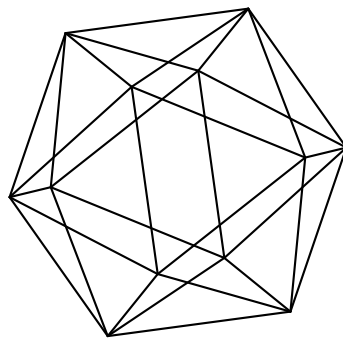# Max-Planck-Institut für Mathematik Bonn

An explicit conductor formula for $\mathrm{GL}(n) \times \mathrm{GL}(1)$ and functorial depth preservation

by

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AN EXPLICIT CONDUCTOR FORMULA FOR $\mathrm{GL}(n) \times \mathrm{GL}(1)$ AND FUNCTORIAL DEPTH PRESERVATION

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ABSTRACT. We prove an explicit formula for the conductor of an irreducible, admissible representation of $\mathrm{GL}(n, F)$ twisted by a character of $F^\times$ where the field F is local and non-archimedean. The components of our formula are analysed, both explicitly and on average, for application to the analytic study of automorphic forms on $\mathrm{GL}(n)$. We also demonstrate that "functoriality preserves depth" for such Rankin–Selberg convolutions and give a precise conjecture in the $\mathrm{GL}(n) \times \mathrm{GL}(m)$ case.

1. THE TWISTED CONDUCTOR PROBLEM

Let F denote a non-archimedean local field of characteristic zero and let $n \geq 2$. For an irreducible, admissible representation π of $\mathrm{GL}(n, F)$ and a quasi-character χ of $F^\times$, form the twist $\chi\pi = (\chi \circ \det) \otimes \pi$. Our main result (Theorem 1.6) is an explicit formula for the conductor $a(\chi\pi)$, c.f. the Artin conductor, as defined in §2.1. This formula is given by

$$a(\chi\pi) = a(\pi) + \Delta_\chi(\pi) - \delta_\chi(\pi) \tag{1}$$

where $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ are non-negative integers as defined in Theorem 1.6; they denote a *dominant* and a *non-twist-minimal interference* term, respectively. We give detailed analysis of these terms in §2.3, answering questions such as "for how many χ is there interference?"

Our primary motivation was originally to formulate a tactile formula for $a(\chi\pi)$, sensitive to fluctuations in the analytic behaviour of automorphic forms on $\mathrm{GL}(n)$. A second objective soon became to prove an explicit formula for the *depth* of any irreducible, admissible representation of $\mathrm{GL}(n, F)$.

The depth $\rho(\pi)$ is a rational number defined analogously the integer $a(\pi)$, when considered from the point of view of newform theory (see §2.1). In fact in Theorem 3.3 we prove a formula explicitly relating the two. However, unlike the conductor, the depth is widely expected to be preserved under Langland's functoriality conjecture and in particular the transfers of representations therein (see [15–17, 19]).

In the case at hand, the Rankin–Selberg transfer from $\mathrm{GL}(n) \times \mathrm{GL}(1)$ to $\mathrm{GL}(n)$, we provide an affirmative answer to this question: the depth $\rho(\pi \boxtimes \chi) = \rho(\chi\pi)$ *is* preserved (see Theorem 3.6). Moreover, our results encourage us to predict an explicit formula asserting depth preservation under the $\mathrm{GL}(n) \times \mathrm{GL}(m)$ to $\mathrm{GL}(mn)$ Rankin–Selberg transfer for all $m \geq 1$ (see Conjecture 3.7).

As an example, computing $a(\chi\pi)$ in the limit $a(\chi) \to \infty$ is straightforward: from Proposition 1.2 and Equation (5) we deduce

$$a(\chi\pi) = na(\chi) \tag{2}$$

whenever $a(\chi) > a(\pi)$. In this case $\Delta_\chi(\pi) = na(\chi) - a(\pi)$ and $\delta_\chi(\pi) = 0$. Bushnell–Henniart extend (2) by proving the upper bound

$$a(\chi\pi) \leq \max\{a(\pi), a(\chi)\} + (n-1)a(\chi), \tag{3}$$

permitting extra summands in the presence of smaller values of $a(\chi)$. This is a special case[1] of [4, Theorem 1], and indeed our own Theorem 1.6. In fact this bound is sharp in that it is attained for *some* π and χ; as in (2) for example.

However, in general such examples become sparse, rendering (3) as rather coarse as one averages over χ for large $a(\pi) \to \infty$. In such cases, understanding the integers $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ *exactly* is of crucial importance for numerous problems in analytic number theory. For instance, when investigating the analytic behaviour of automorphic forms and their L-functions, such a formula is the unavoidable consequence of two techniques: taking harmonic $\mathrm{GL}(1)$-averages combined with the use of the functional equation for $\mathrm{GL}(n) \times \mathrm{GL}(1)$-$L$-functions.

Such character twists arise in the work of Nelson–Pitale–Saha [18], who address the quantum unique ergodicity conjecture for holomorphic cusp forms with "powerful" level (see [18, Remarks 1.9 & 3.16]).

The current record for upper and lower bounds for the sup-norm of a Maaß-newform on $\mathrm{GL}(2)$ in the level aspect [24–26] depends crucially on the $n = 2$ case of Theorem 1.6. An advantage of working locally there is that such conductor formulae automatically hold in the number field setting, where the strongest bounds for the sup-norm are proved in a forthcoming work of Edgar Assing.

Of a more constructive flavour, in [1], Brunault computed the value of ramification indeces of modular parameterisation maps of various elliptic curves (of conductor N) over $\mathbb{Q}$. Whenever a newform attached to E is "twist minimal", Brunault could prove that this index was trivial (equal to 1), holding in particular whenever N is square-free. This problem was recently solved by Saha and the present author [6] in full generality. In our solution, the subtleties behind evaluating conductors of twists explicitly gives rise to the few examples of non-trivial ramification indices.

These results all concern the case $n = 2$, where the conductor formula for twists of supercuspidal representations was given by Tunnell [30, Proposition 3.4] in his thesis; see [6, Lemma 2.7] for the general case. Tunnell himself applied his formula to count isomorphism classes of supercuspidal representations of fixed (odd) conductor (see [30, Theorem 3.9]). He used this observation to prove the local Langlands correspondence for $\mathrm{GL}(2, F)$ in the majority of cases.

The present result is suggestive of similar applications: a bound for local Whittaker newforms (and a corresponding global sup-norm bound) in the level aspect;

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[1]The purpose of [4] is to establish a variant of (3) for $\mathrm{GL}(n) \times \mathrm{GL}(m)$-conductors where, more generally, $n, m \geq 1$.

bounds for matrix coefficients of local representations, and estimates relating to the Voronoĭ summation problem for $\mathrm{GL}(n)$, to name a few.

