K-theory of braided monoidal categories

A.A.Davydov

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Department of Mathematics, Division of Information and Communication Sciences, Macquarie University, Sydney, NSW 2109, Australia

davydov@math.mq.edu.au

Abstract

The commutativity constraint of braided category gives rise to an algebraic structure on its K-theory known as Gerstenhaber algebra. If, in addition, the braiding has a compatible balanced structure the Gerstenhaber bracket on K-theory is generated by a Batalin-Vilkovisky differential.

We use these algebraic structures to prove a generalization of Anderson-Moore-Vafa theorem Anderson-Moore-Vafa which says that the order of the twist in a semi-simple ribbon (tortile) category with finitely many simple objects is finite.

1 Introduction

Categories equipped with tensor products, called monoidal or tensor categories, have been studied and used extensively (see [9] and references therein). Informally speaking, monoidal category is a "categorification" of the notion of monoid (or algebra in the linear case). Thinking of elements of the set an objects of a category we have to replace the operation on the set by a functor between appropriate categories; the relations for the operation give rise to isomorphisms of functors (*constraints*). The remarkable thing about this categorification process is that the constraints can be chosen to be *coherent*, i.e. satisfy to natural self-compatibility conditions. The presence of coherent constraints explains the following phenomenon: going back with inverse "decategorification" process such as K-theory we will have richer structure than we started with. In this paper we examine the extra structure appearing on the K-theory of commutative (braided) monoidal categories. It was proved in [8] that the K-theory spectrum of a braided monoidal category is an E_2-ring spectrum, i.e. an algebra over the little cube operad e_2. Here we note that the K-groups of a braided monoidal category (the stable homotopy of the corresponding K-theory spectrum) form an algebra over the homology of the operad e_2.

The commutativity constraint (braiding) for a monoidal category was introduced in [9] (which circulated as a preprint since mid eigties). Immediately after, it found applications in the theory of quantum groups [7], low-dimensional topology and mathematical phisics (see [17]). The algebraic structure on K-theory induced by braiding appeared first in the deformation theory of associative algebras [13, 14] and now known as Gerstenhaber algebra. A *Gerstenhaber* algebra is a graded commutative algebra equipped with the bracket $\{\ ,\ \}$ of degree 1 which gives a Lie algebra structure on the suspension

$$\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\},$$
$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}$$

and obey a version of graded Leibniz identity:

$$\{a,bc\} = \{a,b\}c + (-1)^{(|a|+1)|b|}b\{a,c\}. \tag{1}$$

When the braiding has a compatible balanced structure (which is a sort of coboundary condition) the Gerstenhaber bracket on K-theory extends to a richer structure appeared first in mathematical physics [4, 5] and known as Batalin-Vilkovisky algebra. A *Batalin-Vilkovisky* or BV-algebra is a Gerstenhaber algebra with an additional operator Δ of degree 1 which is a derivation with respect to the bracket such that $\Delta^2 = 0$ and

$$\{a,b\} = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b). \tag{2}$$

Tha paper is organised as follows. Using relations between braided categories and braid groups we present braided (balanced) monoidal categories as an algebras over certain categorical operads (the section 2). Calculating K-theory of these operads we prove that algebras over K-theory operads are exactly Gerstenhaber (BV-) algebars (the section 3). As an application we prove a generalization of a theorem of Anderson-Moore and Vafa Anderson-Moore-Vafa [1, 18] saing that the order of the twist in a semi-simple balanced monoidal category with finitely many simple objects is finite (the section 4, Theorem 4.6).

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2 Braided monoidal categories as algebras over an operad

2.1 braided and balanced categories

Throughout the paper we mean by a monoidal category a strict monoidal category, e.g. we assume that associativity constraint is identity. Although it seems restrictive, the coherence theorem (see [9], Theorem 1.2) says that any monoidal category is monoidal equivalent to a strict one.

