Zagier Formula and a new kind of Selberg Zeta Functions

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1. INTRODUCTION

We start from formulating the results. Let $\Gamma_{\mathbf{Z}}$ be the modular group $\Gamma_{\mathbf{Z}} = PSL(2, \mathbb{Z})$ and let $Z(s; \Gamma_{\mathbf{Z}})$ be the Selberg zeta function.In the half-plane Res > 1 it is defined by means of the absolute convergent product

$$Z(s;\Gamma) = \prod_{\{P\}_{\Gamma_{\mathbf{Z}}}} \prod_{k=0}^{\infty} (1 - \mathcal{N}(P)^{-k-s})$$

where $\{P\}_{\Gamma_{\mathbf{Z}}}$ run through the set of all primitive hyperbolic conjugacy classes in $\Gamma_{\mathbf{Z}}$, $\mathcal{N}(P)$ is a norm of P. It is well known that $Z(s;\Gamma)$ can be extended meromorphically to $s \in \mathbb{C}$ and $Z(s;\Gamma)$ satisfies some functional equation. All these properties of $Z(s;\Gamma)$ are related to the Selberg trace formula.

The main result of this paper is the definition of a new kind of zeta functions. They depend on the integer parameter $n \geq 1$. For Res $\geq A > 1$ they are defined by means of the absolute convergent product

$$Z(s;\Gamma;n) = \prod_{\{P\}_{\Gamma_{\mathbf{Z}}}} (1 - \mathcal{N}(P)^{-s})^{\frac{\zeta_P(2n+1)}{\ln \mathcal{N}(P)} c(n)}$$

$\zeta_P(t)$ is some zeta function (see the main part of the paper) $c(n)$ is a constant depending on n only.

We prove that the logarithmic derivative

$$Z^{-1}(s;\Gamma_{\mathbf{Z}};n) \quad \frac{d}{ds} \quad Z(s;\Gamma_{\mathbf{Z}};n)$$

can be extended meromorphically to the half-plane Res > 0 at least. We avoid in this paper some analytic difficulties which are not principal. We consider mainly the case $n = 1$ and instead of $\Gamma_{\mathbf{Z}}$ we examine some special cycloidal subgroups Γ.

These zeta functions $Z(s;\Gamma;n)$ are related to special Zagier formula which we call to mind now.

In the paper [1] Don Zagier derived the Selberg trace formula for the modular group examined the regularized integral

$$\text{reg} \int_{F_{\mathbf{Z}}} \sum_{\gamma \in \Gamma_{\mathbf{Z}}} k(z, \gamma z) E(z, s) d\mu(z) \tag{1}$$

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for $s \in \mathbb{C}$ which is close to the pole of $E(z,s)$ $s=1$. Here $k(z,z')$ is a $PSL(2,\mathbb{R})$ invariant kernel on the hyperbolic plane $H, E(z,s)$ is the Eisenstein-Maass series. We integrate in (1) over the fundamental domain of the modular group in H. The measure μ is defined by the Poincaré metric on H.

The left hand side of the Zagier formula is as in the case of the Selberg trace formula the sum over the conjugacy classes in $\Gamma_{\mathbf{Z}}$.

$$\operatorname{reg} \sum_{\{\gamma\}\Gamma_{\mathbf{Z}}} \int_{F_\gamma} k(z,\gamma z) E(z,s) d\mu(z)$$

From the other side we make use of the expansion in eigenfunctions of the automorphic Laplacian $A(\Gamma_{\mathbf{Z}})$ for the Poincaré series

$$(2) \quad K_\Gamma(z,z') = \sum_{\gamma\in\Gamma} k(z,\gamma z'),$$

$$\sum_{\gamma\in\Gamma} k(z,\gamma z') = \sum_j h(\lambda_j) v_j(z) \overline{v_j(z')} +$$
$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(1/4 + r^2) E(z, 1/2 + ir) \overline{E(z', 1/2 + ir)} dr$$

$\{v_j\}$ is the basis of all eigenfunctions of the discrete spectrum of $A(\Gamma_{\mathbf{Z}})$, $k(z,z') = k(u(z,z')), u(z,z')$ is the Selberg fundamental invariant of two points $z, z' \in H, h(\lambda)$ is the Selberg transformation of function $k(t)$.

