $p = XY, q = YX$, then also $p = (pXq)(qYp)$, $q = (qYp)(pXp)$, so that the matrices pXq and qYp determine mutually inverse isomorphisms of pA^∞ and qA^∞. This completes the proof.

It is also convenient to introduce the parallel formalism for left projective A–modules: here we consider the right $\mathcal{M}A$–module A_∞ of rows (a_i), $i \geq 1$, $a_i = 0$ for large i, and map a projector p to the left A–module $A_\infty p$.

Replacing A by the opposite ring A^{op} switches these two constructions.

A trace of A is any homomorphism of additive groups $t : A \to G$ vanishing on commutators; by definition, it factors through the universal trace $A \to A/[A, A]$. Combining it with the matrix trace, we get its canonical extension to $\mathcal{M}A$. From the Proposition 1.4.1 (b) it follows that $t(p)$ depends only on the isomorphism class of pA^∞. The class $p \bmod [A, A]$ is called *the Hattori–Stallings rank* of pA^∞.

We define $K_0(A)$ as the Grothendieck group of Pr_A. If $N_A \in Pr_A$, $[N_A]$ denotes its class in $K_0(A)$. If N_A is the range of an idempotent p and t is a trace, $t(p)$ depends only on $[N_A]$ and is additive on exact triples, hence t becomes a homomorphism of $K_0(A)$ (it is called dimension in the theory of von Neumann algebras).

The crucial role of traces in the theory of quantum tori (and in more general functional analytic situations) is explained by the fact that for irrational tori projective modules are exactly classified by the value of the (unique) normalized trace of the respective projector (Rieffel).

The following simple Lemma on traces is a useful technical tool. We assume in it that we work with algebras over a ground field.

1.4.2. Lemma. *Consider two unital algebras A, B and an A–B–bimodule ${}_AM_B$ which is projective as a module over A and over B. Assume that the (dual) space of traces $A/[A, A]$ of A is one–dimensional, whereas that of B is ≥ 1–dimensional, and that $1 \notin [A, A]$. Choose non–zero traces t_A and t_B. Then there exists such a constant c that for any $N_A \in Pr_A$ we have*

$$t_B([N \otimes_A M_B]) = c\, t_A([N_A])). \tag{1.4}$$

The value of this constant is obtained by putting $N_A = A_A$ in (1.4):

$$c = t_B([{}_A M_B]) t_A([A_A])^{-1}. \tag{1.5}$$

Proof. Let the modules N_A, M_B be given as ranges of idempotents $q \in \mathcal{M}A$, $p \in \mathcal{M}B$ respectively.

Put $A_1 := \mathrm{End}_B(M_B)$ and identify $p\mathcal{M}Bp$ with A_1 as above. The structure of left A–module on M_B is given by a ring homomorphism $\varphi: A \to A_1$. The trace t_B induces a trace on $p\mathcal{M}Bp$ and thus a trace t_1 on A_1. In turn, t_1 induces via φ a trace on A, and since the latter is unique, there exists a constant $c \neq 0$ such that we have identically $t_1(\varphi(a)) = c\, t_A(a)$ for all $a \in A$. By the associativity of tensor multiplication, we have

$$N \otimes_A M_B = (N \otimes_A A_1) \otimes_{A_1} M_B.$$

As a right A_1–module, $N \otimes_A A_1$ is isomorphic to $q_1 A_1^\infty$ where $q_1 = \varphi(q)$. We have $q_1^2 = q_1$ and since $q_1 \in p\mathcal{M}Bp$, $q_1 p = pq_1 = q_1$. Hence finally $N \otimes_A M_B$ as a right B–module is isomorphic to $q_1 B^\infty$. Thus

$$t_B([N \otimes_A M_B]) = t_B(q_1) = t_{A_1}(q_1) = c\, t_A(q) = c\, t_A([N_A]).$$

1.5. Involutions and scalar products. Assume now that A is endowed with an additive (linear or antilinear in the case of algebras) involution $a \mapsto a^*$, $(ab)^* = b^* a^*$, $a^{**} = a$. It extends to matrix algebras: $(B^*)_{ij} := B^*_{ji}$. Similarly, it extends to $A^\infty \to A_\infty$ and $A_\infty \to A^\infty$, compatibly with the module structures.

In such a context, it makes sense to consider only those projective modules which are ranges of *projections*, that is, $*$–invariant idempotents. In fact, in the case of C^*–algebras the resulting subcategory of projective modules is equivalent to the full category, because *every idempotent is von Neumann equivalent to a projection.* In fact, if p is an idempotent, then

$$P := pp^* \, [1 - (p - p^*)^2]^{-1}$$

is an equivalent projection. The C^*–structure is used to ensure the invertibility of $1 - (p - p^*)^2$; the rest is pure algebra: see e. g. [Da], IV.1.

Taking into account the involution, we get an additional structure on our modules and bimodules consisting of scalar products and identities relating them. This is a simple but important formalism made explicit by M. Rieffel.

1.5.1. Lemma. *Let M_B be a projective module (over a ring with involution B) isomorphic to $p\mathcal{M}B$ with $p^* = p$. Put $A = \mathrm{End}_B(M_B)$, identify this ring with $p\mathcal{M}Bp$ as above, and consider M as an A–B bimodule. The involution on pBp is induced by that on B.*

Define two scalar products ${}_A\langle *, * \rangle : M \times M \to A$ and $\langle *, * \rangle_B : M \times M \to B$:

$$ {}_A\langle pb, pc\rangle := (pb)(pc)^* = pbc^*p \in p\mathcal{M}Bp = A, \tag{1.6}$$

$$ \langle pb, pc\rangle_B := (pb)^*pc = b^*pc \in B. \tag{1.7}$$

Then the following identities hold, in which $l, m, n \in M$, $a \in A, b \in B$:

$$ {}_A\langle m, n\rangle^* = {}_A\langle n, m\rangle, \quad \langle m, n\rangle_B^* = \langle n, m\rangle_B, \tag{1.8}$$

$$ {}_A\langle am, n\rangle = a\,{}_A\langle m, n\rangle, \quad {}_A\langle m, an\rangle = {}_A\langle m, n\rangle\, a^*, \tag{1.9}$$

$$ \langle mb, n\rangle_B = b^*\langle m, n\rangle_B, \quad \langle m, nb\rangle_B = \langle m, n\rangle_B b, \tag{1.10}$$

$$ {}_A\langle l, m\rangle n = l\langle m, n\rangle_B. \tag{1.11}$$

We omit the checks which are straightforward.

1.6. Rieffel's projections. As we will see in §3, over toric C^*–algebras many bimodules ${}_AM_B$ are constructed directly, by inducing them from a Heisenberg representation, and the scalar products with the properties summarized in the Lemma 1.4.2 are introduced by an ad hoc formula.

In this case it is useful to know that, conversely, projections can be produced from such a setup. The following Lemma due to Rieffel ([Ri3]) furnishes them.

1.6.1. Lemma. *Assume that ${}_AM_B$ is a bimodule over two rings with involution, endowed with two scalar products satisfying the formalism (1.8)–(1.11). Let $m \in {}_AM_B$.*

(a) If $m\langle m, m\rangle_B = m$, then $p := {}_A\langle m, m\rangle$ is a projection in A.

(b) Conversely, assume that from ${}_A\langle n, n\rangle = 0$ it follows that $n = 0$. In this case, if p as above is a projection, then $m\langle m, m\rangle_B = m$.

Proof. (a) Using (1.6) and (1.8), we obtain

$$ p^2 = {}_A\langle m, m\rangle\, {}_A\langle m, m\rangle = {}_A\langle {}_A\langle m, m\rangle m, m\rangle = {}_A\langle m\langle m, m\rangle_B, m\rangle = {}_A\langle m, m\rangle = p.$$

From (1.3) it follows that $p^* = p$.

(b) Conversely, if p is a projection, then we get similarly

$$ {}_A\langle m\langle m, m\rangle_B - m, m\langle m, m\rangle_B - m\rangle = 0.$$

This completes the proof.

Rieffel also remarks that if $x \in {}_AM_B$ is such an element that one can construct an invertible *–invariant square root $\langle x, x\rangle_B^{1/2}$, then $m := x\,\langle x, x\rangle_B^{-1/2}$ satisfies 1.6.1(a).

F. Boca in [Bo2] takes for x a Gaussian element in the relevant Heisenberg module. Then $\langle x, x\rangle_B$ turns out to be a quantum theta in the sense of [Ma3]. We develop this remark in §3 for multidimensional case.

The net result is that we have a supply of explicit projections in toric algebras which in the notation of §3 below are given by the formulas

$$p_T := {}_D\langle f_T\,\Theta_{D^!}^{-1/2}, f_T\,\Theta_{D^!}^{-1/2}\rangle \in C(D,\alpha).$$

For notation, see (3.20), Theorem 3.7, and section 3.3.

This formula (and its generalizations) relates the representation theory of quantum tori to the theory of quantum thetas which has a distinct flavor of non–commutative *algebraic* geometry. Notice however that the existence of $\Theta_{D^!}^{-1/2}$ is not established in full generality.

The whole formalism sketched in 1.3–1.6.1 is a simple algebraic version of some basic machinery in the theory of von Neumann and C^*–algebras. In particular, see [Co1] and [Jo2] and original papers by A. Connes and A. Wassermann who overcame some highly nontrivial complications arising in the operator context.

