Ball model for Hilbert's twelvth problem

by

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Name des Hauses * = siehe Anzeige	Inhaber	Straße	Ruf Vorwahl (02644)	Ruhetag	P = Parkplatz	MA = Mittag und Abendtisch K = Kegelbahn	Saal von bis Personen	WP = Weinprobe	Tanz f. Gesell.
estaurants und Cafés									
Hotel Weinstock*	Rolf Geller	Linzhausenstraße 38	24 59	Mo	P	MA	10–400		n.V.
Hotel Palm*	Illa Schlang	Vor dem Leetor 13	25 32		P	MA, K	50–500	WP	n.V.
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Hotel-Restaurant Gut Frühscheid	Familie Wiedemeyer	Am Ronigerweg	70 15	Di	P	MA	10–50		n.V.
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Zum Köbes	Jakob Lück	Buttermarkt 9	26 53	Mi					
Weinhaus zur Linde	Familie Hesse	Hospitalstraße 12	23 92						
Brückenschänke	Anneliese Seinsche	Linzhausenstraße 72	31 23	Do	P				
Ristorante Pizzeria Franco	Franco Soravia	Klosterstraße 5	22 00						
Restaurant Burg Linz*	Robert J. Schwaab	Burgplatz 4	70 21		P	MA	bis 200		n.V.
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Zur Alten Post*	Rudi Stein	Rheinstraße 5	25 17			MA, K	20–90		n.V.
Zur Bunten Stadt	Günter Wagner	Neustraße 23	33 07	Do		MA	40–180		n.V.
Bacchus-Keller*	Ingeborg Probst	Marktplatz 17	24 80	Mo	P		50–80	WP	n.V.
La Campagnola	Fam. Di Lauro	Strohgasse 6	78 00	Di		MA			
Sirtaki	N. Gionarkias	Strohgasse 8	78 40	Mo		MA	30		
Rheintor	Christine Klinkusch	Burgplatz 10	27 38	Mi		MA			
Em Stropp	Klaus Staby	Neustraße 13	26 24	Mo					
Zur Mühle*	Tina Paffhausen	Mühlengasse 17	32 24	Di		MA	30		
Schmeck'Lecker*	Flesche-Aimola	Burgplatz 13	79 37			MA			
Zum alten Stern	Toni Euskirchen	Asbacher Straße 139	26 59	Do	P	MA	20–100		n.V.
„Go-In"	Helene Nietzard	Asbacher Straße 17	29 94	Sa/So					Disco
Em Pözje	Jürgen Kryll	Neustraße 17	17 10	Di		MA	40		
Zum Neutor	Chr. Holberg	Neustraße 27	57 97	Mi					
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Weinhaus Winzen	Willi Winzen	Asbacher Straße 74	32 55	Di				WP	
Zum Markus	Salvatore Globbe	Klosterstraße 8 a	32 85	Di		MA			
Nibelungen im Rheinischen Hof	Heinrich Klein	Rheinstraße 13	25 78			MA			
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Linzer Café-Restaurant	Dietmar Brietzke	Kanzlerstraße	18 82			M.	90		n.V.
Café Schneider	P. Schneider	Neustraße 15	25 02						
Peppers Café, Galerie	Klaus Pepper	Asbacher Straße 25	47 07				F.-Wohnung u. Kleinkunst		
Martins Stuben	Martin Scherer	Am Gestade 2	52 53	Mi	P				
Em Stüffje	Jürgen Kryll	Buttermarkt 10	34 70	Mo					
Jägerklause	Kurt Kurz	Marktplatz 27	69 18	Di					
Grill am Markt	Hartmut Rüdiger	Strohgasse 18	12 70			MA			
Ital. Eiscafé	Aimola-Soravia	Burgplatz 12	41 00						
Eis-Salon Dolomiten	Julio Corazza	Rheinstraße 15	57 35						
Ital. Eisdiele am Markt	Elisabeth Sagui	Marktplatz 22	43 08		P				
Eis-Salon Cordella	Modesto Cordella	Klosterstraße 12	36 89						
Bahnhofsgaststätte	Katharina Schmidt	Bahnhofsplatz 1	34 28		P	MA			n.V.
Kaktus	Jean-Regis Dibus	Commenderiestraße 2	14 74	Mo		MA			

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R.-P. Holzapfel

Contents

0. Introduction.

Until now one misses a clear solution of Hilbert's 12-th problem entitled "Ausdehnung des Kroneckerschen Satzes ueber abelsche Koerper auf einen beliebigen algebraischen Rationalitaetsbereich". This problem plays a central role among the 23 Hilbert problems because it joins some of them with each other, namely the problems 7, 9, 21 and 22. The 12-th problem is based on Kroneckers work on the explicit description of abelian number fields over the field $\mathbb{Q}$ of rational numbers or over imaginary quadratic number fields, respectively, by means of special values of special transcendent functions of one complex variable. The Theorem of Kronecker-Weber asserts that each absolute abelian number field is generated by a rational expression of a unit root over $\mathbb{Q}$, where unit roots are understood as special values of the exponential function.

As a counter part appears Hilbert's 7-th problem. It asks for the quality of values of the shifted exponential function

$$e(z) = \exp(\pi i z),\ i = \sqrt{-1},$$

at algebraic arguments outside of the rationals $\mathbb{Q}$ and conjectures that all these values are transcendent numbers (see [10]). This problem has been solved affirmatively and independently by Gelfond [7] and Schneider [19] in 1934. Altogether we know

I. $e(z)$ has algebraic values on $\mathbb{Q}$;

II. $e(z)$ has transcendent values on $\overline{\mathbb{Q}} - \mathbb{Q}$ ($\overline{\mathbb{Q}}$ the field of algebraic numbers);

III. the number-theoretic meaning of the values $e(q)$, $q \in \mathbb{Q}$.

Substituting the base field $\mathbb{Q}$ by an imaginary quadratic number field one needs special values of Weierstrass' $\wp$-function (at torsion points of an elliptic curve) and special values (singular moduli) of the elliptic modular function j in order to generate all abelian extensions. For a precise formulation of this Main Theorem of Complex Multiplication we refer to Shimura's book [35], Ch.5. Historically, this main theorem is known as "Kroneckers Jugendtraum". It appears in Hilbert's programm as "Aufgabe" (Kronecker's problem) preparing the 12-th problem itself.

On the other hand C.L.Siegel [31] proved in 1949 that j takes transcendent values at algebraic points on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}\,\tau > 0\}$ which are not singular. In analogy with the exponential function we can summerize the situation in the following manner. Let

$$\mathbb{H}_{sing} = \{\tau \in \mathbb{H};\ [\mathbb{Q}(\tau):\mathbb{Q}] = 2\}$$

be the set of singular moduli. The transcendent function j has a well-known Fourier series

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^{2} + \ldots$$

with integral coefficients and $q = \exp(2\pi i\tau)$, see [30]. One knows

I. j has algebraic values on $\mathbb{H}_{sing}$;

II. j takes transzendent values on $\mathbb{H}(\bar{\mathbb{Q}}) \setminus \mathbb{H}_{sing}$, where $\mathbb{H}(\bar{\mathbb{Q}})$ denotes the set $\mathbb{H} \cap \bar{\mathbb{Q}}$ of algebraic numbers on the upper half plane;

III. the number-theoretic construction / quality / meaning of $j(\sigma),\ \sigma \in \mathbb{H}_{sing}$.

Hilbert asked in his 12-th problem for transcendent functions of s e v e r a l variables with properties corresponding to those of the exponential function and the elliptic modular function: "Von der hoechsten Bedeutung endlich erscheint mir die Ausdehnung des Kroneckerschen Satzes auf den Fall, dass an Stelle des Bereichs der rationalen Zahlen oder des imaginaeren Zahlenbereiches ein beliebiger algebraischer Zahlkoerper als Rationalitaetsbereich zugrunde gelegt wird; ich halte dies Problem fuer eines der tiefgehendsten und weittragendsten Probleme der Zahlen- und Funktionentheorie"; and a littlebit later: "Wie wir sehen, treten in dem eben gekennzeichneten Problem die drei grundlegenden Disziplinen der Mathematik, naemlich Zahlentheorie, Algebra und Funktionentheorie in die innigste gegenseitige Beruehrung, und ich bin sicher, dass insbesondere die Theorie der analytischen Funktionen m e h r e r e r Variablen eine wesentliche Bereicherung erfahren wuerde, wenn es gelaenge, diejenigen Funktionen aufzufinden und zu diskutieren, die fuer einen beliebigen algebraischen Zahlkoerper die entsprechende Rolle spielen, wie die Exponentialfunktion fuer den Koerper der rationalen Zahlen und die elliptische Modulfunktion fuer den imaginaeren quadratischen Zahlkoerper".

We found only a few places in the mathematical literature with an explicit reference to the twelvth problem of Hilbert. On first place we remember to Hecke's thesis [8] and habilitation [9]. They are closely connected with the creation of the theory of Hilbert modular surfaces. This work is difficult to understand and it would be nice to clarify the situation from a modern point of view. The next important place where the twelvth problem is mentioned one can find in the book [36] of Shimura-Taniyama. Indeed, Shimura's theory of complex multiplication is an important tool for finding solutions of the problem. The latest hint to Hilbert's twelvth problem we found is due to Tate [37] in connection with the Stark conjecture. It touches the problem but will not solve it in the original sense of Hilbert. Manin accepts in his review [12] only Hecke's work of 1912, 1913 as a finer approximation to a solution of the 12-th problem. Langlands announced in [21] some doubts of Hilbert's formulations. In our opinion the twelvth problem needs a stronger formulation in order to catch solutions. With regard to the transcendental functions e,j above and their properties I.,II.,III. we propose the following definition.

A solution model for Hilbert's twelvth problem is a triple (V, V_{sing}, f) consisting of

(i) a (non-compact) complex manifold V with fixed analytic embedding into a complex projective space $\mathbb{P}^M(\mathbb{C})$;

(ii) a subset V_{sing} of the algebraic points $V(\bar{\mathbb{Q}}) = V \cap \mathbb{P}^M(\bar{\mathbb{Q}})$ lying dense in V;

(iii) A transcendent holomorphic map

$$f = (f_0 : f_1 : \dots : f_N) : V \longrightarrow \mathbb{P}^N(\mathbb{C});$$

satisfying the postulates I.,II.,III. below.

Remark. We call f transcendent if f is not the restriction of a rational map in the sense of algebraic geometry. The elements of V_{sing} are called the singular points of V.

I. $f(\sigma) = (f_0(\sigma) : \dots : f_N(\sigma))$ is algebraic, that means $f(\sigma) \in \mathbb{P}^N(\bar{\mathbb{Q}})$, for $\sigma \in V_{sing}$;

II. $f(\tau)$ is transcendent, that means $f(\tau) \notin \mathbb{P}^N(\bar{\mathbb{Q}})$, for

$\tau \in V(\bar{\mathbb{Q}}) \setminus V_{sing}$;

III. one has a number-theoretic construction / quality / meaning of field extensions

$$F'_\sigma(f(\sigma)) = F'_\sigma(\ldots, f_i(\sigma)/f_j(\sigma), \ldots)$$

for suitable well-defined "elementary" number fields F'_σ, $\sigma \in V_{sing}$.

Of course, we assume that V_{sing} is given independently of the holomorphic functions $f_0, \ldots, f_N$.

The first two conditions are very sharp but condition III. is free for several interpretations.

A (twodimensional) ball model for Hilbert's twelvth problem is a solution model $(\mathbb{B}, \mathbb{B}_{sing}, f)$, where $\mathbb{B}$ is the complex two-dimensional unit ball. The Main Theorem of section 1 presents a ball model $(\mathbb{B}, \mathbb{B}_{CM}, th)$ for the twelvth problem satisfying I. and III. Recently Shiga proved that also II. is essentially satisfied (see Remark 13.18). The components th_i of $th = (th_1 : th_2 : th_3 : th_4)$ are restrictions of elementary polynomials of theta constants to the ball $\mathbb{B}$ embedded in the generalized Siegel upper half plane $\mathbb{H}_3$, where the theta constants live.

We prefered to formulate the number-theoretic Main Theorem in the first section corresponding to Hilbert's order in his list of problems. Consequently we have to explain immediately after the notions of Shimura's class fields, complex multiplication of abelian varieties, moduli fields in the sections 2.,3.,4. and 5. This prepares at the same time the number-theoretic side of proof of the Main Theorem in section 13.

The geometric and analytic starting point is section 6. For an algebraic geometer it is convenient to begin there. The following sections will demonstrate that the simple configuration of four points and six lines through pairs of them in the projective plane determines completely the construction of our ball model. This is a consequence of some recent developments: A theorem of R.Kobayashi [18] provides the existence of a ball covering of $\mathbb{P}^2$ branched along the quadrilateral introduced above. There is only one possibility. The corresponding ramification indices can be calculated by the effective finiteness theorem for ball lattices due to the author [16]. The corresponding group of the covering has been found in a classification atlas of Picard modular surfaces due to the author and Feustel. This group appears as monodromy group of a Fuchsian system of partial differential equations uniquely determined by the quadrilateral. This system coincides with the Euler-Picard system of an algebraic curve family in the sense of the author's book [12]. The solution consists of variations of integrals of a differential form of first kind along cycles on Picard curves. The Picard curve family studied first by Picard in 1883 plays the same role as the elliptic curve family in Kronecker's problem. Now we discovered that its investigation was absolutely necessary for finding our ball model for Hilbert's twelvth problem. The proof of the Main Theorem is delegated to the fine arithmetic and analytic study of the family of Picard curves. This will be done in the sections 9. - 12. using and explaining available recent results of Shiga [32]. Feustel [6] and the author [13].

The modern tools in the way of proof should also work for other cases, where the starting situation of a branched covering is precisely known. We think of Hilbert modular surfaces, Picard modular surfaces, a Picard modular threefold investigated carefully by Bruce Hunt and the Siegel modular threefold connected with hyperelliptic curves of genus 2. The latter case should be open a door to a precise modern understanding of Hecke's work on Hilbert's twelvth problem.

We close the introduction with two problems. More of them can be found at the end of the final section 13.

0.1 Problem. Study special values of Picard modular functions of higher level in connection with non-abelian class field theory.

0.2 Problem. Generate more (if possible all) abelian extensions of reflex fields of cubic extensions of the Eisenstein numbers by means of special values of some additional transcendent functions.

1. Formulation of the Main Theorem

First we present roughly the basic objects we need in the Main Theorem. More precise definitions are given in the later sections.

0. Basic field: $K = \mathbb{Q}(\sqrt{-3})$ the field of Eisenstein numbers;

1. Geometric object: the ball $\mathbb{B} \cong$ (K-linear equivalent in $\mathbb{P}^2$ to)

$$\mathbb{B}^2 = \{\tau = (\tau_1, \tau_2) \in \mathbb{C}^2;\ |\tau_1|^2 + |\tau_2|^2 < 1\},$$

embedded in $\mathbb{H}_3$ (Siegel domain, see 10.);

2. Analytic functions: $th_1, th_2, th_3, th_4 : \mathbb{B} \longrightarrow \mathbb{C}$ (restricted theta constants);

3. Special arguments (CM-modules): $\sigma \in \mathbb{B}(\bar{\mathbb{Q}})$ (dense in $\mathbb{B}$);

4. Correspondences: $\sigma \longmapsto F_\sigma/K$ (relative cubic number fields)

$\longmapsto \mathfrak{a}_\sigma$, $\mathbb{Z}$-lattice in F_σ

$\longmapsto \Phi_\sigma = \sum_{i=1}^{3} \varphi_i$, $\varphi_i : F_\sigma \hookrightarrow \mathbb{C}$ (field embeddings); $\varphi_i \neq \varphi_j$, $i \neq j$, $\varphi_i \neq \bar{\varphi}_k$;

$\longmapsto F'_\sigma$, the reflex field of (F_σ, Φ_σ);

5. Function field $K(th) = K(th_1/th_2, th_1/th_3, \ldots, th_3/th_4)$;

6. Number fields $K(th(\sigma)) = K(th)(\sigma) = K(f(\sigma);\ f \in K(th))$, where we neglect to adjoint $f(\sigma)$, if $f(\sigma) = \infty$;

7. Symmetric group S_4 acting on $K(th)$ via permutation of indices at the generators th_i/th_j,

$S_4(\sigma) = S_4 \cap Gal(K(th(\sigma))/\mathbb{Q})$ acting on $K(th(\sigma))$;

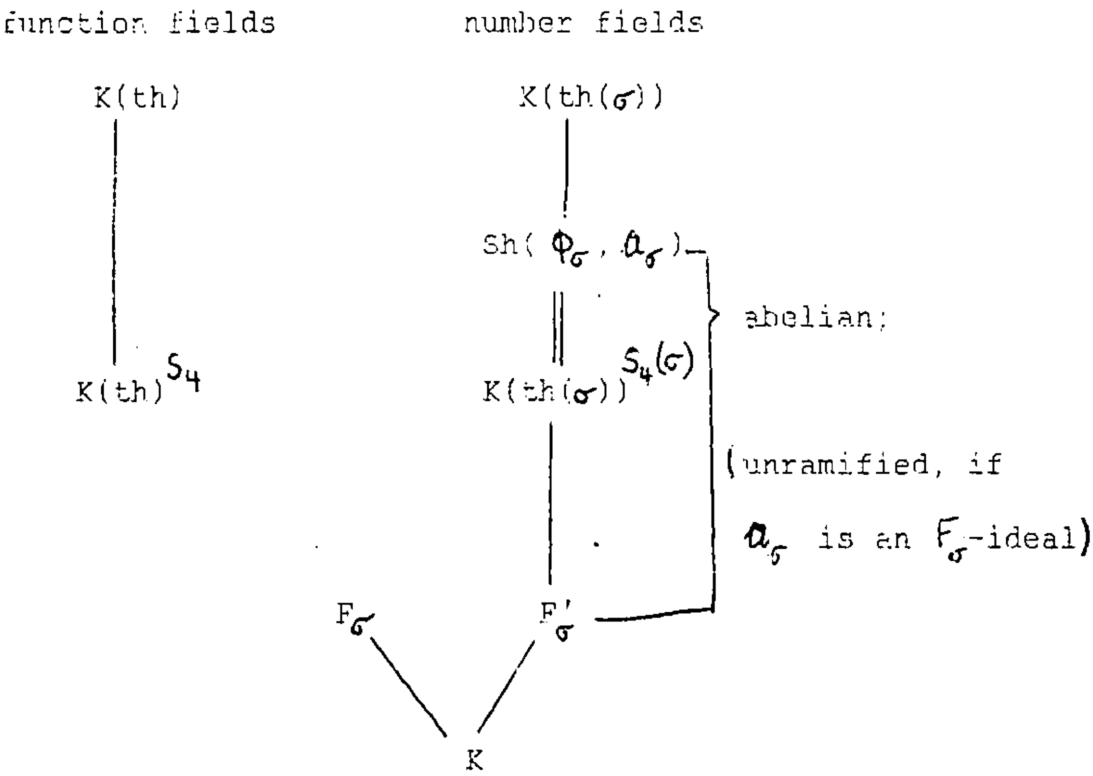
8. Shimura class fields $Sh(\Phi_\sigma, \mathfrak{a}_\sigma)$.

Now we can formulate the

MAIN THEOREM (Construction of Shimura class fields for cubic extensions of Eisenstein numbers via special values of Theta constants).

With the above notations one has for each CM-module $\sigma \in \mathbb{B}$ field towers as described in diagram (1.1):

(1.1)



2. Shimura Class Fields.

We follow the book [20] of S.Lang. First we have to introduce the reflex fields. Fixing notations we let F be a totally imaginary number field of absolute degree 2g and Φ a choice of g embeddings $\varphi_i : F \hookrightarrow \mathbb{C}$ pairwise not cojugated to each other. We write $\Phi = \Phi_F = \sum_{i=1}^{g} \varphi_i$ and call the pair (F, Φ) a CM-type.

If M/F is a finite field extension, then we can lift Φ_F to

$$\Phi_M = \sum_{i=1}^{g} \sum \{\text{all extensions of } \varphi_i \text{ to } M\}$$

So we get a CM-type lifting $(F, \Phi_F) \longmapsto (M, \Phi_M)$. We set

(2.1) $$\mathrm{Stab}(\Phi) = \{\mu \in \mathrm{Aut}(\mathbb{C});\ \mu \circ \Phi = \Phi\},$$

where $\mathrm{Aut}(\mathbb{C})$ denotes the group of all field automorphisms of $\mathbb{C}$. Now assume that M/F as above is a Galois extension. Then we define the r e f l e x f i e l d F' of (F, Φ) as fixed field

(2.2) $$F' = M^{\mathrm{Stab}\,\Phi} = \mathbb{Q}(\mathrm{Tr}_\Phi(F)),$$

where $Tr_{\Phi} : F \longrightarrow F'$ denotes the type trace defined by $Tr_{\Phi}(f) = \sum_{i=1}^{g} \varphi_i(f)$. With $\Phi_M = \sum_j \psi_j$, ψ_j understood as automorphisms of M, we set $\Phi'_M = \sum_j \psi_j^{-1}$. One can show that the type (M, Φ'_M) is the lift of a uniquly determined primitive type (F', Φ'), which is called the r e f l e x t y p e of (F, Φ). A type is called p r i m i t i v e , if it is not lifted from a lower field. If the starting type (F, Φ) is primitive, then the reflex (F'', Φ'') of its reflex (F', Φ') coincides with (F, Φ). In general, the double reflex field F'' is contained in F. Altogether we describe the situation in the following diagram (2.3):

(2.3)

$$\begin{array}{ccc} & M & \\ \text{Gal} \diagup & & \diagdown \\ F & & F' \end{array} , \qquad \begin{array}{ccc} \Phi_M & \longmapsto & \Phi'_M = \sum_j \psi_j^{-1} \\ \uparrow & & \downarrow \\ \Phi = \Phi_F & & \Phi'_{F'} = \Phi' \end{array}$$

Fixing F, Φ, the t y p e n o r m N_{Φ} or the r e f l e x n o r m $N' = N_{\Phi'}$ are respectively defined by

$$N_{\Phi} : F \longrightarrow F' , \qquad N' = N_{\Phi'} : F' \longrightarrow F'' \subseteq F$$
$$f \longmapsto \prod_{i=1}^{g} \varphi_i(f)$$

Both, N_{Φ} and N', can be extended to the idele groups of the fields F or F', respectively:

$$N_{\Phi} : \mathbb{A}_F^* \longrightarrow \mathbb{A}_{F'}^* , \qquad N' : \mathbb{A}_{F'}^* \longrightarrow \mathbb{A}_F^*$$

Now we are well-prepared to define the Shimura class fields mentioned several times above. For this purpose we let $\mathfrak{a}$ a $\mathbb{Z}$-lattice in F. The absolute norm of ideles s is denoted by $\mathbb{N}(s)$. Now we define the idele group $U(\phi, \mathfrak{a}) \subseteq \mathbb{A}^*_{F'}$ of an (extended) type $(F, \phi, \mathfrak{a})$ by

(2.4) $\quad U(\phi, \mathfrak{a}) = \{s \in \mathbb{A}^*_{F'} ;\ N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a},\ \beta\bar{\beta} = \mathbb{N}(s^{-1}) \in \mathbb{Q}$

for a suitable $\beta \in F\}$

We remark that the multiplication of an idele $t \in \mathbb{A}^*_F$ with $\mathfrak{a}$ is defined componentwise on the finite part $t_{fin} = (t_p) \in \mathbb{A}^*_{F,fin}$: There is a unique $\mathbb{Z}$-lattice $t\mathfrak{a}$ in F with local components $(t\mathfrak{a})_p = t_p\mathfrak{a}_p$ for all $p \in \operatorname{Spec} \mathbb{Z}$, $\mathfrak{a}_p = \mathbb{Z}_p \otimes \mathfrak{a}$.

Now we apply global abelian class field theory in order to define $Sh(\phi, \mathfrak{a})$ as class field of the reflex field F'. For details we refer to the monograph [25] of Neukirch.

Let M be a finite abelian field extension of F'. Then there is an exact sequence

(2.5)

$$1 \longrightarrow U/F'^{\times} \longrightarrow \mathbb{A}^*_{F'}/F'^{\times} \longrightarrow \mathrm{Gal}(M/F') \longrightarrow 1$$

where $(t, M/F')$, $t \in \mathbb{A}^*_{F'}$, is the global norm rest symbol locally defined by Frobenius automorphisms. The idele group $U = U_M$ is equal to the extended norm group $N_{M/F'}(\mathbb{A}^*_M) \cdot F'^{\times}$. Conversely, if U is a cofinite subgroup of the idele group A of F' containing $F'^{\times}$, then there exists a unique finite abelian extension $M_U = M$, the class field of F belonging to U, such that the above sequence (2.5) is exact. So there is a biunivoque correspondence $U_M = U \longleftrightarrow M = M_U$ (reciprocity).

Now we take the projective limit of our finite abelian groups $\mathrm{Gal}(M/F')$ along all finite abelian extensions M of F'. On this way we obtain the Galois group $\mathrm{Gal}(F'^{ab}/F')$ of the maximal abelian extension F'^{ab} of F'. The norm rest maps in (2.5) yields an injective map $(.,F'): \mathbb{A}^*_{F'}/F'^{\times} \longrightarrow Gal(F'^{ab}/F')$. Via the norm rest symbols (s,F') the idele group $\mathbb{A}^*_{F'}$ acts on F, and the finite abelian extension fields M appear as fixed fields of the corresponding subgroups U of $\mathbb{A}^*_{F'}$. So we can write

(2.6) $$M_U = (F'^{ab})^{(U,F')} = \mathbb{C}^{\widetilde{(U,F')}} ,$$

where $\widetilde{(U,F')}$ denotes the group of all extensions of elements $(s,F') \in Gal(F'^{ab}/F')$, $s \in U$, to automorphisms of $\mathbb{C}$.

2.7 Definition. The S h i m u r a   c l a s s   f i e l d $Sh(\Phi,\mathfrak{a})$ of the type $(F,\Phi,\mathfrak{a})$ is the class field of F' corresponding to $U(\Phi,\mathfrak{a})$ defined in (2.4):

$$Sh(\Phi,\mathfrak{a}) = (F'^{ab})^{(U(\Phi,\mathfrak{a}),F')} = \mathbb{C}^{\widetilde{(U(\Phi,\mathfrak{a}),F')}}.$$

3. Complex multiplication.

Let A be an abelian variety over the complex numbers and F a finite field extension of Q. We say that A has F - m u l t i - p l i c a t i o n , if there is an embedding $: F \hookrightarrow \mathbb{Q}\otimes End\ A$ into the endomorphism algebra $\mathbb{Q}\otimes End\ A$ of A. In this case the degree $[F:Q]$ is not greater than 2g, g = dim A. If $[F:\mathbb{Q}] = 2g$, then we say that A has c o m p l e x   m u l t i p l i c a - t i o n . In this case F acts on the tangent space T_A of A (at 0). So we have an embedding $F^{\times} \hookrightarrow Gl(T_A) = Gl(\mathbb{C})$. This representation splits into g one-dimensional representations. The corresponding characters are understood as embeddings $\varphi_i : F \hookrightarrow \mathbb{C}$, $i = 1, \ldots, g$. Setting $\Phi = \sum_{i=1}^{g} \varphi_i$ we get a type (F,Φ). Then A is called an a b e l i a n   v a r i e t y   o f   t y p e

(F, Φ).

One can prove that an abelian variety with complex multiplication of type $(F, \Phi, \mathfrak{a})$ is isomorphic to $\mathbb{C}^g/\Phi(\mathfrak{a})$, where $\Phi(\mathfrak{a})$ is the $\mathbb{Z}$-lattice in $\mathbb{C}^g$ generated by the vectors

$$\Phi(a_j) = {}^t(\varphi_1(a_j), \ldots, \varphi_g(a_j)), \; j = 1, \ldots, 2g$$

and $a_1, \ldots, a_{2g}$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$. For $f \in F$ we let $D_\Phi(f)$ the diagonal matrix with the elements $\varphi_i(f)$ in the diagonal. The representation $D_\Phi : F^* \longrightarrow Gl_g(\mathbb{C})$ defines a complex multiplication on $\mathbb{C}^g/\Phi(\mathfrak{a})$. Under the isomorphism with A both multiplications are compatible. The *multiplication ring* is:

$$\mathcal{O} = F \cap \operatorname{End} A = [\mathfrak{a} : \mathfrak{a}]_F = \{f \in F;\; f\cdot\mathfrak{a} \subseteq \mathfrak{a}\}.$$

Refining our language we will say that A is of type $(\mathcal{O}, F, \Phi)$ and we will call $\mathbb{C}^g/\Phi(\mathfrak{a})$ together with the complex multiplication defined by D_Φ the *standard torus model* of A or of the types $(F, \Phi, \mathfrak{a}), (\mathcal{O}, \Phi, \mathfrak{a})$. Two standard models of type $(\mathcal{O}, \Phi)$ are isomorphic (with compatibility of multiplication) if and only if the corresponding $\mathcal{O}$-modules $\mathfrak{a}, \mathfrak{a}'$ are isomorphic. The standard torus $\mathbb{C}^g/\Phi(\mathfrak{a})$ of given multiplication type $(F, \Phi, \mathfrak{a})$ is an abelian variety iff Φ is lifted from a primitive type. For proofs we refer to [ZD] again.

Let F be a number field and $\mathcal{O}$ an order in F (multiplicatively closed $\mathbb{Z}$-lattice in F). We denote by $L(\mathcal{O})$ the set of $\mathcal{O}$-lattices $\mathfrak{a}$ in F ($\mathbb{Z}$-lattices in F which are $\mathcal{O}$-modules) with factor ring $[\mathfrak{a} : \mathfrak{a}]_F = \mathcal{O}$. The set $cl(\mathcal{O})$ of $\mathcal{O}$-isomorphy classes of $L(\mathcal{O})$ is finite (see e.g. [3], II.6, Th.3). Its number is denoted by $h(\mathcal{O})$.

For instance, if $\mathfrak{O} = \mathfrak{O}_F$ is the ring of integers in F, then $L(\mathfrak{O})$ is the ideal group of F, $cl(\mathfrak{O})$ the ideal class group and $h(\mathfrak{O})$ coincides with the class number $h(F)$ of F. Altogether we can count now the isomorphy classes of abelian CM-varieties of given types:

3.1 Propositon. Let (F, Φ) be a complex multiplication type, $\mathfrak{O}$ an order in F. The number of isomorphy classes of abelian varieties with complex multiplication of type $(\mathfrak{O}, \Phi)$ is equal to $h(\mathfrak{O})$, if (F, Φ) is a lifted type, or equal to 0, otherwise.