Here, in §3, we give an application of our formula to the study of depth of irreducible, admissible representations of $\mathrm{GL}(n, F) \times \mathrm{GL}(1, F)$. In §2 we provide a detailed analysis of the terms $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ in (1). Lastly, in §4, we give a uniform proof of (1) for all quasi-square-integrable representations (see Proposition 1.2); we use such representations as building blocks in the general case, which we now establish in §1.1.

1.1. **A minimal classification and the explicit conductor formula.**

1.1.1. *The Langlands classification for* $\mathrm{GL}(n, F)$. Let $\mathscr{A}_F(n)$ denote the set of (equivalence classes of) the irreducible, admissible representations of $\mathrm{GL}(n, F)$. The natural building blocks that describe $\mathscr{A}_F(n)$ are the *quasi-square-integrable* representations; that is, the $\pi \in \mathscr{A}_F(n)$ for which there exists an $\alpha \in \mathbb{R}$ such that the matrix coefficients of $|\cdot|^\alpha \pi$ are square-integrable on $\mathrm{GL}(n, F)$ modulo its centre.

The so-called 'Langlands classification' (due to Berstein–Zelevinsky) describes the structure of each representation in the graded ring $\mathscr{A}_F = \oplus_{n \geq 1} \mathscr{A}_F(n)$ in terms of the subset $\mathscr{S}\mathscr{G}_F$ of quasi-square-integrable representations. By [33, Theorems 9.3 & 9.7], one deduces an addition law $\boxplus$ on $\mathscr{S}\mathscr{G}_F$, by which $\mathscr{S}\mathscr{G}_F$ generates a free commutative monoid Λ. The classification is then the assertion that there is a bijection between $\mathscr{A}_F$ and the semi-group of non-identity elements in Λ, thus endowing $\mathscr{A}_F$ with the addition law $\boxplus$. Crucially, the maps $(\mathscr{A}_F, \boxplus) \to (\mathbb{C}, \cdot)$ given by applying L- or ε-factors are homomorphisms of semi-groups (see [31, §2.5] for their definitions). Both expositions [20, 31] provide excellent background on this topic.

The upshot of this classification being that for *any* $\pi \in \mathscr{A}_F(n)$ there exists a unique partition $n_1 + \cdots + n_r = n$ alongside a collection of quasi-square-integrable representations $\pi_i \in \mathscr{S}\mathscr{G}_F \cap \mathscr{A}_F(n_i)$ for $1 \leq i \leq r$ such that

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r \tag{4}$$

and, for any quasi-character χ of $F^\times$,

$$a(\chi\pi) = a(\chi\pi_1) + \cdots + a(\chi\pi_r). \tag{5}$$

Equation (5) follows from the definition of the ε-factor and formula (11) in §2.1.

1.1.2. *Minimality and the formula for quasi-square-integrable representations.*

Definition 1.1. An irreducible, admissible representation π of $\mathrm{GL}(n, F)$ is called *twist minimal* if $a(\pi)$ is the least integer amongst the conductors $a(\chi\pi)$, ranging over all quasi-characters χ of $F^\times$.

In particular, if a quasi-square-integrable representation $\pi \in \mathscr{S}\mathscr{G}_F$ is not twist minimal then $n \mid a(\pi)$. For these representations, the notion of twist-minimality is sufficient to give an exact formula.

Proposition 1.2. *Let π be an irreducible, admissible, square-integrable representation of $\mathrm{GL}(n, F)$ and let χ be a quasi-character of $F^{\times}$. Then*

$$a(\chi\pi) \leq \max\{a(\pi),\, na(\chi)\} \tag{6}$$

with equality in (6) whenever π is twist minimal or $a(\pi) \neq na(\chi)$.

Remark 1.3. In practice, one handles those π which are *not* twist minimal as follows: tautologically, write $\pi = \mu\pi^{\mathrm{min}}$ where μ is a quasi-character of $F^{\times}$ and π^{min} *is* twist minimal. Then Proposition 1.2 implies $a(\chi\pi) = \max\{a(\pi^{\mathrm{min}}),\, na(\chi\mu)\}$. It is the collusion of the characters χ and μ that give rise to any degeneracies.

Let us draw attention to the conductor formula of Bushnell–Henniart–Kutzko [5, Theorem 6.5] for the (more general) $\mathrm{GL}(n) \times \mathrm{GL}(m)$-pairs of *supercuspidal* representations. There they deploy the full structure theory of supercuspidal representations to proving an outstanding identity, relating the conductor to the respective inducing data. Proposition 1.2 may indeed be derived from their work.

However, in our case with $m = 1$, the formula is more simple and holds uniformly for the larger set $\mathscr{SG}_F$, which contains the supercuspidals (c.f. the known formula for the Steinberg representation [23, p. 18]). Indeed, we are able to give an elementary proof of Proposition 1.2. This promotes our philosophy that, as far as the conductor is concerned, the set $\mathscr{SG}_F$ (and in particular the subset of twist minimal elements) contains sufficient and necessary information to explicitly determine the conductor via the decomposition given in (4).

We defer our proof of Proposition 1.2 until §4.4. The arguments made there are also used determine a result on the central character.

Proposition 1.4. *Let π be an irreducible, admissible, square-integrable representation of $\mathrm{GL}(n, F)$ with central character $\pi|_{F^{\times}} = \omega_{\pi}$. Then*

$$a(\omega_{\pi}) \leq \frac{a(\pi)}{n}. \tag{7}$$

Remark 1.5. As a rule of thumb, problems arising in the "highly ramified central character aspect" come from interactions between the components in $\pi_1 \boxplus \cdots \boxplus \pi_r$ for $r \geq 2$. We mean this in the sense that otherwise $a(\omega_{\pi})$ does not influence $a(\pi)$. As such it should be handled in a separate fashion, as we do in the present work.

1.1.3. *The general formula.* We arrive at our main result, having defined a sufficient and necessary set of properties of representations $\pi \in \mathscr{A}_F$ in order to give a fully explicit formula for the conductor.

Theorem 1.6. *Let π be an irreducible, admissible representation of $\mathrm{GL}(n, F)$ given in terms of quasi–square–integrable representations π_i of $\mathrm{GL}(n_i, F)$, as described in (4), where $n = n_1 + \cdots + n_r$; write $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$. Let χ be a quasi-character of $F^{\times}$. Then*

$$a(\chi\pi) = a(\pi) + \Delta_{\chi}(\pi) - \delta_{\chi}(\pi)$$

where Δ_χ and δ_χ are defined by the semi-group homomorphisms $(\mathscr{A}_F, \boxplus) \to (\mathbb{Z}_{\geq 0}, +)$ given by their values on the representations $\pi_i = \mu_i \pi_i^{\min}$, the twist of a minimal representation $\pi_i^{\min}$ and a quasi-character μ of $F^\times$, as follows:

$$\Delta_\chi(\pi_i) = \begin{cases} \max\{n_i a(\chi) - a(\pi_i),\, 0\} & \text{if } a(\chi) \neq a(\mu_i) \\ 0 & \text{if } a(\chi) = a(\mu_i) \end{cases}$$

and

$$\delta_\chi(\pi_i) = \begin{cases} a(\pi_i) - \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\} & \text{if } a(\chi) = a(\mu_i) \\ 0 & \text{if } a(\chi) \neq a(\mu_i). \end{cases}$$

Both terms are non-negative for any π and χ.