1.6.2. Morita category and two–dimensional quantum tori. By definition, two–dimensional quantum tori are objects of the category $\mathcal{QT}$ whose morphisms are isomorphism classes of projective bimodules ${}_AM_B$ corresponding to projections, so that the formalism of the previous subsections is readily applicable. In particular, A_θ has a unique trace t_A which is normalized by the condition $t_A(1) = 1$ and which vanishes on any frame.

We will see in 1.7 below that there is a functorial correspondence between $\mathcal{QT}$ and pseudolattices which is fairly similar to the correspondence between elliptic curves and lattices. In particular, Real Multiplication of pseudolattices is reflected in $\mathcal{QT}$.

In order to achieve arithmetical applications of Real Multiplication, one has to find still smaller rings and modules, perhaps finitely generated in an algebraic sense and admitting models over rings of algebraic integers. Their definition remains the central unsolved problem in our approach. Since the points of finite order m on an elliptic curve E/K are in fact points of a finite group scheme over K acting upon E, it is conceivable that in the C^*–world the relevant finite objects should be seeked among weak Hopf algebras (or weak quantum groupoids) acting upon C^*–algebras: see recent reports [NiVa], [KaNi1], [KaNi2], and the references quoted therein.

The famous paper [Jo1] shows how a spectrum of algebraic numbers can be generated from such a setting. Jones's discrete spectrum of indices of subfactors is $\{4\cos^2\frac{\pi}{n}\,|\,n\geq 3\}$, whereas Stark's numbers in the cyclotomic case are $4\sin^2\frac{\pi m}{n}$. Both generate the maximal real subextension of $\mathbf{Q}^{ab}$.

Is this only a coincidence?

Returning to the C^* (or smooth) context, notice in conclusion that a bimodule ${}_AM_B$ can be treated as an $A\otimes B^{op}$–left module (completed tensor product). If it were projective, we could classify bimodules for toric A,B using the fact that $A\otimes B^{op}$ is again toric: their invariants would come from $K_0(A)\otimes K_0(B^{op})$ (the "trivial part") and from $K_1(A)\otimes K_1(B^{op})$ (the really interesting correspondences). However, intuitively it seems clear that such bimodules are much smaller than projective modules because they are separately A– and B^{op}–projective and hence, like Rieffel's elementary modules, should be realisable in functions of $\dim A=\dim B$ variables, whereas $A\otimes B^{op}$–projective modules are realizable only by functions of the doubled number of variables.

Therefore several questions arise about a possible extension of the classification theory of modules.

Question. Is any Morita morphism bimodule a maximal quotient of a unique projective $A\otimes B^{op}$–module ?

More generally, toric projective modules can have nontrivial maximal quotients, like in the situation with highest weight and Verma modules. One should seek for canonical projective resolutions of such modules.

The algebraic machinery might be connected with the fact that $A\otimes B^{op}$ contains large commutative subalgebras, so that a module can be decomposed according to their characters. E. g. if A,B are two–dimensional quantum tori, $A\otimes B^{op}$ contains two–dimensional classical tori, and prescribing their characters may produce the interesting quotients.

Question. Can one find a description of the derived category of perfect complexes over toric algebras?

1.7. Two functors relating $\mathcal{QT}$ to $\mathcal{PL}$. We start with defining a functor $K:\mathcal{QT}\to\mathcal{PL}$. Let the torus T be represented by an algebra A. On objects, we put:

$$K(T)=(L_A,V_A,j_A,s_A). \tag{1.12}$$

Here $L_A:=K_0(A)$, the K_0–group of the category of right projective A–modules (as above, given by projections in finite matrix algebras over A); V_A is the target group of the universal trace on A, that is, the quotient space of A modulo the completed commutator subspace $[A,A]$. Furthermore, $j_A=t_A:\,K_0(A)\to V_A$ is this universal trace extended to matrix algebras; its value on the class of a module, as we already explained, is its value at the respective projection. Finally s_A is taken in such a way that positive elements in $K_0(A)$ become represented by the classes of actual (not virtual) projective modules.

On morphisms, we define directly the left vertical arrow of the respective diagram (1.1):

$$K({}_AM_B)([N_A]) := [N \otimes_A M_B]. \tag{1.13}$$

The existence of the right vertical arrow follows from the Lemma 1.4.1.

1.7.1. Theorem. *(a) The family of maps (1.12), (1.13) can be uniquely completed to a functor $K : \mathcal{QT} \to \mathcal{PL}$.*

(b) This functor is essentially surjective on objects and (strict) morphisms.

(c) Assume that two bimodules ${}_AM_B$ and ${}_AM'_B$ considered as morphisms in $\mathcal{QT}$ become equal after applying K. Put $A_1 := \mathrm{End}_B(M_B)$ and consider ${}_AM_B$ as an A_1–B bimodule ${}_{A_1}M_B$.

There exist two ring homomorphisms $\varphi, \psi : A \to A_1$ such that if one considers A_1 as an A–A_1 bimodule ${}_\varphi A_{1,A_1}$, (resp. ${}_\psi A_{1,A_1}$) using φ (resp. ψ) to define the left action, and the ring structure of A_1 to define the right action, one obtains

$${}_\psi A_1 \otimes_{A_1} M'_B \cong {}_\varphi A_1 \otimes_{A_1} M_B \tag{1.14}$$

as A–B–bimodules.

In particular, if $\otimes_A M_B$ and $\otimes_A M'_B$ produce Morita equivalences, these functors differ by an automorphism of the category Mod_A which is induced by an automorphism of the ring A.

Comments. This result should be compared to the easy Theorem 0.3.1 which provides the geometric basis of the Complex Multiplication. The statement about quantum tori sounds less neat, however in 1.7.2 we will complement it by the construction of a functor in the reverse direction defined only on isomorphisms, which should suffice for the envisioned applications to Real Multiplication.

Proof. (a) Lemma 1.4.2 shows that, after passing to traces, (1.13) becomes the multiplication by a positive number representing a (strict) morphism of pseudolattices $K(m)$. The compatibility with the composition of morphisms is straightforward.

(b) It remains to establish the following three facts.

(i) *Every object of $\mathcal{PL}$ is isomorphic to an object lying in the image of K.*

In fact, the pseudolattice denoted $(L_\theta, 1)$ in 1.1(b) is isomorphic to $K(A_\theta)$ where A_θ is the respective rotation algebra. This is the main result of the theory, due to Connes, Rieffel, Pimsner–Voiculescu, Elliott. It is worth recalling here one of the several known strategies for proving it (cf. [Da], Ch. VI).

First, one checks that for any $\alpha \in [0,1] \cap \mathbf{Z}+\mathbf{Z}\theta$ there exists a projection $p_\alpha \in A_\theta$ with the normalized trace $\tau(p_\alpha) = \alpha$. Using functional calculus, one can directly construct such projections of the form $f(U)V + g(U) + h(U)V^*$ (Rieffel–Powers, see [Da], p. 171.) It follows than $\tau(K_0(A_\theta)) \supset \mathbf{Z} + \mathbf{Z}\theta$.

Second, one shows that A_θ can be embedded into an approximately finite algebra $\mathcal{A}_\theta$ which is the completed inductive limit of A_{p_n/q_n}, where p_n/q_n are consecutive convergents to θ. This embedding allows one to calculate $\tau(K_0(A_\theta))$ as the inductive limit of ordered groups $\sigma(K_0(\mathcal{A}_{p_n/q_n}))$, and this inductive limit is explicitly identified with $\mathbf{Z} + \mathbf{Z}\theta$.

This last argument can be read as a weak continuity property of A_θ with respect to θ varying in the set of cusps. In 1.8 below, we will discuss in what sense A_θ can be regarded as a limit of E_τ when τ tends to θ from the upper half–plane.

(ii) *Every morphism $K(T_\theta) \to K(T_{\theta'})$ in $\mathcal{PL}$ is of the form $K(m)$ where m is the tensor multiplication by an appropriate bimodule.*

Clearly, it suffices to choose a generating family of morphisms in $\mathcal{PL}$ (such that any morphism is a composition of members of this family) and to show that each generator can be lifted to $\mathcal{QT}$.

Any morphism of pseudolattices restricted upon L–components is a composition of an injection and an isomorphism (respecting ordering); moreover, this restriction uniquely determines it. Any injection can be decomposed into product of two injections with cyclic quotients.

Isomorphisms between pseudolattices $L_{\theta'} \to L_\theta$ can be decomposed into a sequence of transformations of the form $\theta \mapsto -\theta$, $\theta \mapsto \theta + 1$, $\theta \mapsto \theta^{-1}$. The map $(U,V) \mapsto (V',U')$ produces an isomorphism $T_\theta \to T_{-\theta}$, whereas T_θ and $T_{\theta+n}$ are obviously the same. The only non–trivial problem is to find a Morita equivalence $T_\theta \to T_{\theta^{-1}}$. Its solution was given in [Co2] and generalized to multidimensional tori in a series of works of Rieffel and his collaborators, see [Ri5], [RiSch].

Alternatively, in [CoDSch] one can find a direct description of a bimodule furnishing a Morita equivalence between T_θ and $T_{\theta'}$, where θ and θ' are related by a transformation from $PGL(2,\mathbf{Z})$. We will reproduce it in 1.7.2 below.

It remains to treat the case of embedding of pseudolattices. Now choose $n > 0$ and consider the embedding of toric algebras $B := A_{n\theta} \hookrightarrow A := A_\theta$ where in self–evident notation $U_B = U_A^n, V_B = V_A$. For ${}_AM_B$ take the bimodule ${}_AA_B$. It is free of rank 1 (resp. n) as A– (resp. B–) module. Denote by t_A, t_B the normalized traces (taking value 1 on 1). Then the constant (1.5) is n, so that the tensor multiplication by ${}_AM_B$ produces the morphism of pseudolattices $L_\theta \mapsto L_{n\theta}$: $\theta \mapsto n\theta$, $1 \mapsto n$. Clearly, any embedding of pseudolattices with cyclic quotient is isomorphic to such one.

