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Abstract

In this paper we give a survey of the modern theory of birational rigidity for Fano fiber spaces over a base of positive dimension. The paper is a follow up of the previous survey on birational rigidity of Fano varieties. We describe the techniques of the method of maximal singularities for Fano fiber spaces.

Bibliography: 54 items.

Introduction

The present paper is a follow up of the survey [1], devoted to the theory of birational rigidity of higher-dimensional Fano varieties: we consider the "relative" version of the theory, that is, we study Fano *fiber spaces* over a positive dimensional base. Whereas for typical Fano varieties the phenomenon of birational rigidity meant the *absence* of non-trivial structures of a rationally connected fiber space, for typical Fano fiber spaces birational rigidity means the *uniqueness* of the default structure. Both in the absolute case (Fano varieties) and the relative case (Fano fiber spaces) the theory is based in the technical sense on the method of maximal singularities, however the fiber spaces over a non-trivial base are essentially harder to work with in many respects, they require additional technical means and for these reasons the theory of Fano fiber spaces needs to be considered separately (from the absolute case). Geometry of Fano varieties and fiber spaces form two branches of the theory of birational rigidity, they developed parallel to each other. It is the technical difficulties that are responsible for the fact that up to the mid-nineties the Sarkisov theorem [2] remained the one and only result on birational rigidity of Fano fiber spaces, and the situation changed only when the paper [3] appeared and made it possible to study successfully new classes of fiber spaces.

In this paper we assume that the reader is familiar with the previous survey [1]. In particular, such crucial concepts as rationally connected variety, threshold of canonical adjunction, maximal singularity etc. are assumed to be known and are used without special explanations. This also applies to the technique of hypertangent divisors which was presented in [1] in full detail.

The structure of the present survey is similar to that of [1]: the first chapter is less formal, its aim is to explain by examples, what types of varieties are studied, which problems are considered and what approaches are available today for their solution. The second chapter deals with birational geometry of Fano fiber spaces over $\mathbb{P}^1$. In the third chapter we discuss varieties with many non-trivial structures of a rationally connected fiber space. In particular, we consider briefly the double spaces of index two: they make the first example of a large class of non-rigid Fano varieties, for which the method of maximal singularities makes it possible to give a complete description of their birational geometry. Note that the recent survey [4] was devoted to fibrations into del Pezzo surfaces of degrees 1 and 2 and for that reason we do not consider this class of varieties.

Everywhere in the sequel "Fano fiber space" means a fiber space over a positive dimensional base, that is, a non-trivial fiber space.

The claims, definitions, remarks, etc., are numbered in this paper in the same way as in [1]: theorem (definition, lemma, remark, ...) $a.b$ is the theorem (...) b of chapter a; in each chapter the numbering is independent; when we refer to a section §a or subsection $a.b$, and do not specify the chapter, the current chapter is meant.

Chapter 1. Birational geometry of rationally connected fiber spaces

§1. Rationally connected fiber spaces

1.1. Fano fiber space: definitions and examples. First of all, recall the well known

Definition 1.1. A surjective morphism of smooth projective varieties $\pi\colon V \to S$ with connected fibers of the same dimension is called a *Fano fiber space*, if the anticanonical class $(-K_V)$ is relatively ample, that is, ample on the fibers of the projection π. A Fano fiber space is said to be *standard*, if

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S, \tag{1}$$

in particular, the relative Picard number is $\rho(V/S) = 1$.

In the present survey we consider only Fano fiber spaces over a rationally connected base S; in that case the variety V is automatically rationally connected. If the variety V has $\mathbb{Q}$-factorial terminal singularities, S is normal and the condition (1) is replaced by the equality $\rho(V/S) = 1$, then this more general object is often called a *Mori fiber space*, see §2. However, the term "a Fano fiber space with singularities" is also justified.

The fibers of a Fano fiber space $\pi\colon V \to S$ are varieties from some family $\mathcal{F}$, the general element of which is a smooth Fano variety, so that the projection π generates a map $S \to \mathcal{F}$, associating to a point $s \in S$ the corresponding fiber. In a sufficiently typical situation the sheaf $\pi_*(-K_V)$ is locally free and allows one to give an explicit construction of a Fano fiber space, which is shown by the following two examples.

Example 1.1 (fibrations into Fano complete intersections). Let S be a smooth projective rationally connected variety of positive dimension, $\mathcal{E}$ a locally free sheaf on S of rank $M + k + 1$, where $M, k \geq 1$ are positive integers, $X = \mathbb{P}(\mathcal{E})$ the corresponding projective bundle in the sense of Grothendieck (X is the projectivisation of the vector bundle, the sheaf of sections of which is $\mathcal{E}^*$, the sheaf, dual to $\mathcal{E}$). The projection $\pi\colon X \to S$ is a locally trivial $\mathbb{P}^{M+k}$-fibration. If $L_X \in \operatorname{Pic} X$ is the class of the tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, then

$$\operatorname{Pic} X = \mathbb{Z}L_X \oplus \pi^* \operatorname{Pic} S,$$

so that X/S is a Fano fiber space. For the canonical class of the variety X there is a well known formula

$$K_X = -(M + k + 1)L_X + \pi^*(\det \mathcal{E} + K_S).$$

Let $(d_1, \dots, d_k) \in \mathbb{Z}_+^k$ be a k-uple of integers, satisfying the conditions

$$d_k \geq \dots \geq d_1 \geq 2 \quad \text{and} \quad d_1 + \dots + d_k = M + k.$$

Consider, furthermore, the set of classes $A_i \in \operatorname{Pic} S$, $i = 1, \dots, k$, and assume that there are irreducible divisors

$$G_i \sim d_i L_X + \pi^* A_i, \quad i = 1, \dots, k,$$

such that the complete intersection

$$V = G_1 \cap \ldots \cap G_k \subset X$$

is a smooth variety, the fibers of which $V \cap \pi^{-1}(s)$, $s \in S$, are of the same dimension M. We denote the restriction of the projection π onto V by the same symbol $\pi\colon V \to S$. The restriction of the tautological class $L_X|_V$ is denoted by the symbol L_V. Obviously,

$$K_V = -L_V + \pi^*\Delta$$

for some $\Delta \in \operatorname{Pic} S$, so that V/S is a Fano fiber space. Moreover, for a point of general position $s \in S$ the fiber $\pi^{-1}(s) \subset V$ is a smooth Fano complete intersection of the type $d_1 \cdot \ldots \cdot d_k$ in $\mathbb{P}^{M+k}$. For $M \geq 3$ this implies that

$$\operatorname{Pic} V = \mathbb{Z}L_V \oplus \pi^* \operatorname{Pic} S = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S, \tag{2}$$

that is, V/S is a standard Fano fiber space. If $M = 2$, then V/S is a fibration into del Pezzo surfaces of degree 3 ($k = 1$), that is, cubic surfaces in $\mathbb{P}^3$, or degree 4 ($k = 2$, $d_1 = d_2 = 2$), that is, complete intersections of the type $2 \cdot 2$ in $\mathbb{P}^4$. If $M = 1$, then V/S is a conic bundle: in that case we require in addition that $\dim S \geq 2$. For $M \in \{1, 2\}$ the condition (2) does not hold automatically and we should require it as an extra. However, if the divisors $G_i, 1 \leq i \leq k \in \{1, 2\}$, are ample, then (2) holds by the Lefschetz theorem. The construction above gives the most "classical" example of a standard Fano fiber space.

Example 1.2 (fibrations into Fano double covers). Let $S, \mathcal{E}, X, \pi$ be the same as in the previous example, $k = 1$. Consider a pair of integers $m \geq 2, l \geq 2$, satisfying the equality $m+l = M+1$. Assume that for some classes $A_Q, A_W \in \operatorname{Pic} S$ there are divisors

$$Q \sim mL_X + \pi^*A_Q \quad \text{and} \quad W_X \sim 2(lL_X + \pi^*A_W),$$

where $Q \subset X$ is a smooth subvariety, intersecting each fiber $\pi^{-1}(s)$, $s \in S$, by a hypersurface (that is, Q does not contain entire fibers of the projection π), and W_X cuts out on Q a smooth divisor that does not contain fibers of the projection $\pi_Q = \pi|_Q$. This collection of data determines the double cover $\sigma\colon V \to Q$, branched over the divisor $W = W_X \cap Q$. The fibers of the projection $\pi_Q \circ \sigma\colon V \to S$ are M-dimensional varieties and the general fiber is a smooth primitive Fano variety, which implies that

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \sigma^*\pi_Q^* \operatorname{Pic} S,$$

that is, V/S is a standard Fano fiber space. A particular case of this construction (corresponding to the value $m = 1$) is a double cover $\sigma\colon V \to X$, branched over a smooth divisor $W_X \sim 2((M+1)L_X + \pi^*A_W)$ that does not contain fibers of the projection π (there is no divisor Q); this is a standard fibrations into Fano double spaces of index one.

In each of the considered examples the fibers of the Fano fiber space $\pi\colon V \to S$ belong to a family $\mathcal{F}$ of M-dimensional schemes, the general element of which is a smooth Fano variety of index one. Each of these families admits a natural structure of a projective variety, so that to the fiber space V/S we can associate a map

$$S \to \mathcal{F},$$

sending a point $s \in S$ to the fiber $\pi^{-1}(s) \in \mathcal{F}$. In this connection, we get a geometric characterization of "twistedness" of the fiber space V/S as complexity of the map $S \to \mathcal{F}$. For instance, if $V \subset \mathbb{P}^1 \times \mathbb{P}^3$ is a smooth divisor of bidegree $(d, 3)$, so that the projection $\pi: V \to \mathbb{P}^1$ onto the first factor realizes V as a fibration into del Pezzo surfaces of degree 3, then the twistedness of this fiber space over the base $\mathbb{P}^1$ is easy to define as the value $d \geq 1$. The twistedness over the base is an intuitively clear degree of complexity of a fiber space, and it is natural to expect that the more twisted the fiber space is, the more rigid is its structure. Now let us give an example of a least twisted fiber space.

Example 1.3 (Fano fiber spaces). Let $F_1, \dots, F_k$, $k \geq 2$, be primitive Fano varieties,

$$V = F_1 \times \ldots \times F_k$$

their direct product, $\pi_i: V \to S_i = \prod\limits_{j \neq i} F_j$ the projection along $F_i, i \in \{1, \dots, k\}$. By assumption, $\pi_i: V \to S_i$ is a standard Fano fiber space. The corresponding map of twistedness $S_i \to \mathcal{F}_i$ is a map to the point.