A monoidal category $\mathcal{G}$ is *braided* [9] if it is equipped with a natural family of isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X, \quad \forall X, Y \in \mathcal{G},$$

called *commutativity constrains*, which satisfy the equations:

$$c_{X,Y\otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z) \ c_{X\otimes Y,Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \tag{3}$$

A braided monoidal category $\mathcal{G}$ is *balanced* if there given a natural family of automorphisms $\tau_X : X \to X$ such that for any $X, Y \in \mathcal{G}$

$$\tau_{X\otimes Y} = c_{Y,X} c_{X,Y} (\tau_X \otimes \tau_Y). \tag{4}$$

The name braided motivated by the relation with Artin braid groups B_n which algebraically can be given by generators and relations:

$$B_n =< s_i, i = 1, ..., n-1 : s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i, |i-j| \geq 2 > .$$

Any element σ of the braid group B_n defines an isomorphism $c(\sigma) : X_1 \otimes ... \otimes X_n \to X_{\sigma(1)} \otimes ... \otimes X_{\sigma(n)}$ natural in objects $X_1, ..., X_n$ of the braided category $\mathcal{G}$. This assignment has the following properties:

$$c(\sigma\tau) = c(\sigma)c(\tau), \ c(s_i) = I \otimes c_{X_i,X_{i+1}}) \otimes I$$

which actually allow to define it uniquely. There are natural surjections from the braid groups B_n to symmetric groups S_n sending generators s_i to the transpositions $(i, i+1)$. The kernel P_n of the surjection $B_n \to S_n$ is called the *pure braid group*. Note that pure braids defines natural automorphisms in a braided category.

In the balanced case the above assignment extends from the braid groups to the semi-direct products (the *balanced* group Bal_n) of the the braid group B_n by the free abelian group $\mathbf{Z}^n$ with respect to the permutation action of B_n on $\mathbf{Z}^n$. The generator t_i of $\mathbf{Z}^n$ corresponds to the natural automorphism $I \otimes \tau_{X_i} \otimes I$ of $X_1 \otimes ... \otimes X_n$. Note that the surjection $B_n \to S_n$ lifts to the surjection $Bal_n \to S_n$ with the kernel $Pb_n = P_b \times \mathbf{Z}^n$ (the *pure balanced* group).

2.2 braided and balanced operads

Consider an operad structure on the collection $P_n, n \geq 0$ of pure braid groups which sends the collection of braids $(\sigma_1, ..., \sigma_n, \sigma)$ from $P_{m_1} \times ... \times P_{m_n} \times P_n$ to the braid $\mu(\sigma_1, ..., \sigma_n, \sigma)$ in $P_{m_1+...+m_n}$ which is a product of replacement of strings of σ by $\sigma_1, ..., \sigma_n$. This operad is not symmetric, but conjugation actions of braid groups on its components are compatible with the operadic substitutions. The composition maps are group homomorphisms so we can think of this group-operad as a Cat-operad $\mathcal{P}$ of one-object groupoids with morphisms groups P_*.

The braid-operad structure on the collection $P_n, n \geq 0$ can be extended to $Pb_n, n \geq 0$. To define the operad extension is enough to specify the image of the pair (I, t) under $\mu : Pb_2 \times Pb_1 \to Pb_2$. We set it to be $s_1^2 t_1 t_2$ in accordance with the equation 4. Again the operadic substitutions will be a group homomorphisms which give us a Cat-operad of one-object groupoids $\mathcal{P}b$ with compatible braid groups actions. The incliusions $P_n \subset Pb_n$ induce a homomorphism of operads $\mathcal{P} \to \mathcal{P}b$.

We define an *End-operad* $\mathcal{E}nd(\mathcal{A})$ of a category $\mathcal{A}$ as a Cat-operad which n-th component is the category of functors $\mathcal{F}unct(\mathcal{A}^{\times n}, \mathcal{A})$ from the cartesian product of n copies of $\mathcal{A}$ to $\mathcal{A}$. Direct product and composition of functors will give an operadic substitution in the usual way. We say that the category $\mathcal{A}$ is an *algebra* over the Cat-operad $\mathcal{O}$ if there given a homomorphism of Cat-operads $\mathcal{O} \to \mathcal{E}nd(\mathcal{A})$ which is a collection of functors $\mathcal{O}_n \to \mathcal{F}unct(\mathcal{A}^{\times n}, \mathcal{A})$ compatible with the operadic substitutions. In other words, the action is a collection of functors

$$\mathcal{O}_n \times \mathcal{A} \times ... \times \mathcal{A} \to \mathcal{A}, \tag{5}$$

compatible with operadic composition of $\mathcal{O}_*$.