(iii) *If $K(m) = K(m')$, the respective B–modules M_B, M'_B are isomorphic.*

In fact, from (1.4) and (1.5) it follows that the B–traces of them coincide, and for two–dimensional irrational tori this means that they are isomorphic.

The remaining argument is straightforward. Choose and fix an isomorphism of M and M' as B–modules. Actions of A upon M and M' correspond to two different homomorphisms $A \to A_1$. This is the essence of (1.14).

This finishes the proof.

1.7.2. The functor $E : \mathcal{PL}_{iso} \to \mathcal{QT}_{iso}$. In this section we rephrase the content of §2 of the recent preprint [DiSch].

Denote by $\mathcal{PL}_{iso}$ the category whose objects are pseudolattices $L_\theta = \mathbf{Z} + \mathbf{Z}\theta$, $\theta \in \mathbf{R} \setminus \mathbf{Q}$, oriented by their embedding into $\mathbf{R}$, and whose morphisms are strict isomorphisms, that is, multiplications by a positive number identifying two pseudolattices. According to (1.2), such isomorphisms $L_{\theta'} \to L_\theta$ are represented by matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL\,(2, \mathbf{Z})$$

such that

$$\theta' = g\theta = \frac{a\theta + b}{c\theta + d}, \quad c\theta + d > 0. \tag{1.15}$$

Denote by $\mathcal{QT}_{iso}$ the category whose objects are irrational toric algebras A_θ and whose morphisms are bimodules inducing Morita equivalences.

Given θ', θ, and g satisfying (1.15), construct an $(A_{\theta'}, A_\theta)$–bimodule ${}_{\theta'}E_\theta(g^{-1})$ (notice the inversion g^{-1}) by the following prescription. The smooth part of ${}_{\theta'}E_\theta(g^{-1})$ consists of functions $f(x, \mu)$ in the Schwartz's space $\mathcal{S}(\mathbf{R} \times \mathbf{Z}_c)$. The generators U, V of A_θ act upon these functions from the right as follows:

$$(fU)(x, \mu) = f(x - \frac{c\theta + d}{c}, \mu - 1),$$

$$(fV)(x, \mu) = e^{2\pi i(x - \mu d/c)} f(x, \mu).$$

The generators U', V' of $A_{\theta'}$ act from the left:

$$(U'f)(x, \mu) = f(x - \frac{1}{c}, \mu - a),$$

$$(V'f)(x, \mu) = \exp\left[2\pi i \left(\frac{x}{c\theta + d} - \frac{\mu}{c}\right)\right] f(x, \mu).$$

To become a bimodule over the respective C^* algebras, the Schwartz space must be appropriately completed, cf. Theorem 3.4.1 below.

1.7.3. Theorem. *The map $E : \mathcal{PL}_{iso} \to \mathcal{QT}_{iso}$ defined on objects by $L_\theta \mapsto A_\theta$ and sending the isomorphism (1.15) to the Morita isomorphism $[{}_{\theta'}E_\theta(g^{-1})]$ is a well defined functor. The composition $K \circ E$ is isomorphic to the identical functor on $\mathcal{PL}_{iso}$.*

This theorem rephrases the main result of [DiSch], section 2, which in our notation establishes an explicit isomorphism of bimodules

$$s_{g,h} : {}_{\theta'}E_\theta(g^{-1}) \otimes_{A_\theta} {}_\theta E_{\theta''}(h^{-1}) \to {}_{\theta'}E_{\theta''}((gh)^{-1})$$

and thus shows that E is multiplicative on isomorphisms of pseudolattices. Here $\theta = g^{-1}(\theta')$ as above, and $\theta'' = h^{-1}(\theta)$ so that $\theta'' = (gh)^{-1}(\theta')$.

This isomorphism is constructed in [DiSch] in the smooth setting. According to [Co2], extension of rings induces a bijection between the set of isomorphism classes of projective modules of finite type over A_θ and over its smooth subring respectively. Moreover, the trace (dimension) of ${}_{\theta'}E_\theta(g^{-1})$ as a right A_θ–module equals $|c\theta + d|$ ([Co2], Theorem 7): since ${}_{\theta'}E_\theta(g^{-1})$ is not given as the image of a projection, Connes develops differential geometric methods for calculating this trace. As an exercise, the reader can check that the dimension of the tensor product indeed equals the product of dimensions of factors.

Notice in conclusion that our version of Morita category using isomorphism classes of bimodules as morphisms is a truncation of a finer notion which treats bimodules as functors and leads to the notion of Morita 2–category. A refinement of the Dieng–Schwarz's result in this direction requires an explicitation of the associativity isomorphism connecting $s_{gh,k} \circ (s_{g,h} \otimes \mathrm{id})$ to $s_{g,hk} \circ (\mathrm{id} \otimes s_{h,k})$ which replaces the straightforward associativity of the triple multiplication of morphisms in 1–categories. This looks like a nice exercise.

1.8. Quantum tori as "limits" of elliptic curves. Reading parallelly subsections 0.1 and 1.1, we see that pseudolattices are in a very precise sense limits of lattices, at least, if one forgets orientation; or else one can add orientation to the definition of a lattice, as the choice of a generator of $\wedge^2(\Lambda)$.

Passing to the isomorphism classes of lattices/pseudolattices does not seem to change this impression: compare (0.3) and (1.3).

Comparison of the relevant geometric categories suggests that two–dimensional quantum tori can be thus considered as limits of elliptic curves. More specifically, take a family of Jacobi parametrized curves $E_\tau = \mathbf{C}/(e^{2\pi i\tau})$ with $\mathrm{Im}\,\tau > 0$ and $\tau \to \theta \in \mathbf{R}$. It is then natural to imagine T_θ as a limit of E_τ.

Fixing a Jacobi uniformization of an elliptic curve (or abelian variety of any dimension) as a part of its structure is necessary, for example, in problems connected with mirror symmetry. In such contexts our intuition seemingly provides a sound picture (cf. a similar discussion in [So], pp. 100, 113–114).

However, limitations of this viewpoint become quite apparent if one has no reason to keep a Jacobi uniformization as a part of the structure, and is interested only in the isomorphim classes of elliptic curves, perhaps somewhat rigidified by a choice of a level structure.

In this case one must contemplate the dynamics of the limiting process not on the closed upper half–plane but on a relevant modular curve X. Letting τ tend to θ along a geodesic, we get a parametrized real curve on X which, when θ is irrational, does not tend to any limiting point. The following lemma shows what can happen.

1.8.1. Lemma. *(a) Let θ be a real quadratic irrationality, θ' its conjugate. Consider the oriented geodesic in H joining θ' to θ. The image of this geodesic on any modular curve X is supported by a closed loop, which we denote $(\theta',\theta)_X$.*

(b) Let θ be as above, and let τ tend to θ along an arbitrary geodesic. Then the image of this geodesic on X has $(\theta',\theta)_X$ as a limit cycle (in positive time).

(c) Each closed geodesic on X is the support of a closed loop $(\theta',\theta)_X$. The union of them is dense in X. It is a strange attractor for the geodesic flow in the following sense. Having chosen a sequence of loops $(\theta_i',\theta_i)_X$, a sequence of integers $n_i \geq 1$, and a sequence of real numbers $\epsilon_i > 0$, $i = 1, 2, \dots$, one can find an oriented geodesic winding $\geq n_i$ times in the ϵ_i–neighborhood of $(\theta_i',\theta_i)_X$ for each i, before jumping to the next loop.

Proof. We will only sketch a couple of arguments.

For (a), notice that θ' and θ are respectively the attracting and the repelling points of a hyperbolic fractional linear transformation $g \in SL(2,\mathbf{Z})$. This transformation maps into itself the whole geodesic joining θ' to θ and acts upon it as a shift by the distance $\log\varepsilon$ where $\varepsilon > 1$ is a unit in the quadratic field generated by θ (cf. formula (1.16) below). If $X = \Gamma \subset H$, where Γ is a subgroup of finite index of the modular group, then $g^n \in \Gamma$ for an appropriate $n \geq 1$. Therefore the geodesic in question will close to a loop on X.

The distance between two geodesics tending to the same θ in H tends to zero; this shows (b).

Finally, (c) is based upon an elementary argument involving continued fractions and diophantine approximations. The Lemma is proved.

Now let us imagine that we have constructed a certain object $R(E_\tau)$ depending on the isomorphism class of E_τ (perhaps, with rigidity). This object can be a number, a function of the lattice, a linear space, a category ... Suppose also that we have constructed a similar object $\mathcal{R}(T_\theta)$ depending on the isomorphism class of T_θ, and that we want to make sense of the intuitive notion that $\mathcal{R}(T_\theta)$ is “a limit of

$R(E_\tau)$." Since in the most interesting for us case (a) of the Lemma 1.8.1 E_τ keeps rotating around the same loop, there are two natural possibilities:

(i) *The object $R(E_\tau)$ actually "does not depend on τ", and $\mathcal{R}(T_\theta)$ is its constant value. Here independence generally means a canonical identification of different $R(E_\tau)$, e.g. via a version of flat connection defined along the loop.*

(ii) *The object $R(E_\tau)$ does depend on τ, and $\mathcal{R}(T_\theta)$ is obtained by a kind of integrating or averaging various $R(E_\tau)$ along the loop.*

The second case looks more interesting, however, it is not immediately obvious that such objects occur in nature. Remarkably, they do, and precisely in the context of real multiplication and Stark's conjecture. In fact, this is how we will interpret the beautiful old calculational tricks due to Hecke: see [He1], [He2], [Her], [Z]. See also [Dar] for a similar observation related to what Darmon calls Stark–Heegner points of elliptic curves.