Recall [1], that by a *structure of a rationally connected fiber space* (or, briefly, a rationally connected structure) on a rationally connected variety V we mean a birational map $V \overset{\chi}{\dashrightarrow} V^+$, where on the variety V^+ a morphism $\pi^+: V^+ \to S^+$ is fixed, which is a rationally connected fiber space. A Fano fiber space is obviously a rationally connected fiber space. Now we can formulate the main problem of birational geometry of Fano fiber spaces as follows:

> *fir a given standard Fano fiber space $\pi: V \to S$ describe all structures of a rationally connected fiber space on the variety V modulo the relation of fiber-wise birational equivalence.*

Informally speaking, the more rationally connected structures there are on a given variety V, the more complicated is its birational geometry (in particular, the projective space $\mathbb{P}^n$ has the most complicated birational geometry). The group of birational self-maps $\operatorname{Bir} V$ acts on the set $RC(V)$ of rationally connected structures and of the greatest interest is the quotient set $\overline{RC}(V)$, introduced in [1]. If V/S is a Fano fiber space over a base of positive dimension, then there is an important subgroup

$$\operatorname{Bir}(V/S) \subset \operatorname{Bir} V$$

of fiber-wise (with respect to π) birational self-maps, equipped with the natural homomorphism

$$\operatorname{Bir}(V/S) \to \operatorname{Bir} S.$$

The kernel of the latter homomorphism is the group $\operatorname{Bir}(F_\eta)$ of the generic fiber (over the generic point of S with the residue field $\mathbb{C}(S)$).

As the first stage in the solution of the general problem that was formulated above, it is natural to identify those Fano fiber spaces V/S, that have exactly one structure of a rationally connected fiber space $\pi\colon V \to S$, or, at least, those, for which that structure is unique modulo the action of the group $\mathrm{Bir}\, V$. Such a uniqueness in most cases is a consequence of the fundamental property of *birational (super)rigidity*. As it was pointed out in [1, Chapter 1, Sec. 3.2], the informal general principle is that a sufficiently high twistedness over the base implies birational rigidity, the uniqueness of the structure V/S and reduces birational geometry of the variety V to birational geometry of the fiber F_η over the generic point. This principle will be realized below in various examples that are by now completely studied.

The method of maximal singularities, which forms a basis of the proof of almost all results of the current survey, as a by-product solves other problems as well: describes the group $\mathrm{Bir}\, V$ and the group $\mathrm{Bir}(V/S)$. The modern version of the method makes it possible to start investigating varieties with several and even many *non-equivalent* structures of a rationally connected fiber space. Chapter 2 is devoted (mainly) to varieties with a unique rationally connected structures, Chapter 3 is devoted to varieties with many structures.

In [1, Chapters 1,2] we explained and illustrated the idea of *untwisting* of a birational map (or *untwisting* maximal singularities). Essentially, this is the universal idea of simplification of a complex object by means of subsequent elementary steps. For Fano varieties such steps are usually birational involutions [1, Chapter 1, §3]. For Fano fiber spaces the most natural candidates for simplifying modifications are *fiber-wise modifications*: here, apart from self-maps, operations of a different type emerge. The informal working principle is that a birational map $\chi\colon V \dashrightarrow V'$ of the total space of a Fano fiber space V/S onto the total space of the fiber space V'/S' should be simplified by fiber-wise modifications until either it becomes, in a certain sense, "simple", or a new structure of a rationally connected fiber space appears on the modified space V, which is not compatible with the original structure V/S. In the latter case replacing the old structure by the new one makes the simplifying step.

Below we consider examples of simplifying modifications of both types.

1.2. Fiber-wise birational maps. In the study of birational geometry of a Fano fiber space $\pi\colon V \to S$ of great importance are the fiber-wise birational modifications, that is, commutative diagrams of maps

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \pi & \downarrow & & \downarrow & \pi^+ \\ & S & \stackrel{\varphi}{\dashrightarrow} & S^+, & \end{array}$$

where χ and φ are birational maps. The most elementary type of such modifications is given by birational self-maps of the generic fiber, $S^+ = S$, $\varphi = \mathrm{id}$, $V^+ = V$. Following [5,3], consider

Example 1.4 (fiber-wise birational self-maps of a pencil of cubic surfaces). In this example the Fano fiber space is a particular case of the construction of Example 1.1. Let $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^1$ be the projectivization of a locally free sheaf of rank 4 on $\mathbb{P}^1$, $V \subset \mathbb{P}(\mathcal{E})$ a smooth divisor, intersecting each fiber $\pi^{-1}(t)$ by a cubic surface $F_t \subset \mathbb{P}^3 = \pi^{-1}(t)$; assume that $V/\mathbb{P}^1$ is a standard Fano fiber space. Consider an arbitrary section $C \subset V$ of the projection $\pi\colon V \to \mathbb{P}^1$. For a general point $t \in \mathbb{P}^1$ a general line $L \subset \mathbb{P}^3 = G_t$, containing the point $C \cap F_t$, meets the cubic surface F_t at two more distinct points, say x, y. Set

$$\tau_C(x) = y.$$

Obviously, this defines a birational involution $\tau_C \in \operatorname{Bir} F_\eta \subset \operatorname{Bir} V$. Let $\alpha\colon V^* \to V$ be the blow up of the curve C, $E = \alpha^{-1}(C)$ the exceptional divisor, $\operatorname{Pic} V^* = \mathbb{Z}h \oplus \mathbb{Z}e \oplus \mathbb{Z}F$, where $h = -K_V$.

Lemma 1.1. *The birational involution τ_C extends to a biregular involution of an invariant open set $V^*\backslash Y$, $\operatorname{codim} Y \geq 2$, and its action on $\operatorname{Pic} V^*/\mathbb{Z}F \cong \mathbb{Z}\bar{h} \oplus \mathbb{Z}\bar{e}$ is given by the relations*

$$\tau_C^*\bar{h} = 3\bar{h} - 4\bar{e},$$

$$\tau_C^*\bar{e} = 2\bar{h} - 3\bar{e}.$$

Proof. See [5].

Now let us consider an arbitrary bi-section $C \subset V$, that is, an irreducible curve which is a two-sheeted cover of the base $\mathbb{P}^1$. We define the involution τ_C by its action on the generic fiber F in the following way (see [5,3]). Let $\{a, b\} = C \cap F$, and $q = L_{ab} \cap F$ be the third point of intersection of the line in $\mathbb{P}^3$ that joins the points a and b, with the cubic surface F. The points q sweep out a curve $C^* \subset V$, a section of the morphism π, that is, $q = C^* \cap F$. The pencil of planes P in $\mathbb{P}^3$, containing the line L_{ab}, generates a pencil of elliptic curves $Q_P = P \cap F$ on the surface F. Set

$$\tau_C\,|_{Q_P}(x) = y,$$

where

$$x + y \sim 2q$$

on Q_P, that is, τ_C is the reflection on the elliptic curve Q_P from the point q. This defines the involution $\tau_C \in \operatorname{Bir} F_\eta \subset \operatorname{Bir} V$.

Let $\alpha\colon V^* \to V$ be the blow up of the curve C, $E = \alpha^{-1}(C)$ the exceptional divisor, $\operatorname{Pic} V^* = \mathbb{Z}h \oplus \mathbb{Z}e \oplus \mathbb{Z}F$, where again $h = -K_V$.

Lemma 1.2. *The birational involution τ_C extends to a biregular involution of an invariant open set $V^*\backslash Y$, $\operatorname{codim} Y \geq 2$, and its action on $\operatorname{Pic} V^*/\mathbb{Z}F \cong \mathbb{Z}\bar{h} \oplus \mathbb{Z}\bar{e}$ is given by the relations*

$$\tau_C^*\bar{h} = 5\bar{h} - 6\bar{e},$$

$$\tau_C^*\bar{e} = 4\bar{h} - 5\bar{e}.$$

Proof: straightforward computations [5].

The birational involutions constructed above are used for the study of geometry of the variety V in the following way. Let $\Sigma \subset |-nK_V + lF|$ be a movable linear system on V, where $n \geq 1$, that is, Σ is not composed from the pencil of fibers. The curve $C \subset V$ is called a *maximal curve* of the system Σ, if the inequality

$$\operatorname{mult}_C \Sigma > n$$

holds. Assume that C is a *horizontal* maximal curve, that is, $\pi(C) = \mathbb{P}^1$ (in other words, C is not contained in a fiber F_t). It is easy to show that

$$(K_V^2 \cdot F) = 3 > \deg(\pi|_C) \in \{1, 2\},$$

that is, C is a section or bi-section of the projection π. By Lemmas 1.1 and 1.2 it is easy to see that the strict transform $\Sigma_1 = (\tau_C)_* \Sigma$ of the linear system Σ with respect to the involution τ_C satisfies the relation $\Sigma_1 \subset |-n_1 K_V + l_1 F|$, where $n_1 < n$. More precisely,

$$n_1 = 3n - 2 \operatorname{mult}_C \Sigma,$$

if C is a section, and

$$n_1 = 5n - 4 \operatorname{mult}_C \Sigma,$$

if C is a bi-section of the projection π. Since $n \in \mathbb{Z}_+$, in finitely many steps we come to a system without maximal curves. This is the first, easier step in the study of birational geometry of the variety V and in the proof of the main theorem on birational rigidity of the fiber space $V/\mathbb{P}^1$ [3], which was formulated in [1] (Theorem 1.7). The second, harder, step (the exclusion of infinitely near maximal singularities) is discussed in §3 of Chapter 2 of the present survey.

Example 1.5 (fiber-wise birational modifications of conic bundles). Let $\pi\colon V \to S$ be a standard conic bundle. Some examples of birational self-maps preserving the fibers were given in [1, Sec. 2.3]. However, what turns out to be productive to describe birational geometry of the variety V (see the Sarkisov theorem and its discussion that were given in [1, Sec. 2.3]) is not simplifying a linear system by birational self-maps as above but another approach, which we will now briefly describe. Consider the simplest situation: let the fiber $C = \pi^{-1}(p)$ over a point $p \in S$ be a non-singular conic, $C \cong \mathbb{P}^1$. Let us blow up simultaneously the point p on the surface S and the curve C on the variety V:

$$\begin{array}{ccccc} & V & \leftarrow & V^+ & \\ \pi & \downarrow & & \downarrow & \pi \\ & S & \leftarrow & S^+, & \end{array}$$

the projection $V^+ \to S^+$ is for simplicity denoted by the same symbol π. Let $E \subset V^+$ and $E_S \subset S^+$ be the exceptional divisors of these blow ups. Note that V^+/S^+ is again a standard conic bundle. Obviously, $E = E_S \times \mathbb{P}^1$. Let $\Sigma \subset |-nK_V + \pi^* A|$ be a movable linear system, $\mu = \operatorname{mult}_C \Sigma$ and

$$\Sigma^+ \subset |-nK_{V^+} + \pi^*(A + (n - \mu)E_S)|$$

its strict transform on V^+. The fiber C is a *maximal curve* of the system Σ, if $\mu > n$, and this example shows how to remove all maximal curves of that type, modifying the conic bundle (if C is a component of a reducible fiber or the support of a non-reduced fiber, then the construction of the birational modification is much more complicated, see [2]). Let us now assume that some curve $\Gamma \subset E$, which is not a fiber of π, satisfies the condition

$$\nu + \mu > 2n,$$

where $\nu = \mathrm{mult}_\Gamma \Sigma^+ > n$, that is, it is an *infinitely near maximal curve* of the original system Σ. Since Σ^+ has no fixed components and for a general divisor $D^+ \in \Sigma^+$ we have