Finally, the integral (1) is computed by means of the zeta functions of Rankin-Selberg type. We still remember that the integral (1) diverges and we make use of appropriate its regularization.

In this way obtained Zagier formula for general "s" looks complicated but it is interesting and very important formula. The Selberg trace formula follows from the Zagier formula as the residue at pole $s=1$.

The important observation is that Zagier formula (general) is simplified for $s = 2k+1, k \in \mathbb{Z}, k \geq 1$. For these values of s we define zeta functions and we prove their analytic continuation theorem.

In principle we can define zeta functions of this type from more general integral

$$\int_F f(z) \sum_{\gamma\in\Gamma} k(z,\gamma z) d\mu(z)$$

in the following cases 1) $f(z) = E(z, 2k+1), k \in \mathbb{Z}, k \geq 1$. 2) $f(z) = v_j(z)$ is an eigenfunction of the discrete spectrum of $A(\Gamma)$. 3)

$$f(z) = (y^2 q(z) \frac{\partial}{\partial \bar{z}})^n, n \in \mathbb{Z}, n \geq 2$$

is the operator on the upper half plane, $y = \operatorname{Im} z > 0, q(z)$ is an analytic form of the weight 2, the dash means the complex conjugation, Γ is some subgroup of a finite index $\Gamma \subset \Gamma_{\mathbf{Z}}$.

We consider here only the case 1) and in more details $k = 1$.

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2. The main part. Let H be the upper half plane $H = \{z \in \mathbb{C} | y = \mathrm{Im} z > 0\}$. For beginning we do not suppose that group Γ is very special. The following assumption is sufficient. Γ is a cofinite group acting on $\mathbb{H}$, Γ has no elliptic elements and Γ has only one parabolic generator $S : z \to z + 1$.

We start now from the regularization of the Zagier integral. (see [1]) Let $\Theta(z, \psi_k)$ be some special incomplete theta-series

$$\Theta(z, \psi_k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_k(y(\gamma z)), y(z) = \mathrm{Im} z,$$

$$\psi_k(k) = \begin{cases} y^{2k+1}, & 1/4 \leq y \leq Y \\ 0, & \text{otherwise} \end{cases}$$

$Y > 0$ is some fixed number, $\Gamma_\infty \subset \Gamma$ is the subgroup generated by S.

Let $k(t) \in C_0^\infty(0, \infty)$. The following assertions are valid (see [2])

1) The series $\sum_{\gamma \in \Gamma} k(u(z, \gamma z'))$ converges absolutely for all $z, z' \in H$.
2) There is the spectral decomposition (2) (we formaly change $\Gamma_{\mathbf{Z}}$ tò Γ).
3) The integral

$$\int_F \sum_{\gamma \in \Gamma} k(u(z, \gamma z))\Theta(z, \psi_k) d\mu(z) \tag{3}$$

converges absolutely. Besides, this integral is equal to

$$\begin{aligned} &\sum_{\{\gamma\}_\Gamma} \int_{F_\gamma} k(u(z, \gamma z))\Theta(z, \psi_k) d\mu(z) = \\ &= \sum_j h(\lambda_j) \int_F |v_j(z)|^2 \Theta(z, \psi_k) d\mu(z) + \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(1/4 + \tau^2) \int_F E(z, 1/2 + i\tau)|^2 \Theta(z, \psi_k) d\mu(z) d\tau, \end{aligned} \tag{4}$$

where $\{\gamma\}_\Gamma$ run through the set of all conjugacy classes in Γ with representatives γ, F_γ is the fundamental domain of the centralizer Γ_γ of the element $\gamma \in \Gamma$ in $\Gamma, u(z, z') = \frac{|z - z'|^2}{yy'}$.