In this section we will only explain the geometric meaning of Hecke's substitution, whereas the (slightly generalized) calculation itself will be treated in the next section.

1.8.2. Hecke's lift of closed geodesics to the space of lattices. Let $K \subset \mathbf{R}$ be a real quadratic subfield of $\mathbf{R}$ and $L \subset K$ an RM pseudolattice. From now on, we denote by $l \mapsto l'$ the nontrivial element of the Galois group of $K/\mathbf{Q}$.

For any real t, consider the following subset of $\mathbf{C}$:

$$\Lambda_t = \Lambda_t(L) := \{\lambda_t = \lambda_t(l) := le^{t/2} + il'e^{-t/2} \,|\, l \in L\} \tag{1.16}$$

Lemma 1.8.3. *(a) $\Lambda_t(L)$ is a lattice.*

(b) Any isomorphism $a : L_1 \to L$ in the narrow sense induces isomorphisms $\Lambda_t(L_1) \to \Lambda_{t+c}(L)$ where c is a constant depending only on a and t is arbitrary.

(c) The image of the curve $\{\Lambda_t \,|\, t \in \mathbf{R}\}$ on the modular curve (0.3) (or any modular curve) is a closed geodesic. The affine coordinate t along this curve is the geodesic length.

Proof. (a) is evident; moreover, if l_1, l_2 form a basis of L, then $\lambda_t(l_1), \lambda_t(l_2)$ form a basis of Λ_t.

For (b), consider an isomorphism $L \mapsto L_1 : l \mapsto al$, $a \in K$, $aa' > 0$. It induces a map $\Lambda_t(L) \to \Lambda_t(L_1)$:

$$\lambda_t(l) \mapsto al\, e^{t/2} + ia'l'e^{-t/2} =$$

$$\sqrt{aa'}\left(\sqrt{\frac{a}{a'}}\, le^{t/2} + \sqrt{\frac{a'}{a}}\, l'e^{-t/2}\right) = \sqrt{aa'}\, \lambda_{t+\log \frac{a}{a'}}(l). \tag{1.17}$$

This produces an isomorphism of $\Lambda_t(L_1)$ with $\Lambda_{t+\log\frac{a}{a'}}(L)$.

For (c), it suffices to consider pseudolattices L generated by 1 and $\theta \in K$ with $\theta' > \theta$. Then $\Lambda_t(L)$ is generated by $e^{t/2} + ie^{-t/2}$ and $\theta\, e^{t/2} + i\theta'\, e^{-t/2}$, and hence isomorphic to the lattice generated by 1 and

$$\tau_t := \frac{\theta\, e^{t/2} + i\theta'\, e^{-t/2}}{e^{t/2} + ie^{-t/2}} = \frac{\theta\, e^t + \theta'\, e^{-t}}{e^t + e^{-t}} + i\,\frac{\theta' - \theta}{e^t + e^{-t}}. \tag{1.18}$$

A straightforward computation shows that

$$\left| \tau_t - \frac{\theta + \theta'}{2} \right|^2 = \left(\frac{\theta' - \theta}{2} \right)^2.$$

Hence τ_t runs over a semicircle in the upper half plane connecting θ' to θ. A further calculation shows that the geodesic length element $\dfrac{|\,d\tau|}{\operatorname{Im}\tau}$ restricted to this semicircle coincides with dt. The normalization of t has a simple geometric meaning: $t = 0$ is the upper point of the geodesic semicircle.

§2. Stark's numbers and theta functions for real quadratic fields

2.1. Stark's numbers at $s = 0$. In this section we fix a real quadratic subfield $K \subset \mathbf{R}$. Denote by $l \mapsto l'$ the action of the nontrivial element of the Galois group of K, and by O_K the ring of integers of K, and put $N(l) = ll'$.

Let L be an arbitrary integral ideal of K which, together with its embedding in $\mathbf{R}$ and the induced ordering, will be considered as a pseudolattice.

Choose also an $l_0 \in O_K$ so that the pair (L, l_0) satisfies the following restrictions:

(i) *The ideals* $\mathfrak{b} := (L, l_0)$ *and* $\mathfrak{a}_0 := (l_0)\mathfrak{b}^{-1}$ *are coprime with* $\mathfrak{f} := L\mathfrak{b}^{-1}$.

(ii) *Let* ε *be a unit of* K *such that* $\varepsilon \equiv 1 \bmod \mathfrak{f}$. *Then* $\varepsilon' > 0$.

Put now

$$\zeta(L, l_0, s) := \operatorname{sgn} l_0'\, N(\mathfrak{b})^s \sum_{l \in L}^{(u)} \frac{\operatorname{sgn}(l_0 + l)'}{|N(l_0 + l)|^s} \tag{2.1}$$

where (u) at the summation sign means that one should take one representative from each coset $(l_0{+}l)\varepsilon$ where ε runs over all units $\equiv 1 \bmod \mathfrak{f}$. Notice that $(l_0{+}L)\varepsilon = l_0 + L$ precisely for such units.

With this conventions, our $\zeta(L, l_0, s)$ is exactly Stark's function denoted $\zeta(s, \mathfrak{c})$ on the page 65 of [St1]: our $\mathfrak{a}_0, \mathfrak{b}, \mathfrak{f}$ have the same meaning in [St1], and our l_0 is Stark's γ. The meaning of Stark's $\mathfrak{c}$ is explained below.

The Stark number of (L, l_0) is defined as

$$S_0(L, l_0) := e^{\zeta'(L,l_0,0)} \tag{2.2}$$

(cf. the general discussion in 0.6).

The simplest examples correspond to the cases when $(L, l_0) = (1)$, $\mathfrak{f} = L$, in particular, $l_0 = 1$.

Notice that pseudolattices which are integral ideals have conductor $f = 1$ in the sense of Lemma 1.1.1.

2.2. Stark's conjecture for real quadratic fields. In [St1], Stark conjectures that $S_0(L, l_0)$ are algebraic units generating abelian extensions of K. To be more precise, let us first describe an abelian extension M/K associated with (L, l_0) using the classical language of class field theory. (Our M is Stark's K, whereas our K corresponds to Stark's k.)

In 2.1 above we constructed, starting with (L, l_0), the ideals $\mathfrak{f}$ and $\mathfrak{b}$ in O_K. Let $I(\mathfrak{f})$ be the group of fractional ideals of K generated by the prime ideals of K not dividing $\mathfrak{f}$, and $S(\mathfrak{f})$ be its subgroup called the principal ray class modulo $\mathfrak{f}$. Then Artin's reciprocity map identifies $G(\mathfrak{f}) := I(\mathfrak{f})/S(\mathfrak{f})$ with the Galois group of M/K.

Consider all pairs (L, l_0) as above with fixed $\mathfrak{f}$. It is not difficult to establish that on this set, $S_0(L, l_0)$ in fact depends only on the class $\mathfrak{c}$ of $\mathfrak{a}_0 = (l_0)\mathfrak{b}^{-1}$ in $G(\mathfrak{f})$. Denote the respective number $E(\mathfrak{c})$.

2.2.1. Conjecture. *The numbers $E(\mathfrak{c})$ are units belonging to M and generating M over K. If the Artin isomorphism associates with $\mathfrak{c}$ an automorphism σ, we have $E(1)^\sigma = E(\mathfrak{c})$.*

(We reproduced here the most optimistic form of the Conjecture 1 on page 65 of [St1] involving $m = 1$ and Artin's reciprocity map).

2.3. Hecke's formulas. In this subsection we will work out Hecke's approach to the computation of sums of the type (2.1), cf. [He2]. It starts with a Mellin transform so that instead of Dirichlet series (2.1) we will be dealing with a version of theta–functions for real quadratic fields. We start with introducing a class of such theta functions more general than strictly needed for dealing with (2.1) (and more general than Hecke's one).

2.4. Theta functions of pseudolattices. Let $K \subset \mathbf{R}$ be as in 2.1. We choose and fix the following data: a pseudolattice $L \subset K$, two numbers $l_0, m_0 \in K$ and a number $\eta = \eta_0 + i\eta_1 \in \mathbf{C}$. A complex variable v will take values in the upper half plane; $\sqrt{-iv}$ is the branch which is positive on the upper part of the imaginary axis.

Finally, choose an infinite cyclic group U of totally positive units in K such that the following conditions hold:

(a) $u(l_0+L)=l_0+L$ for all $u\in U$.

(b) $\operatorname{tr} ulm_0 \equiv \operatorname{tr} lm_0 \bmod \mathbf{Z}$, $\operatorname{tr} ul_0m_0 \equiv \operatorname{tr} l_0m_0 \bmod 2\mathbf{Z}$ for all $l\in L$, $u\in U$, where $\operatorname{tr}:=\operatorname{tr}_{K/\mathbf{Q}}$.

Put now

$$\Theta^U_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v):=$$

$$\sum_{l_0+l \bmod U} (\eta_0 \operatorname{sgn}(l_0'+l')+\eta_1\operatorname{sgn}(l_0+l))\, e^{2\pi i\, v|(l_0+l)(l_0'+l')|} e^{-2\pi i \operatorname{tr} lm_0} e^{-\pi i \operatorname{tr} l_0 m_0}. \tag{2.3}$$

Notation $l_0+l \bmod U$ means that we sum over a system of representatives of orbits of U acting upon l_0+L.