The construction of the natural automorphisms $c(\sigma)$ implies the following

Proposition 2.3 *Braided strict monoidal categories are algebras over the Cat-operad $\mathcal{P}$.*
Balanced strict monoidal categories are algebras over the Cat-operad $\mathcal{P}b$.

We can not speak about K-theory of the operads $\mathcal{P}, \mathcal{P}b$ for their components are not abelian (exact) categories. To do so we need to replace them by their abelian variants.

2.4 linearisation

Recall that a *pseudo-abelian* (or *Cauchy complete*) category is an additive category containing images of all its idempotent endomorphisms. Here we describe k-linear pseudo-abelian Cat-operads $\mathcal{P}_k, \mathcal{P}b_k$. The components of the operads $\mathcal{P}_k, \mathcal{P}b_k$ are k-linear pseudo-abelian envelopes of the components of $\mathcal{P}, \mathcal{P}b$. By k-linear abelian envelope of $\mathcal{G}$ we mean an initial object in the category of functors from $\mathcal{G}$ to k-linear pseudo-abelian categories. For a small category $\mathcal{G}$ with the set of morphisms Mor its k-linear pseudo-abelian envelope can be identified with the category of projective modules over the semigroup ring $k[Mor]$. In particular, the k-linear pseudo-abelian envelope of a one-object groupoid with the morphism group G is the category of projective modules over the group ring $k[G]$.

We denote by $\mathcal{P}_k$ the collection of the categories of finitely generated projective modules over the group rings $k[P_*]$. The operadic composition of pure braid groups induces a Cat-operad structure on the colection $\mathcal{P}_k$. Analogously, the collection of the categories of finitely generated projective modules over the group rings $k[Pb_*]$ forms an operad $\mathcal{P}b_k$. Note that the components of $\mathcal{P}_k$ and $\mathcal{P}b_k$ are exact categories and the operadic compositions are exact functors.

Now we can formulate the linear analog of the proposition 2.3.

Proposition 2.5 *Any k-linear abelian braided strict monoidal category is an algebra over the Cat-operad $\mathcal{P}_k$.*
Any k-linear abelian balanced strict monoidal category is an algebra over the Cat-operad $\mathcal{P}b_k$.

Sckech of the proof:
In a braided category $\mathcal{G}$ the natural automorphisms $c(\sigma)$ defines the action of the pure braid group P_n on the product $X_1 \otimes ... \otimes X_n$ of any objects

$X_1, ..., X_n \in \mathcal{G}$. When $\mathcal{G}$ is abelian we can define the product $M \otimes_{P_n} (X_1 \otimes ... \otimes X_n)$ for any finitely generated projective $k[P_n]$-module M. Indeed, presenting M as the image of an idempotent $p \in M_m(k[P_n])$ we set $M \otimes_{P_n} (X_1 \otimes ... \otimes X_n)$ to be the image of $c(p) \in End_{\mathcal{G}}(X_1 \otimes ... \otimes X_n)$. The setting does not depend (upto an isomorphism) on the choice of the idempotent p. The functors

$$(X_1, ..., X_n) \mapsto M \otimes_{P_n} (X_1 \otimes ... \otimes X_n) \tag{6}$$

will give the action of the operad $\mathcal{P}_k$ on $\mathcal{G}$.
The balanced case can be treated similarly. $\square$

Note that when the tensor product of the category $\mathcal{G}$ is bi-exact the functors (6) will be poly-exact.