The functions $h(\lambda), k(t)$ are connected with each other by the following transformation

$$\begin{cases} \int_w^\infty \frac{k(t)dt}{\sqrt{t-w}} = Q(w), & k(t) = -\frac{1}{\pi} \int_t^\infty \frac{dQ(w)}{\sqrt{w-t}} \\ Q(e^u + e^{-u} - 2) = g(u) & \\ h(1/4 + \tau^2) = \int_{-\infty}^\infty g(u) e^{i\tau u} du, & g(u) = \frac{1}{2\pi} \int_{-\infty}^\infty h(1/4 + r^2) e^{-iu\tau} d\tau \end{cases} \tag{4}$$

For deriving the desired formula we have to find the asymptotic behavior at $Y \to \infty$ of each term in the formula (4) and thus we obtain the regularization of the Zagier integral.

We consider (4) in more details. We start from the left hand side. The set of all conjugacy classes in γ is subdivided for the identity, hyperbolic classes and parabolic classes.

a) **The identity term** in the formula (4) is equal to

$$k(0)\int_F \Theta(z,\psi_k)d\mu(z) = k(0)\int_0^\infty \frac{dy}{y^2}\int_0^1 dx\psi_k(y) = k(0)\frac{Y^{2k}}{2k} + \underset{Y\to\infty}{o(1)}$$

b) **The parabolic terms in formula (4)** The zero coefficient of the Fourier series expansion of the Eisenstein-Maass series is equal to

$$\int_0^1 E(x+iy, 2k+1)dx = y^{2k+1} + \varphi(2k+1)y^{-2k}$$

where $\varphi(s)$ is the automorphic scattering matrix (one dimensional) (see [2]). The sum of all parabolic terms in (4) is

$$2\sum_{n=1}^{\infty}\int_0^Y k(\frac{n^2}{y^2})y^{2k-1}dy + 2\varphi(2k+1)\sum_{n=1}^{\infty}\int_0^\infty k(\frac{n^2}{y^2})y^{-2k-2}dy + \underset{Y\to\infty}{o(1)} \tag{6}$$

The second sum in (6) is equal to

$$2\varphi(2k+1)\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\int_0^\infty k(t^2)t^{2k}dt = 2\varphi(2k+1)\zeta(2k+1)\int_0^\infty k(t^2)t^{2k}dt$$

$\zeta(s)$ is the Riemann zeta function.

The first sum in (6) is computed in the following way

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\int_0^Y k(\frac{n^2}{y^2})y^{2k-1}dy - k(0)\frac{Y^{2k}}{2k} &= -k(0)\frac{Y^{2k}}{2k} + \\ +\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dx\int_0^Y y^{2k-1}dy\,k(\frac{x^2}{y^2})e^{2\pi i n x} &= \frac{Y^{2k+1}}{2k+1}\cdot 2\int_0^\infty k(t^2)dt - \\ -\frac{Y^{2k}}{2k}k(0) + \sum_{n\neq 0}\int_{-\infty}^{\infty}dt\int_0^Y y^{2k}dy\,k(t^2)e^{2\pi i n y t}\end{aligned} \tag{7}$$

Making use of the integral Green formula $2k$ times at "t" we continue the equality (7)

$$\begin{aligned}&= \text{main terms} + \sum_{n\neq 0}\frac{1}{(2\pi i n)^{2k}}\int_{-\infty}^{\infty}dt\left(\frac{d^{2k}}{dt^{2k}}k(t^2)\right)\frac{e^{\pi i n t Y}-1}{2\pi i n t} + \underset{Y\to\infty}{o(1)} \\ &= \text{main terms} + 2\sum_{n=1}^{\infty}\frac{1}{(2\pi i n)^{2k+1}}\int_0^Y M_{2k}\left(\frac{x^2}{Y^2}\right)\frac{e^{2\pi i n x}-e^{-2\pi i n x}}{x}dx + \underset{Y\to\infty}{o(1)} \\ &= \frac{Y^{2k+1}}{2k+1}\cdot 2\int_0^\infty k(t^2)dt - k(0)\frac{Y^{2k}}{2k} + \frac{(-1)^k}{(2\pi)^{2k}}M_{2k}(0)\zeta(2k+1) + \underset{Y\to\infty}{o(1)}\end{aligned}$$

where $M_{2k}(t^2) = \frac{d^{2k}}{dt^{2k}} k(t^2)$ and $M_{2k}(0) = \frac{(2k)!}{k!} k^{(k)}(0)$.