Notice that such U always exists, and that if we choose a smaller subgroup $V\subset U$, then

$$\Theta^V_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v)=[U:V]\,\Theta^U_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v).$$

In order to relate these thetas to Stark's numbers, consider the function

$$\Theta^U_{L,1}\begin{bmatrix} l_0\\ 0\end{bmatrix}(v)=\sum_{l_0+l \bmod U}\operatorname{sgn}(l_0'+l')\, e^{2\pi i\, v|(l_0+l)(l_0'+l')|}. \tag{2.4}$$

Then we have

$$\sum_{l_0+l \bmod U}\frac{\operatorname{sgn}(l_0'+l')}{|N(l_0+l)|^s}=\frac{(2\pi)^s}{\Gamma(s)}\int_0^{i\infty}(-iv)^s\Theta^U_{L,1}\begin{bmatrix} l_0\\ 0\end{bmatrix}(v)\,\frac{dv}{v}. \tag{2.5}$$

We will now show that these RM thetas can be obtained by averaging some theta constants (related to the complex lattices) along the closed geodesics described in 1.8 above.

2.5. Theta constants along geodesics. Starting with the same data as in 2.4, we introduce first of all a family of lattices $\Lambda_t=\Lambda_t(L)$ defined by (1.16). From l_0 which was used to shift L, we will produce a shift of Λ_t:

$$\lambda_{0,t}:=l_0\,e^{t/2}+il_0'\,e^{-t/2}.$$

The number m_0 determines a character of L appearing in (2.3): $l\mapsto e^{-2\pi i \operatorname{tr} lm_0}$. Similarly, we will produce a character of Λ_t from

$$\mu_{0,t}:=m_0\,e^{t/2}+im_0'\,e^{-t/2}$$

by using the scalar product on $\mathbf{C}$

$$(x \cdot y) = \operatorname{Im} xy = x_0 y_1 + x_1 y_0 \tag{2.6}$$

where $x = x_0 + ix_1$, $y = y_0 + iy_1$. Since $l_0, m_0 \in L \otimes \mathbf{Q}$, we have similarly $\lambda_{0,t}, \mu_{0,t} \in \Lambda_t \otimes \mathbf{Q}$. Omitting t for brevity, we put:

$$\theta_{\Lambda,\eta} \begin{bmatrix} \lambda_0 \\ \mu_0 \end{bmatrix} (v) := \sum_{\lambda \in \Lambda} ((\lambda_0 + \lambda) \cdot \eta)\, e^{\pi i v |\lambda_0 + \lambda|^2} e^{-2\pi i (\lambda \cdot \mu_0) - \pi i (\lambda_0 \cdot \mu_0)}. \tag{2.7}$$

The two types of thetas are related by Hecke's averaging formula:

2.6. Proposition. *We have*

$$\Theta^U_{L,\eta} \begin{bmatrix} l_0 \\ m_0 \end{bmatrix} (v) = \sqrt{-iv} \int_{-\log \varepsilon}^{\log \varepsilon} \theta_{\Lambda_t,\eta} \begin{bmatrix} \lambda_{0,t} \\ \mu_{0,t} \end{bmatrix} (v)\, dt \tag{2.8}$$

where $\varepsilon > 1$ is a generator of U.

Proof. The following formulas are valid for $\operatorname{Im} v > 0$:

$$e^{2\pi i v |mm'|} = \sqrt{-iv}\, |m'| \int_{-\infty}^{\infty} e^{-t/2} e^{\pi i v (m^2 e^t + m'^2 e^{-t})}\, dt =$$

$$\sqrt{-iv}\, |m| \int_{-\infty}^{\infty} e^{t/2} e^{\pi i v (m^2 e^t + m'^2 e^{-t})}\, dt \tag{2.9}$$

(see e. g. [La], pp. 270–271). In the rhs of (2.3), replace the first exponent by its integral versions (2.9), using the first version at η_0 and the second at η_1. We get:

$$\Theta^U_{L,\eta} \begin{bmatrix} l_0 \\ m_0 \end{bmatrix} (v) =$$

$$\sqrt{-iv} \int_{-\infty}^{\infty} \sum_{l_0 + l \bmod U} (\eta_0\, (l'_0 + l')\, e^{-t/2} + \eta_1\, (l_0 + l)\, e^{t/2}) \times$$

$$e^{\pi i v ((l_0+l)^2 e^t + (l'_0 + l')^2 e^{-t})} e^{-2\pi i \operatorname{tr} l m_0} e^{-\pi i \operatorname{tr} l_0 m_0}\, dt\,. \tag{2.10}$$

In view of (1.16) and (2.6) we have

$$\eta_0\, (l'_0 + l')\, e^{-t/2} + \eta_1\, (l_0 + l)\, e^{t/2} = ((\lambda_{0,t} + \lambda_t) \cdot \eta),$$

$$(l_0 + l)^2 e^t + (l'_0 + l')^2 e^{-t} = |\lambda_{0,t} + \lambda_t|^2,$$

and similarly

$$\operatorname{tr} lm_0 = (\lambda_t \cdot \mu_{0,t}), \quad \operatorname{tr} l_0 m_0 = (\lambda_{0,t} \cdot \mu_{0,t}).$$

Inserting this into (2.10), we obtain

$$\sqrt{-iv} \int_{-\infty}^{\infty} dt \sum_{l_0 + l \bmod U} ((\lambda_{0,t} + \lambda_t) \cdot \eta) e^{\pi i\, v |\lambda_{0,t} + \lambda_t|^2}\, e^{-2\pi i\, (\lambda_t \cdot \mu_{0,t})} e^{-\pi i\, (\lambda_{0,t} \cdot \mu_{0,t})}. \tag{2.11}$$

Replacing $l_0 + l$ by $\varepsilon(l_0 + l)$ is equivalent to replacing t by $t + 2\log\varepsilon$. Hence finally the right hand side of (2.11) can be rewritten as

$$\sqrt{-iv} \int_{-\log\varepsilon}^{\log\varepsilon} dt \sum_{\lambda_t \in \Lambda_t} ((\lambda_{0,t} + \lambda_t) \cdot \eta)\, e^{\pi i\, v |\lambda_{0,t} + \lambda_t|^2} e^{-2\pi i\, (\lambda_t \cdot \mu_{0,t}) - \pi i\, (\lambda_{0,t} \cdot \mu_{0,t})} \tag{2.12}$$

which is the same as (2.8).

We will now apply Poisson formula in order to derive functional equations for Hecke's thetas.

2.7. Poisson formula. Let V be a real vector space, $\widehat{V}$ its dual. We will denote by $(x \cdot y) \in \mathbf{R}$ the scalar product of $x \in V$ and $y \in \widehat{V}$. Choose a lattice (discrete subgroup of finite covolume) $\Lambda \subset V$ and put

$$\Lambda^! := \{\mu \in \widehat{V} \mid \forall \lambda \in \Lambda,\, (\lambda \cdot \mu) \in \mathbf{Z}\}. \tag{2.13}$$

Choose also a Haar measure dx on V and define the Fourier transform of a Schwarz function f on V by

$$\widehat{f}(y) := \int_V f(x)\, e^{-2\pi i (x \cdot y)} dx. \tag{2.14}$$

If $f(x)$ in this formula is replaced by $f(x + x_0)\, e^{-2\pi i (x \cdot y_0) - \pi i (x_0 \cdot y_0)}$ for some $x_0 \in V$, $y_0 \in \widehat{V}$, its Fourier transform $\widehat{f}(y)$ gets replaced by $\widehat{f}(y + y_0)\, e^{2\pi i (x_0 \cdot y) + \pi i (x_0 \cdot y_0)}$.

The Poisson formula reads

$$\sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{\int_{V/\Lambda} dx} \sum_{\mu \in \Lambda^!} \widehat{f}(\mu), \tag{2.15}$$

and for shifted functions as above

$$\sum_{\lambda \in \Lambda} f(\lambda_0 + \lambda)\, e^{-2\pi i (\lambda \cdot \mu_0) - \pi i (\lambda_0 \cdot \mu_0)} = \frac{1}{\int_{V/\Lambda} dx} \sum_{\mu \in \Lambda^!} \widehat{f}(\mu_0 + \mu)\, e^{2\pi i (\lambda_0 \cdot \mu) + \pi i (\lambda_0 \cdot \mu_0)}. \tag{2.16}$$

2.8. Functional equations for θ and Θ. In order to transform (2.12) using the Poisson formula, we put

$$V = \mathbf{C} = \{x_0 + i x_1\},\ \widehat{V} = \mathbf{C} = \{y_0 + i y_1\}, \tag{2.17}$$

and take (2.6) for the scalar product.

2.8.1. Lemma. *Let the lattice $\Lambda_t \subset \mathbf{C}$ be given by (1.15). Then the dual lattice $\Lambda_t^!$ with respect to the pairing (2.6) has the similar structure*

$$\Lambda_t^! = \Lambda_t(M) := \{me^{t/2} + im'e^{-t/2} \,|\, m \in M\} \tag{2.18}$$

where we denoted by $M = L^?$ the pseudolattice

$$M := \{m \in K \,|\, \forall l \in L,\, \mathrm{tr}_{K/\mathbf{Q}}(l'm) \in \mathbf{Z}.\}.$$

Proof. Denote by Γ the lattice (2.18). For any $\lambda = le^{t/2} + il'e^{-t/2} \in \Lambda_t$ and $\mu = me^{t/2} + im'e^{-t/2} \in \Gamma$ we have

$$(\lambda \cdot \mu) = \mathrm{Im}\,\lambda\mu = lm' + l'm = \mathrm{tr}_{K/\mathbf{Q}}(lm'). \tag{2.19}$$

Therefore this scalar product lies in $\mathbf{Z}$ if $m \in M$ so that $\Gamma \subset \Lambda_t^!$. Clearly, then, Γ must be commensurable with $\Lambda_t^!$, so that the right hand side of (2.19) can be used for computing $(\lambda \cdot \mu)$ on the whole $\Lambda_t^!$. This finishes the proof.