The number of isomorphy classes of abelian CM-varieties A with CM-ring $\mathfrak{O} = F \cap \mathrm{End}\, A$ is equal to $h(\mathfrak{O})\cdot l(F)$, where $l(F)$ denotes the number of lifted types (F, Φ). It cannot be greater than $2^g \cdot h(\mathfrak{O})$, $g = [F:\mathbb{Q}]/2 = \mathrm{rank}_{\mathbb{Z}}(\mathfrak{O})/2 = \dim A$.

3.2 Remarks. "Lifted type" means: lifted from a primitive type. Especially, primitive types are understood as special cases of lifted types. The number fields E of primitive types are well-understood. These are the so-called C M - f i e l d s characterized as imaginary quadratic extensions of totally real number fields. So the set of multiplication fields of complex CM-varieties coincides with the set of extensions of CM-fields. Namely, each CM-type (E, Ψ) of a CM-field E is lifted from a primitive type (see [10], I.2, Lemma 2.2), hence E is a multiplication field of an abelian CM-variety by the above torus construction. This is true for any lifted type. Especially, each cubic extension F of $K = \mathbb{Q}(\sqrt{-3})$ appears as multiplication field of a suitable abelian CM-threefold.

Each abelian variety A can be decomposed up to isogeny into a product of simple abelian varieties. S i m p l e a b e l i a n v a r i e t i e s are defined as indecomposable ones in this sense. If A is an abelian CM-variety, then the isogeny decomposition of A into simple abelian varieties is a power $A \approx B \times \ldots \times B$. The multiplication type (F, Φ) of A is lifted from a (uniquely determined) multiplication type (E, Ψ) of B. Especially, the simple factor B of A is a CM-variety. The corresponding CM-algebra $\mathbb{Q} \otimes \mathrm{End}\, B$ is isomorphic to E. Looking back to A one checks easily that the CM-algebra $\mathbb{Q} \otimes \mathrm{End}\, A$ is isomorphic to the matrix algebra $\mathrm{Mat}_s(E)$, where s denotes the number of the decomposing factors B of A.

4. Moduli fields.

Let X be a complex projective variety (subvariety of $\mathbb{P}^N$, say), $\mu \in \mathrm{Aut}(\mathbb{C})$ a field automorphism of $\mathbb{C}$. Applying μ to point coordinates we obtain the μ-transform X^μ of X embedded also in $\mathbb{P}^N$. If X is defined by the homogeneous equation system $F_1 = \ldots = F_m = 0$, then X^μ is defined by $F_1^\mu = \ldots = F_m^\mu = 0$, where F_i^μ arises from F_i by applying μ to all the coefficients of the polynomial F_i. We denote by cl(X) the the class of models X' of X. The projective variety X' is a model (or $\mathbb{C}$-model) of X, if it is isomorphic to X in the analytic category (that means over $\mathbb{C}$). If the model X' of X is defined over the subfield k of $\mathbb{C}$, then we call X' a k - m o d e l of X. This means that X' is defined by equations with coefficients in k. In this case k is called a *definition field* of cl(X) (or of X, if X is understood as scheme without specification of embedding). In arithmetic geometry one looks for small fields of definition.

The correspondence $X \longmapsto X^\mu$ is functorial: For each rational map $f: X \longrightarrow Y$ of complex projective varieties the transform $f^\mu: X^\mu \longrightarrow Y^\mu$ is well-defined via μ-transformation of constants. On this way we can correctly define $cl(X)^\mu$ by representants. We set

$$\mathrm{Stab}\ cl(X) = \{\mu \in \mathrm{Aut}(\mathbb{C});\ X^\mu \cong X\}.$$

4.1 Definition. The fixed field of Stab cl(X) in $\mathbb{C}$ is called the *moduli field* of X (or cl(X)). It is denoted by

$$M(X) = M(cl(X)) = \mathbb{C}^{\mathrm{Stab}\ cl(X)} .$$

We come back now to complex abelian varieties A with complex multiplication of type (F, Φ), say. For $\mu \in \mathrm{Aut}(\mathbb{C})$ the μ-transform A^μ of A is also an abelian variety. The endomorphism rings of A and A^μ are isomorphic by functoriality of μ. The isomorphism extends to the algebras of complex multiplication. Therefore A^μ has F-multiplication, too. Looking at the representations on the tangent spaces it is easy to see that A^μ is of type $(F, \mu\circ\Phi)$. So the type doesn't change if and only if μ belongs to $\mathrm{Aut}(\mathbb{C}/F')$ by definition (2.2) of the reflex field F'. We come to the first comparision playing a role in the Main Diagram (1.1).

4.2 Lemma. For a complex CM-variety of type (F, Φ) with reflex field F' and moduli field M(A) it holds that $F' \subseteq M(A)$.

Proof. It suffices to check that Stab cl(A) $\subseteq$ Stab(Φ). If μ stabilizes cl(A), then the representations of F in the tangent spaces T_A or T_{A^μ}, respectively, are equivalent. Therefore A^μ has the same type (F, Φ) as A, hence $\mu \in$ Stab(Φ), which was to be proved.

In the theory of abelian varieties it is useful to specify classes of projective embeddings translated to the internal geometry of A. A *polarized* abelian variety is a pair (A, $\mathfrak{C}$) consisting of an abelian variety A and a $\mathbb{Q}$-line in $\mathbb{Q} \otimes \mathrm{Pic}^{\alpha}(A)$ containing an ample divisor class; $\mathrm{Pic}^{\alpha}(A)$ denotes the group of algebraic equivalence classes of divisors on A. We say that (A, $\mathfrak{C}$) is defined over k, if A is and if $\mathfrak{C}$ can be represented by an ample divisor C defined over k by the k-embedding of A used just before. If A is defined over k, then we can find a polarization $\mathfrak{C}$ of A also defined over k. Namely, choose an ample divisor D on A defined over the algebraic closure $\bar{k}$ of k. It is really defined already over a finite Galois extension of k. The sum of all Galois conjugates of D represents obviously a polarization $\mathfrak{C}$ defined over k.

In obvious manner one introduces the μ-transforms $(A, \mathfrak{C})^\mu$ for $\mu \in \mathrm{Aut}(\mathbb{C})$, $\mathrm{cl}(A, \mathfrak{C})^\mu$, Stab cl(A, $\mathfrak{C}$) and the *moduli-field* of a polarized abelian variety

$$M(A, \mathfrak{C}) = M(\mathrm{cl}(A, \mathfrak{C})) = \mathbb{C}^{\mathrm{Stab}\, \mathrm{cl}(A, \mathfrak{C})} .$$

4.3 Lemma. Let k be a definition field of the polarized abelian variety (A, $\mathfrak{C}$). Then it holds that $M(A, \mathfrak{C}) \subseteq k$.

Proof (see [36], I.4.2, Prop.14). For $\mu \in \mathrm{Aut}(\mathbb{C}/k)$ we have an obvious isomorphism $(A, \mathcal{C})^{\mu} \cong (A, \mathcal{C})$, hence $\mu \in \mathrm{Stab}(A, \mathcal{C})$. The rest is clear. ∎

The next result prepares the drawing-up of the field tower on the right-hand side of the main diagram (1.1).

4.4 Proposition. Let A be a complex CM-variety. Then we can choose algebraic number fields k as definition fields of A or $(A, \mathcal{C})$, respectively. For each such field one has the field tower

(4.5) $$F' \subseteq M(A) \subseteq M(A, \mathcal{C}) \subseteq k \subseteq \bar{\mathbb{Q}},$$

where F' is the reflex field of the type (F, Φ) of A.

Proof. The existence of a small definition field $k \subseteq \bar{\mathbb{Q}}$ has been verified by Shimura-Taniyama in [36]. The second inclusion follows from $\mathrm{Stab}\ \mathrm{cl}(A) \supseteq \mathrm{Stab}\ \mathrm{cl}(A, \mathcal{C})$. The remaining inclusions come from the Lemmas 4.2 and 4.3. ∎

5. Main theorem of complex multiplication.

We want to connect the moduli field of a polarized CM-variety $(A, \mathcal{C})$, A of type $(F, \Phi, \mathfrak{a})$ with the Shimura class field of the same type introduced in 2. For this purpose we refine the notion of types again taking into account the polarization. Via projectively embedding Theta functions one corresponds to the polarization $\mathcal{C}$ a (unique, up to $\mathbb{Q}^{\times}$-multiplication,) Riemann form $E: T_A \times T_A \longrightarrow \mathbb{C}$ ($\mathbb{R}$-bilinear, scew-symmetric, non-degenerate with rational values on $\Phi(\mathfrak{a} \times \mathfrak{a})$). It is useful to choose a basic form E of this class. It takes integral values on $\phi(\mathfrak{a}\times\mathfrak{a})$ and is not an integral multiple of a form of the same kind. With these notations the polarized abelian CM-variety $(A,\mathcal{C})$ is said to be of type $(F,\Phi,\mathfrak{a},E)$. If there is no danger of misunderstandings, then we identify A with its standard torus model $\mathbb{C}^g/\phi(\mathfrak{a})$, see 3. Since $F = \mathbb{Q}\otimes\mathfrak{a}$ the embedding ϕ of F into $\mathbb{C}^g$ induces an embedding $F/\mathfrak{a} \longrightarrow A_{tor}$ into the torsion points of $A(\mathbb{C})$. We will denote this embedding shortly also by Φ.

The Riemann form E is said to be Φ - admissible, if $E(D_\Phi(f)z,w) = E(z,D_\Phi(\bar{f})w)$ for all $f \in F$, $z,w \in \mathbb{C}^g = T_A$. In this case also the polarization $\mathcal{C}$ corresponding to E is called admissible. From now on we assume that the multiplication field F is a CM-field; A needs not to be simple. Then there exists an admissible polarization on A (see [20], I.4, Thm. 4.5). We will work only with polarized abelian CM-varieties of admissible type $(F,\Phi,\mathfrak{a},E)$. The following Main Theorem of complex multiplication holds for them:

5.1 Theorem ([20], III.6). With the above assumptions and notations let $\mu \in \mathrm{Aut}(\mathbb{C}/F')$ with restriction $\mu|F'^{ab} = (s,F')$ for a suitable $s \in \mathbb{A}_{F'}^*$. Then it holds that:

(i) $(A,\Phi)^\mu$ is of type $(F,\Phi,N'(s^{-1})\mathfrak{a}, \mathbb{N}(s)E)$.

(ii) With the componentwise action of the finite part of the idele $N'(s^{-1}) \in \mathbb{A}_F^*$ on $F/\mathfrak{a} \cong \bigoplus_{\mathfrak{p}} F_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ the following diagram is commutative:

$$(5.2)\qquad \begin{array}{ccc} F/\mathfrak{a} & \longrightarrow & A_{tor} \\ {\scriptstyle N'(s^{-1})}\big\downarrow & & \big\downarrow {\scriptstyle \mu \in \widetilde{(s,F')}} \\ F/N'(s^{-1})\mathfrak{a} & \longrightarrow & A^{\mu}_{tor} \end{array} \qquad \blacksquare$$

As an immediate consequence we get the relation with moduli fields.

5.3 Theorem (Shimura [36]; [35], V.5.5; see also [20], V.4). Let $(A, \mathcal{C})$ be a polarized abelian variety of admissible CM-type $(F, \Phi, \mathfrak{a}, E)$. Then the corresponding Shimura class field and moduli field coincide:

$$Sh(\Phi, \mathfrak{a}) = M(A, \mathcal{C}).$$

Proof. We remember that $Sh := Sh(\Phi, \mathfrak{a}) = \mathbb{C}^{\widetilde{(U,F')}}$, $U = U(\Phi, \mathfrak{a})$, see (2.6). For $M = M(A, \mathcal{C})$ we first show that $M \subseteq Sh$. This follows immediately from

(5.4) $$\widetilde{(U,F')} \subseteq \text{Stab cl}(A, \mathcal{C}).$$

So we take an automorphism $\mu \in \widetilde{(s,F')}$ for $s \in U$. By the Main Theorem of complex multiplication 5.1 (i) the μ-transform $(A, \mathcal{C})^{\mu}$ is of type $(F, \Phi, N'(s^{-1})\mathfrak{a}, \mathbb{N}(s)E)$. By definition of U in (2.4) there is a $\beta \in F$ such that $N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a}$ and $\mathbb{N}(s^{-1}) = \beta\bar{\beta} \in \mathbb{Q}^{\times}$. Comparing the standard torus models of $(A, \mathcal{C})$ and $(A^{\mu}, \mathcal{C}^{\mu})$ we get

$$A \cong \mathbb{C}/\Phi(\mathfrak{a}) \cong \mathbb{C}/\Phi(\beta\mathfrak{a}) = A^{\mu}, \quad \mathbb{N}(s)E = (\beta\bar{\beta})^{-1}\cdot E \in \mathbb{Q}^{\times}\cdot E.$$

Therefore $(A, \mathcal{C}) \cong (A, \mathcal{C})^{\mu}$, hence $\mu \in \text{Stab cl}(A, \mathcal{C})$.

Conversely, take $\mu \in \text{Stab cl}(A, \mathcal{C})$; then we dispose on an isomorphism $(A, \mathcal{C}) \overset{\cong}{\dashrightarrow} (A, \mathcal{C})^{\mu}$. On the torsion level it has been made precise multiplying $F/\mathfrak{a}$ by $N'(s^{-1})$, $\mu \in \widetilde{(s,F')}$, see 5.1 (ii). On the other hand A and A^{μ} have equivalent standard torus models. More precisely, the isomorphism $A \xrightarrow{\sim} A^{\mu}$ corresponds to a β-multiplication $\mathbb{C}^{g}/\Phi(\mathfrak{a}) \xrightarrow{\sim} \mathbb{C}^{g}/\Phi(\beta\mathfrak{a})$ for a suitable $\beta \in F^{\times}$. But $(A,\mathcal{C})^{\mu}$ is of type $(F,\Phi,N'(s^{-1})\mathfrak{a},\mathbb{N}(s)E)$ by 5.1 (i). Comparing both presentations we get

$$N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a}, \quad \mathbb{N}(s^{-1}) = \beta\bar{\beta} \in \mathbb{Q}^{\times}.$$

Therefore s belongs to U and $\mu \in \widetilde{(s,F')}$, hence

(5.5) $$\mathrm{Stab}\ \mathrm{cl}(A,\mathcal{C}) \subseteq (U.F')$$

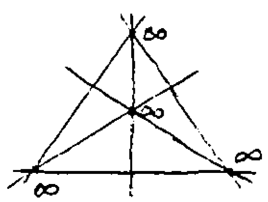
in contrast to (5.4). It follows that $Sh \subseteq M$. The identity we looked for is proved. ∎

6. The geometric starting point, the projective plane covered by the ball.

In order to generate Shimura class fields of cubic extensions F of the Eisenstein numbers by special values of transcendent functions we need special Theta constants essentially defined by certain special functional equations. In order to find and understand them deeply we enclose them into the most general and actual framework hoping for similar applications to other interesting cases in the future. We want also justify Hilbert's imagination about the "innigste Beruehrung". Consider the picture/diagram

(6.1)
$$\begin{array}{ccc} \mathbb{H} & & \mathbb{B} \\ j\downarrow \mathrm{Sl}(\mathbb{Z}) & & \pi\downarrow \Gamma' \\ \mathbb{P}^{1} & & \mathbb{P}^{2} \end{array}$$

branch locus: ∞, 2, 3



The left-hand side is well-known from the theory of elliptic curves. It describes the quotient map of the modular group from Poincaré's upper half plane $\mathbb{H}$: Im z > 0 to the projective line $\mathbb{P}^1(\mathbb{C})$. There are three branch points: two in the ordinary sense with ramification indices 2 or 3, respectively. The third is a cusp point coming from the boundary of $\mathbb{H}$. therefore it has been weighted by ∞ . The quotient map can be realized by the elliptic modular function

$$j(\tau) = q^{-1} + 744q^0 + 196884q + \sum_{n=2}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}, \ a_n \in \mathbb{Z}.$$

We are well-prepared for the understanding of the right-hand side of (6.1) by the previous sections and chapters: We looked for a ball covering of the projective plane $\mathbb{P}^2$ with discrete covering group $\Gamma' \subset \mathrm{Aut}_{hol}(\mathbb{B}) = \mathbb{P}U((2,1),\mathbb{C})$ branched precisely along the six lines of the complete quadrilateral with triple points as cusp points. We denote by π the corresponding (analytic) quotient map.

6.2 Theorem. Up to linear isomorphy ($\mathbb{P}U((2,1),\mathbb{C})$-conjugation for Γ') there exists one and only one such covering.

The uniqueness has been proved in [17], IV.11 via orbital heights, the proportionality conditions and their translation into a solvable system of diophantine equations. Moreover, the proportionality test is positive: The only solution of the diophantine equation system yields the ramification index 3 for all six lines. The most general result providing the existence of the covering is due to R.Kobayashi [18]. He proved under geometric conditions including the wights found in our proportionality test the existence of a ball covering over a given surface with prescribed branch locus. This general result can be applied to our situation described in (6.1). We will not use Kobayashi's result because the existence of π has been proved by another more arithmetic method in the framework of classification of Picard modular surfaces, see Prop.V.1.3 in [17] or [12], where we started from the arithmetic group (congruence Eisenstein lattice) $\Gamma' = \Gamma(\sqrt{-3})$, $\Gamma = U((2,1),\mathcal{O}_K)$. The proof involved orbital hights calculated as volumes of a fundamental domain by means of a special L-series value (see [15]). The advantage is to dispose explicitly on the discrete group of the covering π. This will be important for finding the functional equations for Theta constants we look for.

6.3 Remark. The covering problem is related with Hilbert's 22-th problem "Uniformisierung analytischer Beziehungen mittels automorpher Funktionen". Looking for uniformizations of two-dimensional analytic varieties ("Gebilde") Hilbert says: "Vielmehr scheinen ...., abgesehen von den Verzweigungspunkten, noch gewisse andere, im allgemeinen unendlich viele diskrete Stellen des vorgelegten analytischen Gebildes ausgenommen zu sein, zu denen man nur gelangt, indem man die neue Variable gewissen Grenzstellen der Funktionen naehert. Eine Klaerung und Loesung dieser Schwierigkeit scheint mir in Anbetracht der fundamentalen Bedeutung der Poincareschen Fragestellung aeusserst wuenschenswert". At the end Hilbert refers to: ... "die neueren Untersuchungen von Picard ueber algebraische Funktionen von zwei Variablen als willkommene und bedeutsame Vorarbeiten ..."

7. Differential equations.

In [18] M.Yoshida succeeded to solve a higher-dimensional version of the Riemann-Hilbert problem. The background is Hilbert's 21-st problem "Beweis der Existenz linearer Differentialgleichungen mit vorgeschriebener Monodromiegruppe" set up for functions of one variable, "...welches darin besteht zu zeigen, dass es stets eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulaeren Stellen und einer gegebenen Monodromiegruppe gibt". It should be remarked that the final solution of this Hilbert problem has been given by H. Roehrl [28] in 1957.

7.1 Theorem (M.Yoshida). Let X be an orbifold (complex manifold with prescribed wighted branch locus) with realizing quotient map $p: \mathbb{B}^n \longrightarrow X$, $\mathbb{B}^n$ the n-dimensional complex ball, and covering group $\Delta \subset U((n,1),\mathbb{C})$. Then the inverse p^{-1} of p is a (multivalued) developing map of a Fuchsian system of linear partial differential equations. ∎

This means that there is locally a fundamental system of solutions $I_0, I_1, \ldots, I_n$ extending analytically to $X \setminus B$, B the branch locus of p, such that the multivalued map

$$(I_0 : I_1 : \ldots : I_n) : X \setminus B \dashrightarrow \mathbb{B}^n \subset \mathbb{P}^n$$

$P \longmapsto (I_0(P): \ldots : I_n(P))$, coincides with p^{-1} on $X \setminus B$. The Fuchsian system is called the **uniformizing equation** of the orbifold and Δ is the **monodromy group** of the system not depending on the special choice of solutions of the system. Especially in our situation described in (6.1) with $n = 2$, $B =$  (quadrilateral), it is important to remark that there is a surjective group homomorphism

(7.2) $\pi_1(\mathbb{P}^2 \setminus$ $) \longrightarrow\!\!\!\!\rightarrow \Gamma'$

describing the unitary monodromy representation of the fundamental group $\pi_1(\mathbb{P}^2 \setminus$ $)$. Yoshida found also an effective method in order to determine a corresponding Fuchsian system (see [38], ch.s 10, 12). It turns out that these equations and also their analytic solutions (Appell series) are well-known long time ago. Working with affine coordinates u,v one can take the following system (7.3) of differential equations:

(7.3) $D_{ij} F(u,v) = 0$ on $\mathbb{C}^2 \setminus$  $= \mathbb{P}^2 \setminus$  with

$$D_{11} = \frac{\partial}{\partial u^2} + [9(u-1)u(v-u)]^{-1} \{3(-5u + 4uv + 3u - 2v)\frac{\partial}{\partial u} + 3(v-1)v\frac{\partial}{\partial v} + (u-v)\},$$

$$D_{12} = \frac{\partial^2}{\partial u \partial v} + [3(u-v)]^{-1} \{\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\},$$

$$D_{22} = \frac{\partial}{\partial v^2} + [9(v-1)v(u-v)]^{-1} \{3(u-1)u\frac{\partial}{\partial u} + 3(-5v + 4uv + 3v - 2u)\frac{\partial}{\partial v} + (u-v)\}.$$

7.4 Remark. Yoshida's general approach lifting the Gauss-Schwarz theory of Fuchsian equations to higher dimensions has a classical origin in the work of Picard and Appell. Especially for the situation of (6.1) a more immediate extension of explicit classical results known as PTDM-Theorem (due to Picard, Terada, Mostow, De-ligne) would be sufficient for our purposes. We refer to [38], [2] and further literature given there.

8. Gauss-Manin connection.

The analytic theory presents analytic solutions of the system (7.3) of partial differential equations. We look for "algebraic solutions" represented by integrals on algebraic curves along cycles depending on parameters u,v. The general framework of the corresponding algebraic theory is known as **Gauss-Manin connection** of algebraic families of algebraic manifolds. We refer to [12] in order to understand the rather explicit approach for algebraic families of curves involving differential equations.

Let $\mathcal{C}/T$ be a smooth algebraic family of smooth algebraic varieties all defined over the complex numbers, say. The relative de Rham complex is a sequence

$$\Omega^{\cdot}_{\mathcal{C}/T}: \quad \mathcal{O}_T \xrightarrow{d} \Omega^1_{\mathcal{C}/T} \xrightarrow{d} \Omega^2_{\mathcal{C}/T} \xrightarrow{d} \cdots .$$

Using open (affine, say) coverings one defines the Čech complexes

$$C^{\cdot}(\Omega^q_{\mathcal{C}/T}): \quad C^0(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^1(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^2(\Omega^q_{\mathcal{C}/T}) \ \ldots$$

in the usual manner. Taking the limit along refinements of open coverings one gets the Čech - de Rham bicomplex $C^{\cdot\cdot}(\Omega_{\mathcal{C}/T})$. The de Rham cohomology groups $\mathbb{H}^i_{DR}(\mathcal{C}/T)$ of the family $\mathcal{C}/T$ are the hypercohomology groups of $C^{\cdot\cdot}(\Omega_{\mathcal{C}/T})$ defined as cohomology groups of the corresponding total Čech - de Rham complex $C^{tot}(\Omega_{\mathcal{C}/T})$. The construction applies to all restricted families $\mathcal{C}_U/U$, U an open part of T. On this way one gets the **de Rham coho-**

h o m o l o g y   s h e a v e s $\mathcal{H}^i_{DR}(\mathcal{C}/T)$ on T.

We restrict ourselves now to curve families $\mathcal{C}/T$. For our purposes it suffices to assume that T is an affine part of a projective space $\mathbb{P}^N$. Let $\mathcal{D}_T$ be the sheaf of differential operators on T. Then the de Rham cohomology sheaf $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ is not only an $\mathcal{O}_T$-module but also a $\mathcal{D}_T$-module sheaf. Looking for a family with a section ϖ in $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ satisfying the differential equations (7.3) with ϖ instead of F one can take the P i c a r d   c u r v e   f a m i l y

$$\mathcal{C}/\mathbb{P}^2 \setminus \blacktriangle \ : \ Y^3 = X(X-1)(X-u)(X-v)$$

and ϖ represented by the differential form $\omega = dx/y$ depending on u,v. For details we refer to [12], II, 1.5. Taking integrals over cycles one gets an "algebraic" fundamental system of solutions

(8.1)

$$I_k(t) = \int_{\alpha_k(t)} \omega(t), \ k = 1,2,3, \ t = (u,v) \in \mathbb{P}^2 \setminus \blacktriangle, \quad \omega = dx/y$$

We refer to [ ], II.2.5, Theorem 2.5.2. Altogether we found the developing map of the Fuchsian system (7.3) in an explicit and algebraic manner. Looking back to the geometric starting point (6.1) and to the theorems 6.2 and 7.1 we receive

8.2 Theorem. The quotient map $\pi: \mathbb{B} \longrightarrow \mathbb{P}^2$ with covering group $\Gamma' = \Gamma(\sqrt{-3})$ is inverted by $(I_1:I_2:I_3): \mathbb{P}^2 \setminus \blacktriangle \dashrightarrow \mathbb{B}$ on $\mathbb{P}^2 \setminus \blacktriangle$ with cycloelliptic integrals $I_k(t)$ described in (8.1) along independent cycle families $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$. ∎

Another proof based on the PTDM-theorem can be found in [12], I.6.3. There has been used also a finer analysis of the Picard curve family, which is useful for our number theoretic ambitions. The next three sections are devoted to this theme.

9. Moduli space for Picard curves.

We investigate the Picard curve family in more detail. A curve C (algebraic, complex, compact) is called a *Picard curve*, if it is isomorphic to a plane projective curve C′ of affine equation type C′: $Y^3 = p_4(X)$, where $p_4(X)$ is a polynomial of degree 4 in X. We exclude subsequently curves C with model C′: $Y^3 = X^4$ because they will get lost in our moduli space below. Via projective Tschirnhaus transformation any Picard curve has a model of equation type

(9.1)

$$Y^3 = \prod_{i=1}^{4} (X - e_i) = X^4 + G_2X^2 + G_3X^3 + G_4 \qquad \text{(affine)},$$

$$WY^3 = \prod_{i=1}^{4} (X - e_iW) = X^4 + G_2W^2X^2 + G_3W^3X + G_4W^4 \qquad \text{(projective)},$$

$$\sum_{i=1}^{4} e_i = 0.$$

The corresponding equations are called *normal forms* of Picard curves. A Picard curve is smooth if and only if for one (each) of its normal forms (9.1) it holds that $e_i \neq e_j$ for $i \neq j$. We correspond to the normal form (9.1) the point

$$(e_1 : e_2 : e_3 : e_4) \in \mathbb{P}^2 = \mathbb{P}_0^3 = \{(z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3;\ \sum_{i=1}^{4} z_i = 0\}.$$

The following result is do to the author. We refer again to the monograph [12], ch.I, 5.2. It asserts that the correspondence

$$\text{Picard curve } C \longmapsto (e_1 : e_2 : e_3 : e_4) \text{ via normal form (9.1)}$$

is correctly defined (at least for smooth curves) up to symmetry interchanging zeros of the normal form polynomial $p_4(X)$ and that non-isomorphic (smooth) curves cannot be represented by the same point in $\mathbb{P}^2$. Obviously, the smooth Picard curves are represented by points not belonging to the six lines $e_i = e_j$, $i \neq j$.

"

9.2 Proposition. The above correspondence induces a bijective map

$$\{cl(C); C \text{ smooth Picard curve}\} \longleftrightarrow (\mathbb{P}^2 \setminus \mathbb{A})/S_4 ,$$

where the symmetric group S_4 acts via natural permutations of the $\mathbb{P}_0^3$-coordinates. ∎

We remark that the proof given in [12] uses geometric invariants, for instance the Hessean of a homogeneous nomal form polynomial. We call the surface $\mathbb{P}^2/S_4$ the (compactified) m o d u l i s p a c e of Picard curves and $(\mathbb{P}^2 \setminus \mathbb{A})/S_4$ the moduli space of smooth Picard curves by a slight abuse of language.

10. The relative Schottky problem for Picard curves.

A smooth Picard curve C has genus 3. Therefore its Jacobian variety J(C) is an abelian threefold. We want to determine in an effective manner the polarized abelian threefolds, which are Jacobian varieties of Picard curves. The period lattice Λ of an abelian variety A is abstractly defined by the exact sequence

(10.1) $$0 \dashrightarrow \Lambda \longrightarrow T_A \longrightarrow A \longrightarrow 0,$$

T_A the tangent space of A (at 0). Choosing coordinates and a basis of Λ we identify T_A with $\mathbb{C}^g$, and Λ is generated by the columns of a $g\times 2g$-matrix Π called a p e r i o d m a t r i x of A. Taking into consideration all possible base changes in T_A and Λ we see that a period matrix of A is unique up to $Gl_g(\mathbb{C})$-multiplication from the left and $Gl_{2g}(\mathbb{Z})$-multiplication from the right-hand side. Let E be a primitive Riemann form on T_A representing a polarization of A. By a theorem of Frobenius there exists a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ such that

(10.2)

$$(E(\lambda_i,\lambda_j)) = \begin{pmatrix} 0 , D \\ -D , 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_g \end{pmatrix}, \quad d_i \in \mathbb{N},\ 1=d_1 | d_2 | \ldots | d_g .$$

If the above diagonal matrix D is the unit matrix E_g, then the corresponding polarization is called p r i n c i p a l . We will only consider principally polarized abelian varieties in this section. A basis of Λ satisfying (10.2) with

$$I = \begin{pmatrix} 0 , & E \\ -E , & 0 \end{pmatrix} = \begin{pmatrix} 0 , & D \\ -D , & 0 \end{pmatrix}$$

is called a n o r m a l b a s i s . A period matrix of a principally polarized abelian variety is always constructed by use of a normal basis of Λ. For a fixed basis of T_A it is uniquely determined up to right-multiplication with elements of the s y m - p l e c t i c (modular) g r o u p

$$Sp(2g,\mathbb{Z}) = \{S \in Gl_{2g}(\mathbb{Z});\ SI^tS = I\}.$$

The lattice in $\mathbb{C}^g$ generated by the the colums of Π is denoted by Λ_Π. The coordinate version of (10.1) is the exact sequence

(10.3) $$0 \longrightarrow \Lambda_\Pi \longrightarrow \mathbb{C}^g \longrightarrow A \longrightarrow 0.$$

A matrix $\Pi \in \mathrm{Mat}_{g\times 2g}(\mathbb{C})$ is called a Riemann matrix (of principal type), if it satisfies the following two conditions known as Riemann's first or second relation, respectively:

(10.4) (R 1) $\quad \Pi \cdot I \cdot {}^t\Pi = 0,$

(R 2) $\quad i\Pi I {}^t\overline{\Pi} > 0$ (positive definit) , $i = \sqrt{-1}$.