Finally, the whole contribution to (4) from the parabolic conjugacy classes is

$$2\frac{Y^{2k+1}}{2k+1}\int_0^\infty k(t^2)dt - k(0)\frac{Y^{2k}}{2k} + \frac{(-1)^k}{(2\pi)^{2k}}\frac{(2k)!}{k!}k^{(k)}(0)\zeta(2k+1) + \\ + 2\varphi(2k+1)\zeta(2k+1)\int_0^\infty k(t^2)t^{2k}dt + \underset{Y\to\infty}{o(1)}$$

c) **The hyperbolic terms in (4)**

Let P be a primitive hyperbolic element. We let $\mathcal{N}(P)$ denote its norm. There exists the element $g(P) \in PSL(2,\mathbb{R})$ with the property

$$\mathcal{N}(P)z = g(P^{-1})Pg(P)z$$

for all $z \in H$. The whole contribution to (4) from the hyperbolic classes is equal to

(8)
$$\sum_{\{P\}_\Gamma}\sum_{m=1}^{\infty}\int_0^\pi \frac{d\varphi}{\sin^2\varphi}\int_1^{\mathcal{N}(P)}\frac{d\rho}{\rho}k\left(\frac{\mathcal{N}(P)^m + \mathcal{N}(P)^{-m} - 2}{\sin^2\varphi}\right)E(g(P)z, 2k+1) + \underset{Y\to\infty}{o(1)}$$

where $\{P\}$ run through the set of all primitive hyperbolic conjugacy classes in Γ.

We denote $w(P,m) = \mathcal{N}(P)^m + \mathcal{N}(P)^{-m} - 2$. We let $\sum'_{\{P\}_\Gamma}$ denote the sum in (8) over the all pairs $\{P\}_\Gamma, \{P^{-1}\}_\Gamma$. We rewrite (8) now

(9)
$$\sum_{\{P\}_\Gamma}{}'\sum_{m=1}^{\infty}\int_0^\pi \frac{d\varphi}{\sin^2\varphi}k\left(\frac{w(P,m)}{\sin^2\varphi}\right)\int_1^{\mathcal{N}(P)}\frac{d\rho}{\rho}\left(E(g(P)z, 2k+1) + \right. \\ \left. + E(g(P^{-1})z, 2k+1)\right) + \underset{Y\to\infty}{o(1)}, z = \rho e^{i\varphi}.$$

It follows from the definition of the Eisenstein-Maass series

(10)
$$\int_1^{\mathcal{N}(P)}\frac{d\rho}{\rho}(E(g(P)z, 2k+1) + E(g(P^{-1})z, 2k+1)) = \\ = (\sin^{2k+1}\varphi)\int_0^\infty dt \cdot t^{2k}\left(\frac{1}{(1+2t\cos\varphi+t^2)^{2k+1}} + \right. \\ \left. + \frac{1}{(1-2t\cos\varphi+t^2)^{2k+1}}\right)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_P}\frac{1}{|cd|^{2k+1}}$$

where γ run through the set of all double cosets $\Gamma_\infty\gamma\Gamma_P$, c, d are the matrix elements

$$\gamma g(P) = \begin{pmatrix} \star & \star \\ c & d \end{pmatrix}$$

We let $\zeta_P(s)$ denote the corresponding Zagier zeta function

$$\zeta_P(s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_P}\frac{1}{|cd|^s}$$