3 K-theory of braided monoidal categories

3.1 K-theory of Cat-operad action

Remind that for a bi-exact functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ Waldhausen [19] assigned a homomorphism $K_*(\mathcal{A}) \otimes K_*(\mathcal{B}) \to K_*(\mathcal{C})$. We will need the multi-version of this which sets a homomorphism $K_*(\mathcal{A}_1) \otimes ... \otimes K_*(\mathcal{A}_n) \to K_*(\mathcal{C})$ in correspondance to a poly-exact functor $\mathcal{A}_1 \times ... \times \mathcal{A}_n \to \mathcal{C}$ which can be acheived by applying the arguments of [19] to poly-exact functors. Applying it to the exact operadic compositions of a Cat-operad (with exact components) we see that its K-theory is naturally an operad of graded abelian groups. Moreover, when such a Cat-operad $\mathcal{O}$ acts on an exact category $\mathcal{A}$ with exact action functors (5) its K-theory acts on the K-theory $K_*(\mathcal{A})$ of its algebra. In particular, homomorphisms in K-theory induced by the functors (6) give the following.

Proposition 3.2 *The operad $K_*(kP_*)$ acts on K-theory of any k-linear abelian braided monoidal category with exact tensor product.*

Similarly, the operad $K_*(kPb_*)$ acts on K-theory of any k-linear abelian balanced monoidal category with exact tensor product.

To calculate the K-theory of pure braid groups we use the assembly map $h_*(BG, K(k)) \to K_*(kG)$ defined by J.-L. Loday [15]. Here $h_*(BG, K(k)) = \pi_*(BG \wedge K(k))$ is the generalized homology associated to the classifying space BG of a group G with coefficients in the K-theory spectrum $K(k)$ of the ground ring k. The Atiyah-Hirzebruch spectral sequence

$$E_{p,q}(2) = H_p(G, K_q(k)) \Longrightarrow h_{p+q}(BG, K(k))$$

gives a way of calculating generalized homology $h_*(BG, K(k))$ in terms of group homology $H_*(G, K_*(k))$.

3.3 homology of braid groups

Cohomology of pure braid groups were calculated by V. I. Arnold in [2]. In particular, his result implies that homology of pure braid groups are free of finite rank as abelian groups. Thus the operad structure on the collection $\{P_*\}$ gives rise to the graded operad on homologies $H_*(P_*)$. Operad composition for $H_*(P_*)$ was described explicitly by F. Cohen [6]. As was noticed by P. Deligne, it follows from F. Cohen description that $H_*(P_*)$ is the operad of Gerstenhaber algebras. It is generated by two elements: a generator of $H_0(P_2)$ (binary operation of degree zero, the product) and a generator $[s_1^2]$ of $H_1(P_2)$ (binary operation of degree one, the bracket). It follows easily from the presentation for the Gerstenhaber operad, it has no non-trivial endomorphism. This fact allow us to prove the following

Lemma 3.4 *For any multiplicative spectre E with $\pi_0(E) = \mathbf{Z}$ the Atiyah-Hirzebruch spectral sequence for pure braid group P_n*

$$E_{p,q}(2) = H_p(P_n, \pi_q(E)) \Longrightarrow h_{p+q}(BP_n, E)$$

degenerates and gives an isomorphism $H_*(P_n) \otimes \pi_*(E) \to h_*(BP_n, E)$.

Proof:
We prove that all differentials of Atiyah-Hirzebruch spectral sequences for pure braid groups P_* are zero. Since the cohomologies $H_*(P_*)$ are free abelian, the Universal Coefficients Formula implies that $H_*(P_*, \pi_*(E))$ is isomorphic to $H_*(P_*) \otimes \pi_*(E)$. Again since $H_*(P_*)$ is free abelian, differentials can be considered as elements of $End(H_*(P_*)) \otimes End(\pi_*(E))$. By the naturality of differentials with respect to group homomorphisms between pure braid groups, their components in $End(H_*(P_*))$ should preserve the operad structure on $H_*(P_*)$. As we have seen there are no such maps in non-zero degree.