Proof. By applying Proposition 1.2 to Equation (4) we have

$$a(\chi\pi) = \sum_{i=1}^r \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\}. \tag{8}$$

In particular, if $a(\chi) \neq a(\mu_i)$ for a given $1 \leq i \leq r$ then the respective summand in (8) is equal to $\max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\} = \max\{a(\pi_i),\, n_i a(\chi)\}$. Whenever we have $a(\chi) \neq a(\mu_i)$ we also have $a(\pi_i) \geq \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\}$. $\square$

Remark 1.7. In the special case $n = 2$, we prove Theorem 1.6 in [6, Lemma 2.7]. In general, one should understand the non-vanishing of $\delta_\chi(\pi)$ as occurring rarely. Whereas, $\Delta_\chi(\pi)$ describes the dominant or "usual" behaviour of $a(\chi\pi)$. We make these statements explicit and quantitative in §2.3.

Corollary 1.8. *Let $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ and χ be as in Theorem 1.6 with $\pi_i = \mu_i \pi_i^{\min}$ for twist minimal representations $\pi_i^{\min}$. Form the "totally minimal" representation $\pi^{\mathrm{tot}} = \pi_1^{\min} \boxplus \cdots \boxplus \pi_r^{\min}$ and let $\Omega_\chi(\pi) = \{1 \leq i \leq r \,:\, a(\pi_i) > n_i a(\chi)\}$. Then*

$$a(\pi^{\min}) \leq a(\chi\pi) \leq a(\pi) + a(\chi)\left(n - \sum_{i \in \Omega_\chi(\pi)} n_i\right). \tag{9}$$

Proof. The lower bound of (9) follows immediately from (8). For $i \in \Omega_\chi(\pi)$ we have $\Delta_\chi(\pi_i) = \delta_\chi(\pi_i) = 0$. The upper bound is then obtained by estimating $\Delta_\chi(\pi_i)$ and $\delta_\chi(\pi_i)$ respectively by $a(\pi_i) + n_i a(\chi)$ for $i \notin \Omega_\chi(\pi)$. $\square$

Proof of Inequality (3). We recover Bushnell–Henniart's bound (3) using Corollary 1.8. If $a(\chi) > a(\pi)$ then $a(\pi) = n a(\chi)$ by (8), whilst if $a(\chi) \leq a(\pi)$ then (3) is a special case of (9) since $\Omega_\chi(\pi) \neq \emptyset$ in this case and each $n_i \geq 1$. $\square$

Remark 1.9 (Generic representations). Legibility permitting, the letter $\mathscr{G}$ indicates that all representations in $\mathscr{SG}_F$ are *generic*. By showing so for the regular representation of $\mathrm{GL}(n, F)$, of a given central character, Jacquet shows that all discrete series representations are generic [11, Theorem 2.1, (3)]. By the Langlands classification (denoting as in (4)), any $\pi \in \mathscr{A}_F(n)$ as is generic (a.k.a.

"non-degenerate") if and only if π is equivalent to the (irreducible) representation induced from the *external* tensor product $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ on the parabolic subgroup associated to $n_1 + \cdots + n_r$ (by [33, Theorem 9.7, (a)]). The elements of $\mathscr{SG}_F$ correspond to those irreducible representations with $r = 1$.

2. THE SIZE OF SPACES OF TWIST-FIXING CHARACTERS

The goal of this section is twofold: in §2.2 we count the number of characters χ that fix the conductor $a(\chi\pi)$ at a given value. Then, in §3.3, we explicitly analyse the behaviour of the dominant and interference terms in Theorem 1.6. These questions are motivated by their applications to analytic number theory.

2.1. **Basic notations: characters and conductors.** Let π denote an irreducible, admissible representation of $\mathrm{GL}(n, F)$ and χ a quasi-character of $F^\times$. Let $\tilde{\pi}$ be the contragredient representation and ω_π the central character of π, respectively.

2.1.1. *The non-archimedean local field.* We denote by $\mathfrak{o}$ the ring of integers of F; $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$; ϖ a choice of uniformising parameter, that is a generator of $\mathfrak{p}$, and $q = \#(\mathfrak{o}/\mathfrak{p})$. Let $|\cdot|$ denote the absolute value on F, normalised so that $|\varpi| = q^{-1}$ and v_F the valuation on F defined via $|x| = q^{-v_F(x)}$ for $x \in F$. We define open neighbourhoods $U_F(m)$ of 1 in $U_F(0) = \mathfrak{o}^\times$ by $U_F(m) = 1 + \varpi^m \mathfrak{o}$ for $m > 0$. Define the integer $a(\chi)$ to be the least $m \geq 0$ such that $\chi(U_F(m)) = \{1\}$. Let $K = \mathrm{GL}(n, \mathfrak{o})$ and for each $m \geq 0$ let $K_1(m)$ be the subgroup of K that stabilises the row vector $(0, \ldots, 0, 1)$, from the right, modulo $\mathfrak{p}^m$.

2.1.2. *The floor and ceiling functions.* For each $\alpha \in \mathbb{R}$ let $\lfloor \alpha \rfloor$ denote the *floor* of α, defined via $\lfloor \alpha \rfloor = m$ if and only if $m \in \mathbb{Z}$ and $m \leq \alpha < m + 1$. Similarly, let $\lceil \alpha \rceil$ denote the *ceiling* of α if $\lceil \alpha \rceil \in \mathbb{Z}$ such that $\lceil \alpha \rceil - 1 < \alpha \leq \lceil \alpha \rceil$.

2.1.3. *Epsilon constants and the conductor.* Here we define the integer $a(\pi)$, the conductor of π. This subsumes the definition of $a(\chi)$ when $n = 1$, for which the following notions were originally founded in Tate's thesis [27].

Let ψ be an additive character of F of exponent $n(\psi) = \min\{m : \psi|_{\mathfrak{p}^m} = 1\}$. Godement–Jacquet prove the existence of ε-factors $\varepsilon(s, \pi, \psi) \in \mathbb{C}[q^{-s}, q^s]$ in [9, Theorem 3.3, (4)]. Applying the functional equation there twice one obtains

$$\varepsilon(s, \pi, \psi)\varepsilon(1 - s, \tilde{\pi}, \psi) = \omega_\pi(-1). \tag{10}$$

Hence $\varepsilon(s, \pi, \psi)$ is a unit in $\mathbb{C}[q^{-s}, q^s]$; that is, a $\mathbb{C}^\times$-constant multiple of an integral power of q^{-s}. Explicitly, using [9, (3.3.5)] we prove

$$\varepsilon(s, \pi, \psi) = \varepsilon(1/2, \pi, \psi)\, q^{(a(\pi) - n(\psi)n)(\frac{1}{2} - s)}, \tag{11}$$

in which we *implicitly define the conductor* $a(\pi)$. After the proof of the local Langlands corresponds for $\mathrm{GL}(n, F)$ [10], the conductor $a(\pi)$ coincides with the Artin conductor of an n-dimensional Weil–Deligne representation. A fundamental property of ε-factors is that $\varepsilon(s, \chi\pi, \psi) = \prod_{i=1}^{r} \varepsilon(s, \chi\pi_i, \psi)$ for $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$, as in (4) (see [9, Theorem 3.4]). This observation proves (5) by applying (11). Moreover, if π is generic, the conductor $a(\pi)$ may be interpreted in terms of newform theory as we immanently explain.