The integral

$$(11)\quad (\sin^{2k+1}\varphi)\int_0^\infty dt\cdot t^{2k}\left(\frac{1}{(1+2t\cos\varphi+t^2)^{2k+1}}+\frac{1}{(1-2t\cos\varphi+t^2)^{2k+1}}\right)$$

is computed by means of the residues theory and it is equal to

$$\sum_{\ell=0}^{\infty} a_{k\ell} ctg^{2\ell}\varphi$$

$a_{k\ell}$ are some constants. We write down these constants later for $k=1$. After some obvious transformations we find that the formula (9) is an expression of the form

$$\sum_{\{P\}_\Gamma}' \zeta_P(2k+1)\sum_{m=1}^{\infty}\sum_{l=0}^{k}\frac{a_{k\ell}}{w(P,m)^{\ell+1/2}}\cdot\int_0^\infty t^{\ell-1/2}k(t+w(P,m))dt+\underset{Y\to\infty}{o(1)}$$

We introduce some notation

$$\int_0^\infty t^{\ell-1/2}k(t+w)dt=Q_\ell(w)$$

Obviously, $Q_o(w)=Q(w)$ from the formula (5).

Therefore we computed the asymptotic behavior at $Y\to\infty$ of the left hand side of the formula (4). We proceed now to the right hand side of this formula. Here we shall limit ourselves to more special discrete groups.

We suppose that eigenfunctions of the discrete spectrum of the automorphic Laplacian $A(\Gamma)$ are only constant and the cusp forms which have property to be even or odd functions

$$\pm v_j(-\bar{z})=v_j(z),$$

and there is no other eigenfunctions of the discrete spectrum of $A(\Gamma)$. It is the serious restriction of the group Γ but, in particular, some arithmetical groups Γ have this property. The right hand side of the formula (4) is an expression of the form

$$(12)\quad \frac{h(0)}{\mu(F)}\int_F\Theta(z,\psi_k)d\mu(z)+\sum_{j>0}h(\lambda_j)\int_F|v_j(z)|^2\Theta(z,\psi_k)d\mu(z)+$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}h(1/4+r^2)\int_F|E(z,1/2+ir)|^2\Theta(z,\psi_k)d\mu(z)$$

where $\lambda_0=0$, $v_0(z)=\mu(F)^{-1/2}$, $j>0$ run through the set of all eigenfunctions of the discrete spectrum of $A(\Gamma)$ which are cusp forms.

For $j>0$ we have $\lambda_j>0$ and

$$v_j(z)=\sum_{n\neq 0}\rho_j(n)\sqrt{y}K_{ir_j}(2\pi|n|y)e^{2\pi inx}$$

$\rho_j(n)$ are Fourier coefficents, $K_s(z)$ is the modified Bessel function, $1/4 + r_j^2 = \lambda_j$.

For the Eisenstein-Maass series we have the same Fourier decomposition

$$E(z,s) = y^s + \varphi(s)y^{1-s} + \sum_{n\neq 0} d_n(s)\sqrt{y}K_{s-1/2}(2\pi|n|y)e^{2\pi i n x}$$

We make use of these decompositions for the transformation of (12). We have

a) **The contribution of v_0 to (12)**

From the definition of $\Theta(z,\psi_k)$ we have

$$\frac{h(0)}{\mu(F)}\int_F \Theta(z,\psi_k)d\mu(z) = \frac{h(0)}{\mu(F)}\int_0^\infty \frac{dy}{y^2}\int_0^1 dx\psi_k(y) = \frac{h(0)}{\mu(F)}\cdot\frac{Y^{2k}}{2k} + \underset{Y\to\infty}{o(1)}$$

b) **The contribution of the cusp forms to (12)**

$$\sum_{j>0} h(\lambda_j)\int_F \Theta(z,\psi_k)|v_j(z)|^2 d\mu(z) = \tag{13}$$

$$=\sum_{j>0} h(\lambda_j)\int_F E(z,2k+1)|v_j(z)^2|d\mu(z) + \underset{Y\to\infty}{o(1)}$$