$$(D^+ \cdot \pi^{-1}(s)) = -n(K_{V^+} \cdot \pi^{-1}(s)) = 2n,$$

the curve Γ is a section of the ruled surface E/E_S, in particular, it is a smooth rational curve. Let $\varphi: V^\sharp \to V^+$ be the birational modification which is a composition of two operations: the blow up of the curve Γ and the subsequent fiber-wise contraction of the strict transform of the ruled surface E. We obtain a new standard conic bundle $\pi: V^\sharp \to S^+$, for the strict transform of the linear system Σ on $V^\sharp$ we get

$$\Sigma^\sharp \subset | - nK_{V^\sharp} + \pi^*(A + (3n - 2\mu - \nu)E_S)|.$$

Note once again, that if a maximal singularity of the system Σ lies over a point of a singular or non-reduced fiber of the projection π, then the required modification is much more complicated of the construction described above. However, what has been said already makes it possible to explain the idea of the proof of the Sarkisov theorem [2] (in [1] it is Theorem 1.6). Let

$$\begin{array}{cccc} & V & \stackrel{\chi}{\dashrightarrow} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & & S' & \end{array}$$

be a birational map between two conic bundles¡ where V satisfies the condition $|4K_S+\Delta| \neq \emptyset$, and $\Delta \subset S$ is the discriminant divisor. Pulling back from the base S' a very ample linear system, we get a movable system Σ' on V'. Let $\Sigma \subset |-nK_V + \pi^* A|$ be its strict transform on V. If χ is not fiber-wise, then $n \geq 1$. So assume that $n \geq 1$. It is easy to check that the condition $|4K_S + \Delta| \neq \emptyset$ is invariant under fiber-wise modifications (this is obvious in the example above). Therefore, applying fiber-wise modifications, we may assume that the linear system Σ has no maximal singularities. Now it is easy to obtain from the condition of termination of canonical adjunction that in this case the class $A \in \mathrm{Pic}\, S$ is not effective (more precisely, not pseudo-effective). Let us compute the self-intersection

$$Z = (D_1 \circ D_2), \quad D_i \in \Sigma,$$

of the linear system Σ and push it down on S:

$$\pi_* Z \sim n^2 \pi_* K_V^2 + 4nA \sim -n^2(4K_S + \Delta) + 4nA.$$

From here by assumption it follows that

$$A \sim \frac{n}{4}(4K_S + \Delta) + \frac{1}{4n}\pi_* Z$$

is an effective class. This contradiction proves the Sarkisov theorem. For the details, see [2].

1.3. Replacing the structure of a Fano fiber space. In the class of Fano fiber spaces of dimension three and higher, fiber-wise modifications appear much more seldom. The following example shows how much more rigid is fiber-wise birational geometry in higher dimensions.

Example 1.6 (the absence of non-trivial fiber-wise modifications in higher dimensions [6]). Let C be a smooth affine curve with a marked point $p \in C$, and $C^* = C \backslash \{p\}$ the punctured curve. Let $\mathcal{V}(d), d \geq 2$, be the set of smooth divisors $V \subset X = C \times \mathbb{P}^M$, $M \geq 3$, each fiber of which

$$F_x = V \cap \{x\} \times \mathbb{P}^M,$$

$x \in C$, is a hypersurface of degree d. Set

$$X^* = C^* \times \mathbb{P}^M, \quad V^* = V \cap X^*,$$

so that V^* is obtained from V by removing the fiber F_p over the marked point. Assume that $d \geq 3$ and $V_1, V_2 \in \mathcal{V}(d)$. The following fact is true [6].

Theorem 1.1. *Let $\chi^* \colon V_1^* \to V_2^*$ be a fiber-wise isomorphism. Then χ^* extends to a fiber-wise isomorphism $\chi \colon V_1 \to V_2$.*

Proof is given in §2 of Chapter 2.

The next example deals with varieties with two structures of Fano fiber spaces. The transition from one structure to another is the required simplifying modification. For other similar examples, see [7,8].

Example 1.7. Consider the following particular case of the construction of Example 1.1: let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus(M-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ be a locally free sheaf of rank $(M+1)$ on $\mathbb{P}^1$, $X = \mathbb{P}(\mathcal{E})$ the corresponding $\mathbb{P}^M$-bundle over $\mathbb{P}^1$, $\operatorname{Pic} X = \mathbb{Z}L_X \oplus \mathbb{Z}R$, where R is the class of a fiber of the projection $\pi_X \colon X \to \mathbb{P}^1$, L_X the tautological class. Let $V \sim ML_X$ be a general divisor. It is a smooth variety, fibered by the projection $\pi = \pi_X|_V \colon V \to \mathbb{P}^1$ into Fano hypersurfaces of index one. Obviously, $V/\mathbb{P}^1$ is a standard Fano fiber space, $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where F is the class of a fiber of the projection π and $K_V = -L$, $L = L_X|_V$. On the variety V, however, there is another structure of a rationally connected fiber space.

Consider the following locally free subsheaves:

$$\mathcal{E}_0 = \mathcal{O}_{\mathbb{P}^1}^{\oplus(M-1)} \hookrightarrow \mathcal{E} \quad \text{and} \quad \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \hookrightarrow \mathcal{E}.$$

Obviously, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$. Let $\mathcal{L}_X$ be the tautological sheaf of Grothendieck of the bundle $\mathbb{P}(\mathcal{E})$ and $\Pi_0 \subset H^0(X, \mathcal{L}_X)$ the subspace, corresponding to the space of sections of the subsheaf $H^0(\mathbb{P}^1, \mathcal{E}_0) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{E})$. Set also

$$\Pi_1 = H^0(X, \mathcal{L}_X \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{E}_1(-1)).$$

Let $x_0,\dots,x_{M-2}$ be some basis of the space Π_0, y_0,y_1 a basis of the space Π_1. Then the sections

$$x_0,\dots,x_{M-2},y_0t_0,y_0t_1,y_1t_0,y_1t_1, \tag{3}$$

where t_0,t_1 is a system of homogeneous coordinates on $\mathbb{P}^1$, forms a basis of the space $H^0(X,\mathcal{L}_X)$. It is easy to see that this complete linear system defines the morphism

$$\xi\colon X\to\overline{X}\subset\mathbb{P}^{M+2},$$

the image of which $\overline{X}$ is a quadric cone with the vertex space $\mathbb{P}^{M-2}=\xi(\Delta_X)$, the base of which is a non-singular quadric in $\mathbb{P}^3$, isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, where $\Delta_X=\mathbb{P}(\mathcal{E}_0)$ is the base set of the pencil $|L_X-R|$, $\Delta_X=\mathbb{P}^{M-2}\times\mathbb{P}^1$, contracted by the map ξ onto the first factor. The morphism ξ is birational, more precisely,

$$\xi\colon X\setminus\Delta_X\to\overline{X}\setminus\xi(\Delta_X)$$

is an isomorphism, and moreover, ξ contracts $\Delta_X=\mathbb{P}^{M-2}\times\mathbb{P}^1$ onto the vertex space of the cone. Let

$$u_0,\dots,u_{M-2},u_{00},u_{01},u_{10},u_{11}$$

be the homogeneous coordinates on $\mathbb{P}^{M+2}$, corresponding to the ordered set of sections (3). The cone $\overline{X}$ is given by the equation

$$u_{00}u_{11}=u_{01}u_{10}.$$

On the cone $\overline{X}$ there are two pencils of M-planes, corresponding to the two pencils of lines on the smooth quadric in $\mathbb{P}^3$. Let $\tau\in\operatorname{Aut}\mathbb{P}^{M+2}$ be the automorphism, permuting the coordinates u_{01} and u_{10}, and not changing the other coordinates. Obviously, $\tau\in\operatorname{Aut}\overline{X}$ is an automorphism of the cone $\overline{X}$, permuting the pencils of M-planes. One of those pencils is the image of the pencil of fibers of the projection π, that is, the pencil $\xi(|R|)$. For the other pencil we have the obvious equality

$$\tau\xi(|R|)=\xi(|L_X-R|).$$

The automorphism τ induces an involutive birational self-map

$$\tau^+\in\operatorname{Bir}X.$$

More precisely, τ^+ is a biregular automorphism outside a closed subset Δ_X of codimension two. Let $\varepsilon\colon\widetilde{X}\to X$ be the blow up of the smooth subvariety Δ_X. Obviously, the variety $\widetilde{X}$ is isomorphic to the blow up of the cone $\overline{X}$ at its vertex space $\xi(\Delta_X)$. It is easy to check that τ^+ extends to a biregular automorphism of the smooth variety $\widetilde{X}$. The linear systems $|kL_X|$, $k\in\mathbb{Z}_+$, are invariant under τ^+. In particular, for a general divisor $V\in|ML_X|$ its τ^+-image $V^+=\tau^+(V)$ is a general divisor of the same linear system, in particular, V^+ is a smooth variety. Note that if $V\in|ML_X|$ is given by an equation

$$h(u_0,\dots,u_{M-2},u_{00},u_{01},u_{10},u_{11}),$$

then its image V^+ is given by the equation

$$h^+(u_*) = h(u_0, \dots, u_{M-2}, u_{00}, u_{10}, u_{01}, u_{11}),$$

where the coordinates u_{01} and u_{10} are permuted.

Therefore, we obtain two Fano fiber spaces, $V/\mathbb{P}^1$ and $V^+/\mathbb{P}^1$, related via the birational isomorphism $\tau^+\colon V \dashrightarrow V^+$, which is not fiber-wise. That birational map is biregular in codimension one and acts on the Picard group in the following way:

$$(\tau^+)^* K_{V^+} = K_V, \quad (\tau^+)^* F^+ = -K_V - F,$$

where F^+ is the class of a fiber of the projection $V^+ \to \mathbb{P}^1$, so that $\operatorname{Pic} V^+ = \mathbb{Z}K_{V^+} \oplus \mathbb{Z}F^+$. By construction, the construction is involutive, that is, $(V^+)^+ = V$.

Note that the birational map $\tau^+\colon V \dashrightarrow V^+$ is the composition of the blow up of the subvariety $\Delta = \Delta_X \cap V$ of codimension two and the subsequent contraction of the exceptional divisor onto the subvariety $\Delta^+ = \Delta_X \cap V^+$.