For a proof we refer to [19], IV, App.I.

Now we turn our attention to (smooth) Picard curves and their (principally polarized) Jacobian threefolds. A period matrix of a Picard curve can be written as

(10.5)

$$\Pi = \int_{\mathfrak{b}} \vec{\omega} = \left(\int_{\beta_j} \omega_i \right) = \begin{pmatrix} A_1 & A_2 & A_4 & A_3 & A_5 & A_6 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_4 & \overline{B}_3 & \overline{B}_5 & \overline{B}_6 \\ \overline{C}_1 & \overline{C}_2 & \overline{C}_4 & \overline{C}_3 & \overline{C}_5 & \overline{C}_6 \end{pmatrix}$$

where $\vec{\omega} = {}^t(\omega_1, \omega_2, \omega_3)$ is a basis of $H^0(C,\Omega_C) \cong H^1(C,\mathcal{O}_C)$ and $\mathfrak{b} = (\beta_1, \beta_2, \beta_4, \beta_3, \beta_5, \beta_6)$ is a $\mathbb{Z}$-basis of the homology group $H_1(C,\mathbb{Z})$. The relative Schottky proplem for Picard curves asks for an affective criterion characterizing period matrices of Picard curves among all period matrices of (principally polarized) abelian threefolds. The idea of constructing typical period matrices described below goes back to Picard [26]. In order to formulate the theorem we need a linear embedding

$$*: \mathbb{C}^3 \hookrightarrow \mathbb{C}^6$$

$$(A,B,C) \longmapsto (A,B,-\bar{\rho}A,C,\bar{\rho}B,\rho C) , \qquad \rho = e^{2\pi i/3} ,$$

and the hermitian form $\langle\ ,\ \rangle$ of signature (2,1) on $\mathbb{C}^3$ represented by the anti-diagonal matrix $R = \begin{pmatrix} 0 & 0 & \bar{\varrho} \\ 0 & 1 & 0 \\ \varrho & 0 & 0 \end{pmatrix}$. It defines at the same time explicitly a ball $\mathbb{B}$ in $\mathbb{P}^2$ and a disc $\mathbb{D} = \mathbb{D}_R \subset \mathbb{B}$:

$$\mathbb{B} = \{\mathbb{P}\mathfrak{a} \in \mathbb{P}^2;\ \langle\mathfrak{a},\mathfrak{a}\rangle < 0\},$$

$$\mathbb{D} = \mathbb{D}_R = \{\tau = \mathbb{P}\mathfrak{a} \in \mathbb{B};\ R\cdot\tau = \tau\}.$$

10.6 Theorem (relative Schottky for Picard curves).
The matrix Π in (10.5) is a period matrix of a smooth Picard curve if and only if the following conditions are satisfied:

Π belongs to $Sl_3(\mathbb{C})\cdot\begin{pmatrix} *\,\mathfrak{a} \\ \hline *\,\mathfrak{b} \\ \hline *\,\mathfrak{r} \end{pmatrix}\cdot Sp(6,\mathbb{Z})$; $\langle\mathfrak{a},\mathfrak{a}\rangle < 0$ (ball condition);

$\mathfrak{b}, \mathfrak{r}$ is a basis of $\mathfrak{a}^\perp$ (orthogonal condition); $\tau = \mathbb{P}\mathfrak{a}$ does not belong to $\bigoplus = \Gamma\mathbb{D}_R$ ("non-degenerate"-condition), where Γ denotes the full Eisenstein lattice

$$\Gamma = U(\langle\ ,\ \rangle,\mathcal{O}_K) = U((2,1),\mathcal{O}_K)\ ,\ \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\varrho \subset K = \mathbb{Q}(\sqrt{-3}).$$

Proof (sketch). Let C be a smooth Picard curve with normal form (9.1). The projection $(x,y) \longmapsto x$ defines a cubic Galois covering $C \longrightarrow \mathbb{P}^1$ with Galois group $G \cong \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ generated by g, say. The homology group $H_1(C,\mathbb{Z})$ is a $\mathbb{Z}[G]$-module. A normal basis $\mathfrak{b} = (\beta_1, \ldots, \beta_6)$ of $H_1(C,\mathbb{Z})$ is a $\mathbb{Z}$-basis satisfying $(\beta_i \circ \beta_j) = I$, $\circ$ the intersection product of cycles. A t y p i c a l basis of $H_1(C,\mathbb{Z})$ is a normal basis of the form

$$*\vec{\alpha} = *(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, -g\alpha_1, \alpha_3 \cdot g\alpha_2, g^2\alpha_3).$$

The existence of at least one typical homology basis on a smooth Picard curve has been proved by Picard himself in [26], see also Alezais [1]. A reproduced version can be found in Picture 6.3.A of ch.I in [12].

A G-(i s o)t y p i c a l basis of $H^1 \underset{Df}{=} H^0(C, \Omega_C)$, Ω_C the sheaf of regular (holomorphic) differential forms (of degree 1) on C, is a basis consisting of G-eigenvectors of H^1. One can take simultaneously for all Picard curves with normal form equation (9.1) the differential forms of first kind $\omega_1 = dx/y$, dx/y^2, xdx/y. The use of a typical basis $*\vec{\alpha}$ of $H_1(C,Z)$ and a isotypical basis $\vec{\omega} = {}^t(\omega_1, \omega_2, \omega_3)$ has the advantage that the corresponding period matrix (10.5) is essentially determined by the three entries A_1, A_2, A_3. With $\mathfrak{a} = (A_1, A_2, A_3)$ and the above notations one obtains t y p i c a l p e r i o d m a t r i c e s

$$(10.7) \qquad \Pi = \int_{*\vec{\alpha}} \vec{\omega} = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{v}} \end{pmatrix}$$

We check now the conditions of Theorem 10.6. The first of them is satisfied because we can choose a typical period matrix. The second and third conditions are translations of Riemann's period relations (R 1), (R 2) in (10.4). The "non-degenerate"-condition is delegated to the if-part of proof below.

Conversely, we assume that a smooth curve C has a typical period matrix as described on the right-hand side of (10.7). We have to show that this can only happen for a Picard curve. For this purpose we look at the moduli space of smooth Picard curves $(\mathbb{P}^2 \setminus \mathbf{\Delta})/S_4$, see Prop. 9.2. We construct a commutative diagram

(10.8)
$$\begin{array}{ccccc}
\mathbb{B} \setminus \oplus & & \overset{*}{\hookrightarrow} & & \mathbb{H}_3 \\
 & \searrow^{\Gamma'} & & & \\
\Gamma \downarrow & & \mathbb{P}^2 \setminus \mathbf{\Delta} & & \downarrow Sp(6,\mathbb{Z}) \\
 & \swarrow_{S_4} & & & \\
(\mathbb{P}^2 \setminus \mathbf{\Delta})/S_4 & \hookrightarrow & \mathcal{M}_3 & \hookrightarrow & \mathcal{A}_3
\end{array}$$

where $\mathcal{M}_3$ or $\mathcal{A}_3$ are the moduli spaces of smooth curves of genus 3 or principally polarized abelian threefolds, respectively. The Torelli embedding $\mathcal{M}_3 \hookrightarrow \mathcal{A}_3$ is represented by the correspondence $cl(C) \mapsto cl(J(C))$, C a curve of genus 3 and $J(C)$ its Jacobian threefold. The upper row is correctly explained by the chain of the following (partly multivalued) correspondences:

(10.9)
$$\tau = \mathbb{P}\mathfrak{a} \mapsto \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{t}} \end{pmatrix} = (\Pi_1 | \Pi_2) \xrightarrow[\Pi_1^{-1}\cdot]{} (E_3 | \tau) \mapsto \tau .$$

The point $*\tau \underset{Df}{=} \tau$ belongs to the generalized Siegel upper half plane of polarized abelian threefolds

$$\mathbb{H}_3 = \{ \tau \in Gl_3(\mathbb{C});\ {}^t\tau = \tau,\ \operatorname{Im} \tau > 0 \}.$$

The vertical arrows in diagram (10.8) represent analytic quotient maps: The Eisenstein lattice $\Gamma = U((2,1),\mathcal{O}_K)$ acts on $\mathbb{B}$. This action can be extended to an action on $\mathbb{H}_3$ along the geometric embedding $*: \mathbb{B} \hookrightarrow \mathbb{H}_3$ by the following s y m p l e c t i c r e p r e p r e s e n t a t i o n of Γ also denoted by $*$:

(10.10) $$\Gamma = U((2,1),\mathcal{O}_K) \overset{*}{\hookrightarrow} Sp(6,\mathbb{Z})$$

For an explicit definition we refer to [14]. More intrinsically one endows $\mathbb{C}^6$ with the hermitian product [,] of signature (3,3) represented by $J = I/\sqrt{-3}$. Its restriction to $\mathbb{C}^3$ along the $*$-embedding coincides with $\langle\ ,\ \rangle$ introduced above, that means:

$$[*\mathfrak{a}, *\mathfrak{b}] = \langle \mathfrak{a}, \mathfrak{b} \rangle \text{ for all } \mathfrak{a}, \mathfrak{b} \in \mathbb{C}^3.$$

The modular symplectic group $Sp(6,\mathbb{Z})$ consists of all linear transformations of $\mathbb{C}^6$, which are compatible with [,] and preserve $\mathbb{Z}^6$. All those elements of $Sp(6,\mathbb{Z})$ preserving additionally $*\mathbb{C}^3$ are collected in a group $Sp'(6,\mathbb{Z})$. Their pull backs to $\mathbb{C}^3$ are compatible with $\langle\ ,\ \rangle$ and preserve $\mathcal{O}_K^3$. One can check that the homomorphism $Sp'(6,\mathbb{Z}) \longrightarrow \Gamma$ defined on this way is an isomorphism. Its inverse map yields the embedding (10.10). Changing over from the right-action on vector rows to the transposed left-action on the the columns we notice that

$$(10.11) \qquad *(\gamma(\mathfrak{a})) = (*\gamma)(*\mathfrak{a}) \quad , \quad \mathfrak{a} \in \mathbb{C}^3, \ \gamma \in \Gamma \ ;$$
$$*(\gamma(\tau)) = (*\gamma)(*\tau) \quad , \quad \tau = \mathbb{P}\mathfrak{a} \in \mathbb{B} \ .$$

Since $\mathcal{A}_3 = \mathbb{H}_3/Sp(6,\mathbb{Z})$ it follows that the diagram (10.8) is commutative.

We are now able to verify the if-part of Theorem 10.6. Let C be a smmoth curve of genus 3 with a typical period matrix. The preimage of $cl(J(C)) \in \mathcal{A}_3$ in $\mathbb{H}_3$ is represented by an element uniquely determined up to $Sp(6,\mathbb{Z})$-multiplication. Without loss of generality we can assume that $\mathcal{T} = *\tau$ because of the correspondence (10.9) going through all typical period matrices. Thus

J(C) must be the (generalized) Jacobian threefold of a possibly degenerate (non-smooth) Picard curve. But C is smooth of genus 3. Therefore C has to be a smooth Picard curve. It remains to check that τ does not belong to $\boldsymbol{\Phi}$. In 9. we proved that $\mathbb{P}^2 \setminus \boldsymbol{\Delta}$ is the complete S_4-preimage of the moduli space of smooth Picard curves. The symmetric group S_4 appears as factor group Γ / Γ' in diagram (10.8). For the coincidence proof of the symmetric action of S_4 on $\mathbb{P}^2$ and the arithmetic action of Γ / Γ' on $\mathbb{P}^2$ we refer to [12]. We dispose also on the knowledge of the preimage of $\boldsymbol{\Delta}$ in $\mathbb{B}$. This is the branch locus of the covering $\pi: \mathbb{B} \longrightarrow \mathbb{B}/\Gamma'$ we started with in (6.1), see also Theorem 6.2 and the explanation thereafter. The ramification locus of π has been carefully analyzed in [12], I.3., especially diagram 3.3.b. It consists of all (infinitely many) $\Gamma(\sqrt{-3})$-reflection discs. This set coincides with the Γ-transforms of one of them, say of $\mathbb{D}_R$. So, if τ belongs to a smooth Picard curve, then it cannot belong to $\boldsymbol{\Phi}$. The Theorem 10.6 is proved. ∎

11. Effective Torelli theorem for Picard curves via Picard modular forms.

Tacitly we used already Torelli's theorem. It appears in diagram (10.8) asserting that $\mathcal{M}_3 \longrightarrow \mathcal{A}_3$ is an embedding or, more generally, the isomorphy class of a smooth curve is uniquely determined by its (polarized) Jacobian variety. We look for a precise pointwise version of this theorem in the case of Picard curves:

11.1 Find for given $\tau \in \mathbb{B}$ (or $*\tau \in *\mathbb{B} \subset \mathbb{H}_3$) the normal form of a Picard curve C_τ corresponding to the moduli point $\pi(\tau) \in \mathbb{P}^2$.

In analogy to the Weierstrass normal form of elliptic curves we can find holomorphic functions $t_1, t_2, t_3, t_4 : \mathbb{B} \longrightarrow \mathbb{C}$ such that the normal forms we look for can be written as

(11.2) $$C_\tau : Y^3 = \prod_{i=1}^{4} (X - t_i(\tau))$$

simultaneously for all $\tau \in \mathbb{B}$. In other words we try to describe the quotient map π we started with in (6.1) in terms of holomorphic functions identifying the quotient map π with

(11.3) $$(t_1 : t_2 : t_3 : t_4) : \mathbb{B} \longrightarrow \mathbb{P}^2$$
$$\tau \ : \longmapsto (t_1(\tau) : t_2(\tau) : t_3(\tau) : t_4(\tau)).$$

The existence proof for these holomorphic functions was the main result of chapter I in [12]. We refer the reader to section 6.3 there entitled 'Inversion of the Picard integral map by means of automorphic forms', especially to Theorem 6.3.12.

Since we need the quality of the functions t_i for finding explicit Fourier series of them, we repeat the way of their construction in [12] without proofs.

11.4 Definition. A holomorphic function $f: \mathbb{B} \longrightarrow \mathbb{C}$ is a *Picard modular form* of the imaginary quadratic number field K and of *weight* m, if there exists a sublattice Γ'' of $U((2,1), \mathcal{O}_K)$ such that the following *functional equations* are satisfied:

(11.5) $\gamma^*(f) = j_\gamma^m \cdot f$ for all $\gamma \in \Gamma''$,

where $\gamma^*(f)(\tau) = f(\gamma(\tau))$ and $j_\gamma(\tau)$ is the Jacobi determinant of $\gamma : \mathbb{B} \xrightarrow{\sim} \mathbb{B}$ at τ.

If (11.5) is satisfied, then we shortly call f a Γ'' - m o - d u l a r f o r m (of weight m). These functions form a finite-dimensional vector space denoted by $[\Gamma'', m]$. We come back now to the Eisenstein numbers, especially to $\Gamma = U((2,1), \mathcal{O}_K)$, $\Gamma' = \Gamma(\sqrt{-3})$ and define the s p e c i a l modular groups by

$$S\Gamma = \Gamma \cap Sl_3(\mathbb{C}) \ , \ S\Gamma(\sqrt{-3}) = \Gamma(\sqrt{-3}) \cap Sl_3(\mathbb{C}) \ , \ \ldots$$

We have three exact sequences

$$(11.6) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & S\Gamma(\sqrt{-3}) & \longrightarrow & \Gamma(\sqrt{-3}) & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & \Gamma(\sqrt{-3}) & \longrightarrow & \Gamma & \longrightarrow & S_4 \times \mathbb{Z}_6 & \longrightarrow & 1 . \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & S\Gamma(\sqrt{-3}) & \longrightarrow & S\Gamma & \longrightarrow & S_4 & \longrightarrow & 1 \\ & & & & \ni & & \ni & & \\ & & & & \gamma & \longmapsto & \bar{\gamma} & & \end{array}$$

The group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ comes from the element $-id \in \Gamma$, $(\mathbb{Z}_2 \subset \mathbb{Z}_6)$.

We look for Picard modular forms t_1, t_2, t_3, t_4 satisfying the following conditions (11.7) and 11.8:

(11.7) $$t_1 + t_2 + t_3 + t_4 = 0,$$

t_1, t_2, t_3 are linearly independent.

11.8 S p e c i a l F u n c t i o n a l E q u a t i o n s .

(i) $\gamma^*(t_i) = j_\gamma \cdot t_i$ for $i = 1,2,3,4$, $\gamma \in S\Gamma(\sqrt{-3})$

(ii) $\gamma^*(t_i/t_j) = t_{\bar{\gamma}(i)}/t_{\bar{\gamma}(j)}$ for $i = 1,2,3,4$, $\gamma \in S\Gamma$.

(iii) $\delta^*(t_i) = (\det \delta)^2 \cdot j_\delta \cdot t_i$

for $i = 1,2,3,4$, δ representing $\Gamma(\sqrt{-3})/S\Gamma(\sqrt{-3})$.

11.9 Theorem (Holzapfel [12]). There exist four Picard modular forms t_1, t_2, t_3, t_4 satisfying the properties (11.7) and 11.8. They are uniquely determined up to the numeration and a common constant factor. The condition (iii) is a consequence of all the previos conditions (11.7), 11.8 (i),(ii).

11.10 Remark. The last statement has been proved first by Feustel [6] by an analytic argument using the Theta presentation of the modular functions t_i we look for. An algebraic proof has been found by the author [13] by means of the dimension formulas for cusp forms of ball lattices [11]. We can give a more precise version of Theorem 11.9 which brings the three conditions (i), (ii),(iii) of 11.8 together. Denoting the image of $\gamma \in \Gamma$ along $\Gamma \longrightarrow S_4 \times Z_6 \longrightarrow S_4$ (see diagram (11.6)) by $\bar{\gamma}$ we get

11.11 Corollary. The four Picard modular functions t_1, t_2, t_3, t_4 are characterized (up to a constant factor) by (11.7) and the functional equations

$$\gamma^*(t_i) = (\det \gamma)^2 \operatorname{sgn}(\bar{\gamma}) j_\gamma \cdot t_{\bar{\gamma}(i)} \quad \text{for } i = 1,2,3,4, \quad \gamma \in \Gamma.$$

Main idea of proof (see [12]). Basically one has to classify the surface $\hat{X} = \widehat{\mathbb{B}/S\Gamma(\sqrt{-3})}$. The group $S\Gamma(\sqrt{-3})$ acts almost freely on $\mathbb{B}$, that means that the non-trivially acting elements have at most isolated fixed points on $\mathbb{B}$. It turns out that

$$\widehat{\mathbb{B}/S\Gamma(\sqrt{-3})} = \hat{X} \dashrightarrow \mathbb{P}^2 = \widehat{\mathbb{B}/\Gamma(\sqrt{-3})}$$

is the unique (cyclic) Z_3-covering of $\mathbb{P}^2$ branched along $\mathbb{A}$ (see the diagrams (11.6) and (6.1). It is not difficult to describe the surface $\hat{X}$ by an equation (see [12], I.4.3).

(11.12) $$\hat{X} : Z^3 = (Y_2^2 - Y_1^2)(Y_2^2 - Y_0^2)(Y_1^2 - Y_0^2)$$

This is a weighted equation with Z of weight 2 and Y_0, Y_1, Y_2 of weight 1. More precisely, this means that $\hat{X}$ is the projective spectrum of the corresponding graded ring

$$\mathbb{C}[Y_0, Y_1, Y_2, Z]/(Z^3 - \prod_{0\le i<j\le 2} (Y_j^2 - Y_i^2)).$$

11.13 Remark. On this place we remember again to the 22-nd Hilbert problem mentioned in 6.3. It turns out that the algebraic relation in (11.12) is satisfied by Picard modular forms y_0, y_1, y_2, z substituting the variables Y_0, Y_1, Y_2 or Z, respectively. The knowledge of the uniformization $\mathbb{B} \longrightarrow \hat{X}$ together with the corresponding arithmetic uniformizing lattice $S\Gamma(\sqrt{-3})$ becomes important for this purpose as it has been predicted by Hilbert in general.

The key point is to understand automorphic forms as sections of logarithmic pluricanonical sheaves. In [12], I.4.3 we proved

(11.14) $$\bigoplus_{m=0}^{\infty} [S\Gamma(\sqrt{-3}), m] = \bigoplus_{m=0}^{\infty} H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + mT)),$$

where $\overline{X}$ is the minimal resolution of singularities of $\hat{X}$, T the compactification divisor resolving the cusp singularities of $\hat{X}$. It consists of four disjoint elliptic curves. As usual $K_{\overline{X}}$ denotes a canonical divisor and $\mathcal{O}(D)$ is the sheaf corresponding to the divisor D. A careful geometric analysis (explicit knowledge of a canonical divisor, vanishing theorem on surfaces) accomplished in [12] yields the ring structure

(11.15) $$\bigoplus_{m=0}^{\infty} H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + mT)) = \mathbb{C}[s_0, s_1, s_2, s]$$

with s_0, s_1, s_2 of weight 1, s of weight 2 and the generating relation

(11.16) $$s^3 = (s_2^2 - s_1^2)(s_2^2 - s_0^2)(s_1^2 - s_0^2)$$

Together with (11.14) we found generators η_0, η_1, η_2 of $[S\Gamma(\sqrt{-3}),1]$ and a $\Gamma(\sqrt{-3})$-modular form η of weight 2 such that η_0, η_1, η_2 and η generate the ring $\bigoplus_{m=0}^{\infty}[S\Gamma(\sqrt{-3}),m]$ of $S\Gamma(\sqrt{-3})$-modular forms satisfying the relation

(11.17) $$\eta^2 = (\eta_2^2 - \eta_1^2)(\eta_2^2 - \eta_0^2)(\eta_1^2 - \eta_0^2)$$

we looked for.

If Γ'' is an arbitrary ball lattice, then it acts on the space of holomorphic functions on the ball $\mathbb{B}$ via

(11.18) $$f \longmapsto j_\gamma^{-m} \cdot \gamma^*(f), \quad f \in H^0(\mathbb{B},\mathcal{O}_{\mathbb{B}}), \quad \gamma \in \Gamma''.$$

The Γ''-invariant functions are the Γ''-modular forms (compare with (11.5)). Especially the lattice $S\Gamma$ acts on $[S\Gamma(\sqrt{-3}),1]$ with ineffective kernel $S\Gamma(\sqrt{-3})$. With the last row of (11.6) we get a three-dimensional representation of S_4. In [12] we proved that this representation is irreducible. It induces a projective representation of S_4 on $\mathbb{P}[S\Gamma(\sqrt{-3}),1] \cong \mathbb{P}^2$. There is only one such representation. Explicitly it can be described by

$$(x_1 : x_2 : x_3 : x_4) \longmapsto (x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)} : x_{\sigma(4)}),$$
$$\sigma \in S_4, \; x_i \in \mathbb{C}, \; \sum x_i = 0.$$

Looking back to $[S\Gamma(\sqrt{-3}),1]$ one finds four Picard modular forms t_1, t_2, t_3, t_4 satisfying (11.7) and 11.8 (i),(ii).

It remains to verify the property (iii) of 11.8. This is much more difficult than it looks like at the first glance. There exist two proofs of different kind. The first has been found by

Feustel in [6]. He used a transcendental method: The modular forms t_i can be understood as restrictions of explicitly known theta constants on $\mathbb{H}_3$ to $\mathbb{B}$ (see below and also Shiga's article [32]). Then the transformation behaviour described in 11.8 (iii) can be checked directly. An algebraic-geometric proof of the functional equations (iii) can be found in the author's paper [13].

In order to solve the relative Torelli problem 11.1 in an effective manner by means of our modular forms t_1, t_2, t_3, t_4 we go back to the quotient map (11.3). It is realized by the modular forms of Theorem 11.9 for the following reasons (see [12] for more details). From the third row of (11.6) one gets a commutative quotient diagram

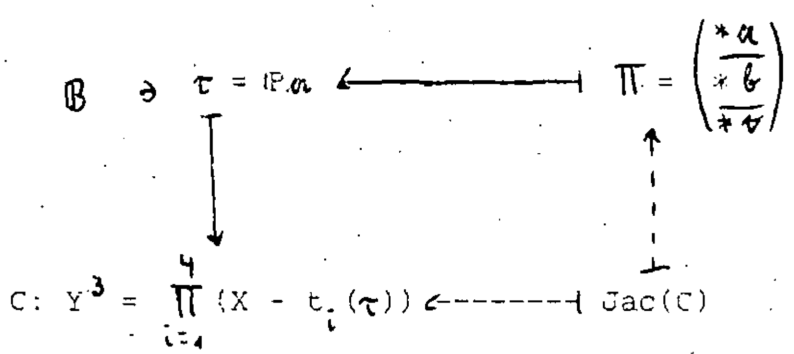
(11.19)
$$\begin{array}{ccc} \mathbb{B} & \xrightarrow[\Gamma(\sqrt{-3})]{(t_1:t_2:t_3)} & \\ {\scriptstyle S\Gamma(\sqrt{-3})}\downarrow & & \searrow \\ \hat{X} & \xrightarrow[Z_3]{(s_0:s_1:s_2)} & \mathbb{P}^2 \end{array}$$

The logarithmic canonical map $\varphi_{K_{\bar{X}}+T}$ goes down to $\hat{X} \longrightarrow \mathbb{P}^2$ and coincides with the Z_3-goutient map on the bottom of diagram (11.19) as has been proved in [12]. Using generators s_i of $H^0(\bar{X}, \mathcal{O}(K_{\bar{X}} + T))$ it can be realized as the projective morphism $(s_0 : s_1 : s_2)$. The sections s_i have been lifted to $S\Gamma(\sqrt{-3})$-modular forms η_i , $i = 0,1,2$ via (11.14). Without loss of generality we can assume that $t_i = \eta_{i-1}$, $i = 1,2,3$. We can identify the big quotient map of (11.19) with the map of (11.3), where t_4 is defined by (11.7).

11.20 Theorem (effective Torelli for Picard curves via modular forms). Let $J(C)$ be the Jacobi threefold of a smooth Picard curve C corresponding to the point $\mathcal{T} = *\tau \in \mathbb{H}_3$. Then the Picard modular forms t_1, t_2, t_3, t_4 defined in theorem 11.9 (up to a common constant factor) yield a normal form of $\dot{C}$ in the following manner:

(11.21) $$Y^3 = (X - t_1(\tau))(X - t_2(\tau))(X - t_3(\tau))(X - t_4(\tau))$$

Proof.: By the relative Schottky theorem 10.6 for Picard curves we have $J(C) = \mathbb{C}^3/\Lambda_{\Pi}$, where Π is given as $(\Pi_1 | \Pi_2)$ in (10.9) connecting $\tau \in \mathbb{B}$ with $\mathcal{T} = *\tau$. The diagram (10.8) with $\Gamma' = \Gamma(\sqrt{-3})$ yields the moduli point of C on $\mathbb{P}^2$ as image of τ along the $\Gamma(\sqrt{-3})$-quotient map. By (11.3) this image is equal to $(t_1(\tau):t_2(\tau):t_3(\tau):t_4(\tau))$. But the normal form of a corresponding Picard curve is given in (11.21), see Prop. 9.2. For the convenience of the reader we present a diagram of correspondences used above in close connection with diagram (10.8).



The theorem is proved. ∎

12. Picard modular forms as theta constants.

Theta functions $\vartheta\left[{a \atop b}\right]$ with characteristics $a,b \in \mathbb{Q}^g$ are holomorphic functions on $\mathbb{C}^g \times \mathbb{H}_g$,

$$\mathbb{H}_g = \{\tau \in Gl_g(\mathbb{C});\ {}^t\tau = \tau,\ \mathrm{Im}\,\tau > 0\}$$

the generalized Siegel upper half plane uniformizing the muduli space of (principally) polarized abelian varieties of dimension g (see e.g. [ ]). Explicitly the theta functions

$$\vartheta\left[{a \atop b}\right]: \mathbb{C}^g \times \mathbb{H}_g \longrightarrow \mathbb{C}$$

are defined by

$$\vartheta\left[{a \atop b}\right](z,\tau) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi i\, {}^t(n+a)\tau(n+a) + 2\pi i\, {}^t(n+a)(z+b)\}$$

The restrictions $\vartheta|0 \times \mathbb{H}_g$

$$\theta\left[{a \atop b}\right](\tau) = \vartheta\left[{a \atop b}\right](0,\tau)$$

are called *theta constants* (with characteristics).

We restrict our attention to the case $g = 3$ and look for extensions of the Picard modular forms t_1, t_2, t_3, t_4 defined in the previous section in $\mathbb{H}_3$ along the embedding $*: \mathbb{B} \hookrightarrow \mathbb{H}_3$ defined in (10.9) and hope to express them in terms of theta constants. Very important for this purpose are the functional equations described in 11.8. So we look for elementary combinations Th of theta constants whose restrictions

$$th(\tau) = Th(*\tau), \quad \tau \in \mathbb{B}$$

satisfy the special functional equations 9.11 (i),(iii):

$$(12.1) \qquad Th(*\gamma) = \det^2\gamma \cdot j_\gamma \cdot Th \quad \text{on } \mathbb{B} \subset \mathbb{H}_3, \qquad \gamma \in (\sqrt{-3}).$$

For the convenience of the reader we summerize the restrictions (or extensions) used above and below in the following diagram.