Finally, the inclusion of the free abelian groups $H_*(P_n) \to H_*(P_n) \otimes \pi_0(E) \to gr_*(h_*(BP_n, E))$ lifts to a map $H_*(P_n) \to h_*(BP_n, E)$. By the multiplicative property of Atiyah-Hirzebruch spectral sequence, its product with the inclusion $\pi_*(E) \to h_*(BP_n, E)$ gives the isomorphism $H_*(P_n) \otimes \pi_*(E) \to h_*(BP_n, E)$. $\square$

Lemma 3.5 *The homology $H_*(Pb_*)$ of the pure balanced groups form an operad of Batalin-Vilkovisky algebras.*

Proof:
Remind that the pure balanced groups are the direct sums of pure braid groups and free abelian groups. Thus, by the Kunneth formula, their homology is a tensor product $H_*(Pb_*) = H_*(P_*) \otimes H_*(\mathbf{Z}^*)$. The operadic composition on it is given by the operad structure on Pb_*. In particular, for the image of the 1-class $\Delta = [t] \in H_1(Pb_1)$ under $\mu : H_*(Pb_2) \otimes H_*(Pb_1) \to H_*(Pb_2)$ we have

$$\mu([t]) = [s_1^2] + [t_1] + [t_2]$$

which is just the equation (2) for the Batalin-Vilkovisky differential. Note that the class $[t]$ generates $H_1(Pb_1) = H_1(\mathbf{Z})$ and together with the bracket $[s_1^2]$ and the product $1 \in H_0(Pb_2)$ generates $H_*(Pb_*)$. $\square$

The composition of the assembly map with an isomorphism from the lemma 3.4 is a homomorphism

$$H_*(P_*) \otimes K_*(k) \to K_*(k[P_*]),$$

which is obviously compatible with the operad composition. Analogously, we have a homomorphism of operads

$$H_*(Pb_*) \otimes K_*(k) \to K_*(k[Pb_*]),$$

Combining these with the proposition 3.2 we have proven the following

Theorem 3.6 *For any braided monoidal abelian category with exact tensor product its K-theory is a Gerstenhaber algebra.*
Similarly, for any balanced monoidal abelian category with exact tensor product its K-theory is a Batalin-Vilkovisky algebra.
The Gerstenhaber bracket and the BV-differential are trivial in the case of symmetric monoidal abelian category.

3.7 explicit formulas

Here we express explicitly the zero degree components of the bracket $\{\ ,\ \} : K_0(\mathcal{G}) \otimes K_0(\mathcal{G}) \to K_1(\mathcal{G})$ and the BV-differential $\Delta : K_0(\mathcal{G}) \to K_1(\mathcal{G})$. Define by $[X, x] \in K_1(\mathcal{G})$ the image of the class of the object $(X, x) \in \mathcal{A}ut(\mathcal{G})$ by the map $K_0(\mathcal{A}ut(\mathcal{G})) \to K_1(\mathcal{G})$. Then

$$\{X, Y\} = [X \otimes Y, \sigma_{X,Y}] \text{ and } \Delta(X) = [X, \tau_X].$$

3.8 semi-simple braided categories

Let now $\mathcal{G}$ be a semi-simple category. Let S be the set of isomorphism classes of simple objects. We identify elements of S with some of their representatives in $\mathcal{G}$. Note that the functor

$$\times_{T\in S} Hom_{\mathcal{G}}(T,\ ) : \mathcal{G} \to \times_{T\in S} End_{\mathcal{G}}(T) - mod$$

is an equivalence between $\mathcal{G}$ and the direct product of categories of finitely generated modules over simple algebras. This equivalence gives the isomorphism in K-theory

$$K_*(\mathcal{G}) \to \oplus_{T\in S} K_*(End_{\mathcal{G}}(T)). \tag{7}$$

The Grothendieck ring $K_0(E)$ of a simple algebra E is isomorphic to the ring of integers. The isomorphism is given by the dimension function which is defined for finitely generated modules. The Whitehead group $K_1(E)$ coincides with the quotient by the commutant of the group of invertible elements in E. The map $K_1(E) \to E^*/[E^*, E^*]$ is called the *generalized determinant.* Combining it with the formulas for the bracket and BV-differential from 3.7 we get the following version for the semi-simple case:

$$\{X, Y\} = \oplus_{T\in S} \det(Hom_{\mathcal{G}}(T, \sigma_{X,Y})), \tag{8}$$

$$\Delta(X) = \oplus_{T\in S} \det(Hom_{\mathcal{G}}(T, \tau_X)), \tag{9}$$

where $\det : GL_n(End_{\mathcal{G}}(T)) \to Aut_{\mathcal{G}}(T)/[Aut_{\mathcal{G}}(T), Aut_{\mathcal{G}}(T)]$ is the generalized determinant.