The transition from the model V to the model V^+ by means of the birational map τ^+ is used as a simplifying modification in the following way. Let

$$\Sigma \subset |-nK_V + lF|$$

be a movable linear system. If $l < 0$, then the linear system $\tau^+_*\Sigma \subset |-n_+K_{V^+} + l_+F^+|$ has parameters

$$n_+ = n + l, \quad l_+ = -l \geq 1.$$

§2. The minimal model program and the Sarkisov program

2.1. Minimal models and Mori fiber spaces. The Minimal Model Program (MMP) generalizes to higher (≥ 3) dimensions the classical theory of minimal models of algebraic surfaces [9]. The purpose of MMP is to associate to every algebraic variety, by means of explicitly described birational modifications, a certain "model" with "good properties" with respect to the canonical class. While for a smooth projective surface it is sufficient to contract the exceptional lines (or (-1)-curves), to obtain either a minimal surface with a numerically effective canonical class or a ruled surface (that is, a $\mathbb{P}^1$-bundle over a smooth projective curve), in dimensions three and higher the situation is much more complicated:

— extremal contractions (the higher-dimensional analogues of the operation of contracting a (-1)-curve) inevitably produce singular varieties, even if the original variety was non-singular;

— a new type of birational modifications emerges, which are isomorphisms in codimension one, that is, outside a closed subset of codimension ≥ 2; it is these modifications that generate the worst technical complications;

— as it became clear starting from the mid-eighties, a technically more natural object is not an algebraic variety X, but a *pair* (or *log pair*) (X, Δ), where Δ is a *boundary*, which is an (effective, as a rule) Weil $\mathbb{Q}$-(or $\mathbb{R}$-)divisor on X, such that $K_X+\Delta$ is a $\mathbb{Q}$-(respectively, $\mathbb{R}$-)Cartier divisor. Working with pairs makes it possible to deal with all types of singularities and is well adjusted to constructing inductive procedures of MMP.

Let X be a normal projective variety, Δ an effective Weil $\mathbb{R}$-divisor, $\Delta = \sum d_i\Delta_i$, where $\Delta_i \subset X$ are distinct prime divisors. Let $f\colon Y \to X$ be a *log resolution* of the pair (X, Δ), that is a sequence of blow ups such that Y is a smooth projective variety and

$$\bigcup \Delta_i^+ \bigcup E_j$$

is a normal crossings divisor, where $\bigcup E_j$ is the exceptional divisor of the morphism f, Δ_i^+ the strict transform of the divisor Δ_i on Y. Writing down

$$K_Y + \Delta^+ = f^*(K_X + \Delta) + \sum a_j E_j$$

(where $\Delta^+ = \sum d_i\Delta_i^+$; recall that $K_X + \Delta$ is a $\mathbb{R}$-Cartier divisor), we say, that the pair (X, Δ) is a klt-*pair* (Kawamata log terminal), if all $d_i < 1$ and all $a_j > -1$.

There are two most important types of klt-pairs:

1) the *minimal models* (or log terminal models), when $K_X + \Delta$ is numerically effective,

2) the *Mori fiber spaces*, when there is a morphism $\varphi\colon X \to S$ onto a normal projective variety S, where $\rho(X/S) = 1$, $\varphi_*\mathcal{O}_X = \mathcal{O}_S$ and $-(K_X + \Delta)$ is φ-ample.

The aim of MMP is to obtain, by means of birational modifications of a special type, from an arbitrary klt-pair (X, Δ) either a minimal model, or a Mori fiber space:

$$X_0 = X \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \dots \overset{f_N}{\dashrightarrow} X_N, \tag{4}$$

$\Delta_0 = \Delta$, $\Delta_i = (f_i)_*\Delta_{i-1}$ and (X_N, Δ_N) is of type 1) or 2). Each birational map $f_{i+1}\colon X_i \dashrightarrow X_{i+1}$ is either a $(K_{X_i}+\Delta_i)$-extremal divisorial contraction, or a flip with respect to a small $(K_{X_i}+\Delta_i)$-extremal contraction $\varphi_i\colon X_i \to Y_i$, that is, the rational map $\varphi_i^+ = \varphi_i \circ f_{i+1}^{-1}\colon X_{i+1} \dashrightarrow Y_i$ is a small birational morphism and $(K_{X_{i+1}} + \Delta_{i+1})$ is relatively φ_i^+-ample. Existence of flips in the arbitrary dimension and the highest generality was proven in [10]. The main difficulty in constructing the MMP was from the start the finiteness problem, that is, the problem of termination of a sequence of flips. Indeed, a sequence of divisorial contractions for obvious reasons can not be infinite. Therefore the algorithm of MMP gives the desired result (a minimal model or a Mori fiber space) provided that a sequence of modifications of the flip type can not be infinite. In [10] the existence of minimal models was proven via a modification of the basic approach: instead of proving the finiteness of *any* sequence of flips, it is sufficient to construct (or prove the existence of) a *certain* sequence, which terminates after finitely many steps. That is what was done in [10]. Now let us formulate the main result of that paper.

Theorem 1.2 [10]. *Let (X, Δ) be a* klt*-pair, where $K_X + \Delta$ is a $\mathbb{R}$-Cartier divisor and $\pi\colon X \to U$ a projective morphism of quasi-projective varieties. Assume that either Δ is π-big, and $K_X + \Delta$ is π-pseudoeffective, or $K_X + \Delta$ is π-big. Then:*

1) the pair (X, Δ) gas a log terminal model over U,

2) if $K_X + \Delta$ is π-big, then (X, Δ) has a log canonical model over U,

3) if $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor, then $\mathcal{O}_U$-algebra

$$\bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)$$

is finitely generated.

This rather technical result implies a number of important geometric facts. We will give three claims, of which the last one is most important for the present paper.

Corollary 1.1 [10]. *Let X be a smooth projective variety of general type. Then X has a minimal model, the canonical ring*

$$\bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X))$$

is finitely generated and X has a model with a Kähler-Einstein metric.

Corollary 1.2 [10]. *Let (X, Δ) be a* klt*-pair and $\pi \colon X \to Z$ a small $(K_X + \Delta)$-extremal contraction. Then for π a flip exists.*

Corollary 1.3 [10]. *Let (X, Δ) be a* klt*-pair, whereas X is a $\mathbb{Q}$-factorial variety. Let $\pi \colon X \to U$ be a projective morphism of normal quasi-projective varieties where $K_X + \Delta$ is not π-pseudo-effective. Then some sequence of MMP modifications*

$$f = f_N \circ \ldots \circ f_1 \colon X \dashrightarrow Y$$

gives a Mori fiber space $g \colon Y \to W$ over U.

The last claim immediately implies that any rationally connected variety X is birationally equivalent to a Mori fiber space

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ & & \downarrow g \\ & & W, \end{array}$$

where Y is a variety with $\mathbb{Q}$-factorial terminal singularities, W is a projective normal variety and $(-K_Y)$ is g-ample. Similarly, for any rationally connected fiber space $\pi \colon X \to S$ there is a commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \pi \downarrow & & \downarrow g \\ S & \leftarrow & W, \end{array}$$

where the top arrow is a birational map and Y/W is a Mori fiber space. (Note that since termination of an *arbitrary* sequence of flips is still an open problem, Corollary 1.3 does *not* claim that *any* sequence of MMP-modifications gives a Mori fiber space.)

It follows from what was said, that all problems of birational geometry of rationally connected varieties could be set within the category of Mori fiber spaces:

instead of finding out whether there is a birational equivalence $X_1 \dashrightarrow X_2$ between two rationally connected varieties, we can replace X_i by birationally equivalent Mori fiber space $\pi_i\colon Y_i \to S_i$ and consider the same problem for Y_1, Y_2. If X_1, X_2 are birationally equivalent, then to describe all birational maps $X_1 \dashrightarrow X_2$ is the same as to describe the group of birational self-maps $\operatorname{Bir} X_i = \operatorname{Bir} Y_i$. If $\pi\colon X \to R$ is a rationally connected fiber space, then the problem of description of the relation between the group $\operatorname{Bir} X$ and the group of fiber-wise birational self-maps $\operatorname{Bir}(X/R)$ is also carried over to the corresponding Mori fiber space Y/S.

Restriction of the problems of birational geometry by the framework of the category of Mori fiber spaces has a number of clear advantages. For instance, it removes the asymmetry of the traditional approach, when birational maps $\chi\colon V \dashrightarrow V'$ are investigated, where the variety V belongs to the category of Fano fiber spaces and V' to the category of rationally connected varieties (or fiber spaces); whereas if we work with Mori fiber spaces (that is, with Fano fiber spaces with $\mathbb{Q}$-factorial terminal singularities and the relative Picard number one) then both varieties belong to the same category. Furthermore, the approach, motivated by MMP, makes it possible to set and solve the general problem of factorization of birational maps in a composition of elementary modifications (links), which we will consider in the next section.

And, nevertheless, it is not clear, to what extent these advantages are essential and whether they justify replacing the traditional approach by the new one, to what extent the main definitions (for instance, that of the key concept of birational rigidity), corresponding to the ideology of MMP, are "better" or "worse"; this issue will be discussed below.

2.2. The problem of factorization of birational maps. The Sarkisov program is the theory of factorization of birational maps between Mori fiber spaces in a composition of elementary modifications (links). Consider the diagram

$$\begin{array}{ccccc} & X & \stackrel{\chi}{\dashrightarrow} & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & & S', & \end{array}$$

where χ is a birational map, X/S and X'/S' are Mori fiber spaces. It is required to construct a sequence of intermediate Mori fiber spaces $\pi_i\colon X_i \to S_i$, $i = 0, 1, \ldots, N$, that starts with $\pi_0 = \pi$ ($X_0 = X$ and $S_0 = S$) and ends with $\pi_N = \pi'$ ($X_N = X'$ and $S_N = S'$), and a sequence of elementary modifications (links) $\tau_i\colon X_{i-1} \dashrightarrow X_i$ such that

$$\chi = \tau_N \circ \ldots \circ \tau_1\colon X_0 = X \dashrightarrow X_N = X'.$$

Recall that a *link* $\tau\colon X \dashrightarrow Y$ between Mori fiber spaces $\pi\colon X \to S$ and $\rho\colon Y \to T$ is a birational map of one of the following four types.

Type I (enlarging the base). There are: an extremal divisorial contraction $\varphi\colon Z \to X$, a birational map $\psi\colon Z \dashrightarrow Y$, which is a composition of flops (in particular, an isomorphism in codimension one) and an extremal contraction $\varepsilon\colon T \to S$ (in particular, $\rho(T/S)=1$) such that the following diagram of maps commutes:

$$\begin{array}{cccccc} & Z & \stackrel{\psi}{\dashrightarrow} & Y & \\ \varphi & \downarrow & & \downarrow & \rho \\ & X & & T & \\ \pi & \downarrow & & \downarrow & \varepsilon \\ & S & = & S, & \end{array} \tag{5}$$

where $\psi\circ\varphi^{-1}=\tau\colon X \dashrightarrow Y$. The simplest exmaple of a link of that type was given above (Example 1.5), it corresponds to the blow up of a point $\varepsilon\colon T\to S$ on the base of the conic bundle and the blow up $\varphi\colon Z\to X$ of the fiber over that point. Let us give one more

Example 1.8. Let $L\subset\mathbb{P}^3$ be a line, $\sigma\colon X\to\mathbb{P}^3$ its blow up, Π_L the pencil of planes in $\mathbb{P}^3$, containing the line L, Π_L^+ its strict transform on X. Obviously, Π_L^+ is the pencil of fibers of the morphism

$$\pi\colon X\to S=\mathbb{P}^1$$

with the fiber $\mathbb{P}^2$, that is, X/S is a Mori fiber space (and a Fano fiber space in the traditional sense). Now let $R\subset X$ be an arbitrary section of the projection π, $\varphi\colon Z\to X$ its blow up. The composite map

$$\pi\circ\varphi\colon Z\to\mathbb{P}^1$$

is a fibration into rational ruled surfaces of type $\mathbb{F}_1$. More precisely, $E\subset Z$ be the exceptional divisor of the blow up φ, that is, $E=\varphi^{-1}(R)$. Obviously the projection $\mathbb{F}_1\to\mathbb{P}^1$ (the regularized projection of $\mathbb{P}^2$ from the point $R\cap\pi^{-1}(s)$, $s\in S$) generates the projection $p\colon Z\to E$, which is a $\mathbb{P}^1$-bundle, that is, Z/E is a Mori (Fano) fiber space. This gives the digram (5) of a link of the first type with $T=E$, where $\varepsilon\colon T\to S$ is the projection $\varphi\colon E\to R$ with respect to the identification $\pi\colon R\to S$ and $Z=Y$, ψ is the identity map. In the described example the Mori fiber space Y/T is obtained from X/S by fibering the fibers $\pi^{-1}(s)$, $s\in S$. Accordingly, the fibers of the new fiber space Y/T are of a smaller dimension. This operation can also be interpreted as fibering the generic fiber F_η of the morphism $X\to S$ over the line $\mathbb{P}^1_\eta$ (which is the generic fiber of the morphism $T\to S$).