(12.2)
$$\begin{array}{ccccc}
\mathbb{B} & \hookrightarrow & \mathbb{H}_3 & & \\
\Gamma(\sqrt{-3}) \ni \gamma \Big\downarrow \gamma & & \Big\downarrow {*\gamma} \in Sp(6,\mathbb{Z}) & & \\
\mathbb{B} & \hookrightarrow & \mathbb{H}_3 & \hookrightarrow & \mathbb{C}^3 \times \mathbb{H}_3 \\
t = \Big\downarrow \theta|_{\mathbb{B}} & & \theta = \Big\downarrow \vartheta|_{0\times\mathbb{H}_3} & & \Big\downarrow \vartheta \\
\mathbb{C} & \xrightarrow{=} & \mathbb{C} & \xrightarrow{=} & \mathbb{C} .
\end{array}$$

12.3 Theorem (Feustel, Shiga).

Let $\theta_i(\tau) = \vartheta_i(0,\tau)$, $i = 0,1,2$, be the theta constants on $\mathbb{H}_3$ restricting the theta functions
$$\vartheta_k = \vartheta\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix}(z,\tau) \;,\; k = 0,1,2,\; z \in \mathbb{C}^3 .$$
Set

(12.4) $\quad Th_1 = \theta_0^3 + \theta_1^3 + \theta_2^3$, $Th_2 = -3\theta_0^3 + \theta_1^3 + \theta_2^3$,
$\quad Th_3 = \theta_0^3 - 3\theta_1^3 + \theta_2^3$, $Th_4 = \theta_0^3 + \theta_1^3 - 3\theta_2^3$,

and

(12.5) $\quad th_i(\tau) = Th_i(*\tau)$, $i = 1,2,3,4$, $\tau \in \mathbb{B}$.

Then the functions $th_i(\tau)$ are the (normalized) Picard modular forms satisfying (11.7) and all the functional equations (i), (ii), (iii) of 11.8 or, equivalently, those of Corollary 11.11.

Proof (main steps). We follow Feustel's proof and refer for explicit calculations to his paper [6] and the related literature given there. The proof summerizes preperatory work of Riemann, Picard [26], [27], Alezais [1], Mumford [24], H.Shiga

[32] and Holzapfel [12] (the functional equations above).

Step 1 (restriction to six functional equations).
Here we go back to the fundamental group $\pi_1(\mathbb{P}^2 \setminus \triangle)$ of the Fuchsian system (7.3) of partial differential equations and the surjective monodromy representation $\pi_1(\mathbb{P}^2 \setminus \triangle) \twoheadrightarrow \Gamma(\sqrt{-3})$, see (7.2). But the fundamental group $\pi_1(\mathbb{P}^2 \setminus \triangle)$ has obviously six generators coming from simple loops in $\mathbb{P}^2$ around each one of the six omitted lines. Therefore also $\Gamma(\sqrt{-3})$ has six generators, say $g_1, \ldots, g_6$. They have been explicitly described already by Picard [26] (with correcture in [27]) and Alezais. Their symplectic lifts $\sigma_i = {}^*g_i \in Sp(6,\mathbb{Z})$, $i = 1,\ldots,6$, can be found explicitly in Feustels paper [6]. In order to check the functional equations 8.11 (i),(iii) for suitable holomorphic functions th on $\mathbb{B}$ it is sufficient to check them for the generators $g_1, \ldots, g_6$ of $\Gamma(\sqrt{-3})$. According to our claim th = Th$|\mathbb{B}$ we have now only to look for holomorphic functions Th on $\mathbb{H}_3$ satisfying the six r e s t r i c t e d f u n c t i o n a l e q u a t i o n s

(12.6) $\quad Th \circ (\sigma_i) = (\det g_i)^2 \cdot j_{\gamma_i} \cdot Th \quad$ on $\mathbb{B} \subset \mathbb{H}_3$, $i = 1,\ldots,6$,

implying (12.1).

Step 2 (Riemann's theorem).
It is a general problem in the theory of algebraic curves to describe a given meromorphic function on a curve C in terms of theta functions on its Jacobian variety by restriction along the Jacobi embedding $C \hookrightarrow J(C)$. This problem has been solved essentially by Riemann. We refer to Mumford's book [24].

12.7 Theorem (Riemann). Let C be a (smooth, compact, complex) curve of positive genus g, $\gamma_1, \ldots, \gamma_{2g}$ a normal basis of $H_1(C,\mathbb{Z})$ and $\vec{\omega} = {}^t(\omega_1, \ldots, \omega_g)$ a basis of $H^0(C,\Omega_C)$ such that the corresponding period matrix has the normalized form

$$\Big(\int_{\gamma_j} \omega_i\Big) = (E_g | \tau), \quad \tau \in \mathbb{H}_g,\ E_g \text{ the unit matrix.}$$

If $f: C \longrightarrow \mathbb{P}^1$ is a meromorphic function with divisor

$$(f) = \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k, \quad a_k, b_k \in C,$$

then it holds that

(12.8)

$$\prod_{i=1}^{g} f(P_i) = \text{const}\cdot\prod_{k=1}^{m}\Big\{\vartheta\Big(\int_{P_0}^{\sum_{i=1}^{g} P_i} \vec{\omega} - \int_{P_0}^{a_k} \vec{\omega} - \Delta\Big)/\vartheta\Big(\int_{P_0}^{\sum_{i=1}^{g} P_i} \vec{\omega} - \int_{P_0}^{b_k} \vec{\omega} - \Delta\Big)\Big\}$$

as meromorphic function on C^g/S_g. One has to use the same pathes in the first integrals of the denominator and numerator.

Notations. Here the Riemann theta function ϑ is considered as holomorphic function on $\mathbb{C}^g$. It coincides with the restriction of $\vartheta\begin{bmatrix}0\\0\end{bmatrix}$ to $\mathbb{C}^g \times \tau$, τ the fixed period matrix defined above, with the notations introduced at the beginning of this section. The auxiliary point $P_0 \in C$ is used to fix the Jacobi embedding

$$C \longrightarrow J(C),\ P \longmapsto \int_{P_0}^{P} \vec{\omega} \bmod \Lambda_\tau, \quad \Lambda_\tau = \mathbb{Z}^g + \tau\cdot\mathbb{Z}^g.$$

Δ denotes the Riemann constant. This is a special well-defined 2-torsion point on J(C) (see [24], ch.II, 3). Both sides are considered as functions on C^g or C^g/S_g, namely $\sum_{i=1}^{g} P_i$ is understood as point on C^g/S_g. On the right-hand-side of (12.8) appears a constant denoted by const. In general one knows only its existence but not its explicit value. Finally, we used the notation

$$\int_{P_0}^{D} \vec{\omega} = \sum_{i=1}^{g} \int_{P_0}^{P_i} \vec{\omega} \quad \text{for } D = \sum_{i=1}^{g} P_i .$$

Remark. For the proof of Riemann's theorem one compares zeros and poles of both sides of (12.8). The key point is the understanding of Riemann's constant Δ. On $\mathbb{C}^g$ it is defined mod Λ_τ by

$$-\Delta + \int_{P_0}^{\sum_{i=1}^{g-1}} \vec{\omega} \bmod \Lambda_\tau \in \Theta, \; P_i \in C,$$

where $\Theta \subset J(C)$ denotes the theta divisor defined by $\vartheta(z) = 0$. Setting e.g. $P_i = a_k$ in (12.8) opens the way of proof of Riemann's theorem in an obvious manner.

Now we apply Riemann's theorem for finding two generators of the field of $\Gamma(\sqrt{-3})$-automorphic functions in theta terms. This has been done already by Picard and Alezais. We write a smooth Picard curve C in the modified normal form

$$C:\; Y^3 = X(X-1)(X-u)(X-v)$$

Then $u = u(\tau)$, $v = v(\tau)$, $\tau \in \mathbb{B}$, generate the field of $\Gamma(\sqrt{-3})$-modular functions. The ramification locus of the $\mathbb{Z}_3$-Galois covering $C \longrightarrow \mathbb{P}^1$ consists of the following five points on C:

$$O = Q_0 = (0,0),\; Q_1 = (1,0),\; Q_u = (u,0),\; Q_v = (v,0),\; \infty .$$

We apply (12.8) to the function $f = x: C \longrightarrow \mathbb{P}^1$ at the points

$$P_1 = Q_1,\; P_2 = Q_2,\; P_3 = Q_u$$

and at the points

$$P_1 = Q_1,\; P_2 = Q_u,\; P_3 = Q_u$$

With $(x) = 3\cdot 0 - 3\cdot\infty$. $P_0 = \infty$ we get with the same constant c

$$u = c\cdot\prod_{1}^{3}\{\vartheta(\int_{\infty}^{2Q_1+Q_u}\vec{\omega} - \int_{\infty}^{0}\vec{\omega} - \Delta)/\vartheta(\int_{\infty}^{2Q_1+Q_u}\vec{\omega} - \Delta)\}$$

$$u^2 = c\cdot\prod_{1}^{3}\{\vartheta(\int_{\infty}^{Q_1+2Q_u}\vec{\omega} - \int_{\infty}^{0}\vec{\omega} - \Delta)/\vartheta(\int_{\infty}^{Q_1+2Q_u}\vec{\omega} - \Delta)\}$$

Since $u = u^2/u$ we get by division of both expressions above a theta formula for $u(\tau)$ without the unknown constant c:

$$u = \prod_{1}^{3}\{\vartheta(. - .)\cdot\vartheta(. - . - .)/\vartheta(. - .)\cdot\vartheta(. - . - .)\}$$

In the same manner using Q_v instead of Q_u we can express v, v^2 and finally $v(\tau) = v^2/v$ in terms of the theta function belonging to $\tilde{\tau} = *\tau$.

Step 3 (Theta constants).

This step is due to Shiga [32]. He calculated explicitly the Riemann constant Δ above using special values. Furthermore he used standard transformation laws to prove that

(12.9) $u(\tau) = \vartheta_1^3(0,*\tau)/\vartheta_2^3(0,*\tau)$, $v(\tau) = \vartheta_0^3(0,*\tau)/\vartheta_2^3(0,*\tau)$

with the notations of Theorem 12.3. The denominator does not vanish identically on $\mathbb{B}$ (Shiga [32]).

Step 4 (automorphic forms).

This last step is due to Feustel [6]. He checked that the dominator and the numerators in (12.9) satisfy the six functional equations (12.6). By step 1 we dispose on three linearly independent $S\Gamma(\sqrt{-3})$-modular forms $\theta_0^3(*\tau)$, $\theta_1^3(*\tau)$, $\theta_2^3(*\tau)$ on $\mathbb{B}$ of wight 1 with the notations of Theorem 12.3. Since dim $[S\Gamma(\sqrt{-3})] = 3$ by the considerations around (11.16) we found up to linear combinations all $S\Gamma(\sqrt{-3})$-modular forms of wight 1. So $\theta_0^3, \theta_1^3, \theta_2^3$ can be identified with η_0, η_1, η_2 in (11.17). Now we remember to the $S_4 = S\Gamma/S\Gamma(\sqrt{-3})$-action on $[\Gamma(\sqrt{-3}),1]$. There must be linear combinations th_1, th_2, th_3, th_4 of $\theta_0^3, \theta_1^3, \theta_2^3$ satisfying the relation (11.7) and the functional equations 11.8 uniquely defined up to a common factor, see Theorem 11.9. The symmetric group S_4 is generated by three tranpositions. It is not difficult to find representants of them in $S\Gamma$ and also their symplectic representations acting on $\mathbb{H}_3$ explicitly. This has been done in [6]. With the definitions (12.4) and (6.12.5) Feustel proved that th_1, th_2, th_3, th_4 are functions with the correct tranformation behaviour we look for. The Theoerem 12.3 is proved. ∎

13. Proof of the Main Theorem.

Now we are able to prove the Main Theorem formulated at the end of 1. We have to concentrate our attention to the verification of the field tower on the right-hand side of diagram (1.1). First we check the list 1.,...,8. of definitions in 1. and fill the gaps. The ball $\mathbb{B}$ is understood as subball of $\mathbb{H}_3$ via the embedding $*$ defined in (10.9). The restricted theta constants th_i, $i = 1,2,3,4$, have been defined in the previous section.

Next we have to give a precise definition for the special arguments used in the Main Theorem. An a r i t h m e t i c p o i n t o n $\mathbb{H}_g$ is a point of

$$\mathbb{H}_g(\bar{\mathbb{Q}}) = \mathbb{H}_g \cap Gl_g(\bar{\mathbb{Q}}).$$

An a r i t h m e t i c b a l l p o i n t is a point of

$$\mathbb{B}(\bar{\mathbb{Q}}) = \mathbb{B} \cap \mathbb{H}_3(\bar{\mathbb{Q}}) = *^{-1}(\mathbb{H}_3(\bar{\mathbb{Q}})).$$

A (principally) polarized abelian variety A of dimension g with complex multiplication determines an arithmetic point $\tau \in \mathbb{H}_g(\bar{\mathbb{Q}})$. Namely we know from section 3 that $A \cong \mathbb{C}^g/\Phi(\mathfrak{a})$, $\mathfrak{a} \in Mat_{g\times 2g}(\bar{\mathbb{Q}})$. On the other hand A is isomorphic to $\mathbb{C}^g/\Lambda_{(E_g|\tau)}$. Therefore $\mathfrak{a}$ and $(E_g|\tau)$ must belong to the same double coset of $Mat_{g\times 2g}(\mathbb{C})$ with respect to $Gl_g(\mathbb{C})$ or $Gl_{2g}(\mathbb{Z})$, respectively. Hence there exist elements $G \in Gl_g(\mathbb{C})$ and $\Sigma \in Gl_{2g}(\mathbb{Z})$ such that $\mathfrak{a}\cdot\Sigma = (G|G\cdot\tau)$. Now it is clear that G, $G\tau$ and finally $\tau = G^{-1}(G\tau)$ belong to $Gl_g(\bar{\mathbb{Q}})$. Especially we dispose on the following well-known

13.1 Lemma. If C is a (smooth) curve of genus g and its Jacbian variety J(C) corresponding to $\tau \in \mathbb{H}$ has complex multiplication, then τ is an arithmetic point of $\mathbb{H}_g$.

13.2 Definition. A point $\sigma \in \mathbb{B}$ is called a CM-module (with respect to Picard curves), if there exists a (smooth) Picard curve C such that σ belongs to the Jacobian threefold J(C) of C (t.m. $C \simeq C_\sigma$), J(C) is simple and has complex multiplication.

From the above considerations it is clear that a CM-module of $\mathbb{B}$ is an arithmetic ball point. There is a dense set of very explicit examples called s t a t i o n a r y m o d u l e s . They are defined as isolated fixed points $\sigma \in \mathbb{B} \setminus \mathfrak{A}$ of elements $\gamma \in U((2,1),K)$ with $[K(\gamma):K] = 3$. For details we refer to [H4].

Now we are able to define F_σ, Φ_σ and $\mathfrak{A}_\sigma$ appearing in the Main Theorem. We have only to connect the above definition with those of section 3. Let $\sigma \in \mathbb{B}(\overline{\mathbb{Q}})$ be a CM-module corresponding to the Picard curve C with Jacobian threefold J(C). It has a CM-type $(F_\sigma, \Phi_\sigma, \mathfrak{A}_\sigma)$ and this is all we need. The corresponding Shimura class field $Sh(\Phi_\sigma, \mathfrak{A}_\sigma)$ has been defined at the end of section 2.

We come to the proof of the Main Theorm. For this purpose we let $\sigma \in \mathbb{B}(\overline{\mathbb{Q}})$ be a CM-module corresponding to the Picard curve C_σ with Jacobian threefold J_σ of type $(F_\sigma, \Phi_\sigma, \mathfrak{A}_\sigma)$.

(13.3) $$C_\sigma \cong: \quad Y^3 = \prod_{i=1}^{4} (X - th_i(\sigma))$$

This comes from the effective Torelli Theorem 11.20 for Picard curves in combination with the theta representation of the Picard modular forms t_i found in section 12, see Theorem 12.3.

13.4 $k_\sigma = K(th(\sigma))$ is a definition field of $cl(C_\sigma)$ and of $cl(J_\sigma)$ in the sense of section 4.

For the curve C_σ this is an immediate consequence of 13.3. If k is a definition field of a curve C, then it is also a definition field of the Jacobian manifold J(C). For this well-known fact we refer the reader to Milne's article [23] in [5]. Consequently, k_σ is a definition field of $cl(J_\sigma)$ or of J_σ itself without loss of generality. The same is true for $(J_\sigma, \mathcal{C}_\sigma)$, $\mathcal{C}_\sigma$ a suitable polarization of J_σ.

(13.5) $$M(J_\sigma, \mathcal{C}_\sigma) = Sh(\Phi_\sigma, \mathfrak{A}_\sigma)$$

This was (the consequence 5.3 of) Shimura's Main Theorem 5.1 of complex multiplication.

(13.6) $$M(J_\sigma) \subseteq M(C_\sigma) = K(th(\sigma))^{S_4(\sigma)}$$

Proof. We remember that

$$(th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma))/S_4 \in \mathbb{P}^2/S_4$$

is the moduli point of the Picard curve C_σ by (13.3) and Proposition 9.2. So for $\mu \in Aut(\mathbb{C})$ one has the following equivalent conditions:

$$\mu \in Stab\ cl(C_\sigma) \longleftrightarrow C^\mu \cong C \longleftrightarrow cl(C^\mu) = cl(C) \longleftrightarrow$$

$$(th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma))^\mu \equiv (th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma)) \mod S_4$$

$$\longleftrightarrow \quad th_i(\sigma)^\mu/th_j(\sigma)^\mu = th_{\pi(i)}(\sigma)/th_{\pi(j)}(\sigma)$$

for all $i,j \in \{1,2,3,4\}$ and a suitable $\pi \in S_4$

$$\longleftrightarrow \quad \mu \in \mathbb{Q}(th(\sigma))^{S_4(\sigma)}.$$

On the other hand we have $Stab\ cl(C_\sigma) \subseteq Stab\ cl(J_\sigma)$, hence $M(C_\sigma) \supseteq M(J_\sigma)$ by the definition 4.1 of moduli fields. It remains to show that K is a subfield of the fields appearing in (13.6). The curve C_σ in (13.3) has an obvious automorphism of order 3. Therefore the field $K = \mathbb{Q}(\rho)$ is a subfield of the endomorphism algebra $\mathbb{Q}\otimes \mathrm{End}(J_\sigma)$. Since J_σ is simple, the field F_σ coincides with this algebra (up to isomorphy). Therefore K is a subfield of F_σ. This is also true if J_σ is not simple and has complex multiplication in the generalized sense of section 3 (DCM = decomposed complex multiplication): Then J_σ is isogeneous to $T \times T \times T$, T an elliptic CM-curve with imaginary quadratic multiplication field E, say. As already mentioned at the end of section 3, the endomorphism algebra of J_σ is isomorphic to $\mathrm{Mat}_3(E)$. The diagonally embedded field E commutes with any other subfield of $\mathbb{Q}\otimes \mathrm{End}(J_\sigma)$, especially with K. Thus the endomorphism algebra contains the subfield K(E). The absolute degree of a subfield of the $\mathbb{Q}$-algebra $\mathbb{Q}\otimes \mathrm{End}(A)$ of an abelian variety A divides $2\cdot\dim A$, see [20], I.1, Th.3.1. Therefore $[K(E):\mathbb{Q}]$ divides $6 = 2\cdot\dim J_\sigma$. This is only possible for K = E. Consequently, K is central in $\mathbb{Q}\otimes \mathrm{End}(J_\sigma)$, and $K\cdot F_\sigma$ is a subfield. Since $[F_\sigma:\mathbb{Q}] = 2\cdot\dim J_\sigma = 6$ by definition of complex multiplication it cannot happen that $K\cdot F_\sigma > F_\sigma$ because F_σ is obviously a maximal subfield of $\mathbb{Q}\otimes \mathrm{End}(J_\sigma)$. So we have $K \subset F_\sigma$, that means that in any case the multiplication field F of a CM-Picard curve is a cubic extension of K. It follows immediately from the trace definition (2.2) of the reflex field F' that $K \subseteq F'$. Now we apply (4.5) to obtain $F'_\sigma \subseteq M(J_\sigma)$. Together with (13.6) and the equivalence considerations above we get

$$K \subseteq M(C_\sigma) = \mathbb{Q}(th(\sigma))^{S_4(\sigma)} .$$

The identity on the right-hand side of (13.6) is verified. By the way we established also the bottom part of diagram (1.1).

13.7 The definition field $k_\sigma = K(th(\sigma))$ of the CM(or DCM)-Picard curve C_σ is an algebraic number field.

Proof. Let $\mathcal{C}$ be a principal polarization of J_σ. The polarized Jacobian variety $Jac(C_\sigma) = (J_\sigma, \mathcal{C})$ has an algebraic definition field k by Proposition 4.4. From Lemma 4.3 we know that

$$M(Jac(C_\sigma)) \subseteq k \subset \bar{\mathbb{Q}} .$$

Obviously, the moduli field $M(C_\sigma)$ coincides with $M(Jac(C_\sigma))$. Together with the identity in (13.6) we see that $K(th(\sigma))^{S_4(\sigma)}$ is a number field. Since $k_\sigma = K(th(\sigma))$ is a finite extension, it is a number field, too. ∎

Altogether we have the following inclusions:

(13.8) $$\begin{array}{l} \qquad\quad M(J_\sigma, \mathcal{C}_\sigma) = Sh(\phi_\sigma, \mathfrak{a}_\sigma) \\ \qquad\quad\;\; \cup\!\! \mid \\ K \subseteq F'_\sigma \subseteq M(J_\sigma) \subseteq M(C_\sigma) = M(Jac(C_\sigma)) \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad K(th(\sigma))^{S_4(\sigma)} \subseteq k_\sigma = K(th(\sigma)) \subset \bar{\mathbb{Q}} \end{array}$$

We restrict ourselves from now on to the case of simple Jacobian threefolds J_σ. For the sake of clear distinction we call σ sometimes in this case a s i m p l e CM-module (SCM).

13.10 Lemma. If A is a simple abelian variety with complex multiplication, then each polarization $\mathcal{C}$ of A is admissible and the moduli field $M(A, \mathcal{C})$ does not depend on the choice of $\mathcal{C}$.

For the proof we refer to Lang's book [10] again. The first statement comes from [10], I.4, Thm. 4.5 (iii) and the condition ADM 2, p. 30. According to the remark in [10], p. 135, the moduli field $M(A, \mathcal{C})$ does not depend on the embedding $\iota: F \hookrightarrow \mathbb{Q} \otimes \mathrm{End}(A)$, F the CM-field of A. The last statement of the lemma is Proposition 1.7 (i) of [10], ch.V. ∎

Let $\sigma \in \mathbb{B}$ be a simple CM-module and $\mathcal{C}$ a principal polarization of J_σ. By Lemma 13.10 $\mathcal{C}$ is admissible and $M(J_\sigma, \mathcal{C})$ coincides with the moduli field $M(J_\sigma, \mathcal{C}_\sigma)$ for any other admissible polarization $\mathcal{C}_\sigma$ of J_σ. Cosequently the inclusions of (13.8) become sharper:

13.11. For simple CM-modules $\sigma \in \mathbb{B}$ it holds that

$$K \subseteq F' \subseteq M(J_\sigma, \mathcal{C}_\sigma) = Sh(\Phi_\sigma, \mathfrak{a}_\sigma) = K(th(\sigma))^{S_4(\sigma)} \subseteq K(th(\sigma)) \subset \bar{\mathbb{Q}}.$$

∎

We established the diagram (1.1) of field towers in the Main Theorem. By the definition 2.7 $Sh(\Phi_\sigma, \mathfrak{a}_\sigma)$ (Shimura class field) is an abelian extension of the reflex field F'_σ. It remains to prove that this extension is unramified, if $\mathfrak{a}_\sigma$ is a (fractional) ideal of F_σ. This follows easily from the construction of Shimura class fields:

13.12 Lemma (see [10], V.4, Thm. 4.1 (ii)). Let A be an abelian CM-variety of type $(F, \Phi, \mathfrak{a})$ such that $\mathfrak{a}$ is a fractional ideal of F. Then the abelian extension $Sh(\Phi, \mathfrak{a})/F'$ is unramified.

Proof. We go back to the construction of the Shimura class field $Sh(\Phi, \mathfrak{a})$ in section 2. Via reciprocity it corresponds to the idele group $U(\Phi, \mathfrak{a})$ defined in (2.4). It suffices to verify that $U(\Phi, \mathfrak{a})$ contains the whole unit group $\mathfrak{o}^*_{F'}$ of F'. This is a well-known necessary and sufficient criterion for the corresponding class field to be unramified (see e.g. [25]). So let ε be a unit of F'. Then, with the notation of (2.4), also $N'(\varepsilon) \in F$ is a unit. Since $\mathfrak{a}$ is a fractional ideal it holds that

$$N'(\varepsilon)\mathfrak{a} = \mathfrak{a} = 1 \cdot \mathfrak{a}.$$

Thus the relations of the right-hand side of (2.4) are satisfied for $s = \varepsilon$, $\beta = 1$. Hence ε belongs to $U(\Phi, \mathfrak{a})$. The lemma is proved, and at the same time we finish the proof of the Main Theorem. ∎

The field of Picard modular functions (of level Γ) is defined to be the field $\mathbb{C}(G_4/G_2^2, G_3^2/G_2^3)$ of Γ-automorphic functions, where $G_1 = 0$, G_2, G_3, G_4 are the elementary symmetric functions of th_1, th_2, th_3, th_4. The subfield of K-modular functions (of the full level Γ) is defined to be $K(G_4/G_2^2, G_3^2/G_2^3)$. It is the subfield of S_4-invariant functions of K(th). For $\tau \in \mathbb{B}$ we define the field of values of Picard K-modular functions (of full level Γ) at τ by

$$K(G_4/G_2^2, G_3^2/G_2^3)(\tau) = \{f(\tau); f \in K(G_4/G_2^2, G_3^2/G_2^3),\ f(\tau) \neq \infty\}$$

13.13. Definition. Let $\sigma \in \mathbb{B}$ be a simple CM-module with J_σ of type $(F_\sigma, \Phi_\sigma, \mathfrak{a}_\sigma)$ such that $\mathfrak{a}_\sigma$ is a fractional ideal of F_σ. Then σ is called an *ideal simple CM-module*.

13.14. Corollary. Let $\sigma \in \mathbb{B}$ be an (ideal) simple CM-module. Then $K(G_4/G_1^2, G_3^2/G_1^3)(\sigma)$ is an (unramified) abelian extension of the reflex field F_σ' of the type (F_σ, Φ_σ).

Proof. Looking at the action of $S_4(\sigma)$ we have the obvious relation

$$K(G_4/G_2^2, G_3^2/G_2^3)(\sigma) \subseteq K(th(\sigma))^{S_4(\sigma)} .$$

Now we can apply the Main Theorem concentrated in diagram (1.1).

13.14. Remark. The celebrated Hilbert class field of a basic number field F' is defined as the maximal abelian extension of F'. It is a finite Galois extension with Galois group isomorphic to the ideal class group of F' (see e.g. [25]). Hilbert class fields play an important role in number theory. The explicit construction by means of special values of transcendent functions can be considered as the essential part of Hilbert's twelvth problem. With our Main Theorem we succeded to construct at least a part of the Hilbert class field of F_σ', if $\sigma \in \mathbb{B}$ is an ideal simple CM-module. Feustel observed that the very explicit stationary modules of elements $\gamma \in U((2,1),K)$ are simple (CM-modules, if $K(\gamma)$ is a cubic extension of K. So we dispose on abelian extensions $M(Jac(C_\sigma)) = K(th(\sigma))^{S_4(\sigma)}$ at all these stationary CM-modules σ. We can produce more abelian extensions of our reflex fields by means of torsion points $T_1, \ldots, T_r$ on J_σ. The moduli field $M(J_\sigma, \mathcal{C}_\sigma; T_1, \ldots, T_r)$ has been defined by Shimura (see [35] or [10]). For any CM-module σ it is an abelian extension of

F'_σ extending $M(J_\sigma, \mathcal{C}_\sigma)$. The corresponding idele group in the sense of class field theory is also well-known. We refer the reader who is interested on these extensions to [10], Ch.V.,Thm.4.2. Unfortunately, until now there exists no description of these extended class fields in terms of special values of analytic functions except for the case of elliptic curves.

3.15. Problem. For which ideal CM-modules is $K(th(\sigma))^{S_4(\sigma)}$ the whole Hilbert class field of F'_σ ?

13.16. Problem. Is the maximal abelian extension F'^{ab}_σ for fixed CM-module σ generated by all the generalized moduli fields of type $M(J_\sigma, \mathcal{C}_\sigma; T_1, \ldots, T_r)$?

It seems to be that the field $K(th(\sigma))$ of K-modular functions of level $\Gamma(\sqrt{-3})$ is not in general an abelian extension of the reflex field F'_σ. This happens certainly, if the subgroup $S_4(\sigma)$ of the symmetric group S_4 is not abelian.

13.17 Problem. For which CM-modules σ is $K(th(\sigma))/F'_\sigma$ an abelian (or non-abelian) field extension ?

Let us change over from the big level groups Γ and $\Gamma(\sqrt{-3})$ to smaller ones, say to normal subgroups Γ'' of finite index of Γ. We denote by $\mathfrak{F}_K(\Gamma'')$ the algebraic closure of $K(G_4/G_2^2, G_3^2/G_2^3)$ in the field $\mathfrak{F}_{\mathbb{C}}(\Gamma'') = \mathbb{C}(\mathbb{B}/\Gamma'')$ of Γ''-automorphic functions. With obvious notations we obtain at each CM-module σ an infinite tree

$\{\mathcal{F}_K(\Gamma'')(\sigma);\ \Gamma''$ normal subgroup of Γ of finite index$\}$

of field extensions of F'_σ. The analogue construction in the theory of elliptic curves yields a generating system of the maximal abelian extension of an imaginary quadratic number fields. It would be interesting to understand our construction in a suitable framework of (non-abelian) class field theory. Especially, one has to investigate the action of (subgroups of) the factor groups Γ/Γ'' in the towers $\mathcal{F}_K(\Gamma'')(\sigma) \supseteq \mathcal{F}_K(\Gamma)(\sigma) \supseteq F'_\sigma$ of number fields at special CM-values σ.

13.18. Remark. Let $\tau \in \mathbb{B}(\bar{\mathbb{Q}})$ be an arithmetic point of the ball. We proved that

$$th(\tau) = (th_1(\tau):th_2(\tau):th_3(\tau):th_4(\tau)) \in \mathbb{P}^2$$

is arithmetic, if τ is a CM (or DCM)-module. The converse implication seems to be true. Very recently Shiga [34] succeeded to prove that at least at simple (simple J_τ) arithmetic modules τ the point $th(\tau)$ is transcendent, that means a non-algebraic point of $\mathbb{P}^2$, if τ is not a CM-module.

13.19. Problem. What happens precisely at non-simple arithmetic modules in both cases, the case of CM-modules and the opposite case ?