2.1.4. *Conductors of generic representations and newform theory.* Now assume that π is *generic* (see Remark 1.9). Then the conductor $a(\pi)$ may be equivalently constructed in a language more familiar to the theory of automorphic forms: let us re-define the conductor $a(\pi)$ of π to be the least non-negative integer m such that π contains a non-zero $K_1(m)$-fixed vector.

The fundamental theorem of newform theory is that the space of $K_1(a(\pi))$-fixed vectors is one-dimensional. This theorem is due to Gelfand–Každan [8] in the present context. Moreover, in the generic case, the coincidence of the definitions for $a(\pi)$ we give in §2.1.3 and §2.1.4 is an important result due to Jacquet–Piatetski-Shapiro–Shalika [13, Théorème (5)].

2.2. **Spaces of twist-fixing characters.**

2.2.1. *Characters of a given conductor.* The valuation v_F defines a split exact sequence $1 \longrightarrow \mathfrak{o}^\times \longrightarrow F^\times \xrightarrow{v_F} \mathbb{Z} \longrightarrow 1$. We thus write any quasi-character χ on $F^\times$ as $\chi(x) = \chi'(x)q^{-v_F(x)m}$ for some $m \in \mathbb{Z}$ and a character χ' of $F^\times$ such that $\chi'(\varpi) = 1$, denoting the space of such χ' by $\mathfrak{X}$ so that the unitary dual of $\mathfrak{o}^\times$ satisfies $\hat{\mathfrak{o}}^\times \cong \mathfrak{X}$. With interest in characters that fix the conductor under twisting, we define the following $\mathfrak{X}$-subsets:

$$\mathfrak{X}(k) = \{\chi \in \mathfrak{X} \,:\, a(\chi) \leq k\}; \quad \mathfrak{X}'(k) = \{\chi \in \mathfrak{X} \,:\, a(\chi) = k\}; \quad (12)$$

and

$$\mathfrak{X}'_\pi(k, j) = \{\chi \in \mathfrak{X} \,:\, a(\chi) = k \text{ and } a(\chi\pi) = j\}. \quad (13)$$

Our present point of departure shall be to give sharp upper bounds for the set $\mathfrak{X}'_\pi(k, j)$ via Corollary 1.8. In contrast, the cardinalities of $\mathfrak{X}(k)$ and $\mathfrak{X}'(k)$ are straightforward to compute.

Lemma 2.1. *For each $k \geq 1$, $\#\mathfrak{X}(k) = q^{k-1}(q - 1)$, $\#\mathfrak{X}'(1) = q - 2$, and for $k \geq 2$, $\#\mathfrak{X}'_k = q^{k-2}(q - 1)^2$.*

Proof. Consider the subgroup series $\{1\} = \mathfrak{X}(0) \leq \mathfrak{X}(1) \leq \cdots \leq \mathfrak{X}(k) \leq \mathfrak{X}$. For $k \geq l \geq k/2 \geq 1$, we have $\mathfrak{X}(k)/\mathfrak{X}(l) \cong U_F(l)/U_F(k) \cong \mathfrak{o}/\mathfrak{p}^{k-l}$. In particular, taking $l = k - 1$ and noting $\mathfrak{X}(1) \cong (\mathfrak{o}/\mathfrak{p})^\times$, one counts the given cardinalities inductively. The number $\#\mathfrak{X}'$ is obtained by subtraction. $\square$

We remark that in [6, Lemmas 2.1 & 2.2] we considered elements of $\mathfrak{X}'(k)$ which fix, or almost fix, the conductor of a given character χ, essentially characterising the existence of elements in $\mathfrak{X}'_\chi(a(\chi), j)$ as q becomes small. Now we consider a "non-Arbelian" variant of this result for the set $\mathfrak{X}'_\pi(k, j)$.

2.2.2. *The quasi-square-integrable case.* First consider those $\pi \in \mathscr{SG}_F \cap \mathscr{A}_F(n)$ so that Proposition 1.2 applies. For integers an $k, j \geq 0$, if either π is minimal or $k \neq a(\pi)/n$ then

$$\mathfrak{X}'_\pi(k, j) = \begin{cases} \mathfrak{X}'_\pi(k) & \text{if } j = \max\{a(\pi),\, nk\} \\ \emptyset & \text{if } j \neq \max\{a(\pi),\, nk\}. \end{cases} \quad (14)$$

These cases correspond to $\ell = 0$ in the following lemma.

Lemma 2.2. *For each $\pi \in \mathscr{SG}_F \cap \mathscr{A}_F(n)$ write $\pi = \mu\pi^{\min}$ for a twist minimal representation $\pi^{\min}$. For integers $j, k \geq 0$ we have $\mathfrak{X}'_\pi(k, j) = \emptyset$ unless $a(\pi^{\min}) \leq j \leq \max\{a(\pi),\, nk\}$, in which case there exists an $\ell \geq 0$ such that $j = \max\{a(\pi),\, nk\} - \ell$ and*

$$\#\mathfrak{X}'_\pi(k, j) \leq \#\mathfrak{X}\left(k - \left\lfloor \tfrac{\ell}{n} \right\rfloor\right). \tag{15}$$

Proof. If either π is minimal or $k \neq a(\pi)/n$ then the lemma follows by (14). Hence assume $a(\pi) = kn$ and $\pi = \mu\pi^{\min}$ where $\pi^{\min}$ is twist minimal with $a(\pi^{\min}) < a(\pi)$ and $\mu \in \mathfrak{X}'(k)$. Then $\mathfrak{X}'_\pi(k, j) = \emptyset$ unless $a(\pi^{\min}) \leq j \leq nk$. In this case, if there exists a $\chi \in \mathfrak{X}'(k)$ such that $\max\{a(\pi^{\min}),\, na(\chi\mu)\} = j$ then there are $\#\mathfrak{X}(\lfloor j/n \rfloor)$ of them as we must have $\chi \in \mu^{-1}\mathfrak{X}(\lfloor j/n \rfloor)$. $\qquad\square$

2.2.3. *The general case.* Lemma 2.2 may be assembled to describe all of $\mathscr{A}_F(n)$.