In the case when the category $\mathcal{G}$ is k-linear over algebraically closed field k, the endomorphism algebras of simple objects coincides with k and we can think of det in the formulas (8),(9) as of ordinary determinant. Moreover, the inverse to the isomorphism (7) which in this case has the form $K_*(\mathcal{G}) \simeq \oplus_{T\in S} K_*(k)$ is given by the multiplication $K_0(\mathcal{G}) \otimes K_*(k) \to K_*(\mathcal{G})$. Since this map is obviously a ring homomorphism and the bracket and the BV-differential are always zero on $K_*(k)$, the formulas (8),(9) define the bracket and BV-differential on the whole $K_*(\mathcal{G})$.

4 Finite semi-simple braided rigid categories; a theorem of Anderson-Moore and Vafa

4.1 bilinear form on Grothendieck ring

From now on by a k-linear category we mean an additive category whose Hom-sets are finite dimensional vector spaces over k.

Let $\mathcal{G}$ be a k-linear semi-simple catgeory. The Grothendieck ring $K_0(\mathcal{G})$ of a semi-simple k-linear monoidal category $\mathcal{G}$ has a bilinear form:

$$([X],[Y]) = dim_k Hom_{\mathcal{G}}(X,Y), \quad \forall X,Y \in \mathcal{G}.$$

Classes of simple objects form an orthogonal basis, proving that $(\ ,\ )$ is symmetric and positively defined. The later condition means that $(x,x)=0$ for $x \in K_0(\mathcal{G})$ if and only if $x=0$.

4.2 rigid categories

Remind that the monoidal category $\mathcal{G}$ is (left and right) *closed* if for any pair of objects $X,Y \in \mathcal{G}$ there exist the terminals among the pairs $Z, z : Z\otimes X \to Y$ and $W, w : X \otimes W \to Y$, correspondingly. The objects of these terminal pairs are called the left and the right *inner Hom-objects* ($\mathcal{H}om^l(X,Y)$ and $\mathcal{H}om^r(X,Y)$) while the morphisms

$$\mathcal{H}om^l(X,Y)\otimes X \to Y, \quad X \otimes \mathcal{H}om^r(X,Y) \to Y$$

are *the evaluation* maps. In the case of *rigid (autonomous)* category the natural maps

$$\begin{aligned} \mathcal{H}om^l(X\otimes Y, Z) &\to \mathcal{H}om^l(X, \mathcal{H}om^l(Y,Z)) \\ \mathcal{H}om^r(X\otimes Y, Z) &\to \mathcal{H}om^r(Y, \mathcal{H}om^r(X,Z)), \end{aligned} \tag{10}$$

defined by the terminality, are isomorphisms. In the case when $Y=1$, the objects $X^* = \mathcal{H}om^l(X,1)$ and $X^\circ = \mathcal{H}om^r(X,1)$ are called left and right *duals* to X. The inner Hom-objects can be expressed in terms of duals since the natural maps

$$Y\otimes X^* \to \mathcal{H}om^l(X,Y), \quad X^\circ \otimes Y \to Hom^r(X,Y) \tag{11}$$

are bijections. We are interested in Hom-objects because of their relations with ordinary Hom-spaces: the natural maps

$$Hom_{\mathcal{G}}(1, \mathcal{H}om^l(X,Y)) \leftarrow Hom_{\mathcal{G}}(X,Y) \to Hom_{\mathcal{G}}(1, \mathcal{H}om^r(X,Y))$$

are isomorphisms. This, together with (10),(11), gives the following multiplicative properties for the bilinear form on the Grothendieck ring of a semi-simple rigid category:

$$(xy, z) = (x, zy^*) = (y, x^\circ z), \tag{12}$$

where $(\ )^*, (\ )^\circ : K_0(\mathcal{G}) \to K_0(\mathcal{G})$ are the (mutually inverse) operators given by the assignments $[X]^* = [X^*], [X]^\circ = [X^\circ]$.

The next result is a variant of the theorem due to E.Artin (Theorem 1.5.3 from [12]).