Type II (fiber-wise modifications). In this case $S=T$. There are extremal divisorial contractions $\varphi\colon Z\to X$ and $\lambda\colon W\to Y$ and a birational isomorphism $\psi\colon Z\dashrightarrow W$, which is a composition of flops (an isomorphism in codimension one) such that the following diagram commutes:

$$\begin{array}{cccccc} & Z & \stackrel{\psi}{\dashrightarrow} & W & \\ \varphi & \downarrow & & \downarrow & \lambda \\ & X & \stackrel{\tau}{\dashrightarrow} & Y & \\ \pi & \downarrow & & \downarrow & \rho \\ & S & = & T. & \end{array}$$

The fiber-wise self-maps of the pencils of cubic surfaces, considered above (Example 1.4) belong to this type.

Let us consider one more

Example 1.9 (replacing the fiber). Let S be a smooth curve, $\pi\colon X \to S$ a $\mathbb{P}^n$-bundle , $x \in X$ an arbitrary point, $F = \pi^{-1}(\pi(x)) \cong \mathbb{P}^n$ the fiber that contains that point. Let

$$\varphi\colon Z \to X$$

be the blow up of the point x, $E = \varphi^{-1}(x) \cong \mathbb{P}^n$ the exceptional divisor, $F^+ \subset Z$ the strict transform of the fiber F. Obviously, $R = F^+ \cap E$ is a hyperplane in $E = \mathbb{P}^n$ and the projection of the fiber $F = \mathbb{P}^n$ from the point x generates a $\mathbb{P}^1$-bundle

$$\lambda_F\colon F^+ \to R.$$

If $L \subset F$ is an arbitrary line, passing through the point x, then its strict transform $L^+ \subset F^+$ is a fiber of the projection λ_F. Since $K_Z = \varphi^* K_X + nE$, the following equalities hold:

$$(K_Z \cdot L^+) = -1, \quad (F^+ \cdot L^+) = -1$$

(taking into account that $(K_X \cdot L) = -(n+1)$), so that the numerical class of the curve L^+ generates the extremal ray $[L^+] \in \overline{NE}(Z)$. Let $\lambda\colon Z \to Y$ be the contraction of that ray. Obviously, $\lambda|_{F^+} = \lambda_F$ and λ contracts the divisor F^+. The image Y is again a $\mathbb{P}^n$-bundle over S, which is birationally isomorphic to the original one:

$$\tau = \lambda \circ \varphi^{-1}\colon X \dashrightarrow Y$$

is a link of the second type. Here $W = Z$ and ψ is an isomorphism. The fiber space Y/S had the same fibers over all points of the curve S, except for the point $\pi(x) \in S$. The fiber over that point is replaced by the exceptional divisor $E = \mathbb{P}^n$. For $n = 1$ the described procedure is the classical modification of a ruled surface.

Finally, the birational modification of a conic bundle, described in Example 1.5, also belongs to type II. (In fact, the links of that type were modelled by those modifications.) Note that by construction, a link of type II always induces a birational isomorphism of the generic fibers $F_\eta \dashrightarrow G_\eta$ of Mori fiber spaces X/S and Y/S.

Type III (shrinking the base). The links of this type are inverse to the links of type I, that is, in the diagram (5) the left hand side and the right hand side are swapped. More precisely, there are: an extremal divisorial contraction $\varphi\colon Z \to Y$, a birational map $\psi\colon X \dashrightarrow Z$, which is a composition of flops (an isomorphism in codimension one) and an extremal contraction $\varepsilon\colon S \to T$, $\rho(S/T) = 1$, such that the following diagram of maps commutes:

$$\begin{array}{ccccc}
 & X & \stackrel{\psi}{\dashrightarrow} & Z & \\
\pi & \downarrow & & \downarrow & \varphi \\
 & S & & Y & \\
\varepsilon & \downarrow & & \downarrow & \rho \\
 & T & = & T, &
\end{array}$$

where $\varphi \circ \psi = \tau \colon X \dashrightarrow Y$. Inverting the construction of Example 1.5, we obtain a link of type III. From the geometric viewpoint, it is most interesting that a composition of two links which are of types I and III, gives, generally speaking, a birational modification which is not compatible with the structures of fiber spaces, that is, fibers are not mapped into fibers, as shows the following

Example 1.10. In the notations of Example 1.8 let us choose the section $R \subset X$ in a special way: let $R = \sigma^{-1}(L^*)$, where $L^* \subset \mathbb{P}^3$ is a line that does not meet L, in particular, $< L, L^* >= \mathbb{P}^3$. In this case the composition $\sigma \circ \varphi \colon Z \to \mathbb{P}^3$ is the blow up of a smooth reducible (non-connected) curve $L \cup L^*$ with the exceptional divisor $E_L \cup E$, where $E_L = \sigma^{-1}(L)$ is the exceptional divisor of the blow up σ. Contracting $E_L \subset Z$, we obtain a link of type III

$$\begin{array}{ccc} Y & \dashrightarrow & X^* \\ \downarrow & & \downarrow \\ T & \to & S^*, \end{array}$$

where X^* is $\mathbb{P}^3$ with the blown up line L^*, the curve $S^* = \mathbb{P}^1$ and $X^* \to S^*$ is the regularized projection from L^*. The composition of this link with the link of Example 1.8 gives a birational map

$$\begin{array}{ccc} X & \dashrightarrow & X^* \\ \downarrow & & \downarrow \\ S & & S^*, \end{array}$$

of $\mathbb{P}^2$-bundles over $S, S^* \cong \mathbb{P}^1$, which is not compatible with the structures of those fiber spaces. Note that in the special case under consideration, $T = S \times S^* = \mathbb{P}^1 \times \mathbb{P}^1$ and the projection $Y \to T$ is the regularization of the rational map

$$\mathbb{P}^3 \dashrightarrow L \times L^*,$$

that maps a point $x \in \mathbb{P}^3 \setminus (L \cup L^*)$ to the unique pair of points $(z, z^*) \in L \times L^*$ such that $x \in < z, z^* >$ (the lines L and L^* identify naturally with the curves S^* and S as sections of the Mori fiber spaces X^*/S^* and X/S, respectively). Note also that the diagram

$$\begin{array}{ccc} T & = & T \\ \downarrow & & \downarrow \\ S & & S^*, \end{array},$$

is an example of a link of type IV (in the dimension two), which we will now describe.

Type IV (replacing the structure of a fiber space). This type is most interesting. There are: a birational map $\psi \colon X \dashrightarrow Y$, which is a composition of flops, and extremal contractions $s \colon S \to R$ and $t \colon T \to R$ such that the following diagram of maps commutes:

$$\begin{array}{ccccc} & X & \stackrel{\psi}{\dashrightarrow} & Y & \\ \pi & \downarrow & & \downarrow & \rho \\ & S & & T & \\ s & \downarrow & & \downarrow & t \\ & R & = & R. & \end{array}$$

The simplest example of a link of this type was given above (replacing one projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ by another). A non-trivial example of a link of type IV was described in Example 1.7. In that example the variety V has precisely two structures of a Fano fiber space and the structures are changed via a flop.

Today the strongest completely proved fact on factorization of birational maps into a composition of links is

Theorem 1.3 [11]. *Let $\pi: X \to S$ and $\pi': X' \to S'$ be Mori fiber spaces with $\mathbb{Q}$-factorial terminal singularities. The varieties X and X' are birationally equivalent if and only if when there exists a sequence of links*

$$\begin{array}{ccc} X_{i-1} & \stackrel{\tau_i}{\dashrightarrow} & X_i \\ \downarrow & & \downarrow \\ S_{i-1} & & S_i, \end{array}$$

$i = 1, \ldots, N$, connecting X/S and X'/S', that is, $X_0/S_0 = X/S$ and $X_N/S_N = X'/S'$.

However, as we mentioned above, the main question is whether an *arbitrary* birational map $\chi: X \dashrightarrow X'$ can be decomposed into a composition of links. In the dimension three the answer if positive [12], in the dimension ≥ 4 the weaker claim, formulated above, is true. This comes from the different approaches to the problem of factorization of birational maps.

2.3. On the proof of Sarkisov program. Let us briefly describe the original Sarkisov's approach [13,14], realized in [12]. Let $\chi: X \dashrightarrow X'$ be a birational map between the total spaces of Mori fiber space X/S and X'/S' of dimension three. One has to show that χ can be decomposed into a composition of elementary links, $\chi = \tau_N \ldots \tau_1$. The proof, based on the original Sarkisov's ideas, is by producing an inductive algorithm, which associates with a birational χ, or, more precisely, a diagram

$$\begin{array}{ccccc} & X & \stackrel{\chi}{\dashrightarrow} & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & & S' & \end{array} \qquad (6)$$

an *untwisting link* of one of the four types I-IV,

$$\begin{array}{cccccccc} & X & \stackrel{\tau}{\dashrightarrow} & X_1 & & \stackrel{\chi\circ\tau^{-1}}{\dashrightarrow} & X' & \\ \pi & \downarrow & & \downarrow & \pi_1 & & \downarrow & \pi' \\ & S & & S_1 & & & S', & \end{array}$$

which decreases a certain invariant of the original birational map,

$$\delta(\chi \circ \tau^{-1}) < \delta(\chi),$$

which can not decrease infinitely.