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BALL MODEL FOR HILBERT'S TWELVTH PROBLEM

R.-P. Holzapfel

Contents

0. Introduction.

Until now one misses a clear solution of Hilbert's 12-th problem entitled "Ausdehnung des Kroneckerschen Satzes ueber abelsche Koerper auf einen beliebigen algebraischen Rationalitaetsbereich". This problem plays a central role among the 23 Hilbert problems because it joins some of them with each other, namely the problems 7, 9, 21 and 22. The 12-th problem is based on Kroneckers work on the explicit description of abelian number fields over the field $\mathbb{Q}$ of rational numbers or over imaginary quadratic number fields, respectively, by means of special values of special transcendent functions of one complex variable. The Theorem of Kronecker-Weber asserts that each absolute abelian number field is generated by a rational expression of a unit root over $\mathbb{Q}$, where unit roots are understood as special values of the exponential function.

As a counter part appears Hilbert's 7-th problem. It asks for the quality of values of the shifted exponential function

$$e(z) = \exp(\pi i z),\ i = \sqrt{-1},$$

at algebraic arguments outside of the rationals $\mathbb{Q}$ and conjectures that all these values are transcendent numbers (see [10]). This problem has been solved affirmatively and independently by Gelfond [7] and Schneider [29] in 1934. Altogether we know

I. $e(z)$ has algebraic values on $\mathbb{Q}$;

II. $e(z)$ has transcendent values on $\bar{\mathbb{Q}} - \mathbb{Q}$ ($\bar{\mathbb{Q}}$ the field of algebraic numbers);

III. the number-theoretic meaning of the values $e(q)$, $q \in \mathbb{Q}$.

Substituting the base field $\mathbb{Q}$ by an imaginary quadratic number field one needs special values of Weierstrass' $\wp$-function (at torsion points of an elliptic curve) and special values (singular moduli) of the elliptic modular function j in order to generate all abelian extensions. For a precise formulation of this Main Theorem of Complex Multiplication we refer to Shimura's book [35], Ch.5. Historically, this main theorem is known as "Kronekkers Jugendtraum". It appears in Hilbert's programm as "Aufgabe" (Kronecker's problem) preparing the 12-th problem itself.

On the other hand C.L.Siegel [31] proved in 1949 that j takes transcendent values at algebraic points on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}\,\tau > 0\}$ which are not singular. In analogy with the exponential function we can summerize the situation in the following manner. Let

$$\mathbb{H}_{sing} = \{\tau \in \mathbb{H};\ [\mathbb{Q}(\tau):\mathbb{Q}] = 2\}$$

be the set of singular moduli. The transcendent function j has a well-known Fourier series

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^{2} + \ldots$$

with integral coefficients and $q = \exp(2\pi i\tau)$, see [30]. One knows

I. j has algebraic values on $\mathbb{H}_{sing}$;

II. j takes transzendent values on $\mathbb{H}(\bar{\mathbb{Q}}) \setminus \mathbb{H}_{sing}$, where $\mathbb{H}(\bar{\mathbb{Q}})$ denotes the set $\mathbb{H}\cap\bar{\mathbb{Q}}$ of algebraic numbers on the upper half plane;

III. the number-theoretic construction / quality / meaning of $j(\sigma)$, $\sigma \in \mathbb{H}_{sing}$.

Hilbert asked in his 12-th problem for transcendent functions of s e v e r a l variables with properties corresponding to those of the exponential function and the elliptic modular function: "Von der hoechsten Bedeutung endlich erscheint mir die Ausdehnung des Kroneckerschen Satzes auf den Fall, dass an Stelle des Bereichs der rationalen Zahlen oder des imaginaeren Zahlenbereiches ein beliebiger algebraischer Zahlkoerper als Rationalitaetsbereich zugrunde gelegt wird; ich halte dies Problem fuer eines der tiefgehendsten und weittragendsten Probleme der Zahlen- und Funktionentheorie"; and a littlebit later: "Wie wir sehen, treten in dem eben gekennzeichneten Problem die drei grundlegenden Disziplinen der Mathematik, naemlich Zahlentheorie, Algebra und Funktionentheorie in die innigste gegenseitige Beruehrung, und ich bin sicher, dass insbesondere die Theorie der analytischen Funktionen m e h r e r e r Variablen eine wesentliche Bereicherung erfahren wuerde, wenn es gelaenge, diejenigen Funktionen aufzufinden und zu diskutieren, die fuer einen beliebigen algebraischen Zahlkoerper die entsprechende Rolle spielen, wie die Exponentialfunktion fuer den Koerper der rationalen Zahlen und die elliptische Modulfunktion fuer den imaginaeren quadratischen Zahlkoerper".

We found only a few places in the mathematical literature with an explicit reference to the twelvth problem of Hilbert. On first place we remember to Hecke's thesis [8] and habilitation [9]. They are closely connected with the creation of the theory of Hilbert modular surfaces. This work is difficult to understand and it would be nice to clarify the situation from a modern point of view. The next important place where the twelvth problem is mentioned one can find in the book [36] of Shimura-Taniyama. Indeed, Shimura's theory of complex multiplication is an important tool for finding solutions of the problem. The latest hint to Hilbert's twelvth problem we found is due to Tate [37] in connection with the Stark conjecture. It touches the problem but will not solve it in the original sense of Hilbert. Manin accepts in his review [22] only Hecke's work of 1912, 1913 as a finer approximation to a solution of the 12-th problem. Langlands announced in [21] some doubts of Hilbert's formulations. In our opinion the twelvth problem needs a stronger formulation in order to catch solutions. With regard to the transcendental functions e, j above and their properties I., II., III. we propose the following definition.

A solution model for Hilbert's twelvth problem is a triple (V, V_{sing}, f) consisting of

(i) a (non-compact) complex manifold V with fixed analytic embedding into a complex projective space $\mathbb{P}^M(\mathbb{C})$;

(ii) a subset V_{sing} of the algebraic points $V(\bar{\mathbb{Q}}) = V \cap \mathbb{P}^M(\bar{\mathbb{Q}})$ lying dense in V;

(iii) A transcendent holomorphic map

$$f = (f_0 : f_1 : \ldots : f_N) : V \longrightarrow \mathbb{P}^N(\mathbb{C});$$

satisfying the postulates I., II., III. below.

Remark. We call f transcendent if f is not the restriction of a rational map in the sense of algebraic geometry. The elements of V_{sing} are called the singular points of V.

I. $f(\sigma) = (f_0(\sigma) : \ldots : f_N(\sigma))$ is algebraic, that means $f(\sigma) \in \mathbb{P}^N(\bar{\mathbb{Q}})$, for $\sigma \in V_{sing}$;

II. $f(\tau)$ is transcendent, that means $f(\tau) \notin \mathbb{P}^N(\bar{\mathbb{Q}})$, for

$\tau \in V(\bar{\mathbb{Q}}) \setminus V_{sing}$;

III. one has a number-theoretic construction / quality / meaning of field extensions

$$F'_\sigma(f(\sigma)) = F'_\sigma(\ldots, f_i(\sigma)/f_j(\sigma), \ldots)$$

for suitable well-defined "elementary" number fields F'_σ, $\sigma \in V_{sing}$.

Of course, we assume that V_{sing} is given independently of the holomorphic functions $f_0, \ldots, f_N$.

The first two conditions are very sharp but condition III. is free for several interpretations.

A (twodimensional) b a l l m o d e l for Hilbert's twelvth problem is a solution model $(\mathbb{B}, \mathbb{B}_{sing}, f)$, where $\mathbb{B}$ is the complex two-dimensional unit ball. The Main Theorem of section 1 presents a ball model $(\mathbb{B}, \mathbb{B}_{CM}, th)$ for the twelvth problem satisfying I. and III. Recently Shiga proved that also II. is essentially satisfied (see Remark 13.18). The components th_i of $th = (th_1 : th_2 : th_3 : th_4)$ are restrictions of elementary polynomials of theta constants to the ball $\mathbb{B}$ embedded in the generalized Siegel upper half plane $\mathbb{H}_3$, where the theta constants live.

We prefered to formulate the number-theoretic Main Theorem in the first section corresponding to Hilbert's order in his list of problems. Consequently we have to explain immediately after the notions of Shimura's class fields, complex multiplication of abelian varieties, moduli fields in the sections 2.,3.,4. and 5. This prepares at the same time the number-theoretic side of proof of the Main Theorem in section 13.

The geometric and analytic starting point is section 6. For an algebraic geometer it is convenient to begin there. The following sections will demonstrate that the simple configuration of four points and six lines through pairs of them in the projective plane determines completely the construction of our ball model. This is a consequence of some recent developments: A theorem of R.Kobayashi [18] provides the existence of a ball covering of $\mathbb{P}^2$ branched along the quadrilateral introduced above. There is only one possibility. The corresponding ramification indices can be calculated by the effective finiteness theorem for ball lattices due to the author [16]. The corresponding group of the covering has been found in a classification atlas of Picard modular surfaces due to the author and Feustel. This group appears as monodromy group of a Fuchsian system of partial differential equations uniquely determined by the quadrilateral. This system coincides with the Euler-Picard system of an algebraic curve family in the sense of the author's book [12]. The solution consists of variations of integrals of a differential form of first kind along cycles on Picard curves. The Picard curve family studied first by Picard in 1883 plays the same role as the elliptic curve family in Kronecker's problem. Now we discovered that its investigation was absolutely necessary for finding our ball model for Hilbert's twelvth problem. The proof of the Main Theorem is delegated to the fine arithmetic and analytic study of the family of Picard curves. This will be done in the sections 9. - 12. using and explaining available recent results of Shiga [32], Feustel [6] and the author [13].

The modern tools in the way of proof should also work for other cases, where the starting situation of a branched covering is precisely known. We think of Hilbert modular surfaces, Picard modular surfaces, a Picard modular threefold investigated carefully by Bruce Hunt and the Siegel modular threefold connected with hyperelliptic curves of genus 2. The latter case should be open a door to a precise modern understanding of Hecke's work on Hilbert's twelvth problem.

We close the introduction with two problems. More of them can be found at the end of the final section 13.

0.1 Problem. Study special values of Picard modular functions of higher level in connection with non-abelian class field theory.

0.2 Problem. Generate more (if possible all) abelian extensions of reflex fields of cubic extensions of the Eisenstein numbers by means of special values of some additional transcendent functions.

1. Formulation of the Main Theorem

First we present roughly the basic objects we need in the Main Theorem. More precise definitions are given in the later sections.

0. Basic field: $K = \mathbb{Q}(\sqrt{-3})$ the field of Eisenstein numbers;

1. Geometric object: the ball $\mathbb{B} \cong$ (K-linear equivalent in $\mathbb{P}^2$ to)

$$\mathbb{B}^2 = \{\tau = (\tau_1, \tau_2) \in \mathbb{C}^2;\ |\tau_1|^2 + |\tau_2|^2 < 1\},$$

embedded in $\mathbb{H}_3$ (Siegel domain, see 10.);

2. Analytic functions: $th_1, th_2, th_3, th_4 : \mathbb{B} \longrightarrow \mathbb{C}$ (restricted theta constants);

3. Special arguments (CM-modules): $\sigma \in \mathbb{B}(\bar{\mathbb{Q}})$ (dense in $\mathbb{B}$);

4. Correspondences: $\sigma \longmapsto F_\sigma/K$ (relative cubic number fields)

$\longmapsto \mathfrak{a}_\sigma$, $\mathbb{Z}$-lattice in F_σ

$\longmapsto \Phi_\sigma = \sum_{i=1}^{3} \varphi_i$, $\varphi_i : F_\sigma \hookrightarrow \mathbb{C}$ (field embeddings); $\varphi_i \neq \varphi_j$, $i \neq j$, $\varphi_i \neq \bar{\varphi}_k$;

$\longmapsto F'_\sigma$, the reflex field of (F_σ, Φ_σ);

5. Function field $K(th) = K(th_1/th_2, th_1/th_3, \ldots, th_3/th_4)$;

6. Number fields $K(th(\sigma)) = K(th)(\sigma) = K(f(\sigma);\ f \in K(th))$, where we neglect to adjoint $f(\sigma)$, if $f(\sigma) = \infty$;

7. Symmetric group S_4 acting on $K(th)$ via permutation of indices at the generators th_i/th_j,

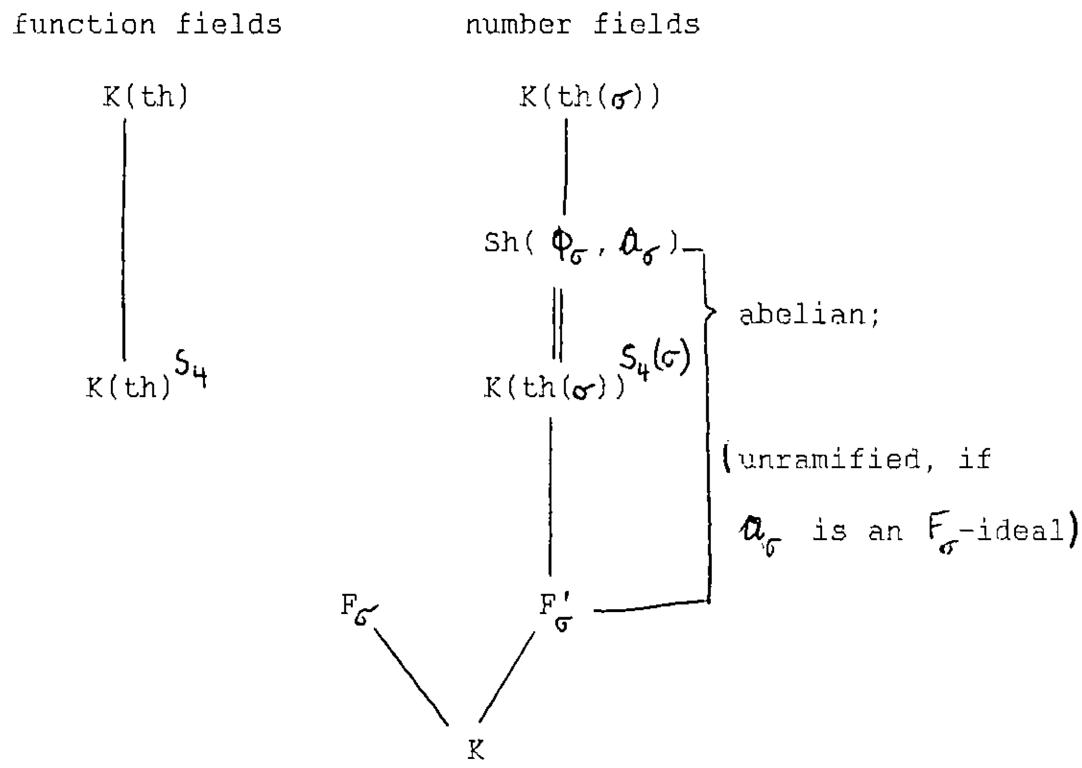
$S_4(\sigma) = S_4 \cap Gal(K(th(\sigma))/\mathbb{Q})$ acting on $K(th(\sigma))$;

8. Shimura class fields $Sh(\Phi_\sigma, \mathfrak{a}_\sigma)$.

Now we can formulate the

MAIN THEOREM (Construction of Shimura class fields for cubic extensions of Eisenstein numbers via special values of Theta constants).

With the above notations one has for each CM-module $\sigma \in \mathbb{B}$ field towers as described in diagram (1.1):



(1.1)

2. Shimura Class Fields.

We follow the book [20] of S.Lang. First we have to introduce the reflex fields. Fixing notations we let F be a totally imaginary number field of absolute degree 2g and Φ a choice of g embeddings $\varphi_i : F \hookrightarrow \mathbb{C}$ pairwise not cojugated to each other. We write $\Phi = \Phi_F = \sum_{i=1}^{g} \varphi_i$ and call the pair (F, Φ) a CM-type.

If M/F is a finite field extension, then we can lift Φ_F to

$$\Phi_M = \sum_{i=1}^{g} \sum \{\text{all extensions of } \varphi_i \text{ to } M\}$$

So we get a CM-type lifting $(F, \Phi_F) \longmapsto (M, \Phi_M)$. We set

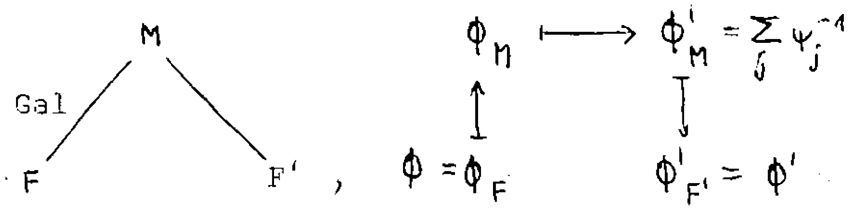
(2.1) $$\mathrm{Stab}(\Phi) = \{\mu \in \mathrm{Aut}(\mathbb{C});\ \mu\circ\Phi = \Phi\},$$

where Aut($\mathbb{C}$) denotes the group of all field automorphisms of $\mathbb{C}$. Now assume that M/F as above is a Galois extension. Then we define the r e f l e x f i e l d F' of (F, Φ) as fixed field

(2.2) $$F' = M^{\mathrm{Stab}\,\Phi} = \mathbb{Q}(\mathrm{Tr}_{\Phi}(F)),$$

where $Tr_\Phi : F \longrightarrow F'$ denotes the type trace defined by $Tr_\Phi(f) = \sum_{i=1}^{g} \varphi_i(f)$. With $\Phi_M = \sum_j \psi_j$, ψ_j understood as automorphisms of M, we set $\Phi'_M = \sum_j \psi_j^{-1}$. One can show that the type (M, Φ'_M) is the lift of a uniquly determined primitive type (F', Φ'), which is called the r e f l e x t y p e of (F, Φ). A type is called p r i m i t i v e , if it is not lifted from a lower field. If the starting type (F, Φ) is primitive, then the reflex (F'', Φ'') of its reflex (F', Φ') coincides with (F, Φ). In general, the double reflex field F'' is contained in F. Altogether we describe the situation in the following diagram (2.3):

(2.3)



Fixing F, Φ , the t y p e n o r m N_Φ or the r e f l e x n o r m $N' = N_{\Phi'}$ are respectively defined by

$$N_\Phi : F \longrightarrow F' \quad , \quad N' = N_{\Phi'} : F' \longrightarrow F'' \subseteq F$$
$$f \longmapsto \prod_{i=1}^{g} \varphi_i(f)$$

Both, N_Φ and N', can be extended to the idele groups of the fields F or F', respectively:

$$N_\Phi : \mathbb{A}_F^* \longrightarrow \mathbb{A}_{F'}^* \quad , \quad N' : \mathbb{A}_{F'}^* \longrightarrow \mathbb{A}_F^*$$

Now we are well-prepared to define the Shimura class fields mentioned several times above. For this purpose we let $\mathfrak{a}$ a $\mathbb{Z}$-lattice in F. The absolute norm of ideles s is denoted by $\mathbb{N}(s)$. Now we define the idele group $U(\phi, \mathfrak{a}) \subseteq \mathbb{A}^*_{F'}$ of an (extended) type $(F, \phi, \mathfrak{a})$ by

(2.4) $$U(\phi, \mathfrak{a}) = \{s \in \mathbb{A}^*_{F'} ;\ N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a},\ \beta\bar{\beta} = \mathbb{N}(s^{-1}) \in \mathbb{Q} \text{ for a suitable } \beta \in F\}$$

We remark that the multiplication of an idele $t \in \mathbb{A}^*_F$ with $\mathfrak{a}$ is defined componentwise on the finite part $t_{fin} = (t_p) \in \mathbb{A}^*_{F,fin}$: There is a unique $\mathbb{Z}$-lattice $t\mathfrak{a}$ in F with local components $(t\mathfrak{a})_p = t_p\mathfrak{a}_p$ for all $p \in \operatorname{Spec} \mathbb{Z}$, $\mathfrak{a}_p = \mathbb{Z}_p \otimes \mathfrak{a}$.

Now we apply global abelian class field theory in order to define $Sh(\phi, \mathfrak{a})$ as class field of the reflex field F'. For details we refer to the monograph [25] of Neukirch.

Let M be a finite abelian field extension of F'. Then there is an exact sequence

(2.5)

$$1 \longrightarrow U/F'^{\times} \longrightarrow \mathbb{A}^*_{F'}/F'^{\times} \longrightarrow \operatorname{Gal}(M/F') \longrightarrow 1$$

where $(t, M/F')$, $t \in \mathbb{A}^*_{F'}$, is the global norm rest symbol locally defined by Frobenius automorphisms. The idele group $U = U_M$ is equal to the extended norm group $N_{M/F'}(\mathbb{A}^*_M)\cdot F'^{\times}$. Conversely, if U is a cofinite subgroup of the idele group A of F' containing $F'^{\times}$, then there exists a unique finite abelian extension $M_U = M$, the class field of F belonging to U, such that the above sequence (2.5) is exact. So there is a biunivoque correspondence $U_M = U \longleftrightarrow M = M_U$ (reciprocity).

Now we take the projective limit of our finite abelian groups Gal(M/F') along all finite abelian extensions M of F'. On this way we obtain the Galois group $\operatorname{Gal}(F'^{ab}/F')$ of the maximal abelian extension F'^{ab} of F'. The norm rest maps in (2.5) yields an injective map $(.,F')\colon \mathbb{A}^*_{F'}/F'^{\times} \longrightarrow \mathrm{Gal}(F'^{ab}/F')$. Via the norm rest symbols (s,F') the idele group $\mathbb{A}^*_{F'}$ acts on F, and the finite abelian extension fields M appear as fixed fields of the corresponding subgroups U of $A^*_{F'}$. So we can write

$$(2.6) \qquad M_U = (F'^{ab})^{(U,F')} = \mathbb{C}^{\widetilde{(U,F')}} ,$$

where $\widetilde{(U,F')}$ denotes the group of all extensions of elements $(s,F') \in \mathrm{Gal}(F'^{ab}/F')$, $s \in U$, to automorphisms of $\mathbb{C}$.

2.7 Definition. The Shimura class field $Sh(\Phi,\mathfrak{a})$ of the type $(F,\Phi,\mathfrak{a})$ is the class field of F' corresponding to $U(\Phi,\mathfrak{a})$ defined in (2.4):

$$Sh(\Phi,\mathfrak{a}) = (F'^{ab})^{(U(\Phi,\mathfrak{a}),F')} = \mathbb{C}^{\widetilde{(U(\Phi,\mathfrak{a}),F')}} .$$

3. Complex multiplication.

Let A be an abelian variety over the complex numbers and F a finite field extension of Q. We say that A has F-multiplication, if there is an embedding $: F \hookrightarrow \mathbb{Q} \otimes \mathrm{End}\, A$ into the endomorphism algebra $\mathbb{Q} \otimes \mathrm{End}\, A$ of A. In this case the degree $[F:Q]$ is not greater than 2g, $g = \dim A$. If $[F:\mathbb{Q}] = 2g$, then we say that A has complex multiplication. In this case F acts on the tangent space T_A of A (at 0). So we have an embedding $F^{\times} \hookrightarrow \mathrm{Gl}(T_A) = \mathrm{Gl}\,(\mathbb{C})$. This representation splits into g one-dimensional representations. The corresponding characters are understood as embeddings $\varphi_i : F \hookrightarrow \mathbb{C}$, $i = 1, \ldots, g$. Setting $\Phi = \sum_{i=1}^{g} \varphi_i$ we get a type (F,Φ). Then A is called an abelian variety of type

(F, Φ).

One can prove that an abelian variety with complex multiplication of type $(F, \Phi, \mathfrak{a})$ is isomorphic to $\mathbb{C}^g/\Phi(\mathfrak{a})$, where $\Phi(\mathfrak{a})$ is the $\mathbb{Z}$-lattice in $\mathbb{C}^g$ generated by the vectors

$$\Phi(a_j) = {}^t(\varphi_1(a_j), \ldots, \varphi_g(a_j)), \; j = 1, \ldots, 2g$$

and $a_1, \ldots, a_{2g}$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$. For $f \in F$ we let $D_\Phi(f)$ the diagonal matrix with the elements $\varphi_i(f)$ in the diagonal. The representation $D_\Phi : F^\times \longrightarrow Gl_g(\mathbb{C})$ defines a complex multiplication on $\mathbb{C}^g/\Phi(\mathfrak{a})$. Under the isomorphism with A both multiplications are compatible. The m u l t i p l i c a t i o n r i n g is:

$$\mathcal{O} = F \cap \operatorname{End} A = [\mathfrak{a} : \mathfrak{a}]_F = \{f \in F;\; f \cdot \mathfrak{a} \subseteq \mathfrak{a}\}.$$

Refining our language we will say that A is of type $(\mathcal{O}, F, \Phi)$ and we will call $\mathbb{C}^g/\Phi(\mathfrak{a})$ together with the complex multiplication defined by D_Φ the s t a n d a r d t o r u s m o - d e l of A or of the types $(F, \Phi, \mathfrak{a}), (\mathcal{O}, \Phi, \mathfrak{a})$. Two standard models of type $(\mathcal{O}, \Phi)$ are isomorphic (with compatibility of multiplication) if and only if the corresponding $\mathcal{O}$-modules $\mathfrak{a}, \mathfrak{a}'$ are isomorphic. The standard torus $\mathbb{C}^g/\Phi(\mathfrak{a})$ of given multiplication type $(F, \Phi, \mathfrak{a})$ is an abelian variety iff Φ is lifted from a primitive type. For proofs we refer to [20] again.

Let F be a number field and $\mathcal{O}$ an order in F (multiplicatively closed $\mathbb{Z}$-lattice in F). We denote by $L(\mathcal{O})$ the set of $\mathcal{O}$-lattices $\mathfrak{a}$ in F ($\mathbb{Z}$-lattices in F which are $\mathcal{O}$-modules) with factor ring $[\mathfrak{a} : \mathfrak{a}]_F = \mathcal{O}$. The set $cl(\mathcal{O})$ of $\mathcal{O}$-isomorphy classes of $L(\mathcal{O})$ is finite (see e.g. [3], II.6, Th.3). Its number is denoted by $h(\mathcal{O})$.

For instance, if $\mathcal{O} = \mathcal{O}_F$ is the ring of integers in F, then $L(\mathcal{O})$ is the ideal group of F, $cl(\mathcal{O})$ the ideal class group and $h(\mathcal{O})$ coincides with the class number h(F) of F. Altogether we can count now the isomorphy classes of abelian CM-varieties of given types:

3.1 Propositon. Let (F, Φ) be a complex multiplication type, $\mathcal{O}$ an order in F. The number of isomorphy classes of abelian varieties with complex multiplication of type $(\mathcal{O}, \Phi)$ is equal to $h(\mathcal{O})$, if (F, Φ) is a lifted type, or equal to 0, otherwise.

The number of isomorphy classes of abelian CM-varieties A with CM-ring $\mathcal{O} = F \cap \text{End } A$ is equal to $h(\mathcal{O}) \cdot l(F)$, where $l(F)$ denotes the number of lifted types (F, Φ). It cannot be greater than $2^g h(\mathcal{O})$, $g = [F:\mathbb{Q}]/2 = \text{rank}_{\mathbb{Z}}(\mathcal{O})/2 = \dim A$.

3.2 Remarks. "Lifted type" means: lifted from a primitive type. Especially, primitive types are understood as special cases of lifted types. The number fields E of primitive types are well-understood. These are the so-called C M - f i e l d s characterized as imaginary quadratic extensions of totally real number fields. So the set of multiplication fields of complex CM-varieties coincides with the set of extensions of CM-fields. Namely, each CM-type (E, Ψ) of a CM-field E is lifted from a primitive type (see [10], I.2, Lemma 2.2), hence E is a multiplication field of an abelian CM-variety by the above torus construction. This is true for any lifted type. Especially, each cubic extension F of $K = \mathbb{Q}(\sqrt{-3})$ appears as multiplication field of a suitable abelian CM-threefold.

Each abelian variety A can be decomposed up to isogeny into a product of simple abelian varieties. *Simple abelian varieties* are defined as indecomposable ones in this sense. If A is an abelian CM-variety, then the isogeny decomposition of A into simple abelian varieties is a power $A \approx B \ldots B$. The multiplication type (F, Φ) of A is lifted from a (uniquely determined) multiplication type (E, Ψ) of B. Especially, the simple factor B of A is a CM-variety. The corresponding CM-algebra $\mathbb{Q} \otimes \mathrm{End}\, B$ is isomorphic to E. Looking back to A one checks easily that the CM-algebra $\mathbb{Q} \otimes \mathrm{End}\, A$ is isomorphic to the matrix algebra $\mathrm{Mat}_s(E)$, where s denotes the number of the decomposing factors B of A.

4. Moduli fields.

Let X be a complex projective variety (subvariety of $\mathbb{P}^N$, say), $\mu \in \mathrm{Aut}(\mathbb{C})$ a field automorphism of $\mathbb{C}$. Applying μ to point coordinates we obtain the μ-transform X^μ of X embedded also in $\mathbb{P}^N$. If X is defined by the homogeneous equation system $F_1 = \ldots = F_m = 0$, then X^μ is defined by $F_1^\mu = \ldots = F_m^\mu = 0$, where F_i^μ arises from F_i by applying μ to all the coefficients of the polynomial F_i. We denote by cl(X) the the class of models X' of X. The projective variety X' is a model (or $\mathbb{C}$-model) of X, if it is isomorphic to X in the analytic category (that means over $\mathbb{C}$). If the model X' of X is defined over the subfield k of $\mathbb{C}$, then we call X' a *k-model* of X. This means that X' is defined by equations with coefficients in k. In this case k is called a *definition field* of cl(X) (or of X, if X is understood as scheme without specification of embedding). In arithmetic geometry one looks for small fields of definition.

The correspondence $X \longmapsto X^\mu$ is functorial: For each rational map $f: X \longrightarrow Y$ of complex projective varieties the transform $f^\mu: X^\mu \longrightarrow Y^\mu$ is well-defined via μ-transformation of constants. On this way we can correctly define $cl(X)^\mu$ by representants. We set

$$\mathrm{Stab}\, cl(X) = \{\mu \in \mathrm{Aut}(\mathbb{C});\ X^\mu \cong X\}.$$

4.1 Definition. The fixed field of Stab cl(X) in $\mathbb{C}$ is called the *moduli field* of X (or cl(X)). It is denoted by

$$M(X) = M(cl(X)) = \mathbb{C}^{\mathrm{Stab}\, cl(X)} .$$

We come back now to complex abelian varieties A with complex multiplication of type (F, Φ), say. For $\mu \in \mathrm{Aut}(\mathbb{C}]$ the μ-transform A^μ of A is also an abelian variety. The endomorphism rings of A and A^μ are isomorphic by functoriality of μ. The isomorphism extends to the algebras of complex multiplication. Therefore A^μ has F-multiplication, too. Looking at the representations on the tangent spaces it is easy to see that A^μ is of type $(F, \mu\circ\Phi)$. So the type doesn't change if and only if μ belongs to $\mathrm{Aut}(\mathbb{C}/F')$ by definition (2.2) of the reflex field F'. We come to the first comparision playing a role in the Main Diagram (1.1).