For example, $O_K^? = \mathfrak{d}^{-1}$ where $\mathfrak{d}$ is the different. In fact, this is the standard definition of the different.

Now let l_1, l_2 be two generators of the pseudolattice L. Put

$$\Delta(L) := |l_1 l_2' - l_1' l_2|. \tag{2.20}$$

Clearly, this number does not depend on the choice of generators.

2.8.2. Lemma. *Let the Haar measure on V be $dx = dx_0\, dx_1$. Choose generators l_1, l_2 of L. Then*

$$\int_{V/\Lambda_t} dx = \Delta(L). \tag{2.21}$$

Proof. If Λ_t is generated by ω_1, ω_2, then the volume (2.21) equals

$$|\mathrm{Re}\,\omega_1\,\mathrm{Im}\,\omega_2 - \mathrm{Re}\,\omega_2\,\mathrm{Im}\,\omega_1|.$$

Taking

$$\omega_1 = l_1 e^{t/2} + il_1' e^{-t/2}, \quad \omega_2 = l_2 e^{t/2} + il_2' e^{-t/2},$$

we get (2.20).

2.8.3. Lemma. *The Fourier transform of*

$$f_{v,\eta}(x) := (x \cdot \eta)\, e^{\pi i v |x|^2},\ \ \eta = \eta_0 + i\eta_1 \tag{2.22}$$

equals

$$g_{v,\eta}(y) := \frac{i}{v^2}\,(y \cdot i\bar{\eta})\, e^{-\frac{\pi i}{v}|y|^2} \tag{2.23}$$

Proof. Putting $w = -iv$ we have

$$f_{v,\eta}(x) = (x_0\eta_1 + x_1\eta_0)\, e^{-\pi w (x_0^2 + x_1^2)},$$

so that its Fourier transform by (2.13) and (2.14) is

$$\eta_1 \int_{-\infty}^{\infty} e^{-\pi w x_0^2}\, e^{-2\pi i x_0 y_1}\, x_0\, dx_0 \cdot \int_{-\infty}^{\infty} e^{-\pi w x_1^2}\, e^{-2\pi i x_1 y_0}\, dx_1 +$$

$$\eta_0 \int_{-\infty}^{\infty} e^{-\pi w x_0^2}\, e^{-2\pi i x_0 y_1}\, dx_0 \cdot \int_{-\infty}^{\infty} e^{-\pi w x_1^2}\, e^{-2\pi i x_1 y_0}\, x_1\, dx_1 =$$

$$(\eta_0 y_0 + \eta_1 y_1)\, \frac{1}{i w^2}\, e^{-\pi \frac{y_0^2 + y_1^2}{w}}.$$

This is (2.23).

2.8.4. A functional equation for θ. Let us now write (2.16) for $f = f_{v,\eta}$ and Λ_t:

$$\sum_{\lambda \in \Lambda_t} ((\lambda_{0,t} + \lambda) \cdot \eta)\, e^{\pi i v |\lambda_{0,t} + \lambda|^2}\, e^{-2\pi i (\lambda \cdot \mu_{0,t}) - \pi i (\lambda_{0,t} \cdot \mu_{0,t})} =$$

$$\frac{i}{\Delta(L)\, v^2} \sum_{\mu \in \Lambda_t^!} ((\mu_{0,t} + \mu) \cdot i\bar{\eta})\, e^{-\frac{\pi i}{v}|\mu_0 + \mu|^2}\, e^{2\pi i (\lambda_{0,t} \cdot \mu) + \pi i (\lambda_{0,t} \cdot \mu_{0,t})}.$$

In the notation (2.7) this means:

$$\theta_{\Lambda_t,\eta} \begin{bmatrix} \lambda_{0,t} \\ \mu_{0,t} \end{bmatrix} (v) = \frac{i}{\Delta(L)\, v^2}\, \theta_{\Lambda_t^!, i\bar{\eta}} \begin{bmatrix} \mu_{0,t} \\ -\lambda_{0,t} \end{bmatrix} \left(-\frac{1}{v}\right). \tag{2.24}$$

We now can establish a functional equation for Θ^U as well:

2.9. Proposition. *We have*

$$\Theta^U_{L,\eta}\begin{bmatrix} l_0 \\ m_0 \end{bmatrix}(v) = \frac{1}{\Delta(L)\, v}\,\Theta^U_{L^?,i\bar\eta}\begin{bmatrix} m_0 \\ -l_0 \end{bmatrix}\left(-\frac{1}{v}\right). \tag{2.25}$$

Proof. This is a straightforward consequence of (2.8) and (2.24).

§3. Heisenberg groups, modules over quantum tori, and theta functions

3.0. Introduction. Most of the constructions of this section are explained for the case of tori of arbitrary dimension. In 3.1–3.5 we remind to the reader the approach to the classical theta functions based upon the theory of Heisenberg groups. We closely follow Mumford's presentation in [Mu3], §1 and §2, which ideally suits our goals. The reader can find missing proofs there.

Quantum tori and their representations appear very naturally, when one restricts the basic Heisenberg representation to a lattice. This leads naturally to the emergence of Rieffel's setup as in Lemma 1.5.1, (1.8)–(1.11), although no explicit projections form a part of the picture. A way to remedy this and to construct certain projections starting with theta functions was proposed by F. Boca in [Bo2]. Generalizing his calculation, we prove the Theorem 3.7, which introduces in the context of representation theory of toric algebras *quantum thetas* in the sense of [Ma3]. This is the third type of thetas we meet in this paper (counting Θ^U and θ of Section 2 for the first two), and thanks to Boca's theorem, they can be used to construct morphisms of quantum tori.

The initial motivation of [Ma3] was to produce quantized versions of coordinate rings of abelian varieties, generated by the classical theta constants, i. e. the values of theta fuctions at the toric points of finite order. The way they appear here gives a partial answer to the question raised by A. S. Schwarz in [Sch2].

3.1. Heisenberg groups. We start with a locally compact abelian topological group $\mathcal{K}$ and denote its character group $\widehat{\mathcal{K}} = \mathrm{Hom}\,(\mathcal{K}, \mathbf{C}_1^*)$, $\mathbf{C}_1^* = \{z \in \mathbf{C} \,|\, |z| = 1\}$. We also choose a skew–symmetric pairing $\epsilon:\, \mathcal{K}\times\mathcal{K} \to \mathbf{C}_1^*$ which is *non–degenerate* in the following sense: it induces an isomorphism $\mathcal{K} \to \widehat{\mathcal{K}}$, and $\epsilon(x,x) \equiv 1$.

Moreover, choose a compatible with ϵ cocycle $\psi:\, K\times K \to \mathbf{C}_1^*$:

$$\psi(x,y)\psi(x+y,z) = \psi(x,y+z)\psi(y,z), \tag{3.1}$$

$$\epsilon(x,y) = \frac{\psi(x,y)}{\psi(y,x)}. \tag{3.2}$$

The condition (3.1) holds automatically if ψ is a bicharacter. Hence if one can find a skewsymmetric bicharacter $\epsilon^{1/2}$ which is a square root of ϵ, it can be taken for ψ.

Another useful construction starts with $\mathcal{K}$ which is already represented as $K_0 \times \widehat{K}_0$ for a topological group K_0. Denoting by $\langle *, * \rangle : K_0 \times \widehat{K}_0 \to \mathbf{C}_1^*$ the canonical pairing, we can simultaneously put

$$\psi((x,\widehat{x}),(y,\widehat{y})) := \langle x, \widehat{y}\rangle, \quad \epsilon((x,\widehat{x}),(y,\widehat{y})) = \frac{\langle x, \widehat{y}\rangle}{\langle y, \widehat{x}\rangle}. \tag{3.3}$$

Having chosen $\mathcal{K}$ and ψ, we can construct the following objects:

(i) *The Heisenberg group* $\mathcal{G} = \mathcal{G}(\mathcal{K}, \psi)$.

As a set, $\mathcal{G}$ is $\mathbf{C}_1^* \times \mathcal{K}$, and the composition law is given by

$$(\lambda, y)(\mu, z) = (\lambda\mu\psi(y,z), y+z). \tag{3.4}$$

The associativity is assured by (3.1). The group comes as a central extension

$$1 \to \mathbf{C}_1^* \to \mathcal{G} \to K \to 1. \tag{3.5}$$

If $\mathcal{K}$ and ψ split as in (3.3), both subgroups K_0 and $\widehat{K}_0$ of $\mathcal{K}$ come together with their lifts to $\mathcal{G}$: $x \mapsto (1, x)$.

(ii) *Representations of* $\mathcal{G}$ *on functions on* $\mathcal{K}$.

Consider a linear space of complex "functions" on $\mathcal{K}$ which is stable with respect to all shifts s_x, $(s_x f)(y) = f(x+y)$, $x, y \in \mathcal{K}$. Here the word "functions" should be understood liberally: completions of spaces of usual functions and distributions will do as long as shifts can be extended in such a way that $s_x s_y = s_{x+y}$, and notation $f(x+y)$ does not imply that we want literally take values at points.