Let us describe this invariant. The dimension $\dim X$ is now arbitrary ≥ 3. Since $\rho(X'/S') = 1$, there is a positive integer m and a very ample divisor A' on the base S' such that

$$D' = -mK_X + \pi'^* A'$$

is a very ample divisor on X'. The very ample linear system $|D'| = \mathcal{H}'$ is fixed through the whole procedure of factorization of the map χ. Note that the map χ is fiber-wise (that is, there exists a rational dominant map $\varepsilon: S \dashrightarrow S'$, making the diagram (6) a commutative one, $\varepsilon\pi = \pi'\chi$) if and only if the strict transform of the linear system $|\pi'^*A'|$ is pulled back from the base S, that is,

$$\chi_*^{-1}|\pi'^*A'| \subset |\pi^*A^+|$$

for some (movable) divisor A^+ on the base S. Set

$$\mathcal{H} = \chi_*^{-1}|D'|$$

to be the strict transform of the system Σ' on X. Since X/S is a Mori fiber space, we get

$$\mathcal{H} \subset |-nK_X + \pi^*A|$$

for some $\mathbb{Q}$-Cartier divisor A on S. Since the denominators are bounded, we may assume that $n \in \mathbb{Z}_+$. Obviously, $n \geq 1$. The following claim holds.

Proposition 1.1. *The inequality $n \geq m$ is holds. If $n = m$, then the map χ is fiber-wise.*

Proof (see, for instance, [12]) is elementary. One has to repeat, almost word for word, the arguments that were used in the proof of Proposition 2.6 of the previous survey. Let us outline it: let $\varphi: Z \to X$ be a resolution of singularities of the map χ, $\psi = \chi \circ \varphi: Z \to X'$ the composite map, a birational morphism,

$$\{E_i \,|\, i \in I\} \quad \text{and} \quad \{E'_j \,|\, j \in J\}$$

the sets of φ- and ψ-exceptional divisors, respectively. Arguing word for word as in [1, Proposition 2.6], we get

$$\left(1 - \frac{n}{m}\right)\varphi^*K_X + \frac{1}{m}\varphi^*\pi^*A = \frac{1}{m}\psi^*\pi'^*A' + \sum_{j\in J} a'_j E'_j + \sum_{i \in I}\left(\frac{b_i}{m} - a_i\right)E_i,$$

where $a'_j > 0$, $a_i > 0$ (the singularities are terminal) and $b_i \geq 0$. Since E_i are φ-exceptional, the restriction of this equality onto the fiber of general position of the projection π shows that $n \geq m$, whereas, if $n = m$, then the strict transform of the linear system $|\pi'^*A'|$ with respect to χ is pulled back from the base S, and each ψ-exceptional divisor E'_j either is φ-exceptional, or its image $\varphi(E'_j)$ is pulled back from the base S. Q.E.D. for Proposition 1.1.

Recall [12], that the *canonical* (respectively, *log canonical*) *threshold* of the pair $(X, \mathcal{H})$ is the number

$$\mathrm{ct}(X, \mathcal{H}) = \sup\{\alpha \in \mathbb{Q}_+ | \text{ the pair } (X, \alpha\mathcal{H}) \text{ is canonical}\}$$

(respectively, $\mathrm{lct}(X, \mathcal{H}) = \sup\{\alpha \in \mathbb{Q}_+ | \text{ the pair } (X, \alpha\mathcal{H}) \text{ is log canonical}\}$). If $\varphi: Z \to X$ is a resolution of singularities of the pair $(X, \mathcal{H})$ with the set of exceptional divisors $\{E_i | i \in I\}$, then

$$\frac{1}{\mathrm{ct}(X, \mathcal{H})} = \max_{i \in I} \quad \frac{\mathrm{ord}_{E_i}\varphi^*D}{a(E_i, X)}$$

for a general divisor $D \in \mathcal{H}$ and similarly

$$\frac{1}{\operatorname{lct}(X, \mathcal{H})} = \max_{i \in I} \quad \frac{\operatorname{ord}_{E_i} \varphi^* D}{a(E_i, X) + 1}.$$

In particular, the inequality $\operatorname{lct}(X, \mathcal{H}) > \operatorname{ct}(X, \mathcal{H})$ takes place. If $\operatorname{ct}(X, \mathcal{H}) = \gamma$, then for each exceptional divisor E_i the inequality

$$\gamma \operatorname{ord}_{E_i} \varphi^* \mathcal{H}) \leq a(E_i, X),$$

holds, and, moreover, for at least one divisor E_i this inequality is an equality. Such divisors are called *crepant* divisors; the corresponding discerete valuations of the field of rational functions $\mathbb{C}(X)$ do not depend on the choice of resolution of singularities and are determined by the pair $(X, \mathcal{H})$. Let $e(X, \mathcal{H})$ be the number of crepant valuations ≥ 1.

Definition 1.2. The *degree* (or *Sarkisov degree*) of the pair $(X, \mathcal{H})$ is the triple of numbers

$$\delta(X, \mathcal{H}) = (n, \gamma = \operatorname{ct}(X, \mathcal{H}), e = e(X, \mathcal{H})).$$

The set of values of the degree δ is ordered in the following way (corresponding to the lexicographic order of the triples (n, γ^{-1}, e)):

$$\delta = (n, \gamma, e) > \delta_1 = (n_1, \gamma_1, e_1),$$

if either $n_1 < n$, or $n_1 = n$, but $\gamma_1 > \gamma$, or, finally, $n_1 = n$ and $\gamma_1 = \gamma$, but $e_1 < e$. Since (with the very ample linear system $\mathcal{H}'$ fixed) the map χ is uniquely determined by the system $\mathcal{H}$, we can write $\delta(\chi)$ instead of $\delta(X, \mathcal{H})$. At this stage in Sarkisov program appears the key concept of a maximal singularity.

Definition 1.3. A *maximal singularity* of the linear system $\mathcal{H}$ (or the birational map χ) in the sense of the minimal model program (briefly, a MMP-maximal singularity) is an exceptional divisor $E \subset Z$ of some resolution $\varphi\colon Z \to X$ of the pair $(X, \mathcal{H})$ (or the discrete valuation of the field of rational functions $\mathbb{C}(X)$, corresponding to that divisor), if the *Noether-Fano inequality* holds:

$$\operatorname{ord}_E \varphi^* \mathcal{H} > n a(E, X).$$

In [12] instead of the word combination "maximal singularity" the "base component of high multiplicity" is used. In the survey [16] the "maximal singularity" is returned, however, one should remember, that in the classical theory the parameter n is the threshold of canonical adjunction. In the general situation, considered above, $c(\mathcal{H}) = n$, only if the divisor A is pseudo-effective on the base S. Thus between the concepts of maximal singularity in the traditional approach and MMP there are certain differences (ignored in [12]). To emphasize the point, we speak about MMP-maximal singularities (for the majority of problems, that are solved by now, these differences are inessential).

Obviously, a MMP-maximal singularity exists if and only if the inequality

$$\mathrm{ct}(X, \mathcal{H}) < \frac{1}{n}$$

holds.

Now ([12, Theorem 5.4]), if $\mathcal{H}$ has an MMP-maximal singularity, then applying $(K_Z+\alpha\mathcal{H}_Z)$-MMP to the extremal extraction $\varphi: Z \to X$ of one of the crepant discrete valuations, where the number α is specially selected, one gets a link $\tau: X/S \dashrightarrow X_1/S_1$ of type I or II, such that $(n_1, \gamma_1, e_1) < (n, \gamma, e)$, and moreover, if $n_1 = n$, then τ is fiber-wise and induces birational isomorphisms of the bases $S \dashrightarrow S_1$ and generic fibers,

$$\begin{array}{ccccc} & X & \stackrel{\tau}{\dashrightarrow} & X_1 & \\ \pi & \downarrow & & \downarrow & \pi_1 \ , \\ & S & \dashrightarrow & S_1, & \end{array}$$

that is, makes a "square" in the terminology of [12]. If, however, $\mathcal{H}$ has no MMP-maximal singularities (that is, the inequality $\mathrm{ct}(X, \mathcal{H}) \geq \frac{1}{n}$ holds), then a sequence of links of types III and IV is constructed

$$X/S \stackrel{\tau_1}{\dashrightarrow} X_1/S_1 \stackrel{\tau_2}{\dashrightarrow} \ldots \stackrel{\tau_k}{\dashrightarrow} X_k/S_k,$$

such that either $n = n_1 = \ldots = n_{k-1} > n_k$, or $n = n_1 = \ldots = n_k$ and the induced birational map is an isomorphism of the Mori fiber spaces. Finally, the last necessary fact, the finiteness of this procedure, that is, that the problem of factorization can solved in finitely many steps, is proved via Alexeev's theorem that there are no accumulation points from below for the log canonical thresholds [16]. A complete proof of the theorem on factorization see in [12] or in the survey [15].

The algorithm, described above, in principle does not depend of dimension, however, today not all facts of MMP that are needed to prove it, are shown in dimension ≥ 4 (although the work is in progress and essential advances have been made; for instance, on the termination of thresholds see [17,18]). All those facts are certain claims on finiteness (for instance, the termination of a sequence of flips), they are needed both for constructing links and for proving the finiteness of the factorization procedure. Using the approach that turned out so successful in [10], Hacon and McKernan in [11] proved a weaker version of the Sarkisov program in the arbitrary dimension: instead of consructing an algorithm of factorization and proving its finiteness, they showed existence of *some* sequence of links, the composition of which gives a birational isomorphism $X \dashrightarrow X'$ of the total spaces of Mori fiber spaces X/S and X'/S' (about which it is assumed in advance, that they are birationally isomorphic).

The main result [11] is as follows.

Theorem 1.4. *Assume that the Mori fiber spaces X/S and X'/S' are both products of the $(K_Z+\Phi)$-MMP for some* klt*-pair (Z, Φ). Then the induced birational map $X \dashrightarrow X'$ is a composition of links of types I-IV.*

It should be pointed out that if the tital spaces of two Mori fiber spaces are birationally equivalent and have $\mathbb{Q}$-factorial terminal singularities, then they are products of the K_Z-MMP for some variety Z, so that the corresponding Mori fiber spaces satisfy the assumptions of Theorem 1.4.

The proof of Theorem 1.4 is based on the technique developed in [10]; to the latter paper we refer an interested reader.

2.4. The method of maximal singularities and the factorization theory: a comparison. After what has been said it makes sense to compare the two approaches to investigating birational geometry of rationally connected fiber spaces: the classical method of maximal singularities and the program of factorization of birational maps, based on the theory of minimal models (the Sarkisov program). It should be noted that both approaches go back to the same ideas (developed in the works of M. Noether and his predecessors, and after that in the works of the Italian classics up to Fano, see [1]), and for that reason in many respects are sufficiently close. Between these approaches there are, however, essential differences, on which we will now dwell.

The main idea of the method of maximal singularities, described in details in [1], is to study the maximal singularities of a birational map $\chi\colon V \dashrightarrow V'$ or, what is equivalent, the maximal singularities of a movable linear system Σ on V, that defines this birational map. The method works successfully, if it is possible to describe explicitly the potentially maximal singularities, which are then untwisted, as a rule, by birational self-maps $\tau \in \operatorname{Bir} V$ (in the relative case, by fiber-wise birational self-maps $\tau \in \operatorname{Bir}(V/S)$). The point, why almost all potentially maximal singularities can be untwisted by means of self-maps, is not discussed, as it is an empirical fact. In those successfully studied cases, when the self-maps are not sufficient, each maximal singularity explicitly defines a transition to another structure of a Fano fiber space. The method of maximal singularities was modelled on the pioneer paper [19]; in the past almost forty years the general approach did not change a lot, although the technical side has been transformed and made radically stronger.