Proposition 2.3. *Let π be an irreducible, admissible representation of $\mathrm{GL}(n, F)$ and π^{tot} a totally minimal representation attached to π as in Corollary 1.8. For integers $j, k \geq 0$ we have $\mathfrak{X}'_\pi(k, j) = \emptyset$ unless j satisfies inequality (9), in which case*

$$\#\mathfrak{X}'_\pi(k, j) \leq \#\mathfrak{X}\left(\left\lfloor \tfrac{j}{n} \right\rfloor\right). \tag{16}$$

Proof. Write $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ as in (4). Set $j' = l - \sum_{i \notin \Phi} a(\chi\pi_i)$ where $\Phi = \{1 \leq i \leq r : a(\pi_i) = n_i a(\chi) \text{ and } \pi_i = \mu_i\pi_i^{\min} \text{ for } a(\pi^{\min}) < a(\pi_i)\}$. Then (14) applies when $i \notin \Phi$. We are left answering the question of how many $\chi \in \mathfrak{X}'(k)$ satisfy

$$j' = \sum_{i \in \Phi} \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\}.$$

For each summand, this is answered by Lemma 2.2. We conclude that j' satisfies $\sum_{i \in \Phi} a(\pi_i^{\min}) \leq j' \leq \sum_{i \in \Phi} \max\{a(\pi_i),\, nk\}$, as per (9), and for some $i \in \Phi$ we have

$$\#\mathfrak{X}'_\pi(k, j) \leq \#\mathfrak{X}\left(\max\left\{\left\lfloor \frac{a(\pi_i^{\min})}{n_i} \right\rfloor,\, a(\chi\mu_i)\right\}\right). \tag{17}$$

$\qquad\square$

Remark 2.4. In practice, the estimate $q^{j/n}$ for (16) often occurs elsewhere, often working *against* upper bounds for a given summation problem. Proposition 2.3 acts to counteract such trivially bounded terms. The bound (16) may be improved to (17) with the general notation of $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$.

2.3. **The leading and interference terms.** Here we detail the asymptotic behaviour of $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$. Our first port of call is to describe the rarity with which the interference term satisfies $\delta_\chi(\pi) \neq 0$. The following lemma follows directly from the definition of $\delta_\chi(\pi)$ in Theorem 1.6.

Lemma 2.5 (Absence of interference)**.** *Let π be an irreducible, admissible representation of $\mathrm{GL}(n, F)$ written, as in (4), in terms of irreducible, quasi-square-integrable representations, $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$. Recall that $\pi_i \in \mathscr{SG}_F$ is a representation of $\mathrm{GL}(n_i, F)$ for $1 \leq i \leq r$. Let χ be a quasi-character of $F^\times$.*

(1) *We have $\delta_\chi(\pi) = 0$ if $n_i \nmid a(\pi_i)$ for each $1 \leq i \leq r$.*

(2) *If there exists an $1 \leq i \leq r$ such that $n_i \mid a(\pi)$ then $\delta_\chi(\pi_i) = 0$ if $a(\pi_i) \neq n_i a(\chi)$.*

(3) *Suppose there exists an $1 \leq i \leq r$ such that $a(\pi_i) = n_i a(\chi)$, then $\delta_\chi(\pi_i) = 0$ if and only if $a(\chi\mu_i) = a(\chi)$ where $\pi_i = \mu_i \pi_i^{\min}$ is written as the μ_i-twist of a twist minimal representation $\pi_i^{\min}$.*

Our second port of call is to quantify the rarity described in Lemma 2.5.

Lemma 2.6 (Quantitative interference)**.** *Let $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ as in Lemma 2.5. Suppose $\chi \in \mathfrak{X}$ and that for some $1 \leq i \leq r$ we have $\delta_\chi(\pi_i) \neq 0$. Write $\pi_i = \mu_i \pi_i^{\min}$ as per part (3) of Lemma 2.5. Then, for each $0 < j \leq a(\pi_i) - a(\pi_i^{\min})$ satisfying $j \equiv a(\pi_i) \pmod{n_i}$, there are precisely*

$$\#\mathfrak{X}\left(\frac{a(\pi_i) - j}{n}\right) \tag{18}$$

characters $\chi \in \mathfrak{X}$ such that $\delta_\chi(\pi_i) = a(\pi_i) - j$. The number of $\chi \in \mathfrak{X}(a(\pi_i)/n)$ satisfying $\delta_\chi(\pi_i) = a(\pi_i)$ is

$$(q - 2) \cdot \#\mathfrak{X}\left(\frac{a(\pi_i)}{n} - 1\right). \tag{19}$$

Proof. The number in (18) is determined by the necessity that

$$\chi \in \mu_i^{-1}\mathfrak{X}\left(\frac{a(\pi_i) - j}{n}\right).$$

Similarly, we count to the number in (19) by observing that $\chi \in \mathfrak{X}(a(\pi_i)/n_i)$ such that χ is an element of neither $\mathfrak{X}((a(\pi_i)/n_i) - 1)$ nor $\mu_i^{-1}\mathfrak{X}((a(\pi_i)/n_i) - 1)$. $\square$

Proposition 2.7 (Dominant behaviour)**.** *In each case of Lemma 2.5 for which χ and $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ satisfy $\delta_\chi(\pi) = 0$, we have the "dominant" conductor formula*

$$a(\chi\pi) = \sum_{i=1}^{r} \max\{a(\pi_i), n_i a(\chi)\}. \tag{20}$$

3. THE DEPTH OF TWISTS OF GENERIC REPRESENTATIONS

A corollary of our conductor formula is an explicit formula for the *depth* of any twist of an irreducible, admissible representation of $\mathrm{GL}(n, F)$. We give this formula in Theorem 3.3, expanding the generality vastly over previous known

cases. In particular we include the *non-generic* representations of $\mathrm{GL}(n, F)$, on which the traditional results of newform theory cannot comment.

For an irreducible, admissible representation π, Moy–Prasad [16, Theorem 5.2] prove the existence of a fraction $\rho(\pi) \in \mathbb{Q}_{\geq 0}$, the depth of $\pi \in \mathscr{A}_F(n)$, which essentially detects the smallest parahoric subgroup on which certain characters are sufficient to determine π. This is formulated and proved in the context of more general reductive groups and beckons the popular question: 'To what extent does depth preserve functorial transfer?' We may consider our twisted representations $\chi\pi$ as transfers from (π, χ) on $\mathrm{GL}(n, F) \times \mathrm{GL}(1, F)$ to $\mathrm{GL}(n \times 1, F)$.

In §3.1 we give a short description of Moy–Prasad's construction of $\rho(\pi)$ before proving our formula for $\rho(\chi\pi)$ in §3.2 and exploring its consequences in §3.3.

3.1. **Depth and parahoric subgroups.** For $\mathrm{GL}(n, F)$, the structure theory of supercuspidal representations is best described by the theory of hereditary orders (see [3, §1] or [2, §1]). However, this structure may also be gleaned from unravelling the definition of an *unrefined minimal K-type*, as given in [16, Definition 5.1], in the context of $\mathrm{GL}(n, F)$.