In this case the formula

$$(U_{(\lambda,y)}f)(x) := \lambda\psi(x,y)f(x+y) \tag{3.6}$$

determines a linear representation of $\mathcal{G}$ on this space.

3.2. Basic unitary representation and its various models. In the notation above, a closed subgroup $K_0 \subset \mathcal{K}$ is called *isotropic*, if $\epsilon(x,y) = 1$ for all $x, y \in K_0$, and *maximal isotropic*, if K_0 is maximal with this property. One can then lift K_0 to $\mathcal{G}$, i.e. to find a homomorphism $K_0 \to \mathcal{G} : x \mapsto (\gamma(x), x)$.

Assume that such K_0 and γ are fixed. Consider the subspace $\mathcal{H}(K_0, \gamma) \subset L_2(\mathcal{K})$ consisting of all functions satisfying the condition

$$\forall\, y \in K_0, \quad f(x+y) = \gamma(y)^{-1}\psi(y,x)^{-1}f(x). \tag{3.7}$$

Using (3.6), this can be equivalently written as

$$\forall\, y \in K_0, \quad (U_{(\gamma(y),y)} f)(x) = \epsilon(x,y) f(x). \tag{3.8}$$

A straightforward calculation shows that this space is invariant with respect to the operators (3.6) and therefore determines a unitary representation of $\mathcal{G}$.

In the particular case when $\mathcal{K}$ is $K_0 \times \widehat{K}_0$ and the cocycle is as in (3.3), we can identify $\mathcal{H}(K_0,\gamma)$ with $L_2(\widehat{K}_0)$ because (3.8) allows us to reconstruct any function from its restriction to $\widehat{K}_0$.

This construction plays the central role in the theory of Heisenberg groups because of the following two facts:

3.2.1. Theorem. *(a) $\mathcal{H}(K_0,\gamma)$ is irreducible.*

(b) Any other unitary representation of $\mathcal{G}$ whose restriction on $\mathbf{C}_1^*$ is $U_{(\lambda,0)} = \lambda\,\mathrm{id}$ is isomorphic to the completed tensor product of $\mathcal{H}(K_0,\gamma)$ and a trivial representation. In particular, representations $\mathcal{H}(K_0,\gamma)$ for different choices of (K_0,γ) are all isomorphic.

The non–degeneracy of e is essentially used in the proof of this unicity statement. Everything said in 3.1 holds without any non–degeneracy assumption.

3.3. Heisenberg groups and modules over quantum tori. Since in this section we will be dealing with quantum tori of arbitrary dimension, it is convenient to introduce some invariant notation. Let D be a free abelian group of finite rank and $\alpha : D \times D \to \mathbf{C}_1^*$ a skewsymmetric pairing. The C^* algebra $C(D,\alpha)$ of the quantum torus $T(D,\alpha)$ with the character group D and quantization parameter α is the universal algebra generated by the family of unitaries $e(h) = e_{D,\alpha}(h)$, $h \in D$, satisfying the relations

$$e(g)e(h) = \alpha(g,h)e(g+h). \tag{3.9}$$

(Left) modules over such tori can be obtained by the following construction: choose a Heisenberg group $\mathcal{G}(\mathcal{K},\psi)$ with a bicharacter cocycle ψ and compatible ϵ. Consider a lattice embedding $l : D \hookrightarrow \mathcal{K}$. Denote by α_D the bicharacter on D induced by ϵ. Choose a basic representation U of $\mathcal{G}(\mathcal{K},\psi)$ in the space $\mathcal{H}$ and define the action of $C(D,\alpha)$ on $\mathcal{H}$ by

$$e_{D,\alpha}(h) f := U_{(1,l(h))} f \tag{3.10}$$

It turns out that an appropriate completion of the subspace of smooth functions is a projective module (see Rieffel's Theorem 3.4.1 below).

3.4. Basic representations as toric bimodules. In the setup of the last paragraph, assume to shorten notation that D is a lattice (discrete subgroup with compact quotient) in $\mathcal{K}$ and denote by $D^!$ the the maximal orthogonal subgroup:

$$D^! := \{x \in \mathcal{K} \,|\, \forall\, h \in D, \epsilon(h,x) = 1\}. \tag{3.11}$$

Let $\alpha^!$ be the pairing induced by ϵ on $D^!$. If $D^!$ is free of finite rank, we get similarly the representation of $C(D^!,\alpha^!)$ on $\mathcal{H}$. Moreover, operators from $C(D,\alpha)$ and $C(D^!,\alpha^!)$ pairwise commute. Identifying $C(D^!,\alpha^!)^{op}$ with $C(D^!,\bar{\alpha}^!)$ in an obvious way, we make of $\mathcal{H}$ an $C(D,\alpha)$–$C(D^!,\bar{\alpha}^!)$ bimodule.

Assuming that we are in the situation of (3.3) and taking the space $L_2(K_0)$ (rather than $L_2(\widehat{K}_0)$) for the basic representation, we will construct the Hermitean scalar products with the properties summarized in the Lemma 1.3.1. For further details, see [Ri3]. We will assume that K_0 is a Lie group of the form $\mathbf{R}^p \times \mathbf{Z}^q \times (\mathbf{R}/\mathbf{Z})^r \times F$ where F is a finite group. Then one can define the Schwartz space $S(K_0)$ consisting of C^∞–functions such that any polynomial times any derivative of the function vanishes at infinity. Rieffel's scalar products are first defined on Schwartz's functions on K_0 and $\mathcal{K}$ and then extended to the appropriate completions. We will write elements of $C(D,\alpha)$ as formal series $F = \sum_{h\in D} a_h e_{D,\alpha}(h)$ where a_h are the (non–commutative) Fourier coefficients defined by $a_h = t(Fe(h)^*)$, t is the normalized trace. If Fourier coefficients form a Schwartz function on D, F will be called smooth.

We start with the standard scalar product on $L_2(K_0)$ (antilinear in the second argument) which will be denoted $\langle *,*\rangle_{L_2}$ and put for $\Phi,\Psi\in S(K_0)$:

$$_D\langle\Phi,\Psi\rangle := \sum_{h\in D}\langle\Phi, e_{D,\alpha}(h)\Psi\rangle_{L_2}\, e_{D,\alpha}(h), \tag{3.12}$$

$$\langle\Phi,\Psi\rangle_{D^!} := \sum_{h\in D^!}\langle e_{D^!,\alpha^!}(h)\Psi,\Phi\rangle_{L_2}\, e_{D^!,\bar{\alpha}^!}(h). \tag{3.13}$$

(Notice the appearance of both $\alpha^!$ and $\bar{\alpha}^!$ in the right hand side of (3.13)).

Before summarizing some results due to Rieffel, we have to add a few words about the normalizations of various Haar measures involved. Any Haar measure on K_0 will do; on $\widehat{K}_0$ we take the respective Plancherel measure. For the volumes of the respective fundamental domains we will then have $|\mathcal{K}/D|\cdot|\mathcal{K}/D^!| = 1$.

3.4.1. Theorem. *Denote by M the completion of $S(K_0)$ with respect to the operator norm $\|_D\langle\Phi,\Phi\rangle\|^{1/2}$. Put*

$$A = C(D,\alpha),\ \ B = C(D^!,\bar{\alpha}^!),\ \ {}_A\langle *,*\rangle = |\mathcal{K}/D|\; {}_D\langle *,*\rangle,\ \ \langle *,*\rangle_B = \langle *,*\rangle_{D^!}.$$

Then we have:

(a) M is a finitely generated projective A–B module isomorphic to the range of a projection (both right and left).

(b) A is the complete endomorphism ring of M_B.

(c) The scalar products defined above satisfy all the identities (1.8)–(1.11).

(d) Let t_B be the normalized trace on B (zeroth Fourier coefficient). Then

$$t_B([M_B]) = |\mathcal{K}/D^!|. \tag{3.14}$$

Notice that, contrary to the purely algebraic context of Lemma 1.3.1 where (1.8)–(1.11) followed directly from the definitions (1.6), (1.7), the deduction of (1.11) from (3.12), (3.13) requires application of the Poisson summation formula.

For further details, see [Ri3], sections 2 and 3.

3.5. Vector Heisenberg group and classical theta functions. We return now temporarily to the setup of 3.1–3.2, involving no additional lattice D and explain the appearance of the classical theta functions as matrix coefficients of the basic Heisenberg representation. We closely follow [Mu3], §3.

We choose as $\mathcal{K}$ the real vector space $V = \mathbf{R}^{2N}$. Any element, say, $x \in V$ will be considered as a pair of columns of height N: x_1 consisting of the first N coordinates of x and x_2 consisting of the last N coordinates. Define the standard symplectic form on V and the cocycle ψ (cf. (3.1)) by

$$A(x,y) = x_1^t y_2 - x_2^t y_1, \quad \psi(x,y) = e^{\pi i A(x,y)}, \tag{3.15}$$

so that

$$\epsilon(x,y) = e^{2\pi i A(x,y)}. \tag{3.16}$$

Having chosen a model $\mathcal{H}$ of the basic representation of the resulting Heisenberg group, Mumford defines in $\mathcal{H}$ a finite–dimensional family of vectors $f_T \in \mathcal{H}$ parametrized by the points T in the Siegel upper half space $\mathfrak{H}_N$ consisting of complex symmetric $N \times N$ matrices with positive definite imaginary part.

In abstract terms, this is the space of all flat Kähler structures on $\mathbf{R}^{2N}$ compatible with A. Such a structure can be thought of, for example, as a pair consisting of a complex structure J and a positive definite Hermitean form H with the imaginary part A.