The MMP-approach was modelled on the proof of the Sarikisov theorem [2], taking into account the ideology of the Mori theory. That approach is about fifteen years "younger". Its main idea, described in the previous section, is to simplify a birational map (or a movable linear system) by means of elementary links. A link is constructed by applying MMP to a suitable log pair. A priori a link is a birational map between *distinct* Mori fiber spaces, that is, even in the case of a fiber-wise link one comes over to another model.

The method of maximal singularities is a (technically powerful and successfully working) scheme of arguments leading to very strong individual results, that is, results for particular explicitly given families of Fano varieties and fiber spaces. On the contrary, the Sarkisov program (in the form in which it is known since the paper [12] was published) is a general existence theorem, claiming the very fact that it is possible to factorize a birational map (or a somewhat weaker fact in dimension ≥ 4). Each of these two approaches has its advantages and disadvantages.

An obvious advantage of the Sarkisov program is its generality. From the description of the program and the proof of the main theorem follows a procedure of constructing factorizing links. Taking into account the rapid development of MMP since the mid-eighties, it is not surprising that the proof of Sarkisov program in [12] generated a lot of optimism in respect of three-dimensional birational geometry. In 1993-95 it seemed to many people that solution of the problems of the classical birational geometry in dimension three, even such as description of the Cremona group of rank three or description of the group of birational self-maps of the three-dimensional cubic, or a proof of the rationality criterion for three-fold conic bundles, is a matter of not so distant future. Something was said on that subject (although in a rather cautious way) in [12]. The first thing to be done, was to find new proofs of the known results that were obtained by the method of maximal singularities. After that, it was meant to make further progress in the area where the classical methods did not work.

Today, after more than fifteen years, one can say that these hopes did not realize. The problems of describing birational maps between three-dimensional Mori fiber spaces turned out to be much more difficult than the authors of [12,14] thought. Their optimism of the early nineties is explained rather by their poor aquaintance with the preceding works of the Moscow school of birational geometry; probably, for the same reason in [12] the work of the Moscow school is considered as belonging to the past: it was expected that the new techniques of MMP will make it possible to easily overcome the difficulties that obstructed successful work of the method of maximal singularities. It should be acknowledged that some ideas that came from MMP turned out to be fruitful indeed. It is, in the first place, the so called connectedness principle of Shokurov and Kollár and the inversion of adjunction that follows from that principle (see Sec. 3.3 of this chapter and §2 of Chapter 3), which made it possible to considerably simplify some parts of the classical techniques (cf., for instance, the exception of a maximal singularity over a quadratic point in [20] and in [21]: the latter approach essentially does not need anything but a straightforward application of the inversion of adjunction).

However, as a whole the attempts to realize the Sarkisov program for a wide class of three-dimensional Mori fiber spaces were not successful. Birational maps between particular three-dimensional varieties were in most cases studied using the well trodden approach of the classical method of maximal singularities, sometimes with certain technical improvements. This applies already to the very first, and best known, paper of that series [22], which was originally conceived as a first big-scale stage in the realization of the Sarkisov program. It turned out that MMP-techniques is useless for all 95 types of the weighted Fano hypersurfaces, because all potentially maximal singularities (on many varieties they simply do not exist) are untwisted by birational self-maps, which, in their turn, are constructed by means of the known classical techniques. Nevertheless, [22] was interpreted as a result of application of the new MMP-techniques (see, for instance, the survey [15]), which was, of course, a big exaggeration. The paper [22] was written in the new language (and contains new proofs of certain known facts of the classic theory), but this seems to be the most that can be said. By its contents [22] undoubtedly belongs to the series of papers done by the method of maximal singularities and directly follows the paper [19] on the three-dimensional quartic.

Apart from [22], a few more papers were published, where the scheme of the Sarkisov program was used for studying birational maps between three-dimensional Mori fiber spaces [23,24]. In [21] some known results, that were earlier obtained by the method of maximal singularities, are presented in the language of Sarkisov program. In [23,24] that program really works: non-trivial links are constructed that connect the varieties under investigation with other Mori fiber spaces, however, the existence of such a link does give any really meaningful information about the birational type of a variety, although it does mean that, in the terminology of Reid-Corti, that the variety is not birationally rigid. The paper [25] which joins this series of papers contains certain non-trivial computations and constructions, also a number of interesting conjectures, but no progress at all towards their proof.

The poverty of results that were obtained in the course of attempts to apply the general theory [12] to the study of birational geometry of three-dimensional Mori fiber spaces, is even more impressive if one compares them with the results that were obtained during the same period of time in the framework of the classical method of maximal singularities [3,26-29], speaking not about the results in the arbitrary dimension. The reason of such unsuccessfulness is, it seems, in the fact that to construct links one needs a precise description of all potentially maximal singularities, so that the work of the classical method of maximal singularities is inevitably included into the work of Sarkisov program as the most important part. On the other hand, the above mentioned empirical fact takes place: for almost all (accessible for investigation) varieties the potentially realizable maximal singularities are untwisted by birational self-maps, which makes the MMP techniques unnecessary, as it is sufficient to define a birational map between two *given* varieties at the generic point, not taking care how it could be decomposed into a sequence of elementary contractions/extractions and flips. It seems that these observations explain the extremely low efficiency of the general theory [12], which found almost no applications in fifteen years.

As an illustration, let us consider Example 1.4. Let $V \stackrel{\pi}{\to} \mathbb{P}^1$ be a standard fibration into cubic surfaces, $C \subset V$ a section of the projection π. Assume that C is a maximal curve of a movable linear system $\Sigma \subset |-nK_V + lF|$, where $l \in \mathbb{Z}_+$. The extremal blow up, associated with the curve C, is simply its blow up in the usual sense, $\varphi\colon \widetilde{V} \to V$. Now the general theory guarantees that application of MMP to a suitable pair $(\widetilde{V}, \alpha\widetilde{\Sigma})$ leads to a link $V \dashrightarrow V^+/\mathbb{P}^1$, untwisting the maximal curve C. However, this general claim which uses all the power of MMP, is practically useless, since the construction of the link is obvious from the elementary geometric considerations and its proof does not require any efforts since $V = V^+$ and the link is well defined on the generic fiber. There are finitely many fibers $F_i = \pi^{-1}(t_i)$, $i = 1, \dots, k$, where through the point $C \cap F_i$ there is at least one line on F_i. MMP guarantees that a sequence of flops, starting from $\widetilde{V}$, leads to a model $V^\sharp$, that admits an extremal contraction (so that the birational self-map of Example 1.4, associated with the section C, is a composition of two links, of type I and type III), however, the classical techniques gives at one the final result with considerably less effort. It is especially important here, that using the classical approach, we stay on the same model, whereas the general theory provides an untwisting of the curve C, by means of, generally speaking, a transition to a new Mori fiber space. For that reason, whereas the classical method of maximal singularities requires only information on the biregular geometry of the *given* model of a Mori fiber space, for the Sarkisov program to work effectively one needs (as it is only natural to expect from a theory of such level of generality) information on *all* models of that dimension. Biregular classification of Mori fiber spaces even in dimension three is still very far from completion (in any reasonable sense). And this is the reason why the general theory [12] is so inefficient.

In dimension three there are three classes of Mori fiber spaces: Fano varieties (with $\mathbb{Q}$-factorial terminal singularities), del Pezzo fibrations and conic bundles. By the mid-nineties, a lot was known on birational geometry of varieties of the first and third classes, however del Pezzo fibrations remained a white spot. Attempts to study their geometry by means of the test class technique were made in the course of about ten years, but they turned out to be unsuccessful [30] (as it became clear somewhat later [3], it was impossible in principle). The situation has changed radically when the paper [3] appeared, where the classical approach was essentially re-developed, in particular, the test class technique was replaced by the technique of counting multiplicities. However, as far as the author knows, up to a very recent time attempts were made to obtain results on birational geometry of del Pezzo fibrations by means of methods similar to the proof of Sarkisov theorem on the conic bundles (Example 1.5), that is, by means of fiber-wise modifications. (Since it is precisely the maximal singularities, the center of which is contained in a fiber, present the biggest difficulty, see Chapter 2.) Those attempts were also unsuccessful. In contrast to the conic bundles, even the simplest fiber-wise modifications of a pencil of cubic surfaces lead to varieties with complicated singularities and the study terminates at this point.

This is the situation today; introducing new ideas and new facts in the future can, of course, change it.

Let us emphasize that this section is in no way a survey of the very minimal model program or Sarkisov program. For that reason, we do not mention (and the more so, do not discuss) the main papers on MMP (due to Mori, Kawamata, Shokurov, Kollár and many others), except for the paper [10]. This is the end of our short visit to the general theory of factorization of birational maps (the Sarkisov program). We come back to the theory, techniques and results of the method of maximal singularities, the subject of the present survey.

§3. Birational rigidity of Fano fiber spaces

3.1. The threshold of canonical adjunction. For an arbtrary rationally connected smooth projective variety X we denote by the symbol A^iX the Chow group of algebraic cycles of codimension $i \geq 1$ modulo numerical equivalence, $A^i_{\mathbb{R}}X = A^iX \otimes \mathbb{R}$. By the symbol A^i_+X we denote the closed cone in $A^i_{\mathbb{R}}X$, generated by the classes of effective cycles (the pseudo-effective cone). By the symbol $A^i_{\mathrm{mov}}X$ we denote the closed cone in $A^i_{\mathbb{R}}X$, generated by the classes of movable divisors (that is, such divisors D, that the complete linear system $|D|$ has no fixed components), the *movable* cone.

Let us consider a standard Fano fiber space $\pi\colon V \to S$. Obviously, we get an inclusion $\pi^* A^i_{\mathrm{mov}}S \subset A^i_{\mathrm{mov}}V$. Furthermore,

$$A^1V = \mathbb{R}[K_V] \oplus \pi^* A^1 S.$$

Definition 1.4. We say that the standard Fano fiber space $\pi\colon V \to S$ satisfies the *K-condition*, if

$$A^1_{\mathrm{mov}}V \subset \mathbb{R}_+[-K_V] \oplus \pi^* A^1_+ S.$$

In other words, V/S satisfies the K-condition if and only if for any movable linear system $|-nK_V + \pi^*A|$ the class $A \in \operatorname{Pic} S$ is pseudo-effective. If the pseudo-effective cone A^1_+S has a sufficienty simple structure, for instance, $A^1S = \mathbb{Z}H_S$, where H_S is the ample generator, so that $A^1_+S = \mathbb{R}_+[H_S]$ is the positive ray, or $S = S_1 \times \ldots \times S_k$, where $A^1S_i \cong \mathbb{Z}$, then it is easy to check that, in a certain sense, the "overwhelming majority" of standard Fano fiber spacea with the given fixed base S satisfies this condition. As an illustration let us consider the construction of Example 1.1, assuming that $A^1S = \mathbb{Z}H_S$.