4.2 Lemma. For a complex CM-variety of type (F, Φ) with reflex field F' and moduli field M(A) it holds that $F' \subseteq M(A)$.

Proof. It suffices to check that Stab cl(A) $\leq$ Stab(Φ). If μ stabilizes cl(A), then the representations of F in the tangent spaces T_A or T_{A^μ}, respectively, are equivalent. Therefore A^μ has the same type (F,Φ) as A, hence $\mu \in$ Stab(Φ), which was to be proved.

In the theory of abelian varieties it is useful to specify classes of projective embeddings translated to the internal geometry of A. A *polarized* abelian variety is a pair $(A,\mathfrak{C})$ consisting of an abelian variety A and a $\mathbb{Q}$-line in $\mathbb{Q}\otimes \mathrm{Pic}^a(A)$ containing an ample divisor class; $\mathrm{Pic}^a(A)$ denotes the group of algebraic equivalence classes of divisors on A. We say that $(A,\mathfrak{C})$ is defined over k, if A is and if $\mathfrak{C}$ can be represented by an ample divisor C defined over k by the k-embedding of A used just before. If A is defined over k, then we can find a polarization $\mathfrak{C}$ of A also defined over k. Namely, choose an ample divisor D on A defined over the algebraic closure $\bar{k}$ of k. It is really defined already over a finite Galois extension of k. The sum of all Galois conjugates of D represents obviously a polarization $\mathfrak{C}$ defined over k.

In obvious manner one introduces the μ-transforms $(A,\mathfrak{C})^\mu$ for $\mu \in \mathrm{Aut}(\mathbb{C})$, $\mathrm{cl}(A,\mathfrak{C})^\mu$, Stab cl$(A,\mathfrak{C})$ and the *moduli-field* of a polarized abelian variety

$$M(A,\mathfrak{C}) = M(\mathrm{cl}(A,\mathfrak{C})) = \mathbb{C}^{\mathrm{Stab}\,\mathrm{cl}(A,\mathfrak{C})} .$$

4.3 Lemma. Let k be a definition field of the polarized abelian variety $(A,\mathfrak{C})$. Then it holds that $M(A,\mathfrak{C}) \subseteq k$.

Proof (see [36], I.4.2, Prop.14). For $\mu \in Aut(\mathbb{C}/k)$ we have an obvious isomorphism $(A, \mathcal{C})^{\mu} \cong (A, \mathcal{C})$, hence $\mu \in Stab(A, \mathcal{C})$. The rest is clear. ∎

The next result prepares the drawing-up of the field tower on the right-hand side of the main diagram (1.1).

4.4 Proposition. Let A be a complex CM-variety. Then we can choose algebraic number fields k as definition fields of A or $(A, \mathcal{C})$, respectively. For each such field one has the field tower

(4.5) $$F' \subseteq M(A) \subseteq M(A, \mathcal{C}) \subseteq k \subseteq \bar{\mathbb{Q}},$$

where F' is the reflex field of the type (F, Φ) of A.

Proof. The existence of a small definition field $k \subseteq \bar{\mathbb{Q}}$ has been verified by Shimura-Taniyama in [36]. The second inclusion follows from Stab cl(A) $\supseteq$ Stab cl$(A, \mathcal{C})$. The remaining inclusions come from the Lemmas 4.2 and 4.3. ∎

5. Main theorem of complex multiplication.

We want to connect the moduli field of a polarized CM-variety $(A, \mathcal{C})$, A of type $(F, \Phi, \mathfrak{a})$ with the Shimura class field of the same type introduced in 2. For this purpose we refine the notion of types again taking into account the polarization. Via projectively embedding Theta functions one corresponds to the polarization $\mathcal{C}$ a (unique, up to $\mathbb{Q}^{\times}$-multiplication,) Riemann form $E: T_A \times T_A \longrightarrow \mathbb{C}$ ($\mathbb{R}$-bilinear, scew-symmetric, non-degenerate with rational values on $\Phi(\mathfrak{a} \times \mathfrak{a})$). It is useful to choose a basic form E of this class. It takes integral values on $\Phi(\mathfrak{a}\times\mathfrak{a})$ and is not an integral multiple of a form of the same kind. With these notations the polarized abelian CM-variety $(A,\mathcal{C})$ is said to be of *type* $(F,\Phi,\mathfrak{a},E)$. If there is no danger of misunderstandings, then we identify A with its standard torus model $\mathbb{C}^g/\Phi(\mathfrak{a})$, see 3. Since $F = \mathbb{Q}\otimes\mathfrak{a}$ the embedding Φ of F into $\mathbb{C}^g$ induces an embedding $F/\mathfrak{a} \longrightarrow A_{tor}$ into the torsion points of $A(\mathbb{C})$. We will denote this embedding shortly also by Φ.

The Riemann form E is said to be Φ*-admissible*, if $E(D_\Phi(f)z,w) = E(z,D_\Phi(\bar{f})w)$ for all $f \in F$, $z,w \in \mathbb{C}^g = T_A$. In this case also the polarization $\mathcal{C}$ corresponding to E is called admissible. From now on we assume that the multiplication field F is a CM-field; A needs not to be simple. Then there exists an admissilble polarization on A (see [20], I.4, Thm. 4.5). We will work only with polarized abelian CM-varieties of admissible type $(F,\Phi,\mathfrak{a},E)$. The following *Main Theorem of complex multiplication* holds for them:

5.1 Theorem ([20], III.6). With the above assumptions and notations let $\mu \in \mathrm{Aut}(\mathbb{C}/F')$ with restriction $\mu|F'^{ab} = (s,F')$ for a suitable $s \in \mathbb{A}^*_{F'}$. Then it holds that:

(i) $(A,\Phi)^\mu$ is of type $(F,\Phi,N'(s^{-1})\mathfrak{a},\mathbb{N}(s)E)$.

(ii) With the componentwise action of the finite part of the idele $N'(s^{-1}) \in \mathbb{A}^*_F$ on $F/\mathfrak{a} \cong \bigoplus_{\mathfrak{p}} F_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ the following diagram is commutative:

$$(5.2)\qquad \begin{array}{ccc} F/\mathfrak{a} & \longrightarrow & A_{tor} \\ {\scriptstyle N'(s^{-1})}\downarrow & & \downarrow {\scriptstyle \mu \in \widetilde{(s,F')}} \\ F/N'(s^{-1})\mathfrak{a} & \longrightarrow & A^{\mu}_{tor} \end{array}$$

As an immediate consequence we get the relation with moduli fields.

5.3 Theorem (Shimura [36]; [35], V.5.5; see also [20], V.4). Let $(A,\mathcal{C})$ be a polarized abelian variety of admissible CM-type $(F,\Phi,\mathfrak{a},E)$. Then the corresponding Shimura class field and moduli field coincide:

$$\mathrm{Sh}(\Phi,\mathfrak{a}) = M(A,\mathcal{C}).$$

Proof. We remember that $\mathrm{Sh} = \mathrm{Sh}(\Phi,\mathfrak{a}) = \mathbb{C}^{\widetilde{(U,F')}}$, $U = U(\Phi,\mathfrak{a})$, see (2.6). For $M = M(A,\mathcal{C})$ we first show that $M \subseteq \mathrm{Sh}$. This follows immediately from

$$(5.4)\qquad \widetilde{(U,F')} \subseteq \mathrm{Stab\ cl}(A,\mathcal{C}).$$

So we take an automorphism $\mu \in \widetilde{(s,F')}$ for $s \in U$. By the Main Theorem of complex multiplication 5.1 (i) the μ-transform $(A,\mathcal{C})^{\mu}$ is of type $(F,\Phi,N'(s^{-1})\mathfrak{a},\mathbb{N}(s)E)$. By definition of U in (2.4) there is a $\beta \in F$ such that $N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a}$ and $\mathbb{N}(s^{-1}) = \beta\bar{\beta} \in \mathbb{Q}^{\times}$. Comparing the standard torus models of $(A,\mathcal{C})$ and $(A^{\mu},\mathcal{C}^{\mu})$ we get

$$A \cong \mathbb{C}/\Phi(\mathfrak{a}) \cong \mathbb{C}/\Phi(\beta\mathfrak{a}) = A^{\mu},\quad \mathbb{N}(s)E = (\beta\bar{\beta})^{-1}\cdot E \in \mathbb{Q}^{\times}\cdot E.$$

Therefore $(A,\mathcal{C}) \cong (A,\mathcal{C})^{\mu}$, hence $\mu \in \mathrm{Stab\ cl}(A,\mathcal{C})$.

Conversely, take $\mu \in \mathrm{Stab\ cl}(A,\mathcal{C})$; then we dispose on an isomorphism $(A,\mathcal{C}) \overset{\simeq}{\dashrightarrow} (A,\mathcal{C})^{\mu}$. On the torsion level it has been made precise multiplying $F/\mathfrak{a}$ by $N'(s^{-1})$, $\mu \in \widetilde{(s,F')}$, see 5.1 (ii). On the other hand A and A^{μ} have equivalent standard torus models. More precisely, the isomorphism $A \xrightarrow{\sim} A^{\mu}$ corresponds to a β-multiplication $\mathbb{C}^{g}/\Phi(\mathfrak{a}) \xrightarrow{\sim} \mathbb{C}^{g}/\Phi(\beta\mathfrak{a})$ for a suitable $\beta \in F^{\times}$. But $(A,\mathcal{C})^{\mu}$ is of type $(F, \Phi, N'(s^{-1})\mathfrak{a}, \mathbb{N}(s)E)$ by 5.1 (i). Comparing both presentations we get

$$N'(s^{-1})\mathfrak{a} = \beta\mathfrak{a}, \quad \mathbb{N}(s^{-1}) = \beta\bar{\beta} \in \mathbb{Q}^{\times}.$$

Therefore s belongs to U and $\mu \in \widetilde{(s,F')}$, hence

(5.5) $$\operatorname{Stab} cl(A,\mathcal{C}) \subseteq (U,F')$$

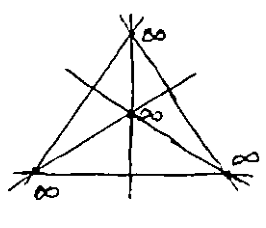
in contrast to (5.4). It follows that $Sh \subseteq M$. The identity we looked for is proved. ∎

6. The geometric starting point, the projective plane covered by the ball.

In order to generate Shimura class fields of cubic extensions F of the Eisenstein numbers by special values of transcendent functions we need special Theta constants essentially defined by certain special functional equations. In order to find and understand them deeply we enclose them into the most general and actual framework hoping for similar applications to other interesting cases in the future. We want also justify Hilbert's imagination about the "innigste Beruehrung". Consider the picture/diagram

$$\begin{array}{ccc} \mathbb{H} & & \mathbb{B} \\ j \downarrow Sl(\mathbb{Z}) & & \pi \downarrow \Gamma' \\ \mathbb{P}^{1} & & \mathbb{P}^{2} \end{array} \qquad (6.1)$$

branch locus: ∞, 2, 3



The left-hand side is well-known from the theory of elliptic curves. It describes the quotient map of the modular group from Poincaré's upper half plane $\mathbb{H}$: Im z > 0 to the projective line $\mathbb{P}^1(\mathbb{C})$. There are three branch points: two in the ordinary sense with ramification indices 2 or 3, respectively. The third is a cusp point coming from the boundary of $\mathbb{H}$. therefore it has been weighted by ∞ . The quotient map can be realized by the elliptic modular function

$$j(\tau) = q^{-1} + 744q^0 + 196884q + \sum_{n=2}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}, \ a_n \in \mathbb{Z}.$$

We are well-prepared for the understanding of the right-hand side of (6.1) by the previous sections and chapters: We looked for a ball covering of the projective plane $\mathbb{P}^2$ with discrete covering group $\Gamma' \subset \mathrm{Aut}_{hol}(\mathbb{B}) = \mathbb{P}U((2,1),\mathbb{C})$ branched precisely along the six lines of the complete quadrilateral with triple points as cusp points. We denote by π the corresponding (analytic) quotient map.

6.2 Theorem. Up to linear isomorphy ($\mathbb{P}U((2,1),\mathbb{C})$-conjugation for Γ') there exists one and only one such covering.

The uniqueness has been proved in [17], IV.11 via orbital heights, the proportionality conditions and their translation into a solvable system of diophantine equations. Moreover, the proportionality test is positive: The only solution of the diophantine equation system yields the ramification index 3 for all six lines. The most general result providing the existence of the covering is due to R.Kobayashi [18]. He proved under geometric conditions including the wights found in our proportionality test the existence of a ball covering over a given surface with prescribed branch locus. This general result can be applied to our situation described in (6.1). We will not use Kobayashi's result because the existence of π has been proved by another more arithmetic method in the framework of classification of Picard modular surfaces, see Prop.V.1.3 in [17] or [12], where we started from the arithmetic group (congruence Eisenstein lattice) $\Gamma' = \Gamma(\sqrt{-3})$, $\Gamma = U((2,1),\mathcal{O}_K)$. The proof involved orbital hights calculated as volumes of a fundamental domain by means of a special L-series value (see [15]). The advantage is to dispose explicitly on the discrete group of the covering π. This will be important for finding the functional equations for Theta constants we look for.

6.3 Remark. The covering problem is related with Hilbert's 22-th problem "Uniformisierung analytischer Beziehungen mittels automorpher Funktionen". Looking for uniformizations of two-dimensional analytic varieties ("Gebilde") Hilbert says: "Vielmehr scheinen ...., abgesehen von den Verzweigungspunkten, noch gewisse andere, im allgemeinen unendlich viele diskrete Stellen des vorgelegten analytischen Gebildes ausgenommen zu sein, zu denen man nur gelangt, indem man die neue Variable gewissen Grenzstellen der Funktionen naehert. Eine Klaerung und Loesung dieser Schwierigkeit scheint mir in Anbetracht der fundamentalen Bedeutung der Poincareschen Fragestellung aeusserst wuenschenswert". At the end Hilbert refers to: ... "die neueren Untersuchungen von Picard ueber algebraische Funktionen von zwei Variablen als willkommene und bedeutsame Vorarbeiten ..."

7. Differential equations.

In [38] M.Yoshida succeeded to solve a higher-dimensional version of the Riemann-Hilbert problem. The background is Hilbert's 21-st problem "Beweis der Existenz linearer Differentialgleichungen mit vorgeschriebener Monodromiegruppe" set up for functions of one variable, "...welches darin besteht zu zeigen, dass es stets eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulaeren Stellen und einer gegebenen Monodromiegruppe gibt". It should be remarked that the final solution of this Hilbert problem has been given by H. Roehrl [28] in 1957.

7.1 Theorem (M.Yoshida). Let X be an orbifold (complex manifold with prescribed wighted branch locus) with realizing quotient map $p: \mathbb{B}^n \longrightarrow X$, $\mathbb{B}^n$ the n-dimensional complex ball, and covering group $\Delta \subset U((n,1),\mathbb{C})$. Then the inverse p^{-1} of p is a (multivalued) developing map of a Fuchsian system of linear partial differential equations. ∎

This means that there is locally a fundamental system of solutions $I_0, I_1, \ldots, I_n$ extending analytically to $X \setminus B$, B the branch locus of p, such that the multivalued map

$$(I_0 : I_1 : \ldots : I_n) : X \setminus B \dashrightarrow \mathbb{B}^n \subset \mathbb{P}^n$$

$P \longmapsto (I_0(P): \dots :I_n(P))$, coincides with p^{-1} on $X \setminus B$. The Fuchsian system is called the u n i f o r m i z i n g e q u a t i o n of the orbifold and Δ is the m o n o d r o m y g r o u p of the system not depending on the special choice of solutions of the system. Especially in our situation described in (6.1) with $n = 2$, $B =$  (quadrilateral), it is important to remark that there is a surjective group homomorphism

(7.2) $$\pi_1(\mathbb{P}^2 \setminus$$  $$) \longrightarrow\!\!\!\!\rightarrow \Gamma'$$

describing the unitary monodromy representation of the fundamental group $\pi_1(\mathbb{P}^2 \setminus$ $)$. Yoshida found also an effective method in order to determine a corresponding Fuchsian system (see [38], ch.s 10, 12). It turns out that these equations and also their analytic solutions (Appell series) are well-known long time ago. Working with affine coordinates u,v one can take the following system (7.3) of differential equations:

(7.3) $D_{ij} F(u,v) = 0$ on $\mathbb{C}^2 \setminus$  $= \mathbb{P}^2 \setminus$  with

$$D_{11} = \frac{\partial}{\partial u^2} + [9(u-1)u(v-u)]^{-1} \{3(-5u +4uv+3u-2v)\frac{\partial}{\partial u}+3(v-1)v\frac{\partial}{\partial v}+(u-v)\},$$

$$D_{12} = \frac{\partial^2}{\partial u \partial v} + [3(u-v)]^{-1} \{\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\},$$

$$D_{22} = \frac{\partial}{\partial v^2} + [9(v-1)v(u-v)]^{-1} \{3(u-1)u\frac{\partial}{\partial u}+3(-5v +4uv+3v-2u)\frac{\partial}{\partial v}+(u-v)\}.$$

7.4 Remark. Yoshida's general approach lifting the Gauss-Schwarz theory of Fuchsian equations to higher dimensions has a classical origin in the work of Picard and Appell. Especially for the situation of (6.1) a more immediate extension of explicit classical results known as PTDM-Theorem (due to Picard, Terada, Mostow, De-ligne) would be sufficient for our purposes. We refer to [38], [2] and further literature given there.

8. Gauss-Manin connection.

The analytic theory presents analytic solutions of the system (7.3) of partial differential equations. We look for "algebraic solutions" represented by integrals on algebraic curves along cycles depending on parameters u,v. The general framework of the corresponding algebraic theory is known as G a u s s - M a n i n c o n n e c t i o n of algebraic families of algebraic manifolds. We refer to [12] in order to understand the rather explicit approach for algebraic families of curves involving differential equations.

Let $\mathcal{C}/T$ be a smooth algebraic family of smooth algebraic varieties all defined over the complex numbers, say. The relative de Rham complex is a sequence

$$\Omega^{\cdot}_{\mathcal{C}/T}: \quad \mathcal{O}_T \xrightarrow{d} \Omega^1_{\mathcal{C}/T} \xrightarrow{d} \Omega^2_{\mathcal{C}/T} \xrightarrow{d} \cdots .$$

Using open (affine, say) coverings one defines the Čech complexes

$$C^{\cdot}(\Omega^q_{\mathcal{C}/T}): \quad C^0(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^1(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^2(\Omega^q_{\mathcal{C}/T}) \cdots$$

in the usual manner. Taking the limit along refinements of open coverings one gets the Čech - de Rham bicomplex $C^{\cdot\cdot}(\Omega_{\mathcal{C}/T})$. The de Rham cohomology groups $\mathbb{H}^i_{DR}(\mathcal{C}/T)$ of the family $\mathcal{C}/T$ are the hypercohomology groups of $C^{\cdot\cdot}(\Omega_{\mathcal{C}/T})$ defined as cohomology groups of the corresponding total Čech - de Rham complex $C^{tot}(\Omega_{\mathcal{C}/T})$. The construction applies to all restricted families $\mathcal{C}_U/U$, U an open part of T. On this way one gets the d e R h a m c o h o -

h o m o l o g y   s h e a v e s $\mathcal{H}^i_{DR}(\mathcal{C}/T)$ on T.

We restrict ourselves now to curve families $\mathcal{C}/T$. For our purposes it suffices to assume that T is an affine part of a projective space $\mathbb{P}^N$. Let $\mathcal{D}_T$ be the sheaf of differential operators on T. Then the de Rham cohomology sheaf $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ is not only an $\mathcal{O}_T$-module but also a $\mathcal{D}_T$-module sheaf. Looking for a family with a section $\bar{\omega}$ in $\mathcal{H}^1_{DR}(\mathcal{C}/T)$ satisfying the differential equations (7.3) with $\bar{\omega}$ instead of F one can take the P i c a r d c u r - v e f a m i l y

$$\mathcal{C}/\mathbb{P}^2 \setminus \mathbb{A} \; : \; Y^3 = X(X-1)(X-u)(X-v)$$

and $\bar{\omega}$ represented by the differential form $\omega = dx/y$ depending on u,v. For details we refer to [12], II, 1.5. Taking integrals over cycles one gets an "algebraic" fundamental system of solutions

(8.1)

$$I_k(t) = \int_{\alpha_k(t)} \omega(t), \; k = 1,2,3, \; t = (u,v) \in \mathbb{P}^2 \setminus \mathbb{A}, \quad \omega = dx/y$$

We refer to [   ], II.2.5, Theorem 2.5.2. Altogether we found the developing map of the Fuchsian system (7.3) in an explicit and algebraic manner. Looking back to the geometric starting point (6.1) and to the theorems 6.2 and 7.1 we receive

8.2 Theorem. The quotient map $\pi: \mathbb{B} \longrightarrow \mathbb{P}^2$ with covering group $\Gamma' = \Gamma(\sqrt{-3})$ is inverted by $(I_1 : I_2 : I_3): \mathbb{P}^2 \setminus \mathbb{A} \dashrightarrow \mathbb{B}$ on $\mathbb{P}^2 \setminus \mathbb{A}$ with cycloelliptic integrals $I_k(t)$ described in (8.1) along independent cycle families $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$. ∎

Another proof based on the PTDM-theorem can be found in [12], I.6.3. There has been used also a finer analysis of the Picard curve family, which is useful for our number theoretic ambitions. The next three sections are devoted to this theme.

9. Moduli space for Picard curves.

We investigate the Picard curve family in more detail. A curve C (algebraic, complex, compact) is called a Picard curve, if it is isomorphic to a plane projective curve C' of affine equation type $C': Y^3 = p_4(X)$, where $p_4(X)$ is a polynomial of degree 4 in X. We exclude subsequently curves C with model $C': Y^3 = X^4$ because they will get lost in our moduli space below. Via projective Tschirnhaus transformation any Picard curve has a model of equation type

(9.1)

$$Y^3 = \prod_{i=1}^{4} (X - e_i) = X^4 + G_2X^2 + G_3X + G_4 \qquad \text{(affine)},$$

$$WY^3 = \prod_{i=1}^{4} (X - e_iW) = X^4 + G_2W^2X^2 + G_3W^3X + G_4W^4 \qquad \text{(projective)},$$

$$\sum_{i=1}^{4} e_i = 0.$$

The corresponding equations are called normal forms of Picard curves. A Picard curve is smooth if and only if for one (each) of its normal forms (9.1) it holds that $e_i \neq e_j$ for $i \neq j$. We correspond to the normal form (9.1) the point

$$(e_1 : e_2 : e_3 : e_4) \in \mathbb{P}^2 = \mathbb{P}_0^3 = \{(z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3;\ \sum_{i=1}^{4} z_i = 0\}.$$

The following result is do to the author. We refer again to the monograph [12], ch.I, 5.2. It asserts that the correspondence

$$\text{Picard curve } C \longmapsto (e_1 : e_2 : e_3 : e_4) \text{ via normal form (9.1)}$$

is correctly defined (at least for smooth curves) up to symmetry interchanging zeros of the normal form polynomial $p_4(X)$ and that non-isomorphic (smooth) curves cannot be represented by the same point in $\mathbb{P}^2$. Obviously, the smooth Picard curves are represented by points not belonging to the six lines $e_i = e_j$, $i \neq j$.

9.2 Proposition. The above correspondence induces a bijective map

$$\{cl(C); C \text{ smooth Picard curve}\} \longleftrightarrow (\mathbb{P}^2 \setminus \Delta)/S_4 ,$$

where the symmetric group S_4 acts via natural permutations of the $\mathbb{P}^3_0$-coordinates. ∎

We remark that the proof given in [12] uses geometric invariants, for instance the Hessean of a homogeneous nomal form polynomial. We call the surface $\mathbb{P}^2/S_4$ the (compactified) moduli space of Picard curves and $(\mathbb{P}^2 \setminus \Delta)/S_4$ the moduli space of smooth Picard curves by a slight abuse of language.

10. The relative Schottky problem for Picard curves.

A smooth Picard curve C has genus 3. Therefore its Jacobian variety J(C) is an abelian threefold. We want to determine in an effective manner the polarized abelian threefolds, which are Jacobian varieties of Picard curves. The period lattice Λ of an abelian variety A is abstractly defined by the exact sequence

$$(10.1) \qquad 0 \dashrightarrow \Lambda \longrightarrow T_A \longrightarrow A \longrightarrow 0,$$

T_A the tangent space of A (at 0). Choosing coordinates and a basis of Λ we identify T_A with $\mathbb{C}^g$, and Λ is generated by the columns of a $g\times 2g$-matrix Π called a *period matrix* of A. Taking into consideration all possible base changes in T_A and Λ we see that a period matrix of A is unique up to $Gl_g(\mathbb{C})$-multiplication from the left and $Gl_{2g}(\mathbb{Z})$-multiplication from the right-hand side. Let E be a primitive Riemann form on T_A representing a polarization of A. By a theorem of Frobenius there exists a $\mathbb{Z}$-basis $\lambda_1, \dots, \lambda_{2g}$ of Λ such that

(10.2)

$$(E(\lambda_i,\lambda_j)) = \begin{pmatrix} 0, & D \\ -D, & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_g \end{pmatrix}, \quad d_i \in \mathbb{N},\ 1=d_1 | d_2 | \dots | d_g .$$

If the above diagonal matrix D is the unit matrix E_g, then the corresponding polarization is called *principal*. We will only consider principally polarized abelian varieties in this section. A basis of Λ satisfying (10.2) with

$$I = \begin{pmatrix} 0, & E \\ -E, & 0 \end{pmatrix} = \begin{pmatrix} 0, & D \\ -D, & 0 \end{pmatrix}$$

is called a *normal basis*. A period matrix of a principally polarized abelian variety is always constructed by use of a normal basis of Λ. For a fixed basis of T_A it is uniquely determined up to right-multiplication with elements of the *symplectic* (modular) *group*

$$Sp(2g,\mathbb{Z}) = \{S \in Gl_{2g}(\mathbb{Z});\ SI\,{}^tS = I\}.$$

The lattice in $\mathbb{C}^g$ generated by the the colums of Π is denoted by Λ_Π. The coordinate version of (10.1) is the exact sequence

(10.3) $$0 \longrightarrow \Lambda_\Pi \longrightarrow \mathbb{C}^g \longrightarrow A \longrightarrow 0.$$

A matrix $\Pi \in Mat_{g\times 2g}(\mathbb{C})$ is called a R i e m a n n m a t r i x (of principal type), if it satisfies the following two conditions known as R i e m a n n 's f i r s t or s e c o n d r e l a t i o n , respectively:

(10.4) (R 1) $\Pi \cdot I \cdot {}^t\Pi = 0,$

(R 2) $i \Pi I {}^t\overline{\Pi} > 0$ (positive definit) , $i = \sqrt{-1}$.

For a proof we refer to [19], IV, App.I.

Now we turn our attention to (smooth) Picard curves and their (principally polarized) Jacobian threefolds. A period matrix of a Picard curve can be written as

(10.5)

$$\Pi = \int_{\mathcal{B}} \vec{\omega} = (\int_{\beta_j} \omega_i) = \begin{pmatrix} A_1 & A_2 & A_4 & A_3 & A_5 & A_6 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_4 & \overline{B}_3 & \overline{B}_5 & \overline{B}_6 \\ \overline{C}_1 & \overline{C}_2 & \overline{C}_4 & \overline{C}_3 & \overline{C}_5 & \overline{C}_6 \end{pmatrix}$$

where $\vec{\omega} = {}^t(\omega_1, \omega_2, \omega_3)$ is a basis of $H^0(C, \Omega_C) \cong H^1(C, \mathcal{O}_C)$ and $\mathcal{B} = (\beta_1, \beta_2, \beta_4, \beta_3, \beta_5, \beta_6)$ is a $\mathbb{Z}$-basis of the homology group $H_1(C,\mathbb{Z})$. The relative Schottky proplem for Picard curves asks for an effective criterion characterizing period matrices of Picard curves among all period matrices of (principally polarized) abelian threefolds. The idea of constructing t y p i c a l period matrices described below goes back to Picard [26]. In order to formulate the theorem we need a linear embedding

$$*: \mathbb{C}^3 \hookrightarrow \mathbb{C}^6$$

$$(A,B,C) \longmapsto (A,B,-\bar{\rho}A,C,\bar{\rho}B,\rho C) , \qquad \rho = e^{2\pi i/3} ,$$

and the hermitian form $\langle\ ,\ \rangle$ of signature (2,1) on $\mathbb{C}^3$ represented by the anti-diagonal matrix $R = \begin{pmatrix} 0 & 0 & \bar{\varsigma} \\ 0 & 1 & 0 \\ \varsigma & 0 & 0 \end{pmatrix}$. It defines at the same time explicitly a ball $\mathbb{B}$ in $\mathbb{P}^2$ and a disc $\mathbb{D} = \mathbb{D}_R \subset \mathbb{B}$:

$$\mathbb{B} = \{\mathbb{P}\mathfrak{a} \in \mathbb{P}^2;\ \langle \mathfrak{a}, \mathfrak{a} \rangle < 0\},$$

$$\mathbb{D} = \mathbb{D}_R = \{\tau = \mathbb{P}\mathfrak{a} \in \mathbb{B};\ R\cdot\tau = \tau\}.$$

10.6 Theorem (relative Schottky for Picard curves).
The matrix Π in (10.5) is a period matrix of a smooth Picard curve if and only if the following conditions are satisfied:

Π belongs to $\mathrm{Gl}_3(\mathbb{C})\cdot\begin{pmatrix} *\,\mathfrak{a} \\ \hline *\,\mathfrak{b} \\ \hline *\,\mathfrak{r} \end{pmatrix}\cdot \mathrm{Sp}(6,\mathbb{Z})$; $\langle \mathfrak{a}, \mathfrak{a} \rangle < 0$ (ball condition);

$\mathfrak{b}, \mathfrak{r}$ is a basis of $\mathfrak{a}^{\perp}$ (orthogonal condition); $\tau = \mathbb{P}\mathfrak{a}$ does not belong to $\bigoplus = \Gamma\cdot\mathbb{D}_R$ ("non-degenerate"-condition), where Γ denotes the full Eisenstein lattice

$$\Gamma = \mathrm{U}(\langle\ ,\ \rangle, \mathcal{O}_K) = \mathrm{U}((2,1), \mathcal{O}_K)\ ,\ \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\varsigma \subset K = \mathbb{Q}(\sqrt{-3}).$$

Proof (sketch). Let C be a smooth Picard curve with normal form (9.1). The projection $(x,y) \longmapsto x$ defines a cubic Galois covering $C \longrightarrow \mathbb{P}^1$ with Galois group $G \cong Z_3 = \mathbb{Z}/3\mathbb{Z}$ generated by g, say. The homology group $H_1(C,\mathbb{Z})$ is a $\mathbb{Z}[G]$-module. A normal basis $\mathfrak{b} = (\beta_1, \ldots, \beta_6)$ of $H_1(C,\mathbb{Z})$ is a $\mathbb{Z}$-basis satisfying $(\beta_i \circ \beta_j) = I$, $\circ$ the intersection product of cycles. A t y p i c a l basis of $H_1(C,\mathbb{Z})$ is a normal basis of the form

$$*\vec{\alpha} = *(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, -g\alpha_1, \alpha_3, g\alpha_2, g^2\alpha_3).$$

The existence of at least one typical homology basis on a smooth Picard curve has been proved by Picard himself in [26], see also Alezais [1]. A reproduced version can be found in Picture 6.3.A of ch.I in [12].