Let $\mathscr{B} = \mathscr{B}(\mathrm{GL}(n), F)$ denote the Bruhat–Tit's building of $\mathrm{GL}(n, F)$ (see [29, §2.9]). The elements of $\mathscr{B}$ may be realised as (projective) *lattice sequences* $\mathcal{L} = (L_\alpha)_{\alpha \in \mathbb{R}}$; these are $\mathfrak{o}$-lattice filtrations satisfying $L_{\alpha+1} = \varpi L_\alpha$ for $\alpha \in \mathbb{R}$.

Bruhat–Tits considered the stabilisers of such lattice sequences $\mathcal{L} = (L_\alpha)_{\alpha \in \mathbb{R}}$, referring to them as *parahoric subgroups*[2] (see [29, §3.1.1]); these are given by $\mathscr{P}(\mathcal{L}, \alpha) = \{g \in \mathrm{GL}(n, \mathfrak{o}/\mathfrak{p}) : g \cdot \lambda = \lambda \pmod{\mathfrak{p}} \text{ for all } \lambda \in L_\alpha\}$. In fact we have already seen an example of such a (filtration of) parahoric subgroups: the $\mathfrak{o}$-lattices L_α generated by $\langle \varpi^{-\lceil\alpha\rceil}e_1, \ldots, \varpi^{-\lceil\alpha\rceil}e_{n-1}, e_n\rangle$ determine the lattice sequence $\mathcal{L} = (L_\alpha)_{\alpha \in \mathbb{R}}$ where $\{e_i\}_i$ denotes the canonical basis of F^n. Then we recover the subgroup series $\mathscr{P}(\mathcal{L}, \alpha) = K_1(\lceil\alpha\rceil)$ as defined §2.1.

Define $\mathscr{P}(\mathcal{L}, \alpha)^+ = \bigcup_{\beta > \alpha} \mathscr{P}(\mathcal{L}, \beta)$. Referring to [17, Theorem 3.5], Moy–Prasad prove that, for each $\pi \in \mathscr{A}_F(n)$, there exists a *minimal* $\alpha = \rho(\pi) \in \mathbb{Q}_{\geq 0}$ such that:

- the space of $\mathscr{P}(\mathcal{L}, \rho(\pi))^+$-fixed vectors is non-zero for some $\mathcal{L} \in \mathscr{B}$;
- if, for any $\mathcal{L} \in \mathscr{B}$, the space of $\mathscr{P}(\mathcal{L}, \rho(\pi))^+$-fixed vectors is non-zero then $\mathscr{P}(\mathcal{L}, \rho(\pi))^+ \neq \mathscr{P}(\mathcal{L}, \rho(\pi))$ and each of its irreducible constituents is either a character, if $\rho(\pi) > 0$, or a cuspidal representation, if $\rho(\pi) = 0$, of the quotient $\mathscr{P}(\mathcal{L}, \rho(\pi))/\mathscr{P}(\mathcal{L}, \rho(\pi))^+$.

Such irreducible constituents of the spaces of $\mathscr{P}(\mathcal{L}, \rho(\pi))^+$-fixed vectors, each coupled with the respective parahoric subgroup $\mathscr{P}(\mathcal{L}, \rho(\pi))$, precisely constitute the *unrefined, minimal K-types* of Moy–Prasad [17, §3.4].

Example 3.1. Let χ be a quasi-character of $F^\times = \mathrm{GL}(1, F)$. We claim that $\rho(\chi) = a(\chi) - 1$. To see this, for each $\alpha \in \mathbb{R}$ consider $\mathfrak{p}^{-\lceil\alpha\rceil}$ as a one-dimensional $\mathfrak{o}/\mathfrak{p}$-lattice and let $\mathcal{L} = (\mathfrak{p}^{-\lceil\alpha\rceil})_{\alpha \in \mathbb{R}}$ be the associated lattice sequence. One recovers $\mathscr{P}(\mathcal{L}, \alpha) = U_F(\lceil\alpha\rceil)$. Then $\alpha = a(\pi) - 1$ is the least such $\alpha \in \mathbb{R}$ for which

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[2]In the context of $\mathrm{GL}(n, F)$, perhaps more intuitively, the parahoric subgroups $\mathscr{P}(\mathcal{L}, \alpha)$ occur as inverse images of finite index subgroups contained in parabolic subgroups of $\mathrm{GL}(n, \mathfrak{o}/\mathfrak{p})$.

$\mathscr{P}(\mathcal{L}, \alpha) \neq \mathscr{P}(\mathcal{L}, \alpha)^+$ and $\chi(\mathscr{P}(\mathcal{L}, \alpha)) \neq \{1\}$. The restriction of χ to $U_F(\lceil \alpha \rceil)$ defines a refined, minimal K-type.

Whilst we only introduced the notion of depth for the group $\mathrm{GL}(n, F)$, the construction of §3.1, and [17, Theorem 3.5], holds essentially verbatim for more general reductive groups. This includes the Cartesian product which we consider presently.

Proposition 3.2. *For each $1 \leq i \leq r$ let π_i be an irreducible, admissible representation of $\mathrm{GL}(n_i, F)$ of depth $\rho(\pi_i)$. Let $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ denote their external tensor product, the natural representation of $\mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F)$ on each component. Then*

$$\rho(\pi_1 \boxtimes \cdots \boxtimes \pi_r) = \max\{\rho(\pi_i) : 1 \leq i \leq r\}. \tag{21}$$

Proof. For each $1 \leq i \leq r$ let $\mathcal{L}^{(i)} = (L_{\alpha_i}^{(i)})_{\alpha_i \in \mathbb{R}}$ be n_i-dimensional lattice sequences such that π_i contains a $\mathscr{P}(\mathcal{L}^{(i)}, \rho(\pi_i))^+$-fixed vector. Construct the lattice sequence $\mathcal{L} = (L_\alpha)_{\alpha \in \mathbb{R}}$ by $L_\alpha = \bigoplus_{i=1}^r L_\alpha^{(i)}$, diagonal in the $\alpha \in \mathbb{R}$ variable. By the minimality of each $\rho(\pi_i)$, the least $\alpha \in \mathbb{R}$ such that there exists a non-zero $\mathscr{P}(\mathcal{L}, \alpha)^+$-fixed vector in $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ is $\alpha = \max\{\rho(\pi_i) : 1 \leq i \leq r\}$. Moreover, each irreducible constituent of the space of such fixed vectors contains an unrefined, minimal K-type, since each component part does. $\qquad\square$

This construction could be understood as a generalisation of newform theory (see §2.1.4). Although contrastingly, when defining the depth of a representation the lattice sequence $\mathcal{L}$ is included in the minimal choice $\rho(\pi)$; in newform theory the choice of congruence subgroup is fixed. (Nevertheless that fixed choice enjoys the property of multiplicity one.) As in newform theory, one hopes to derive an expression for $\rho(\pi)$ directly from the ε-factors of (11). It is then no surprise that, even for non-generic representations, the depth $\rho(\pi)$ and the conductor $a(\pi)$ are intimately related.