The last integral is the Rankin-Selberg convolution and it is equal to

$$\frac{1}{4\pi^{2k+1}}\frac{\Gamma^2(k+1/2)}{\Gamma(k+1)}|\Gamma(k+1/2+ir_j)|^2\zeta(2k+1;v_j)$$

where $\zeta(2k+1;v_j)$ is the Rankin-Selberg zeta function

$$\zeta(s;v_j) = \sum_{n=1}^\infty \frac{|\rho_j(n)|^2}{n^s}$$

$\Gamma(s)$ is the Euler function. As a result (13) is equal to

$$\frac{1}{4\pi^{2k+1}}\frac{\Gamma^2(k+1/2)}{\Gamma(2k+1)}\sum_{j>0} h(\lambda_j)|\Gamma(k+1/2+ir_j)|^2\zeta(2k+1;v_j) + \underset{Y\to\infty}{o(1)}$$

c) **The contribution of the Eisenstein-Maass series to (12)**

We have

(14)

$$\frac{1}{4\pi}\int_{-\infty}^\infty h(1/4+r^2)\int_F \Theta(z,\psi_k)|E(z,1/2+ir)|^2 d\mu(z)dr =$$

$$=\frac{1}{4\pi}\int_{-\infty}^\infty h(1/4+r^2)\int_0^\infty \frac{dy}{y^2}\int_0^1 dx\psi_k(y)|E(z,1/2+ir)|^2 dr =$$

$$=\frac{Y^{2k+1}}{2\pi(2k+1)}\int_{-\infty}^\infty h(1/4+r^2)dr + \frac{1}{4\pi}\int_{-\infty}^\infty h(1/4+r^2)\frac{\varphi(1/2+ir)}{2k+1-2ir}Y^{2k+1-2ir}dr+$$

$$+\frac{1}{4\pi}\int_{-\infty}^\infty h(1/4+r^2)\frac{\varphi(1/2-ir)}{2k+1+2ir}Y^{2k+1+2ir}dr+$$

$$\frac{1}{16\pi^{2k+2}}\frac{\Gamma^2(k+1/2)}{\Gamma(2k+1)}\int_{-\infty}^\infty h(1/4+r^2)|\Gamma(k+1/2+ir)|^2\zeta(2k+1,E(\cdot,1/2+ir))dr$$

$$+ \underset{Y\to\infty}{o(1)}$$

where the Rankin-Selberg zeta function for the Eisenstein-Maass series is defined by the formula

$$\zeta(s; E(\cdot, 1/2+ir)) = \sum_{n=1}^{\infty} \frac{|d_n(1/2+ir)|^2}{n^s}$$

The integrals with the function φ are computed using the theory of residues. The functions h and φ are meromorphic (analytic). The only pole of the $\varphi(s)$ in the half plane Re$s \geq 1/2$ is the simple pole at $s = 1$ with the residue equal to $\mu(F)^{-1}$. We remember also the functional equation $\varphi(s)\varphi(1-s) = 1$. Finally (14) is equal to

$$\frac{Y^{2k+1}}{2k+1}g(0) - \frac{1}{2k}Y^{2k}h(0)\frac{1}{\mu(F)} + \frac{1}{2}h(-k^2-k)\varphi(k+1)+$$
$$+\frac{1}{16\pi^{2k+2}}m\frac{\Gamma^2(k+1/2)}{\Gamma(2k+1)}\int_{-\infty}^{\infty} h(1/4+r^2)|\Gamma(k+1/2+ir)|^2\zeta(2k+1, E(\cdot, 1/2+ir))dr+$$
$$+\underset{Y\to\infty}{o(1)}$$

Therefore we found the asymptotic behavior at $Y \to \infty$ of each term in the formula (4) and as a result we proved the following assertion.

Theorem 1. *Let $k(t)$ be a function $k(t) \in C_0^\infty(\mathbb{R}^+)$ then the following formula is valid*