A G-(iso)typical basis of $H^1 \underset{Df}{=} H^0(C, \Omega_C)$, Ω_C the sheaf of regular (holomorphic) differential forms (of degree 1) on C, is a basis consisting of G-eigenvectors of H^1. One can take simultaneously for all Picard curves with normal form equation (9.1) the differential forms of first kind $\omega_1 = dx/y$, dx/y^2, xdx/y. The use of a typical basis $*\vec{\alpha}$ of $H_1(C,\mathbb{Z})$ and a isotypical basis $\vec{\omega} = {}^t(\omega_1, \omega_2, \omega_3)$ has the advantage that the corresponding period matrix (10.5) is essentially determined by the three entries A_1, A_2, A_3. With $\mathfrak{a} = (A_1, A_2, A_3)$ and the above notations one obtains typical period matrices

$$(10.7) \qquad \Pi = \int_{*\vec{\alpha}} \vec{\omega} = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{r}} \end{pmatrix}$$

We check now the conditions of Theorem 10.6. The first of them is satisfied because we can choose a typical period matrix. The second and third conditions are translations of Riemann's period relations (R 1), (R 2) in (10.4). The "non-degenerate"-condition is delegated to the if-part of proof below.

Conversely, we assume that a smooth curve C has a typical period matrix as described on the right-hand side of (10.7). We have to show that this can only happen for a Picard curve. For this purpose we look at the moduli space of smooth Picard curves $(\mathbb{P}^2\setminus\blacktriangle)/S_4$, see Prop. 9.2. We construct a commutative diagram

(10.8)
$$\begin{array}{ccccc} \mathbb{B}\setminus\oplus & & \overset{*}{\hookrightarrow} & & \mathbb{H}_3 \\ \Gamma\big\downarrow & \overset{\Gamma'}{\searrow}\ \mathbb{P}^2\setminus\blacktriangle & & & \big\downarrow Sp(6,\mathbb{Z}) \\ & \overset{S_4}{\swarrow} & & & \\ (\mathbb{P}^2\setminus\blacktriangle)/S_4 & \hookrightarrow & \mathcal{M}_3 & \hookrightarrow & \mathcal{A}_3 \end{array}$$

where $\mathcal{M}_3$ or $\mathcal{A}_3$ are the moduli spaces of smooth curves of genus 3 or principally polarized abelian threefolds, respectively. The Torelli embedding $\mathcal{M}_3 \hookrightarrow \mathcal{A}_3$ is represented by the correspondence $cl(C) \mapsto cl(J(C))$, C a curve of genus 3 and J(C) its Jacobian threefold. The upper row is correctly explained by the chain of the following (partly multivalued) correspondences:

(10.9)
$$\tau = \mathbb{P}\mathfrak{a} \mapsto \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{t}} \end{pmatrix} = (\Pi_1 \mid \Pi_2) \underset{\Pi_1^{-1}\cdot}{\longmapsto} (E_3 \mid \tau) \longmapsto \tau .$$

The point $*\tau \underset{Df}{=} \tau$ belongs to the generalized Siegel upper half plane of polarized abelian threefolds

$$\mathbb{H}_3 = \{ \tau \in Gl_3(\mathbb{C});\ {}^t\tau = \tau,\ \mathrm{Im}\,\tau > 0\}.$$

The vertical arrows in diagram (10.8) represent analytic quotient maps: The Eisenstein lattice $\Gamma = U((2,1),\mathcal{O}_K)$ acts on $\mathbb{B}$. This action can be extended to an action on $\mathbb{H}_3$ along the geometric embedding $*:\mathbb{B} \hookrightarrow \mathbb{H}_3$ by the following s y m p l e c t i c r e p r e p r e s e n t a t i o n of Γ also denoted by $*$:

(10.10)
$$\Gamma = U((2,1),\mathcal{O}_K) \overset{*}{\hookrightarrow} Sp(6,\mathbb{Z})$$

For an explicit definition we refer to [14]. More intrinsically one endows $\mathbb{C}^6$ with the hermitian product [,] of signature (3,3) represented by $J = I/\sqrt{-3}$. Its restriction to $\mathbb{C}^3$ along the $*$-embedding coincides with $\langle\ ,\ \rangle$ introduced above, that means:

$$[*\mathfrak{a}, *\mathfrak{b}] = \langle \mathfrak{a}, \mathfrak{b} \rangle \text{ for all } \mathfrak{a}, \mathfrak{b} \in \mathbb{C}^3.$$

The modular symplectic group $Sp(6,\mathbb{Z})$ consists of all linear transformations of $\mathbb{C}^6$, which are compatible with [,] and preserve $\mathbb{Z}^6$. All those elements of $Sp(6,\mathbb{Z})$ preserving additionally $*\mathbb{C}^3$ are collected in a group $Sp'(6,\mathbb{Z})$. Their pull backs to $\mathbb{C}^3$ are compatible with $\langle\ ,\ \rangle$ and preserve $\mathcal{O}_K^3$. One can check that the homomorphism $Sp'(6,\mathbb{Z}) \longrightarrow \Gamma$ defined on this way is an isomorphism. Its inverse map yields the embedding (10.10). Changing over from the right-action on vector rows to the transposed left-action on the the columns we notice that

$$\text{(10.11)} \qquad *(\gamma(\mathfrak{a})) = (*\gamma)(*\mathfrak{a}) \quad , \quad \mathfrak{a} \in \mathbb{C}^3, \ \gamma \in \Gamma \ ;$$
$$*(\gamma(\tau)) = (*\gamma)(*\tau) \quad , \quad \tau = \mathbb{P}\mathfrak{a} \in \mathbb{B} \ .$$

Since $\mathcal{A}_3 = \mathbb{H}_3/Sp(6,\mathbb{Z})$ it follows that the diagram (10.8) is commutative.

We are now able to verify the if-part of Theorem 10.6. Let C be a smmoth curve of genus 3 with a typical period matrix. The preimage of $cl(J(C)) \in \mathcal{A}_3$ in $\mathbb{H}_3$ is represented by an element uniquely determined up to $Sp(6,\mathbb{Z})$-multiplication. Without loss of generality we can assume that $\mathcal{T} = *\tau$ because of the correspondence (10.9) going through all typical period matrices. Thus

J(C) must be the (generalized) Jacobian threefold of a possibly degenerate (non-smooth) Picard curve. But C is smooth of genus 3. Therefore C has to be a smooth Picard curve. It remains to check that τ does not belong to $\diamondsuit$. In 9. we proved that $\mathbb{P}^2 \setminus \triangle$ is the complete S_4-preimage of the moduli space of smooth Picard curves. The symmetric group S_4 appears as factor group Γ / Γ' in diagram (10.8). For the coincidence proof of the symmetric action of S_4 on $\mathbb{P}^2$ and the arithmetic action of Γ / Γ' on $\mathbb{P}^2$ we refer to [12]. We dispose also on the knowledge of the preimage of $\triangle$ in $\mathbb{B}$. This is the branch locus of the covering $\pi: \mathbb{B} \longrightarrow \mathbb{B}/\Gamma'$ we started with in (6.1), see also Theorem 6.2 and the explanation thereafter. The ramification locus of π has been carefully analyzed in [12], I.3., especially diagram 3.3.b. It consists of all (infinitely many) $\Gamma(\sqrt{-3})$-reflection discs. This set coincides with the Γ-transforms of one of them, say of $\mathbb{D}_R$. So, if τ belongs to a smooth Picard curve, then it cannot belong to $\diamondsuit$. The Theorem 10.6 is proved. ∎

11. Effective Torelli theorem for Picard curves via Picard modular forms.

Tacitly we used already Torelli's theorem. It appears in diagram (10.8) asserting that $\mathcal{M}_3 \longrightarrow \mathcal{A}_3$ is an embedding or, more generally, the isomorphy class of a smooth curve is uniquely determined by its (polarized) Jacobian variety. We look for a precise pointwise version of this theorem in the case of Picard curves:

11.1 Find for given $\tau \in \mathbb{B}$ (or $*\tau \in *\mathbb{B} \subset \mathbb{H}_3$) the normal form of a Picard curve C_τ corresponding to the moduli point $\pi(\tau) \in \mathbb{P}^2$.

In analogy to the Weierstrass normal form of elliptic curves we can find holomorphic functions $t_1, t_2, t_3, t_4 : \mathbb{B} \longrightarrow \mathbb{C}$ such that the normal forms we look for can be written as

$$(11.2) \qquad C_\tau : Y^3 = \prod_{i=1}^{4} (X - t_i(\tau))$$

simultaneously for all $\tau \in \mathbb{B}$. In other words we try to describe the quotient map π we started with in (6.1) in terms of holomorphic functions identifying the quotient map π with

$$(11.3) \qquad \begin{aligned} (t_1 : t_2 : t_3 : t_4) &: \mathbb{B} \longrightarrow \mathbb{P}^2 \\ \tau &: \longmapsto (t_1(\tau) : t_2(\tau) : t_3(\tau) : t_4(\tau)). \end{aligned}$$

The existence proof for these holomorphic functions was the main result of chapter I in [12]. We refer the reader to section 6.3 there entitled 'Inversion of the Picard integral map by means of automorphic forms', especially to Theorem 6.3.12.

Since we need the quality of the functions t_i for finding explicit Fourier series of them, we repeat the way of their construction in [12] without proofs.

11.4 Definition. A holomorphic function $f: \mathbb{B} \longrightarrow C$ is a *Picard modular form* of the imaginary quadratic number field K and of *weight* m, if there exists a sublattice Γ'' of $U((2,1), \mathcal{O}_K)$ such that the following *functional equations* are satisfied:

(11.5) $\gamma^*(f) = j_\gamma^m \cdot f$ for all $\gamma \in \Gamma''$,

where $\gamma^*(f)(\tau) = f(\gamma(\tau))$ and $j_\gamma(\tau)$ is the Jacobi determinant of $\gamma : \mathbb{B} \xrightarrow{\sim} \mathbb{B}$ at τ.

If (11.5) is satisfied, then we shortly call f a Γ'' - m o - d u l a r f o r m (of weight m). These functions form a finite-dimensional vector space denoted by $[\Gamma'', m]$. We come back now to the Eisenstein numbers, especially to $\Gamma = U((2,1), \mathcal{O}_K)$, $\Gamma' = \Gamma(\sqrt{-3})$ and define the s p e c i a l modular groups by

$$S\Gamma = \Gamma \cap Sl\ (\mathbb{C})\ ,\ S\Gamma(\sqrt{-3}) = \Gamma(\sqrt{-3}) \cap Sl_3(\mathbb{C})\ ,\ \ldots$$

We have three exact sequences

(11.6)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & S\Gamma(\sqrt{-3}) & \longrightarrow & \Gamma(\sqrt{-3}) & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & \Gamma(\sqrt{-3}) & \longrightarrow & \Gamma & \longrightarrow & S_4 \times \mathbb{Z}_6 & \longrightarrow & 1 . \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & S\Gamma(\sqrt{-3}) & \longrightarrow & S\Gamma & \longrightarrow & S_4 & \longrightarrow & 1 \\ & & & & \Psi & & \Psi & & \\ & & & & \gamma & \longmapsto & \bar{\gamma} & & \end{array}$$

The group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ comes from the element $-id \in \Gamma$, $(\mathbb{Z}_2 \subset \mathbb{Z}_6)$.

We look for Picard modular forms t_1, t_2, t_3, t_4 satisfying the following conditions (11.7) and 11.8:

(11.7) $$t_1 + t_2 + t_3 + t_4 = 0,$$

t_1, t_2, t_3 are linearly independent.

11.8 S p e c i a l F u n c t i o n a l E q u a t i o n s .

(i) $\gamma^*(t_i) = j_\gamma \cdot t_i$ for $i = 1,2,3,4$, $\gamma \in S\Gamma(\sqrt{-3})$

(ii) $\gamma^*(t_i/t_j) = t_{\bar{\gamma}(i)}/t_{\bar{\gamma}(j)}$ for $i = 1,2,3,4$, $\gamma \in S\Gamma$.

(iii) $\delta^*(t_i) = (\det \delta)^2 \cdot j_\delta \cdot t_i$

for $i = 1,2,3,4$, δ representing $\Gamma(\sqrt{-3})/S\Gamma(\sqrt{-3})$.

11.9 Theorem (Holzapfel [12]). There exist four Picard modular forms t_1, t_2, t_3, t_4 satisfying the properties (11.7) and 11.8. They are uniquely determined up to the numeration and a common constant factor. The condition (iii) is a consequence of all the previos conditions (11.7), 11.8 (i),(ii).

11.10 Remark. The last statement has been proved first by Feustel [6] by an analytic argument using the Theta presentation of the modular functions t_i we look for. An algebraic proof has been found by the author [13] by means of the dimension formulas for cusp forms of ball lattices [11]. We can give a more precise version of Theorem 11.9 which brings the three conditions (i), (ii),(iii) of 11.8 together. Denoting the image of $\gamma \in \Gamma$ along $\Gamma \longrightarrow S_4 \ltimes Z_i \longrightarrow S_4$ (see diagram (11.6)) by $\bar{\gamma}$ we get

11.11 Corollary. The four Picard modular functions t_1, t_2, t_3, t_4 are characterized (up to a constant factor) by (11.7) and the functional equations

$$\gamma^*(t_i) = (\det_\gamma)^2 \cdot \operatorname{sgn}(\bar{\gamma}) \cdot j_\gamma \cdot t_{\bar{\gamma}(i)} \quad \text{for } i = 1,2,3,4, \quad \gamma \in \Gamma.$$

Main idea of proof (see [12]). Basically one has to classify the surface $\hat{X} = \widehat{\mathbb{B}/S\Gamma(\sqrt{-3})}$. The group $S\Gamma(\sqrt{-3})$ acts almost freely on $\mathbb{B}$, that means that the non-trivially acting elements have at most isolated fixed points on $\mathbb{B}$. It turns out that

$$\widehat{\mathbb{B}/S\Gamma(\sqrt{-3})} = \hat{X} \dashrightarrow \mathbb{P}^2 = \widehat{\mathbb{B}/\Gamma(\sqrt{-3})}$$

is the unique (cyclic) Z_3-covering of $\mathbb{P}^2$ branched along $\mathbb{A}$ (see the diagrams (11.6) and (6.1). It is not difficult to describe the surface $\hat{X}$ by an equation (see [12], I.4.3).

(11.12) $$\hat{X} : Z^3 = (Y_2^2 - Y_1^2)(Y_2^2 - Y_0^2)(Y_1^2 - Y_0^2)$$

This is a weighted equation with Z of weight 2 and Y_0, Y_1, Y_2 of weight 1. More precisely, this means that $\hat{X}$ is the projective spectrum of the corresponding graded ring

$$\mathbb{C}[Y_0, Y_1, Y_2, Z]/(Z^3 - \prod_{0 \le i < j \le 2} (Y_j^2 - Y_i^2)).$$

11.13 Remark. On this place we remember again to the 22-nd Hilbert problem mentioned in 6.3. It turns out that the algebraic relation in (11.12) is satisfied by Picard modular forms y_0, y_1, y_2, z substituting the variables Y_0, Y_1, Y_2 or Z, respectively. The knowledge of the uniformization $\mathbb{B} \longrightarrow \hat{X}$ together with the corresponding arithmetic uniformizing lattice $S\Gamma(\sqrt{-3})$ becomes important for this purpose as it has been predicted by Hilbert in general.

The key point is to understand automorphic forms as sections of logarithmic pluricanonical sheaves. In [12], I.4.3 we proved

(11.14) $$\bigoplus_{m=0}^{\infty} [S\Gamma(\sqrt{-3}), m] = \bigoplus_{m=0}^{\infty} H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + mT)),$$

where $\overline{X}$ is the minimal resolution of singularities of $\hat{X}$, T the compactification divisor resolving the cusp singularities of $\hat{X}$. It consists of four disjoint elliptic curves. As usual $K_{\overline{X}}$ denotes a canonical divisor and $\mathcal{O}(D)$ is the sheaf corresponding to the divisor D. A careful geometric analysis (explicit knowledge of a canonical divisor, vanishing theorem on surfaces) accomplished in [12] yields the ring structure

(11.15) $$\bigoplus_{m=0}^{\infty} H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + mT)) = \mathbb{C}[s_0, s_1, s_2, s]$$

with s_0, s_1, s_2 of weight 1, s of weight 2 and the generating relation

(11.16) $$s^3 = (s_2^2 - s_1^2)(s_2^2 - s_0^2)(s_1^2 - s_0^2)$$

Together with (11.14) we found generators η_0, η_1, η_2 of $[S\Gamma(\sqrt{-3}),1]$ and a $\Gamma(\sqrt{-3})$-modular form η of weight 2 such that η_0, η_1, η_2 and η generate the ring $\bigoplus_{m=0}^{\infty} [S\Gamma(\sqrt{-3}),m]$ of $S\Gamma(\sqrt{-3})$-modular forms satisfying the relation

(11.17) $$\eta^2 = (\eta_2^2 - \eta_1^2)(\eta_2^2 - \eta_0^2)(\eta_1^2 - \eta_0^2)$$

we looked for.

If Γ'' is an arbitrary ball lattice, then it acts on the space of holomorphic functions on the ball $\mathbb{B}$ via

(11.18) $$f \longmapsto j_\gamma^{-m} \cdot \gamma^*(f), \quad f \in H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}}), \quad \gamma \in \Gamma''.$$

The Γ''-invariant functions are the Γ''-modular forms (compare with (11.5)). Especially the lattice $S\Gamma$ acts on $[S\Gamma(\sqrt{-3}),1]$ with ineffective kernel $S\Gamma(\sqrt{-3})$. With the last row of (11.6) we get a three-dimensional representation of S_4. In [12] we proved that this representation is irreducible. It induces a projective representation of S_4 on $\mathbb{P}[S\Gamma(\sqrt{-3}),1] \cong \mathbb{P}^2$. There is only one such representation. Explicitly it can be described by

$$(x_1 : x_2 : x_3 : x_4) \longmapsto (x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)} : x_{\sigma(4)}),$$
$$\sigma \in S_4, \quad x_i \in \mathbb{C}, \quad \sum x_i = 0.$$

Looking back to $[S\Gamma(\sqrt{-3}),1]$ one finds four Picard modular forms t_1, t_2, t_3, t_4 satisfying (11.7) and 11.8 (i),(ii).

It remains to verify the property (iii) of 11.8. This is much more difficult than it looks like at the first glance. There exist two proofs of different kind. The first has been found by

Feustel in [6]. He used a transcendental method: The modular forms t_i can be understood as restrictions of explicitly known theta constants on $\mathbb{H}_3$ to $\mathbb{B}$ (see below and also Shiga's article [32]). Then the transformation behaviour described in 11.8 (iii) can be checked directly. An algebraic-geometric proof of the functional equations (iii) can be found in the author's paper [13].

∎

In order to solve the relative Torelli problem 11.1 in an effective manner by means of our modular forms t_1, t_2, t_3, t_4 we go back to the quotient map (11.3). It is realized by the modular forms of Theorem 11.9 for the following reasons (see [12] for more details). From the third row of (11.6) one gets a commutative quotient diagram

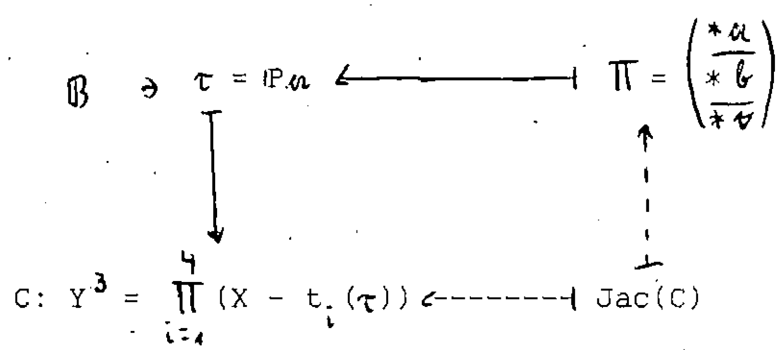
(11.19)
$$\begin{array}{ccc} \mathbb{B} & & \\ {\scriptstyle S\Gamma(\sqrt{-3})}\downarrow & \overset{(t_1:t_2:t_3)}{\underset{\Gamma(\sqrt{-3})}{\searrow}} & \\ \hat{X} & \xrightarrow[Z_3]{(s_0:s_1:s_2)} & \mathbb{P}^2 \end{array}$$

The logarithmic canonical map $\varphi_{K_{\overline{X}}+T}$ goes down to $\hat{X} \longrightarrow \mathbb{P}^2$ and coincides with the Z_3-goutient map on the bottom of diagram (11.19) as has been proved in [12]. Using generators s_i of $H^0(\overline{X}, \mathcal{O}(K_{\overline{X}} + T)$ it can be realized as the projective morphism $(s_0 : s_1 : s_2)$. The sections s_i have been lifted to $S\Gamma(\sqrt{-3})$-modular forms η_i , $i = 0,1,2$ via (11.14). Without loss of generality we can assume that $t_i = \eta_{i-1}$, $i = 1,2,3$. We can identify the big quotient map of (11.19) with the map of (11.3), where t_4 is defined by (11.7).

11.20 Theorem (effective Torelli for Picard curves via modular forms). Let J(C) be the Jacobi threefold of a smooth Picard curve C corresponding to the point $\mathcal{T} = *\tau \in \mathbb{H}_3$. Then the Picard modular forms t_1, t_2, t_3, t_4 defined in theorem 11.9 (up to a common constant factor) yield a normal form of C in the following manner:

(11.21) $$Y^3 = (X - t_1(\tau))(X - t_2(\tau))(X - t_3(\tau))(X - t_4(\tau))$$

Proof.: By the relative Schottky theorem 10.6 for Picard curves we have $J(C) = \mathbb{C}^3/\Lambda_\Pi$, where Π is given as $(\Pi_1 \mid \Pi_2)$ in (10.9) connecting $\tau \in \mathbb{B}$ with $\mathcal{T} = *\tau$. The diagram (10.8) with $\Gamma' = \Gamma(\sqrt{-3})$ yields the moduli point of C on $\mathbb{P}^2$ as image of τ along the $\Gamma(\sqrt{-3})$-quotient map. By (11.3) this image is equal to $(t_1(\tau):t_2(\tau):t_3(\tau):t_4(\tau))$. But the normal form of a corresponding Picard curve is given in (11.21), see Prop. 9.2. For the convenience of the reader we present a diagram of correspondences used above in close connection with diagram (10.8).



The theorem is proved. ∎

12. Picard modular forms as theta constants.

Theta functions $\vartheta\left[{a \atop b}\right]$ with characteristics $a,b \in \mathbb{Q}^g$ are holomorphic functions on $\mathbb{C}^g \times \mathbb{H}_g$,

$$\mathbb{H}_g = \{ \tau \in Gl_g(\mathbb{C});\ {}^t\tau = \tau,\ \mathrm{Im}\,\tau > 0\}$$

the generalized Siegel upper half plane uniformizing the muduli space of (principally) polarized abelian varieties of dimension g (see e.g. [ ]). Explicitly the theta functions

$$\vartheta\left[{a \atop b}\right]: \mathbb{C}^g \times \mathbb{H}_g \longrightarrow \mathbb{C}$$

are defined by

$$\vartheta\left[{a \atop b}\right](z,\tau) = \sum_{n\in\mathbb{Z}^g} \exp\{\pi i\, {}^t(n+a)\tau(n+a) + 2\pi i\, {}^t(n+a)(z+b)\}$$

The restrictions $\vartheta|0\times\mathbb{H}_g$

$$\theta\left[{a \atop b}\right](\tau) = \vartheta\left[{a \atop b}\right](0,\tau)$$

are called t h e t a c o n s t a n t s (with characteristics).

We restrict our attention to the case g = 3 and look for extensions of the Picard modular forms t_1, t_2, t_3, t_4 defined in the previous section in $\mathbb{H}_3$ along the embedding $*: \mathbb{B} \hookrightarrow \mathbb{H}_3$ defined in (10.9) and hope to express them in terms of theta constants. Very important for this purpose are the functional equations described in 11.8. So we look for elementary combinations Th of theta constants whose restrictions

$$th(\tau) = Th(*\tau), \quad \tau \in \mathbb{B}$$

satisfy the special functional equations 8.11 (i),(iii):

(12.1) $\quad Th(*\gamma) = \det^2\gamma \cdot j_\gamma \cdot Th \quad$ on $\mathbb{B} \subset \mathbb{H}_3$, $\quad \gamma \in (\sqrt{-3})$.

For the convenience of the reader we summerize the restrictions (or extensions) used above and below in the following diagram.

(12.2)
$$\begin{array}{ccccc}
\mathbb{B} & \hookrightarrow & \mathbb{H}_3 & & \\
\Gamma(\sqrt{-3}) \ni \gamma \;\downarrow \wr & & \wr \downarrow\; *\gamma \in Sp(6,\mathbb{Z}) & & \\
\mathbb{B} & \hookrightarrow & \mathbb{H}_3 & \hookrightarrow & \mathbb{C}^3 \times \mathbb{H}_3 \\
t = \downarrow \theta|_{\mathbb{B}} & & \theta = \downarrow \vartheta|_{0\times\mathbb{H}_3} & & \downarrow \vartheta \\
\mathbb{C} & \xrightarrow{=} & \mathbb{C} & \xrightarrow{=} & \mathbb{C}
\end{array}$$

12.3 Theorem (Feustel, Shiga).

Let $\theta_i(\tau) = \vartheta_i(0,\tau)$, $i = 0,1,2$, be the theta constants on $\mathbb{H}_3$ restricting the theta functions

$$\vartheta_k = \vartheta\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix}(z,\tau) \ , \ k = 0,1,2, \ z \in \mathbb{C}^3 .$$

Set

(12.4)
$$Th_1 = \theta_0^3 + \theta_1^3 + \theta_2^3 \ , \ Th_2 = -3\theta_0^3 + \theta_1^3 + \theta_2^3 \ ,$$
$$Th_3 = \theta_0^3 - 3\theta_1^3 + \theta_2^3 \ , \ Th_4 = \theta_0^3 + \theta_1^3 - 3\theta_2^3 \ ,$$

and

(12.5) $\quad th_i(\tau) = Th_i(*\tau)$, $i = 1,2,3,4$, $\tau \in \mathbb{B}$.

Then the functions $th_i(\tau)$ are the (normalized) Picard modular forms satisfying (11.7) and all the functional equations (i), (ii), (iii) of 11.8 or, equivalently, those of Corollary 11.11.

Proof (main steps). We follow Feustel's proof and refer for explicit calculations to his paper [6] and the related literature given there. The proof summerizes preperatory work of Riemann, Picard [26], [27], Alezais [1], Mumford [24], H.Shiga

[32] and Holzapfel [12] (the functional equations above).

Step 1 (restriction to six functional equations).

Here we go back to the fundamental group $\pi_1(\mathbb{P}^2 \setminus \triangle)$ of the Fuchsian system (7.3) of partial differential equations and the surjective monodromy representation $\pi_1(\mathbb{P}^2 \setminus \triangle) \twoheadrightarrow \Gamma(\sqrt{-3})$, see (7.2). But the fundamental group $\pi_1(\mathbb{P}^2 \setminus \triangle)$ has obviously six generators coming from simple loops in $\mathbb{P}^2$ around each one of the six omitted lines. Therefore also $\Gamma(\sqrt{-3})$ has six generators, say $g_1, \dots, g_6$. They have been explicitly described already by Picard [26] (with correcture in [27]) and Alezais. Their symplectic lifts $\sigma_i = {*g_i} \in Sp(6,\mathbb{Z})$, $i = 1,\dots,6$, can be found explicitly in Feustels paper [6]. In order to check the functional equations 8.11 (i),(iii) for suitable holomorphic functions th on $\mathbb{B}$ it is sufficient to check them for the generators $g_1,\dots,g_6$ of $\Gamma(\sqrt{-3})$. According to our claim th = Th$|\mathbb{B}$ we have now only to look for holomorphic functions Th on $\mathbb{H}_3$ satisfying the six r e - s t r i c t e d f u n c t i o n a l e q u a t i o n s

(12.6) $$Th \circ (\sigma_i) = (\det g_i)^2 \cdot j_{\gamma_i} \cdot Th \quad \text{on } \mathbb{B} \subset \mathbb{H}_3,\ i = 1,\dots,6,$$

implying (12.1).

Step 2 (Riemann's theorem).

It is a general problem in the theory of algebraic curves to describe a given meromorphic function on a curve C in terms of theta functions on its Jacobian variety by restriction along the Jacobi embedding $C \hookrightarrow J(C)$. This problem has been solved essentially by Riemann. We refer to Mumford's book [24].