3.2. **The depth formula for twists.** Here we deduce a formula for the depth $\rho(\chi\pi)$ in terms of the conductors $a(\pi)$ and $a(\chi)$ where χ is a quasi-character of $F^\times$ and π is an irreducible, admissible, representation of $\mathrm{GL}(n, F)$.

We build on work of Bushnell– Fröhlich [3] and Bushnell [2], who prove an explicit formula for the supercuspidal representations. It is there where the true inner workings of the construction of §3.1 take place.

In [15, Theorem 3.1], Lansky–Raghuram extend the formula to the quasi-square-integrable representations, and thus show that functoriality is preserved under the Jacquet–Langlands transfer. For $\pi \in \mathscr{SG}_F$ they establish

$$\rho(\pi) = \max\left\{\frac{a(\pi) - n}{n}, 0\right\}. \tag{22}$$

The key lemma in their proof is due to Moy–Prasad [17, Theorem 5.2]; it says that 'depth preserves parabolic induction'. We can thus apply (22) to Proposition 3.2 and obtain a general formula for *any* element of $\mathscr{A}_F$.

Theorem 3.3. *Let $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ be an irreducible, admissible representation of $\mathrm{GL}(n, F)$ written, as in (4), in terms of irreducible, quasi-square-integrable representations π_i of $\mathrm{GL}(n_i, F)$ where $n = n_1 + \cdots n_r$. Then*

$$\rho(\pi) = \max \left\{ \frac{a(\pi_i) - n_i}{n_i}, \, 0 \, : \, 1 \leq i \leq r \right\}. \tag{23}$$

Proof. Under the Langlands classification, π corresponds to an irreducible subquotient of the induced representation of the external tensor product $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ on the parabolic subgroup with Levi component isomorphic to $\prod_{i=1}^{r} \mathrm{GL}(n_i, F)$. By [17, Theorem 5.2] we have $\rho(\pi) = \rho(\pi_1 \boxtimes \cdots \boxtimes \pi_r)$. The depth of $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ is computed in Proposition 3.2. $\square$

Remark 3.4. A interesting novelty of this formula is its application to the non-generic representations of $\mathrm{GL}(n, F)$: even outside the realm of newform theory, we have united the exponent $a(\pi)$, in the ε-factor, with the invariant $\rho(\pi)$ associated to fixed-vectors in π. One might hope that a similar study applied to other classical groups might help understand the (absence of) newform theories there.[3]

Corollary 3.5. *Let π be an irreducible, admissible representation of $\mathrm{GL}(n, F)$ and let χ be a quasi-character of $F^{\times}$.*

(1) *Applying the result of Theorem 1.6 to (23) we obtain an an explicit formula for $\rho(\pi)$ in terms of χ and a minimal decomposition for each π_i.*
(2) *In addition, if π is quasi-square-integrable and twist minimal then*

$$\rho(\chi\pi) = \max\{\rho(\pi), \rho(\chi)\}. \tag{24}$$

Proof. Firstly, (1) is apparent. To prove (2), we first have the identity $\rho(\chi) = a(\chi) - 1$, which was the subject of Example 3.1. Then applying Theorem 1.6 to Equation (22) we recover (24). $\square$

3.3. The stability of depth under functorial transfer.
There has been much speculation over whether the depth of a representation is preserved under transfers of representations occurring in Langlands' functoriality conjecture. An example of this is our present pursuit: consider the Rankin–Selberg transfer of ireducible, admissible representations (π, χ) on $\mathrm{GL}(n, F) \times \mathrm{GL}(1, F)$ to $\chi\pi$ on $\mathrm{GL}(n, F)$.

A known case is due to Lansky–Raghuram [15], who prove that depth is preserved under the Jacquet–Langlands transfer (described in §4.3). Another interesting result is one of Pan [19], who proves a remarkable result: depth is preserved for representations lifted via the theta correspondence for any reductive dual pair.

A key lemma in both of these works, and our own, is that depth preserves parabolic induction. We believe that this fundamentally supports the conjecture that depth preserves all functorial liftings. In particular, it supports our explicit conjecture on depth of Rankin–Selberg convolutions for $\mathrm{GL}(n, F) \times \mathrm{GL}(m, F)$.

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[3]The most noteworthy attempt at installing such a newform theory is that of Roberts–Schmidt [21] who describe the (paramodular) 'local newforms for $\mathrm{GSp}(4)$'.

Presently, we are able to provide an affirmative result for the Rankin–Selberg transfer $\mathrm{GL}(n, F) \times \mathrm{GL}(1, F)$ to $\mathrm{GL}(n, F)$. For ease of expression, we restrict to twist minimal elements in $\mathscr{SG}_F$.

Theorem 3.6. *Let π be an irreducible, admissible, quasi-square-integrable representation of $\mathrm{GL}(n, F)$ and, for simplicity, suppose π is twist minimal. Let χ be a quasi-character of $F^\times$ and consider both the external tensor product representation $\pi \boxtimes \chi$ of $\mathrm{GL}(n, F) \times \mathrm{GL}(1, F)$ and the twist $\chi\pi$ of $\mathrm{GL}(n, F)$. Then*

$$\rho(\pi \boxtimes \chi) = \rho(\chi\pi) = \max\{\rho(\pi), \rho(\chi)\}. \tag{25}$$

Proof. We derived $\rho(\chi\pi) = \max\{\rho(\pi), \rho(\chi)\}$ in Corollary 3.5. On the other hand, we computed $\rho(\pi \boxtimes \chi)$ in Proposition 3.2. $\qquad\square$

Furthermore, Proposition 3.2 permits us to explicitly predict the depth of general Rankin–Selberg convolutions.

Conjecture 3.7. *For $i = 1, 2$, let π_i be an irreducible, admissible representation of $\mathrm{GL}(n_i, F)$ and consider the Rankin–Selberg convolution $\pi_1 \times \pi_2$. Then, the depth of the external tensor product $\pi_1 \boxtimes \pi_2$ is preserved upon transfer to $\pi_1 \times \pi_2$. In particular, at least if each π_i is quasi-square-integrable,*

$$\rho(\pi_1 \times \pi_2) = \max\{\rho(\pi_1), \rho(\pi_2)\}. \tag{26}$$

4. CONDUCTORS OF TWISTS VIA DIVISION ALGEBRAS

In this section we provide proofs for Propositions 1.2 and 1.4. These results apply to all quasi-square-integrable representations *uniformly*; this is reflected in our proof. In particular, our conductor formula bypasses many of the subtleties occuring in the formula for supercuspidal representations in [5].