$$\sum_{\{P\}_\Gamma}{}' \zeta_P(2k+1) \sum_{m=1}^{\infty}\sum_{\ell=0}^{k} \frac{a_{k\ell}}{w(P,m)^{\ell+1/2}}Q_\ell(w(P,m))+$$
$$\frac{(-1)^k}{(2\pi)^{2k}}\zeta(2k+1)\frac{(2k)!}{k!}k^{(k)}(0) + 2\varphi(2k+1)\zeta(2k+1)\int_0^\infty t^{2k}k(t)dt =$$
$$=\frac{\varphi(k+1)h(-k^2-k)}{2} + \frac{1}{4\pi^{2k+1}}\frac{\Gamma^2(k+1/2)}{\Gamma(2k+1)}\sum_{j>0} h(\lambda_j)|\Gamma(k+1/2+ir_j)|^2 \cdot \zeta(2k+1, v_j)+$$
$$+\frac{1}{16\pi^{2k+2}}\frac{\Gamma^2(k+1/2)}{\Gamma(2k+1)}\int_{-\infty}^{\infty} h(1/4+r^2)|\Gamma(k+1/2+ir)|^2\zeta(2k+1, E(\cdot, 1/2+ir))dr$$

This Zagier formula is simplified essentially when $k = 1$. Besides we can rewrite it in terms of a function $h(\lambda)$ but not a function $k(t)$ like in the Selberg trace formula.

We will not give here the precise definition of the formula and we formulate the result only

Theorem 2. *In condition of the Theorem 1 the following formula is valid:*

(15)

$$\sum_{\{P\}_\Gamma}' \zeta_P(3) \sum_{m=1}^{\infty} \left\{ \frac{\pi}{8} \cdot \frac{g(mln\mathcal{N}(P))}{\mathcal{N}(P)^{m/2} - \mathcal{N}(P)^{-m/2}} - \frac{9}{32} \cdot \frac{1}{(\mathcal{N}(P)^{m/2} - \mathcal{N}(P)^{-m/2})^3} \cdot \right.$$

$$\left. \cdot \int_{-\infty}^{\infty} h(1/4+r^2)\mathcal{N}(P)^{-irm} \left(\frac{\mathcal{N}(P)^m}{1-ir} + \frac{\mathcal{N}(P)^{-m}}{1+ir} \right) dr \right\} -$$

$$-\frac{\varphi(3)\zeta(3)}{2\pi} \int_{-\infty}^{\infty} \frac{h(1/4+\tau^2)}{1+\tau^2} d\tau + \frac{\zeta(3)}{64\pi^3} \int_{-\infty}^{\infty} (1/4+\tau^2)\tau th(\pi\tau) \cdot h(1/4+\tau^2) d\tau =$$

$$= \frac{\varphi(2)h(-2)}{2} + \frac{1}{32\pi} \sum_{j>0} h(\lambda_j) \frac{\lambda_j}{ch\pi r_j} \zeta(3, v_j) +$$

$$+ \frac{1}{128\pi^2} \int_{-\infty}^{\infty} \frac{h(1/4+\tau^2)(1/4+\tau^2)}{ch\tau\pi} \zeta(3, E(\cdot, 1/2+i\tau)) d\tau$$

We can define now the zeta function of the Selberg type making use of the formula (15). We choose the test function $h(\lambda)$ as follows

$$h(1/4+\tau^2; s; a) = \frac{1}{(s-1/2)^2+\tau^2} + \frac{s(1-s)}{2(2s-1)} \cdot \frac{1}{(s+1/2)^2+\tau^2} -$$

$$-\frac{s(1-s)}{2(2s-1)} \cdot \frac{1}{(s-3/2)^2+\tau^2} - \frac{1}{(a-1/2)^2+\tau^2} - \frac{a(1-a)}{2(2a-1)} \cdot \frac{1}{(a+1/2)^2+\tau^2} +$$

$$+ \frac{a(1-a)}{2(2a-1)} \cdot \frac{1}{(a-3/2)^2+\tau^2}, \qquad Res > 1,\ a > Res$$

It is not hard to see that for given test function $h(\lambda; s; a)$ the formula (15) is valid but the corresponding k function does not satisfy the condition of the Theorem 2.