Any given T determines directly the complex structure J_T: it is given by the-complex coordinates $\underline{x}_1, \dots, \underline{x}_N$ on V:

$$\underline{x}_i = \sum_j T_{ij} x_j^{(1)} + x_i^{(2)} \tag{3.17}$$

where now $x_j^{(1)}$ (resp. $x_i^{(2)}$) are the coordinates of x_1 (resp. x_2). The values of the Hermitean form H_T on the basic vectors $e_j^{(2)}$ of the second half of V are

$$H_T(e_i^{(2)}, e_j^{(2)}) = (\operatorname{Im} T)^{-1}_{ij}. \tag{3.18}$$

We can now define f_T in the Mumford's first realization of the fundamental representation:

$$\mathcal{H} := L_2(\mathbf{R}^N), \qquad (U_{(\lambda,y_1,y_2)}f)(x) = \lambda e^{2\pi i x_1^t y_2 + \pi i y_1^t x_2} f(x_1 + y_1) \tag{3.19}$$

which is a specialization of (3.6) restricted to the subspace (3.8). Namely, we have

$$f_T(x) = e^{\pi i x^t T x}. \tag{3.20}$$

The classical theta function is defined by

$$\theta(\underline{x}, T) := \sum_{n \in \mathbf{Z}^N} e^{\pi i n^t T n + 2\pi i n^t \underline{x}}. \tag{3.21}$$

To express it as a matrix coefficient, Mumford introduces the distribution

$$e_{\mathbf{Z}} := \sum_{n \in \mathbf{Z}^N} \delta_n \tag{3.22}$$

and then checks that

$$\langle U_{(1,x)} f_T, e_{\mathbf{Z}} \rangle = c\, e^{\pi i x_1^t \underline{x}} \theta(\underline{x}, T). \tag{3.23}$$

(See [Ma3], Corollary 3.4).

3.6. Quantum theta functions. In this subsection I give a brief review of the formalism of quantum theta functions introduced in [Ma1] and further studied in [Ma2], [Ma3]. For details and motivation, see [Ma3].

Consider the character group of a quantum torus (D, α). In this subsection we will be interested in the space of formal infinite linear combinations of $e(h) = e_{D,\alpha}(h)$ which we will call formal functions. Theta functions are defined as solutions of functional equations which can be invariantly described in terms of another version of Heisenberg group $\mathcal{G}(D, \alpha)$ acting on this space: it consists of all linear operators on formal functions of the form

$$\Phi \mapsto c\, e(g)\, x^*(\Phi)\, e(h)^{-1}$$

where $c \in \mathbf{C}^*$, $g, h \in D$, $x \in T(D,1)(\mathbf{C}) := \mathrm{Hom}\,(D, \mathbf{C}^*)$ an arbitrary point of the algebraic torus with the character group D, x^* is the shift automorphism multiplying $e_{D,\alpha}(h)$ by $x(h)$.

Notice that such a shift x^* generally does not respect the unitarity of $e_{D,\alpha}(h)$ and cannot be extended to the automorphisms of $C(D, \alpha)$ unless the values of x belong to $\mathbf{C}_1^*$.

We now define a (formal) theta multiplier for (D,α) as an injective homomorphism $\mathcal{L} : B \to \mathcal{G}(D,\alpha)$ where B is a free abelian group of the same rank as D.

A quantum theta function with multiplier $\mathcal{L}$ is a formal function on $T(D,\alpha)$ invariant with respect to the action of (the image of) B.

$\Gamma(\mathcal{L})$ is the linear space of theta functions with multiplier $\mathcal{L}$.

The theta functions constructed below will have coefficients from the Schwartz space of D and therefore will represent smooth elements of $C(D,\alpha)$. Their multipliers will have the property $\dim\Gamma(\mathcal{L}) = 1$. In other words, as the classical $\theta(\underline{x},T)$, our thetas will correspond only to the principal polarizations. In order to get more general thetas one should consider more general Heisenberg groups in the sense of 3.1 into which (D,α) could be embedded.

3.7. Theorem. *We have*

$$ {}_D\langle f_T, f_T\rangle = \frac{1}{\sqrt{2^N \det \operatorname{Im} T}} \sum_{h\in D} e^{-\frac{\pi}{2}\underline{h}^t(\operatorname{Im} T)^{-1}\underline{h}^*} e_{D,\alpha}(h), \tag{3.24} $$

$$ \langle f_T, f_T\rangle_{D^!} = \frac{1}{\sqrt{2^N \det \operatorname{Im} T}} \sum_{h\in D^!} e^{-\frac{\pi}{2}\underline{h}^t(\operatorname{Im} T)^{-1}\underline{h}^*} e_{D^!,\bar\alpha^!}(h). \tag{3.25} $$

Here $\underline{h} := Th_1 + h_2$ *(cf. (3.17)) and* $\underline{h}^* := \overline{T}h_1 + h_2$*. These scalar products are quantum theta functions* $\Theta_D, \Theta_{D^!}$ *satisfying the following functional equations:*

$$ \forall g\in D, \quad c_g e_{D,\alpha}(g) x_g^*(\Theta_D) = \Theta_D, \tag{3.26} $$

$$ \forall g\in D^!, \quad c_g^! e_{D^!,\bar\alpha^!}(g) x_g^{!*}(\Theta_{D^!}) = \Theta_{D^!} \tag{3.27} $$

where

$$ c_g = e^{\frac{3\pi}{2}\underline{g}^t(\operatorname{Im} T)^{-1}\underline{g}^*}, \quad x_g^*(e_{D,\alpha}(h)) = e^{X_g(h)} e_{D,\alpha}(h), $$
$$ X_g(h) = -\pi \operatorname{Re}(\underline{g}^t(\operatorname{Im} T)^{-1}\underline{h}^*) - \pi i\, A(g,h), \tag{3.28} $$

$$ c_g^! = e^{\frac{3\pi}{2}\underline{g}^t(\operatorname{Im} T)^{-1}\underline{g}^*}, \quad x_g^{!*}(e_{D^!,\bar\alpha}(h)) = e^{X_g^!(h)} e_{D^!,\bar\alpha}(h), $$
$$ X_g^!(h) = -\pi \operatorname{Re}(\underline{g}^t(\operatorname{Im} T)^{-1}\underline{h}^*) + \pi i\, A(g,h). \tag{3.29} $$

Proof. We will check (3.24) and (3.26); the other two formulas can be treated similarly.

The general formula (3.12) must be specialized to our case $K_0 = \mathbf{R}^N$, the first half of $\mathcal{K}$. For the L_2–scalar product we take $\int \Phi\overline{\Psi}dx_1$ where dx_1 is the standard Haar measure. From (3.10) and (3.19) it follows that

$$ (e_{D,\alpha}(h)\Psi)(x_1) = e^{2\pi i x_1^t h_2 + \pi i h_1^t h_2}\Psi(x_1+h_1). $$

Hence

$$_D\langle \Phi, \Psi\rangle = \sum_{h\in D} e^{-\pi i h_1^t h_2} \int \Phi(x_1)\overline{\Psi(x_1+h_1)} e^{-2\pi i x_1^t h_2} dx_1 \cdot e_{D,\alpha}(h).$$

Putting here $\Phi = \Psi = f_T$ (see (3.20)), we get:

$$_D\langle f_T, f_T\rangle = \sum_{h\in D} e^{-\pi i h_1^t h_2} \int e^{\pi i [x_1^t T x_1 - (x_1^t + h_1^t)\overline{T}(x_1+h_1) - 2x_1^t h_2]} dx_1 \cdot e_{D,\alpha}(h). \quad (3.30)$$

The exponential expression under the integral sign in (3.30) can be represented as $e^{-(q(x_1)+l_h(x_1)+c_h)}$ where

$$q(x_1) = 2\pi\, x_1^t\, (\operatorname{Im} T)^{-1}\, x_1,\ \ l_h(x_1) = 2\pi i\, (h_1^t\overline{T} + h_2^t)\, x_1,\ \ c_h = \pi i h_1^t \overline{T} h_1. \quad (3.31)$$

Denote

$$\lambda_h = \frac{i}{2\operatorname{Im} T}[\overline{T} h_1 + h_2]. \quad (3.32)$$

Then we have

$$q(x_1+\lambda_h) - q(\lambda_h) = q(x_1) + l_h(x_1)$$

and therefore

$$\int e^{-(q(x_1)+l_h(x_1)+c_h)} dx_1 = e^{-c_h+q(\lambda_h)} \int e^{-q(x_1+\lambda_h)} dx_1 = e^{-c_h+q(\lambda_h)} \frac{\pi^{N/2}}{\sqrt{\det q}}. \quad (3.33)$$

Putting (3.30)–(3.33) together, we get (3.24).

The equation (3.26) is checked by a straightforward computation: putting $Q(h) = \dfrac{\pi}{2}\, h_1^t\, (\operatorname{Im} T)^{-1}\, h_1$ we have

$$c_g e_{D,\alpha}(g) x_g^*(\Theta_D) = c_g \sum_{h\in D} e^{-Q(h)+X_g(h)+\pi i\, A(g,h)} e_{D,\alpha}(g+h) =$$

$$c_g \sum_{h\in D} e^{-Q(h-g)+X_g(h-g)+\pi i\, A(g,h-g)} e_{D,\alpha}(h) =$$

$$c_g e^{-Q(g)+X_g(-g)} \sum_{h\in D} e^{-Q(h)+Q(g)+X_g(h)+\pi i\, A(g,h)} e_{D,\alpha}(h) = \Theta_D.$$

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