Lemma 4.3 *Let $\mathcal{G}$ be a k-linear semi-simple autonomous category with finitely many simple objects. Then its rational Grothendieck algebra $K_0(\mathcal{G}) \otimes$ $\mathbf{Q}$ is semi-simple.*

Proof:
We prove that any ideal of $K_0(\mathcal{G}) \otimes \mathbf{Q}$ is a direct summand (as a subalgebra) which, by induction, means that the Grothendieck algebra is a direct sum of simple subalgebras. Let $I \subset K_0(\mathcal{G}) \otimes \mathbf{Q}$ be a two-sided ideal. Then, by the formulas 12, the compliment

$$I^\perp = \{x \in K_0(\mathcal{G}) \otimes \mathbf{Q} : (I, x) = 0\}$$

is also a two-sided ideal. By the positive definiteness of the form $(\ ,\ )$, the intersection $I \cap I^\perp$ is trivial and the inclusion $I \oplus I^\perp \to K_0(\mathcal{G}) \otimes \mathbf{Q}$ is an isomorphism. Finally, the products $II^\perp, I^\perp I$ are trivial as subspaces of $I \cap I^\perp$ thus turning the additive decomposition $K_0(\mathcal{G}) \otimes \mathbf{Q} = I \oplus I^\perp$ in to a decomposition of algebras. $\Box$

Remark 4.4

Using the first equation of (12) twice and the symmetry of the form $(\ ,\ )$ we can see that

$$(x, x) = (1, xx^*) = (xx^*, 1) = (x, x^{**})$$

for any x. For a simple x, it will give the equality $x = x^{**}$. In particular, left and right duals in semi-simple monoidal category coincide. So, in Lemma 4.3, we can assume only one-side rigidness instead of two-side.

4.5 eigenvalues of double braiding and twist

Now we can prove a generalization of a theorem of Anderson-Moore and Vafa [1, 18] on the rationality of h-phases of RCFT which in categorical terms sais that the twist in modular category has finite order. Remind that modular category is a special kind of balanced monoidal semi-simple category with finitely many simple objects.

Theorem 4.6 *Let $\mathcal{G}$ be a k-linear semi-simple rigid category with finitely many simple objects. Suppose that*
1. $\mathcal{G}$ is braided with the braiding c, then for any $X, Y, Z \in \mathcal{G}$ the determinant $\det(Hom_{\mathcal{G}}(Z, c_{X,Y}c_{Y,X})$ is a root of unity;
2.$\mathcal{G}$ is balanced with the twist τ, then for any $X \in \mathcal{G}$ the determinant $\det(Hom_{\mathcal{G}}(Z, \tau_X)$ is a root of unity.

Proof:
By the lemma 4.3 we know that for such $\mathcal{G}$ the rational Grothendieck ring $K_0(\mathcal{G}) \otimes \mathbf{Q}$ is a semi-simple commutative $\mathbf{Q}$-algebra. In particular, any its derivation is zero. Now the formula (1) shows that the Gertenhaber bracket is a bi-derivation. Thus the value of the bracket is torsion: $\{K_0(\mathcal{G}), K_0(\mathcal{G})\} \subset K_1(\mathcal{G})_{tors}$, which together with (8) gives the first statement.
Since the bracket is zero on $K_0(\mathcal{G})\otimes\mathbf{Q}$ the formula (2) for the BV-differencial becomes a derivation condition and implies the triviality of Δ on $K_0(\mathcal{G})\otimes\mathbf{Q}$. So the image of the BV-differential is also torsion: $\Delta(K_0(\mathcal{G})) \subset K_1(\mathcal{G})_{tors}$, which combined with (9) gives the second part of the theorem. □

Corollary 4.7 *Let $\mathcal{G}$ be a semi-simple rigid balanced category with finitely many simple objects. Let c and τ be the commutativity constraint and the twist correspondingly. Then there is an integer n such that $(c_{Y,X}c_{X,Y})^n = I_{X,Y}$ and $\tau_X^n = I_X$ for all $X, Y \in \mathcal{G}$.*

Proof:
Since $\tau_X = \det(Hom_{\mathcal{G}}(X, \tau_X)$ for a simple $X \in \mathcal{G}$, the part 2 of the theorem 4.6 implies that $\tau_X^n = I_X$ for all simple X and, thus for all X. This implies $(c_{Y,X}c_{X,Y})^n = I_{X,Y}$, since $c_{Y,X}c_{X,Y} = (\tau_X \otimes \tau_Y)\tau_{X\otimes Y}^{-1}$ and $\tau_X \otimes \tau_Y$ commutes with $tau_{X\otimes Y}$. □

The next example shows that the finiteness condition for the number of simple objects is important.