We introduce now two zeta functions

$$Z_1(s) = \prod_{\{P\}_\Gamma}' \prod_{m=0}^{\infty} (1 - \mathcal{N}(P)^{-s-m})^{\frac{\pi}{8} \frac{\zeta_P(3)}{ln\mathcal{N}(P)}}$$

$$Z_2(s) = \prod_{\{P\}_\Gamma}' \prod_{m=0}^{\infty} \prod_{k=0}^{\infty} \prod_{l=0}^{\infty} (1 - \mathcal{N}(P)^{-s-m-k-l})^{\frac{\pi}{8} \frac{\zeta_P(3)}{ln\mathcal{N}(P)}}$$

It is not hard to prove also that for Re$s > A \gg 1$ these products converge absolutely. Besides the contribution of all hyperbolic classes in the formula (15) for the function $h(1/4+\tau^2; s; a)$ is equal to the following expression

(16)

$$\frac{1}{2s-1} \left\{ \frac{Z_1'}{Z_1}(s) + \frac{s(1-s)}{2} \left(\frac{Z_1'}{Z_1}(s+1)\frac{1}{2s+1} - \frac{Z_1'}{Z_1}(s-1)\frac{1}{2s-3} \right) + \right.$$

$$+ \frac{9}{2s-3} \frac{Z_2'}{Z_2}(s) + \frac{9s(1-s)}{2(2s+1)(2s-1)} \frac{Z_2'}{Z_2}(s+1) - \frac{9s(1-s)}{2(2s-3)(2s-5)} \frac{Z_2'}{Z_2}(s-1) +$$

$$+ \frac{9}{2s+1} \frac{Z_2'}{Z_2}(s+2) + \frac{9}{2(2s+1)(2s+3)} \frac{Z_2'}{Z_2}(s+3) -$$

$$\left. - \frac{9s(1-s)}{2(2s-1)(2s-3)} \frac{Z_2'}{Z_2}(s+1) \right\}$$

minus the same expression with the argument a instead of s. The dash in the last formula means the sign of the derivative of course.

We define now the zeta function

$$Z_0(s) = \prod_{\{P\}_\Gamma}' (1 - \mathcal{N}(P)^{-s})^{a(P)}, \quad a(P) = \frac{\pi}{8} \frac{\zeta(3)}{ln \mathcal{N}(P)}$$

This function is generating for $Z_1(s), Z_2(s)$. We have

$$Z_1(s) = \prod_{m=0}^{\infty} Z_0(s+m), Z_2(s) = \prod_{k=0}^{\infty}\prod_{l=0}^{\infty} Z_1(s+k+l) =$$

$$= \prod_{m=0}^{\infty}\prod_{k=0}^{\infty}\prod_{l=0}^{\infty} Z_0(s+m+k+l)$$

Making use of the formulae (15), (16) we can extend meromorphically the logarithmic derivatives

$$\frac{Z_0'}{Z_0}(s), \frac{Z_1'}{Z_1}(s), \frac{Z_2'}{Z_2}(s) \tag{17}$$

to the half plane Res > 0 at least if we can prove the meromorphic continuation of the integral

$$\int_{-\infty}^{\infty} \frac{(1/4+\tau^2)}{ch\pi\tau} h(1/4+\tau^2; s; a)\zeta(3, E(\cdot, 1/2+i\tau))d\tau \tag{18}$$

We can do it for cycloidal subgroups of the modular group $\Gamma_{\mathbf{Z}}$ (or for subgroups with one parabolic generator only) without elliptic generators. As it follows from the paper [3] the integral (18) is reduced to the integral essentially

$$\text{const.} \int_{-\infty}^{\infty} (1/4+\tau^2)(h(1/4+\tau^2; s; a)\frac{\zeta(3-2i\tau)\zeta(3+2i\tau)}{\zeta(1-2i\tau)\zeta(1+2i\tau)}d\tau$$

with the Rieman zeta function for these discrete groups. The last assertion is valid because for any cycloidal subgroup of $\Gamma_{\mathbf{Z}}$ the Eisenstein-Maass series is almost equal to the Eisenstein-Maass series of $\Gamma_{\mathbf{Z}}$. We will not give here the details of the proof and we formulate only the result.

Theorem 3. *Let Γ be a cycloidal subgroup of the modular group without elliptic elements. Then the functions (17) can be extended meromorphically to the half plane $Res > 0$ at least.*

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