4.1. **Central simple division algebras.** Let D be a division algebra over F of dimension $[D : F] = n^2$. Let $\mathrm{Nrd} = \mathrm{Nrd}_D$ denote the reduced norm on D. (See [14, §4.1] for a pleasant construction.) Any valuation on D may be obtained via composing the reduced norm with a valuation on F (see [28, Theorem 1.4]); let us normalise such a choice by $v_D = v_F \circ \mathrm{Nrd}$. Define a basis of neighbourhoods of $1 \in D^\times$ by $U_D(m) = \{x \in D^\times : v_D(x-1) \geq m\}$ for $m > 0$ and let $U_D(0) = \ker(v_D)$. Note that if $n = 1$ (so that $D = F$) we recover $U_D(m) = U_F(m)$. It is an important fact that the norm map $\mathrm{Nrd} \colon D^\times \to F^\times$ is surjective (see [32, Prop. 6, Ch. X-2, p. 195] for instance). Upon restriction to the above neighbourhoods, for each $m \geq 0$ we have $\mathrm{Nrd}(U_D(m)) = U_D(m) \cap F$.

Lemma 4.1. *For each $m \geq 0$ we have $U_D(m) \cap F^\times = U_F(\lceil m/n \rceil)$.*

Proof. For all $a \in F^\times$ we have $v_D(a) = v_F(\mathrm{Nrd}(a)) = v_F(a^n) = nv_F(a)$; the definition of $U_F(\lceil m/n \rceil)$ is then equivalent to that of the intersection. $\qquad\square$

Lemma 4.2. *For each $m \geq 0$ we have $\mathrm{Nrd}(U_D(m)) = U_F(\lceil m/n \rceil)$.*

Proof. This follows by applying Lemma 4.1 to $\mathrm{Nrd}(U_D(m)) = U_D(m) \cap F$. $\qquad\square$

4.2. **Twisting and conductors in a division algebra.** If χ is a quasi-character of $F^\times$ and π' an irreducible, admissible representation of $D^\times$, analogous to the unramified case we form the twist $\chi\pi' = (\chi \circ \mathrm{Nrd}) \otimes \pi'$. Define the *level* $l(\pi')$ of π' to be the least non-negative integer m such that $\pi'(U_D(m)) = \{1\}$.

Lemma 4.3. *Let π' be an irreducible, admissible representation of $D^\times$. The conductor $a(\pi')$ partaking in the Jacquet–Langlands correspondence (defined via the ε-factor) is related to the level $l(\pi)$ by the formula*

$$a(\pi') = l(\pi') + n - 1.$$

Proof. This is proved in [14, §4.3] (and stated explicitly in [14, (4.3.4)]). To assist (mathematical) translation, we remark on the following: their unit groups V_j equal our $U_D(j)$ for $j \geq 0$. Fix their character $\chi \in \mathrm{Hom}(V_j/V_{j+1}, \mathbb{C}^\times)$ by defining $\chi = \pi'|_{V_j}$ and choosing $j = l(\pi') - 1$. Then their $c \in D$, der Kontrolleur von χ, satisfies $v_D(c) = -a(\chi) = -a(\pi')$; it is constructed in [14, (4.3.1)] from where we have $v_D(c) = -n - j$, noting the non-triviality of χ on V_j. All together this implies $a(\pi') = n + j = n + l(\pi') - 1$. □

Lemma 4.4. *Let χ be a quasi-character of $F^\times$. Then*

$$l(\chi \circ \mathrm{Nrd}) = na(\chi) - n + 1.$$

Proof. By Lemma 4.2 we consider χ restricted to $U_F(\lceil m/n \rceil)$ for each $m \geq 0$. By the minimality of the $a(\chi)$, the character $\chi \circ \mathrm{Nrd}$ is trivial on $U_D(m)$ whenever

$$n(a(\chi) - 1) \leq m - 1. \tag{27}$$

By the minimality of the level, we have equality in (27) when $m = l(\chi \circ \mathrm{Nrd})$. □

4.3. **The Jacquet–Langlands correspondence for division algebras.** This special (and local) case of functoriality stipulates a bijection between the following:

- Equivalence classes of irreducible, admissible, unitary representations of $\mathrm{GL}(n, F)$ which are *square-integrable* modulo centre.
- Equivalence classes of irreducible, admissible, unitary representations of $D^\times$ where D is a central-simple F-algebra of dimension n^2.

Remark 4.5. Discrete series representations of the above two types have unitary central characters. In the stated bijection, if π corresponds to π' then their central characters agree; $\omega_\pi = \omega_{\pi'}$. Moreover, $\chi\pi$ corresponds to $\chi\pi'$ for any quasi-character χ. As a consequence of the Peter–Weyl theorem, the representations of $D^\times$ are finite dimensional (since $D^\times$ is compact modulo centre).

The correspondence as stated here is due to Rogawski [22, Theorem 5.8], where the original case $n = 2$ was famously proved by Jacquet–Langlands [12]. The most general statement allows one to replace $D^\times$ with $\mathrm{GL}(m, D)$ where D has dimension d^2 and m must satisfy $n = md$. This is established in [7].

4.4. **A proof of the formula in the quasi-square-integrable case.** We now prove Propositions 1.2 and 1.4. Assume the hypotheses and notations in these propositions; in particular, $\pi \in \mathscr{SG}_F$. We first reduce the proof to the case where π is just square-integrable.

Lemma 4.6. *For all quasi-characters χ with $a(\chi) = 0$ we have $a(\chi\pi) = a(\pi)$.*

Proof. Each $\pi \in \mathscr{SG}_F$ is generic (see Remark 1.9). Then, if $m \geq 0$, the space $\pi^{K_1(m)}$ of $K_1(m)$-fixed vectors in π is non-zero if and only if $(\chi\pi)^{K_1(m)} \neq \{0\}$. $\square$

Henceforth we assume π to be square-integrable. The generalised Jacquet–Langlands correspondence implies $a(\chi\pi) = a(\chi\pi')$ where π' is the irreducible, admissible, unitary representation of $D^\times$ associated to π as determined by [22, Theorem 5.8]. The proof of Propositions 1.2 now follows by applying Lemmas 4.3 and 4.4 to the following.

Lemma 4.7. *Let π' be an irreducible, admissible, unitary representation of $D^\times$ and χ a quasi-character of $F^\times$. Then*

$$l(\chi\pi') \leq \max\{l(\pi'), l(\chi \circ \mathrm{Nrd})\} \tag{28}$$

with equality in (28) *whenever π' is twist minimal or $l(\pi') \neq l(\chi \circ \mathrm{Nrd})$.*

Proof. By definition, $(\chi\pi')(x) = \chi(\mathrm{Nrd}(x))\pi'(x)$ for every $x \in U_D(m)$ with $m \geq 0$. One immediately obtains (28) by minimality. Equality also follows in the given cases, noting that twist minimality in $a(\pi')$ is equivalent to twist minimality in $l(\pi')$ since they are linearly related (by Lemma 4.3). $\square$

Proof of Proposition 1.4. Recall that $\pi'|_{F^\times} = \omega_\pi$. Taking $m = l(\pi')$ in Lemma 4.1 we deduce that

$$a(\omega_\pi) \leq \left\lceil \frac{l(\pi')}{n} \right\rceil < \frac{a(\pi) - n + 1}{n} + 1 = \frac{a(\pi) + 1}{n},$$

implying $na(\omega_\pi) \leq a(\pi)$. $\square$

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