12.7 Theorem (Riemann). Let C be a (smooth, compact, complex) curve of positive genus g, $\gamma_1, \ldots, \gamma_{2g}$ a normal basis of $H_1(C,\mathbb{Z})$ and $\vec{\omega} = {}^t(\omega_1, \ldots, \omega_g)$ a basis of $H^0(C,\Omega_C)$ such that the corresponding period matrix has the normalized form

$$\Big(\int_{\gamma_j} \omega_i\Big) = (E_g|\tau), \quad \tau \in \mathbb{H}_g, E_g \text{ the unit matrix.}$$

If $f: C \longrightarrow \mathbb{P}^1$ is a meromorphic function with divisor

$$(f) = \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k, \quad a_k, b_k \in C,$$

then it holds that

(12.8)

$$\prod_{i=1}^{g} f(P_i) = \text{const}\cdot\prod_{k=1}^{m}\Big\{\vartheta\Big(\int_{P_0}^{\sum_{i=1}^{g}P_i}\vec{\omega} - \int_{P_0}^{a_k}\vec{\omega} - \Delta\Big)/\vartheta\Big(\int_{P_0}^{\sum_{i=1}^{g}P_i}\vec{\omega} - \int_{P_0}^{b_k}\vec{\omega} - \Delta\Big)\Big\}$$

as meromorphic function on C^g/S_g. One has to use the same pathes in the first integrals of the denominator and numerator.

Notations. Here the Riemann theta function ϑ is considered as holomorphic function on $\mathbb{C}^g$. It coincides with the restriction of $\vartheta\begin{bmatrix}0\\0\end{bmatrix}$ to $\mathbb{C}^g \times \tau$, τ the fixed period matrix defined above, with the notations introduced at the beginning of this section. The auxiliary point $P_0 \in C$ is used to fix the Jacobi embedding

$$C \hookrightarrow J(C), \ P \longmapsto \int_{P_0}^{P}\vec{\omega} \bmod \Lambda_\tau, \quad \Lambda_\tau = \mathbb{Z}^g + \tau\cdot\mathbb{Z}^g.$$

Δ denotes the Riemann constant. This is a special well-defined 2-torsion point on J(C) (see [24], ch.II, 3). Both sides are considered as functions on C^g or C^g/S_g, namely $\sum_{i=1}^{g} P_i$ is understood as point on C^g/S_g. On the right-hand-side of (12.8) appears a constant denoted by const. In general one knows only its existence but not its explicit value. Finally, we used the notation

$$\int_{P_0}^{D} \vec{\omega} = \sum_{i=1}^{g} \int_{P_0}^{P_i} \vec{\omega} \quad \text{for } D = \sum_{i=1}^{g} P_i .$$

Remark. For the proof of Riemann's theorem one compares zeros and poles of both sides of (12.8). The key point is the understanding of Riemann's constant Δ. On $\mathbb{C}^g$ it is defined mod Λ_τ by

$$-\Delta + \int_{P_0}^{\sum_{i=1}^{g-1}} \vec{\omega} \mod \Lambda_\tau \in \Theta, \; P_i \in C,$$

where $\Theta \subset J(C)$ denotes the theta divisor defined by $\vartheta(z) = 0$. Setting e.g. $P_i = a_k$ in (12.8) opens the way of proof of Riemann's theorem in an obvious manner.

Now we apply Riemann's theorem for finding two generators of the field of $\Gamma(\sqrt{-3})$-automorphic functions in theta terms. This has been done already by Picard and Alezais. We write a smooth Picard curve C in the modified normal form

$$C: \; Y^3 = X(X-1)(X-u)(X-v)$$

Then $u = u(\tau)$, $v = v(\tau)$, $\tau \in \mathbb{B}$, generate the field of $\Gamma(\sqrt{-3})$-modular functions. The ramification locus of the $\mathbb{Z}_3$-Galois covering $C \longrightarrow \mathbb{P}^1$ consists of the following five points on C:

$$O = Q_0 = (0,0), \; Q_1 = (1,0), \; Q_u = (u,0), \; Q_v = (v,0), \; \infty .$$

We apply (12.8) to the function $f = x: C \longrightarrow \mathbb{P}^1$ at the points

$$P_1 = Q_1, \; P_2 = Q_2, \; P_3 = Q_u$$

and at the points

$$P_1 = Q_1, \; P_2 = Q_u, \; P_3 = Q_u$$

With (x) = 3·0 - 3·∞ , $P_0 = \infty$ we get with the same constant c

$$u = c\cdot\prod_1^3 \{\vartheta(\int_\infty^{2Q_1+Q_u} \vec{\omega} - \int_\infty^0 \vec{\omega} - \Delta)/\vartheta(\int_\infty^{2Q_1+Q_u} \vec{\omega} - \Delta)\}$$

$$u^{\lambda} = c\cdot\prod_1^3 \{\vartheta(\int_\infty^{Q_1+2Q_u} \vec{\omega} - \int_\infty^0 \vec{\omega} - \Delta)/\vartheta(\int_\infty^{Q_1+2Q_u} \vec{\omega} - \Delta)\}$$

Since $u = u^{\lambda}/u$ we get by division of both expressions above a theta formula for $u(\tau)$ without the unknown constant c:

$$u = \prod_1^3 \{\vartheta(.-.)\cdot\vartheta(.-.-.)/\vartheta(.-.)\cdot\vartheta(.-.-.)\}$$

In the same manner using Q_v instead of Q_u we can express v, v^{λ} and finally $v(\tau) = v^{\lambda}/v$ in terms of the theta function belonging to $\mathcal{T} = *\tau$.

Step 3 (Theta constants).

This step is due to Shiga [32]. He calculated explicitly the Riemann constant Δ above using special values. Furthermore he used standard transformation laws to prove that

(12.9) $u(\tau) = \vartheta_1^3(0,*\tau)/\vartheta_2^3(0,*\tau)$, $v(\tau) = \vartheta_0^3(0,*\tau)/\vartheta_2^3(0,*\tau)$

with the notations of Theorem 12.3. The denominator does not vanish identically on $\mathbb{B}$ (Shiga [32]).

Step 4 (automorphic forms).

This last step is due to Feustel [6]. He checked that the dominator and the numerators in (12.9) satisfy the six functional equations (12.6). By step 1 we dispose on three linearly independent $S\Gamma(\sqrt{-3})$-modular forms $\theta_0^3(*\tau)$, $\theta_1^3(*\tau)$, $\theta_2^3(*\tau)$ on $\mathbb{B}$ of wight 1 with the notations of Theorem 12.3. Since dim $[S\Gamma(\sqrt{-3})] = 3$ by the considerations around (11.16) we found up to linear combinations all $S\Gamma(\sqrt{-3})$-modular forms of wight 1. So $\theta_0^3, \theta_1^3, \theta_2^3$ can be identified with η_0, η_1, η_2 in (11.17). Now we remember to the $S_4 = S\Gamma/S\Gamma(\sqrt{-3})$-action on $[\Gamma(\sqrt{-3}), 1]$. There must be linear combinations th_1, th_2, th_3, th_4 of $\theta_0^3, \theta_1^3, \theta_2^3$ satisfying the relation (11.7) and the functional equations 11.8 uniquely defined up to a common factor, see Theorem 11.9. The symmetric group S_4 is generated by three tranpositions. It is not difficult to find representants of them in $S\Gamma$ and also their symplectic representations acting on $\mathbb{H}_3$ explicitly. This has been done in [6]. With the definitions (12.4) and (6.12.5) Feustel proved that th_1, th_2, th_3, th_4 are functions with the correct tranformation behaviour we look for. The Theoerem 12.3 is proved. ∎

13. Proof of the Main Theorem.

Now we are able to prove the Main Theorem formulated at the end of 1. We have to concentrate our attention to the verification of the field tower on the right-hand side of diagram (1.1). First we check the list 1.,...,8. of definitions in 1. and fill the gaps. The ball $\mathbb{B}$ is understood as subball of $\mathbb{H}_3$ via the embedding $*$ defined in (10.9). The restricted theta constants th_i, $i = 1,2,3,4$, have been defined in the previous section.

Next we have to give a precise definition for the special arguments used in the Main Theorem. An a r i t h m e t i c p o i n t o n $\mathbb{H}_g$ is a point of

$$\mathbb{H}_g(\bar{\mathbb{Q}}) = \mathbb{H}_g \cap \mathrm{Gl}_g(\bar{\mathbb{Q}}).$$

An a r i t h m e t i c b a l l p o i n t is a point of

$$\mathbb{B}(\bar{\mathbb{Q}}) = \mathbb{B} \cap \mathbb{H}_3(\bar{\mathbb{Q}}) = *^{-1}(\mathbb{H}_3(\bar{\mathbb{Q}})).$$

A (principally) polarized abelian variety A of dimension g with complex multiplication determines an arithmetic point $\tau \in \mathbb{H}_g(\bar{\mathbb{Q}})$. Namely we know from section 3 that $A \cong \mathbb{C}^g/\Phi(\mathfrak{A})$, $\mathfrak{A} \in \mathrm{Mat}_{g\times 2g}(\bar{\mathbb{Q}})$. On the other hand A is isomorphic to $\mathbb{C}^g/\Lambda_{(E_g|\tau)}$. Therefore $\mathfrak{A}$ and $(E_g|\tau)$ must belong to the same double coset of $\mathrm{Mat}_{g\times 2g}(\mathbb{C})$ with respect to $\mathrm{Gl}_g(\mathbb{C})$ or $\mathrm{Gl}_{2g}(\mathbb{Z})$, respectively. Hence there exist elements $G \in \mathrm{Gl}_g(\mathbb{C})$ and $\Sigma \in \mathrm{Gl}_{2g}(\mathbb{Z})$ such that $\mathfrak{A}\cdot\Sigma = (G|G\cdot\tau)$. Now it is clear that G, $G\tau$ and finally $\tau = G^{-1}(G\tau)$ belong to $\mathrm{Gl}_g(\bar{\mathbb{Q}})$. Especially we dispose on the following well-known

13.1 Lemma. If C is a (smooth) curve of genus g and its Jacbian variety J(C) corresponding to $\tau \in \mathbb{H}$ has complex multiplication, then τ is an arithmetic point of $\mathbb{H}_g$.

13.2 Definition. A point $\sigma \in \mathbb{B}$ is called a CM-module (with respect to Picard curves), if there exists a (smooth) Picard curve C such that σ belongs to the Jacobian threefold J(C) of C (t.m. $C \simeq C_\sigma$), J(C) is simple and has complex multiplication.

From the above considerations it is clear that a CM-module of $\mathbb{B}$ is an arithmetic ball point. There is a dense set of very explicit examples called *stationary modules*. They are defined as isolated fixed points $\sigma \in \mathbb{B} \setminus \mathfrak{A}$ of elements $\gamma \in U((2,1),K)$ with $[K(\gamma):K] = 3$. For details we refer to [14].

Now we are able to define F_σ, Φ_σ and $\mathfrak{a}_\sigma$ appearing in the Main Theorem. We have only to connect the above definition with those of section 3. Let $\sigma \in \mathbb{B}(\overline{\mathbb{Q}})$ be a CM-module corresponding to the Picard curve C with Jacobian threefold J(C). It has a CM-type $(F_\sigma, \Phi_\sigma, \mathfrak{a}_\sigma)$ and this is all we need. The corresponding Shimura class field $Sh(\Phi_\sigma, \mathfrak{a}_\sigma)$ has been defined at the end of section 2.

We come to the proof of the Main Theorm. For this purpose we let $\sigma \in \mathbb{B}(\overline{\mathbb{Q}})$ be a CM-module corresponding to the Picard curve C_σ with Jacobian threefold J_σ of type $(F_\sigma, \Phi_\sigma, \mathfrak{a}_\sigma)$.

$$(13.3) \qquad C_\sigma \cong: \quad Y^3 = \prod_{i=1}^{4} (X - th_i(\sigma))$$

This comes from the effective Torelli Theorem 11.20 for Picard curves in combination with the theta representation of the Picard modular forms t_i found in section 12, see Theorem 12.3.

13.4 $k_\sigma = K(th(\sigma))$ is a definition field of $cl(C_\sigma)$ and of $cl(J_\sigma)$ in the sense of section 4.

For the curve C_σ this is an immediate consequence of 13.3. If k is a definition field of a curve C, then it is also a definition field of the Jacobian manifold J(C). For this well-known fact we refer the reader to Milne's article [23] in [5]. Consequently, k_σ is a definition field of $cl(J_\sigma)$ or of J_σ itself without loss of generality. The same is true for $(J_\sigma, \mathcal{C}_\sigma)$, $\mathcal{C}_\sigma$ a suitable polarization of J_σ.

(13.5) $$M(J_\sigma, \mathcal{C}_\sigma) = Sh(\Phi_\sigma, \mathfrak{A}_\sigma)$$

This was (the consequence 5.3 of) Shimura's Main Theorem 5.1 of complex multiplication.

(13.6) $$M(J_\sigma) \subseteq M(C_\sigma) = K(th(\sigma))^{S_4(\sigma)}$$

Proof. We remember that

$$(th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma))/S_4 \in \mathbb{P}^2/S_4$$

is the moduli point of the Picard curve C_σ by (13.3) and Proposition 9.2. So for $\mu \in Aut(\mathbb{C})$ one has the following equivalent conditions:

$$\mu \in Stab\ cl(C_\sigma) \longleftrightarrow C^\mu \cong C \longleftrightarrow cl(C^\mu) = cl(C) \longleftrightarrow$$

$$(th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma))^\mu \equiv (th_1(\sigma):th_2(\sigma):th_3(\sigma):th_4(\sigma)) \mod S_4$$

$$\longleftrightarrow \quad th_i(\sigma)^\mu/th_j(\sigma)^\mu = th_{\pi(i)}(\sigma)/th_{\pi(j)}(\sigma)$$

for all $i,j \in \{1,2,3,4\}$ and a suitable $\pi \in S_4$

$$\longleftrightarrow \quad \mu \in \mathbb{Q}(th(\sigma))^{S_4(\sigma)}.$$

On the other hand we have $Stab\ cl(C_\sigma) \subseteq Stab\ cl(J_\sigma)$, hence $M(C_\sigma) \supseteq M(J_\sigma)$ by the definition 4.1 of moduli fields. It remains to show that K is a subfield of the fields appearing in (13.6). The curve C_σ in (13.3) has an obvious automorphism of order 3. Therefore the field $K = \mathbb{Q}(\rho)$ is a subfield of the endomorphism algebra $\mathbb{Q}\otimes \mathrm{End}(J_\sigma)$. Since J_σ is simple, the field F_σ coincides with this algebra (up to isomorphy). Therefore K is a subfield of F_σ. This is also true if J_σ is not simple and has complex multiplication in the generalized sense of section 3 (DCM = decomposed complex multiplication): Then J_σ is isogeneous to $T\times T\times T$, T an elliptic CM-curve with imaginary quadratic multiplication field E, say. As already mentioned at the end of section 3, the endomorphism algebra of J_σ is isomorphic to $\mathrm{Mat}_3(E)$. The diagonally embedded field E commutes with any other subfield of $\mathbb{Q}\otimes\mathrm{End}(J_\sigma)$, especially with K. Thus the endomorphism algebra contains the subfield K(E). The absolute degree of a subfield of the $\mathbb{Q}$-algebra $\mathbb{Q}\otimes\mathrm{End}(A)$ of an abelian variety A divides $2\cdot\dim A$, see [20], I.1, Th.3.1. Therefore $[K(E):\mathbb{Q}]$ divides $6 = 2\cdot\dim J_\sigma$. This is only possible for $K = E$. Consequently, K is central in $\mathbb{Q}\otimes\mathrm{End}(J_\sigma)$, and $K\cdot F_\sigma$ is a subfield. Since $[F_\sigma:\mathbb{Q}] = 2\cdot\dim J_\sigma = 6$ by definition of complex multiplication it cannot happen that $K\cdot F_\sigma > F_\sigma$ because F_σ is obviously a maximal subfield of $\mathbb{Q}\otimes\mathrm{End}(J_\sigma)$. So we have $K\subset F_\sigma$, that means that in any case the multiplication field F of a CM-Picard curve is a cubic extension of K. It follows immediately from the trace definition (2.2) of the reflex field F′ that $K \subseteq F'$. Now we apply (4.5) to obtain $F'_\sigma \subseteq M(J_\sigma)$. Together with (13.6) and the equivalence considerations above we get

$$K \subseteq M(C_\sigma) = \mathbb{Q}(\mathrm{th}(\sigma))^{S_4(\sigma)} .$$

The identity on the right-hand side of (13.6) is verified. By the way we established also the bottom part of diagram (1.1).

13.7 The definition field $k_\sigma = K(th(\sigma))$ of the CM(or DCM)-Picard curve C_σ is an algebraic number field.

Proof. Let $\mathcal{C}$ be a principal polarization of J_σ. The polarized Jacobian variety $Jac(C_\sigma) = (J_\sigma, \mathcal{C})$ has an algebraic definition field k by Proposition 4.4. From Lemma 4.3 we know that

$$M(Jac(C_\sigma)) \subseteq k \subset \bar{\mathbb{Q}} .$$

Obviously, the moduli field $M(C_\sigma)$ coincides with $M(Jac(C_\sigma))$. Together with the identity in (13.6) we see that $K(th(\sigma))^{S_4(\sigma)}$ is a number field. Since $k_\sigma = K(th(\sigma))$ is a finite extension, it is a number field, too. ∎

Altogether we have the following inclusions:

$$\begin{array}{lllll}
(13.8) & M(J_\sigma, \mathcal{C}_\sigma) = Sh(\phi_\sigma, \mathfrak{a}_\sigma) & & & \\
& \cup\!\!| & & & \\
K \subseteq F'_\sigma \subseteq & M(J_\sigma) \subseteq M(C_\sigma) = M(Jac(C_\sigma)) & & & \\
& \qquad\qquad\qquad \| & & & \\
& \qquad\qquad K(th(\sigma))^{S_4(\sigma)} \subseteq & k_\sigma = K(th(\sigma)) \subset \bar{\mathbb{Q}} & &
\end{array}$$

We restrict ourselves from now on to the case of simple Jacobian threefolds J_σ. For the sake of clear distinction we call σ sometimes in this case a s i m p l e CM-module (SCM).

13.10 Lemma. If A is a simple abelian variety with complex multiplication, then each polarization $\mathcal{C}$ of A is admissible and the moduli field $M(A, \mathcal{C})$ does not depend on the choice of $\mathcal{C}$.

For the proof we refer to Lang's book [10] again. The first statement comes from [10], I.4, Thm. 4.5 (iii) and the condition ADM 2, p. 20. According to the remark in [10], p. 135, the moduli field $M(A, \mathcal{C})$ does not depend on the embedding $\iota: F \hookrightarrow \mathbb{Q} \otimes End(A)$, F the CM-field of A. The last statement of the lemma is Proposition 1.7 (i) of [10], ch.V. ■

Let $\sigma \in \mathbb{B}$ be a simple CM-module and $\mathcal{C}$ a principal polarization of J_σ. By Lemma 13.10 $\mathcal{C}$ is admissible and $M(J_\sigma, \mathcal{C})$ coincides with the moduli field $M(J_\sigma, \mathcal{C}_\sigma)$ for any other admissible polarization $\mathcal{C}_\sigma$ of J_σ. Cosequently the inclusions of (13.8) become sharper:

13.11. For simple CM-modules $\sigma \in \mathbb{B}$ it holds that

$$K \subseteq F' \subseteq M(J_\sigma, \mathcal{C}_\sigma) = Sh(\Phi_\sigma, \mathfrak{a}_\sigma) = K(th(\sigma))^{S_4(\sigma)} \subseteq K(th(\sigma)) \subset \bar{\mathbb{Q}}.$$

■

We established the diagram (1.1) of field towers in the Main Theorem. By the definition 2.7 $Sh(\Phi_\sigma, \mathfrak{a}_\sigma)$ (Shimura class field) is an abelian extension of the reflex field F'_σ. It remains to prove that this extension is unramified, if $\mathfrak{a}_\sigma$ is a (fractional) ideal of F_σ. This follows easily from the construction of Shimura class fields:

13.12 Lemma (see [10], V.4, Thm. 4.1 (ii)). Let A be an abelian CM-variety of type $(F, \Phi, \mathfrak{a})$ such that $\mathfrak{a}$ is a fractional ideal of F. Then the abelian extension $Sh(\Phi, \mathfrak{a})/F'$ is unramified.

Proof. We go back to the construction of the Shimura class field $Sh(\Phi, \mathfrak{a})$ in section 2. Via reciprocity it corresponds to the idele group $U(\Phi, \mathfrak{a})$ defined in (2.4). It suffices to verify that $U(\Phi, \mathfrak{a})$ contains the whole unit group $\mathcal{O}_{F'}^{*}$ of F'. This is a well-known necessary and sufficient criterion for the corresponding class field to be unramified (see e.g. [25]). So let ε be a unit of F'. Then, with the notation of (2.4), also $N'(\varepsilon) \in F$ is a unit. Since $\mathfrak{a}$ is a fractional ideal it holds that

$$N'(\varepsilon)\mathfrak{a} = \mathfrak{a} = 1 \cdot \mathfrak{a}.$$

Thus the relations of the right-hand side of (2.4) are satisfied for $s = \varepsilon$, $\beta = 1$. Hence ε belongs to $U(\Phi, \mathfrak{a})$. The lemma is proved, and at the same time we finish the proof of the Main Theorem. ∎

The field of Picard modular functions (of level Γ) is defined to be the field $\mathbb{C}(G_4/G_2^2, G_3^2/G_2^3)$ of Γ-automorphic functions, where $G_1 = 0$, G_2, G_3, G_4 are the elementary symmetric functions of th_1, th_2, th_3, th_4. The subfield of K-modular functions (of the full level Γ) is defined to be $K(G_4/G_2^2, G_3^2/G_2^3)$. It is the subfield of S_4-invariant functions of K(th). For $\tau \in \mathbb{B}$ we define the field of values of Picard K-modular functions (of full level Γ) at τ by

$$K(G_4/G_2^2, G_3^2/G_2^3)(\tau) = \{f(\tau); f \in K(G_4/G_2^2, G_3^2/G_2^3),\ f(\tau) \neq \infty\}.$$

13.13. Definition. Let $\sigma \in \mathbb{B}$ be a simple CM-module with J_σ of type $(F_\sigma, \Phi_\sigma, \mathfrak{a}_\sigma)$ such that $\mathfrak{a}_\sigma$ is a fractional ideal of F_σ. Then σ is called an *ideal simple CM-module*.

13.14. Corollary. Let $\sigma \in \mathbb{B}$ be an (ideal) simple CM-module. Then $K(G_4/G_2^2, G_3^2/G_2^3)(\sigma)$ is an (unramified) abelian extension of the reflex field F'_σ of the type (F_σ, Φ_σ).

Proof. Looking at the action of $S_4(\sigma)$ we have the obvious relation

$$K(G_4/G_2^2, G_3^2/G_2^3)(\sigma) \leq K(\mathrm{th}(\sigma))^{S_4(\sigma)} .$$

Now we can apply the Main Theorem concentrated in diagram (1.1).

13.14. Remark. The celebrated H i l b e r t c l a s s f i e l d of a basic number field F' is defined as the maximal abelian extension of F'. It is a finite Galois extension with Galois group isomorphic to the ideal class group of F' (see e.g. [25]). Hilbert class fields play an important role in number theory. The explicit construction by means of special values of transcendent functions can be considered as the essential part of Hilbert's twelvth problem. With our Main Theorem we succeded to construct at least a part of the Hilbert class field of F'_σ, if $\sigma \in \mathbb{B}$ is an ideal simple CM-module. Feustel observed that the very explicit stationary modules of elements $\gamma \in U((2,1),K)$ are simple (CM-modules, if $K(\gamma)$ is a cubic extension of K. So we dispose on abelian extensions $M(\mathrm{Jac}(C_\sigma)) = K(\mathrm{th}(\sigma))^{S_4(\sigma)}$ at all these stationary CM-modules σ. We can produce more abelian extensions of our reflex fields by means of torsion points $T_1, \ldots, T_r$ on J_σ. The moduli field $M(J_\sigma, \mathcal{C}_\sigma; T_1, \ldots, T_r)$ has been defined by Shimura (see [35] or [20]). For any CM-module σ it is an abelian extension of

F'_{σ} extending $M(J_{\sigma}, \mathcal{C}_{\sigma})$. The corresponding idele group in the sense of class field theory is also well-known. We refer the reader who is interested on these extensions to [10], Ch.V.,Thm.4.2. Unfortunately, until now there exists no description of these extended class fields in terms of special values of analytic functions except for the case of elliptic curves.

3.15. Problem. For which ideal CM-modules is $K(th(\sigma))^{S_4(\sigma)}$ the whole Hilbert class field of F'_{σ} ?

13.16. Problem. Is the maximal abelian extension F'^{ab}_{σ} for fixed CM-module σ generated by all the generalized moduli fields of type $M(J_{\sigma}, \mathcal{C}_{\sigma}; T_1, \ldots, T_r)$?

It seems to be that the field $K(th(\sigma))$ of K-modular functions of level $\Gamma(\sqrt{-3})$ is not in general an abelian extension of the reflex field F'_{σ}. This happens certainly, if the subgroup $S_4(\sigma)$ of the symmetric group S_4 is not abelian.

13.17 Problem. For which CM-modules σ is $K(th(\sigma))/F'_{\sigma}$ an abelian (or non-abelian) field extension ?

Let us change over from the big level groups Γ and $\Gamma(\sqrt{-3})$ to smaller ones, say to normal subgroups Γ'' of finite index of Γ. We denote by $\mathfrak{F}_K(\Gamma'')$ the algebraic closure of $K(G_4/G_2^2, G_3^2/G_2^3)$ in the field $\mathfrak{F}_{\mathbb{C}}(\Gamma'') = \mathbb{C}(\mathbb{B}/\Gamma'')$ of Γ''-automorphic functions. With obvious notations we obtain at each CM-module σ an infinite tree

$$\{\mathfrak{F}_K(\Gamma'')(\sigma);\ \Gamma''\text{ normal subgroup of }\Gamma\text{ of finite index}\}$$

of field extensions of F'_σ. The analogue construction in the theory of elliptic curves yields a generating system of the maximal abelian extension of an imaginary quadratic number fields. It would be interesting to understand our construction in a suitable framework of (non-abelian) class field theory. Especially, one has to investigate the action of (subgroups of) the factor groups Γ/Γ'' in the towers $\mathfrak{F}_K(\Gamma'')(\sigma) \supseteq \mathfrak{F}_K(\Gamma)(\sigma) \supseteq F'_\sigma$ of number fields at special CM-values σ.

13.18. Remark. Let $\tau \in \mathbb{B}(\bar{\mathbb{Q}})$ be an arithmetic point of the ball. We proved that

$$th(\tau) = (th_1(\tau):th_2(\tau):th_3(\tau):th_4(\tau)) \in \mathbb{P}^2$$

is arithmetic, if τ is a CM (or DCM)-module. The converse implication seems to be true. Very recently Shiga [34] succeeded to prove that at least at simple (simple J_τ) arithmetic modules τ the point $th(\tau)$ is transcendent, that means a non-algebraic point of $\mathbb{P}^2$, if τ is not a CM-module.

13.19. Problem. What happens precisely at non-simple arithmetic modules in both cases, the case of CM-modules and the opposite case ?

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UEBERSICHT

- Shimura's Class Fields
- Complex Multiplication
- Moduli Fields
- Main Theorem of Complex Multiplication
- The Geometric Starting Point, the Projective Plane Covered by the Ball
- Differential Operators
- Gauss-Manin Connection
- Moduli Space for Picard Curves $Y^3 = P_4(X)$
- The Relative Schottky Problem for Picard Curves
- Effective Torelli Theorem for Picard Curves via Picard Modular Forms
- Picard Modular forms as Theta Constants
- Number fields generated by special values

Definition:

A s o l u t i o n m o d e l for Hilbert's twelvth problem is a triple (V, V_{sing}, f) consisting of

(i) a (non-compact) complex manifold V with fixed analytic embedding into a complex projective space $\mathbb{P}^M(\mathbb{C})$;

(ii) a subset V_{sing} of the algebraic points $V(\bar{\mathbb{Q}}) = V \cap \mathbb{P}^M(\bar{\mathbb{Q}})$ lying dense in V;

(iii) A transcendent holomorphic map

$$f = (f_0 : f_1 : \dots : f_N) : V \longrightarrow \mathbb{P}^N(\mathbb{C});$$

satisfying the postulates I.,II.,III. below.

We call f t r a n s c e n d e n t if f is not the restriction of a rational map in the sense of algebraic geometry. The elements of V_{sing} are called the s i n g u l a r p o i n t s (below: singular moduli) of V.

I. $f(\sigma) = (f_0(\sigma) : \dots : f_N(\sigma))$ is algebraic, that means $f(\sigma) \in \mathbb{P}^N(\bar{\mathbb{Q}})$, for $\sigma \in V_{sing}$;

II. $f(\tau)$ is transcendent, that means $f(\tau) \notin \mathbb{P}^N(\bar{\mathbb{Q}})$, for $\tau \in V(\bar{\mathbb{Q}}) - V_{sing}$;

III. one has a number-theoretic construction / quality / meaning of field extensions

$$F_\sigma'(f(\sigma)) = F_\sigma'(\dots, f_i(\sigma)/f_j(\sigma), \dots)$$

for suitable well-defined "elementary" number fields F_σ', $\sigma \in V_{sing}$.

A (twodimensional) b a l l m o d e l for Hilbert's twelvth problem is a solution model $(\mathbb{B}, \mathbb{B}_{sing}, f)$, where $\mathbb{B}$ is the complex two-dimensional unit ball.

ad II. Recently Shiga (1990) proved that II. is essentially satisfied for our model below (see Main Theorem)

ARBEITSPROBLEME

1. Problem. Study special values of Picard modular functions of higher level in connection with non-abelian class field theory.
2. Problem. Generate more (if possible all) abelian extensions of reflex fields of cubic extensions of the Eisenstein numbers by means of special values of some additional transcendent functions.
3. Problem. What happens at DCM-points (ball points where the Jacobian threefolds of the corresponding Picard curves have "decomposed complex multiplication") ?
4. Problem. Check Vojta's conjectures for Picard curves (consequences: either disproof of Parshin conjecture a la Miyaoka-Yau for arithmetic surfaces, or asymptotic Fermat).

5. Problem. Find more solution models for Hilbert's twelvth problem (along: uniformizations of Picard modular surfaces, Hilbert modular surfaces, Siegel modular threefold; Hecke's thesis).

6. Arithmetic of Picard curves (more general: cyclic curves $y^m = p_n(x)$): Effective proof of Shafarevich conjecture; number of k-rational points of such curves defined over finite fields k (J. Estrada-Sarlabous).