Example 4.8 *Grothendieck ring of quantum $sl(2)$.*

Let x denote the class of fundamental (2-dimensional) representation X of $sl_q(2)$. Then the Grothendieck ring $R(sl_q(2))$ of the catgeory of finite dimensional representations of $sl_q(2)$ is a polynomial ring $\mathbf{Z}[x]$ in it. Since the quantum R-matrix $c_{X,X}$ has two eigenvalues -1 and q and the the eigenspace of the first is the trivial representation we have the formula for the bracket

$$\{x,x\} = (x^2 - 1) \otimes q^2.$$

References

[1] G. Anderson, G. Moore, Rationality in conformal field theory. Comm. Math. Phys. 117 (1988), no. 3, 441–450.

[2] V. I. Arnol'd, The cohomology ring of the group of colored braids. Mat. Zametki 5 (1969) 227–231 (Russian); English transl. in Trans. Moscow Math. Soc. 21 (1970), 30-52.

[3] E. Artin, Theory of braids. Ann. of Math. (2) 48, (1947). 101–126.

[4] I. A. Batalin, G. A. Vilkovisky, Gauge algebra and quantization. Phys. Lett. B 102 (1981), no. 1, 27–31.

[5] I. A. Batalin, G. A. Vilkovisky, Quantization of gauge theories with linearly dependent generators. Phys. Rev. D (3) 28 (1983), no. 10, 2567–2582.

[6] F. R. Cohen, The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$, In "The homology of iterated loop spaces.", Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976. vii+490 pp.

[7] Drinfel'd, V. G. Almost cocommutative Hopf algebras. (Russian) Algebra i Analiz 1 (1989), no. 2,30–46 translation in Leningrad Math. J. 1 (1990), no. 2, 321–342

[8] G. Dunn, K-theory of braided tensor ring categories with higher commutativity. K-Theory 9 (1995), no. 6, 591–605.

[9] A. Joyal, R. Street, Braided tensor categories. Adv. Math. 102 (1993), no. 1, 20–78.

[10] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories. Comm. Math. Phys. 159 (1994), no. 2, 265–285; (preprint hep-th/9212043).

[11] H. Gillet, D. R. Grayson, The loop space of the Q-construction. Illinois J. Math. 31 (1987), no. 4, 574–597.

[12] M. Goto, F. D. Grosshans, Semisimple Lie algebras. Lecture Notes in Pure and Applied Mathematics, Vol. 38. Marcel Dekker, Inc., New York-Basel, 1978. 480 pp.

[13] M. Gerstenhaber, The cohomology structure of an associative ring. Ann. of Math. (2) 78 1963 267–288.

[14] M. Gerstenhaber, On the deformation of rings and algebras. Ann. of Math. (2) 79 1964 59–103.

[15] J.-L. Loday, K-thorie algbrique et reprsentations de groupes. Ann. Sci. cole Norm. Sup. (4) 9 (1976), no. 3, 309–377.

[16] B. H. Lian, G. J. Zuckerman, New perspectives on the BRST-algebraic structure of string theory. Comm. Math. Phys. 154 (1993), no. 3, 613–646; (preprint hep-th/9211072).

[17] Turaev, V. G. Quantum invariants of knots and 3-manifolds. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994. 588 pp.

[18] C. Vafa, Toward classification of conformal theories. Phys. Lett. B 206 (1988), no. 3, 421–426.

[19] F. Waldhausen, K-theory of generalized free products. I, II, III, IV. Ann. of Math. (2) 108 (1978), no. 1, 135–204; no. 2, 205–256.