In the notations of Example 1.1 we have $A_i \sim a_i H_S$ for some $a_i \in \mathbb{Z}$. Set $a = a_1 + \ldots + a_k$. Twisting the locally free sheaf $\mathcal{E}$, we may assume that it is generated by global sections, so that the tautological class L_X is numerically effective. Set also

$$bH_S \sim \det \mathcal{E} + K_S,$$

where the integral parameter b depends on X only. We get

$$K_V = -L + (a+b)H,$$

where for the brevity of notations $L = L_V$ and $H = \pi^* H_S$. Now if the linear system $|-nK_V + lH|$ is non-empty (the more so, movable), then the inequality

$$\begin{array}{cc} (-nK_V + lH) \cdot L^M \cdot H^{\dim S - 1} & \geq 0 \\ \| & \\ nL^{M+1}H^{\dim S-1} + d(l - n(a+b))H^{\dim S} & \end{array}$$

holds, where $d = d_1 \cdot \ldots \cdot d_k$, which immediately implies that for some $a_0 \in \mathbb{Z}$ for $a \geq a_0$ we have $l \in \mathbb{Z}_+$, that is, the K-condition holds for all standard Fano fiber space, satisfying the inequality $a_1 + \ldots + a_k \geq a_0$.

This example shows that a "majority" of standard Fano fiber spaces of Example 1.1 satisfies a stronger condition:

$$A^1_+V \subset \mathbb{R}_+[-K_V] \oplus \pi^* A^1_+ S.$$

The situation is similar for Fano double covers (Example 1.2). On the contrary, it is easy to see that Fano direct products (Example 1.3) do not satisfy the K-condition.

Now let us consider the most interesting case $S = \mathbb{P}^1$. To check the K-condition we use the fact that the *self-intersection* of a movable class $z \in A^1_{\rm mov}X$ is a pseudo-effective class of codimension two: $z^2 \in A^2_+X$. Therefore, if $z^2 \notin A^2_+X$, then $z \notin A^1_{\rm mov}X$.

Definition 1.5. We say that a standard Fano fiber space $\pi: V \to \mathbb{P}^1$ satisfies the K^2-*condition*, if

$$K_V^2 \notin \mathrm{Int}\, A^2_+V.$$

Let $\mathrm{Pic}\, V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where F is the class of a fiber of the projection π.

Proposition 1.2. *If a fiber space $\pi: V \to \mathbb{P}^1$ satisfies the K^2-condition, then it satisfies the K-condition, too.*

Proof. The self-intersection of the class $-nK_V + lF$ is

$$n^2K_V^2 + 2nl(-K_V \cdot F),$$

where in the brackets it is the anticanonical section of the fiber, that is, an effective cycle of codimension two. By the K^2-condition, the self-intersection can not be pseudo-effective for $l < 0$, which is what was required. Q.E.D.

The importance of the K-condition can be seen from the following simple fact.

Proposition 1.3. *Assume that a standard Fano fiber space $\pi: V \to S$ satisfies the K-condition.*

(i) The threshold of canonical adjunction of a movable linear system $\Sigma \subset |-nK_V + \pi^*A|$ is $c(\Sigma, V) = n$.

(ii) If the movable linear system Σ satisfies the equality $c(\Sigma, V) = 0$, then Σ is a π-pull back of a movable linear system Σ_S on the base S.

(iii) Assume in addition that the variety V is birationally superrigid: $c_{\rm virt}(\Sigma) = c(\Sigma, V)$ for any movable linear system Σ. Then every structure of a rationally connected fiber space on V is compatible with $\pi: V \to S$, that is, for any birational map $\chi: V \dashrightarrow V'$, where $\pi': V' \to S'$ is a rationally connected fiber space, there is a rational dominant map $\varepsilon: S \dashrightarrow S'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{\varepsilon}{\dashrightarrow} & S'. & \end{array}$$

Proof. (See [1, §1].) The claim (i) is obvious, because $\pi^*A \in A^1_+V$ is apseudo-effective class, whereas the class $\alpha K_V + \pi^*A$ for $\alpha > 0$ is negative on the fibers and cannot be pseudo-effective. The claim (ii) follows from (i). If the linear system Σ' on the variety V' is movable and pulled back from the base S', then its strict transform $\Sigma = \chi_*^{-1}\Sigma'$ on V is a movable linear system, satisfying the equality $c_{\rm virt}(\Sigma) = c(\Sigma', V') = 0$. By the asuumption on birational superrigidity and the claim (ii), it follows that Σ is a π-pull back of a movable linear system on S. This proves the claim (iii) and completes the proof of the proposition. Q.E.D.

Now let us give an example of a non-trivial computation of the threshold of canonical adjunction on a variety that does not satisfy the K-condition.

Example 1.11. Consider the Fano fiber space $\pi: V \to \mathbb{P}^1$ of Example 1.7 (we use the notations of that example). If a movable linear system Σ is a subsystem of the complete linear system $|-nK_V + lF|$ with $l \in \mathbb{Z}_+$, then, as above, we get $c(\Sigma) = n$. However, if $l < 0$, then the linear system $\Sigma^+ = \tau_*^+\Sigma$ is a subsystem of the complete linear system $|-n_+K_{V^+} + l_+F^+|$ on the standard Fano fiber space $V^+/\mathbb{P}^1$ from the same family. Now since $l_+ = -l \in \mathbb{Z}_+$, we have $c(\Sigma^+, V^+) = n_+$. However, $\tau^+: V \dashrightarrow V^+$ is a biregular isomorphism in codimension one (that is, an isomorphism of complements to closed sets $\Delta \subset V$, $\Delta^+ \subset V^+$ of codimension two). Therefore, $c(\Sigma, V) = c(\Sigma^+, V^+)$, that is, for $l < 0$ we have

$$c(\Sigma, V) = n_+ = n + l.$$

Another similar example is given by Fano fiber spaces $V/\mathbb{P}^1$ with a non-trivial (that is, non-compatible with the structure of the fiber space $\pi: V \to \mathbb{P}^1$) birational involution $\tau \in \mathrm{BirV}$ [31, Sec. 3.1] and [32, Sec. 2.3]; in the latter case they are varieties, described in the part (iii) of Theorem 1.5 below. Computations in those examples are completely similar to those considered above: if $\Sigma \subset |-nK_V + lF|$ $l < 0$ is a movable linear system, then applying the involution τ (which is biregular in codimension one), we transform Σ into a system $\Sigma^+ \subset |-n_+K_V + l_+F|$ with $l_+ \in \mathbb{Z}_+$, which makes it possible to compute the threshold $c(\Sigma, V) = c(\Sigma^+, V) = n_+$. For the details, see the papers mentioned above.

Today there are two main approaches to proving birational rigidity of Fano fiber spaces: the quadratic and the linear ones. The quadratic method is aimed at studying the self-intersection $Z = (D_1 \circ D_2)$ of a movable linear system $\Sigma \ni D_i$; the method extends the techniques of proving birational rigidity of primitive Fano varieties, described in the previous survey. Almost all results on birational geometry of standard Fano fiber spaces $V/\mathbb{P}^1$ are obtained by means of the quadratic method.

The linear method is aimed at studying singularities of an arbitrary divisor $D \in \Sigma$ or its restriction $D|_R$ onto a specially selected algebraic subvariety $R \subset V$. This method works in the proof of the theorem on Fano direct products [33], it is used in the proofs of the theorems on birational rigidity of fiber spaces $V/\mathbb{P}^1$ to simplify certain steps [7,34], also in the proof of the theorem on birational geometry of Fano double spaces of index two [35]. The quadratic technique is considered in detail in Chapter 2, the linear technique in Chapter 3.

3.2. The quadratic method: main results. Consider a standard Fano fiber space $\pi: V \to \mathbb{P}^1$, that is, $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where F is the class of a fiber of the projection π. Assume in addition that the condition

$$A^2V = \mathbb{Z}K_V^2 \oplus \mathbb{Z}H_F, \tag{7}$$

holds, where $H_F = (-K_V \cdot F)$ is the class of an anticanonical section of the fiber. The K^2-condition can be weakened in the following way.

Definition 1.6. A standard Fano fiber space $V/\mathbb{P}^1$ satisfies the K^2-*condition of depth* $\varepsilon \geq 0$, if

$$K_V^2 - \varepsilon H_F \notin \operatorname{Int} A_+^2 V.$$

Sometimes for more clarity the K^2-condition of depth $\varepsilon = 0$ will be called the *strong* K^2-*condition.* If the depth is not specified, then it is assumed that it is equal to zero.

It is easy to see that the K-condition for the class of Fano fiber space under consideration takes the following form

$$-K_V \notin \operatorname{Int} A_{\mathrm{mov}}^1 V.$$

Besides, it is obvious that if $\varepsilon_1 \leq \varepsilon_2$, then the K^2-condition of depth ε_1 implies the K^2-condition of depth ε_2.

Fibrations into Fano complete intersections. This class of varieties corresponds to the base $S = \mathbb{P}^1$ in Example 1.1. Let $a_* = \{0 = a_0 \leq a_1 \leq \ldots \leq a_{M+k}\}$ be a non-decreasing sequence of non-negative integers, $\mathcal{E} = \bigoplus\limits_{i=0}^{M+k} \mathcal{O}_{\mathbb{P}^1}(a_i)$ a locally free sheaf on $\mathbb{P}^1$, $X = \mathbb{P}(\mathcal{E})$ the corresponding projective bundle in the sense of Grothendieck. Obviously, we get

$$\operatorname{Pic} X = \mathbb{Z}L_X \oplus \mathbb{Z}R, \quad K_X = -(M+k+1)L_X + (a_X - 2)R,$$

where L_X is the class of the tautological sheaf, R is the class of a fiber of the morphism $\pi\colon X \to \mathbb{P}^1$, $a_X = a_1 + \ldots + a_{M+k}$. Furthermore, we have $L_X^{M+k+1} = a_X$.

For a set of k integers $(b_1, \ldots, b_k) \in \mathbb{Z}_+^k$ let $G_i \in |\, d_i L_X + b_i R|$ be irreducible divisors such that the complete intersection

$$V = G_1 \cap \ldots \cap G_k \subset X$$

is a smooth subvariety. The projection $\pi\,|_V\colon V \to \mathbb{P}^1$ is denoted by the same symbol π, the fiber $\pi^{-1}(t) \subset V$ by the symbol F_t, the restriction $L_X\,|_V$ by the symbol L.

The fiber space $V/\mathbb{P}^1$ is a standard Fano fiber space, satisfying the condition (7). Obviously, $K_V = -L + (a_X + b_X - 2)F$, where $b_X = b_1 + \ldots + b_k$. It is easy to check the formulas

$$(L^M \cdot F) = (H_F \cdot L^{M-1}) = d, \qquad L^{M+1} = d\left(a_X + \sum_{i=1}^{k} \frac{b_i}{d_i}\right),$$

where $d = d_1 \ldots d_k$ is the degree of the fiber. From here we get:

$$(-K_V \cdot L^M) = d\left(2 - \sum_{i=1}^{k} \frac{d_i - 1}{d_i} b_i\right) \text{ and } (K_V^2 \cdot L^{M-1}) = d\left(4 - a_X - \sum_{i=1}^{k} \frac{2d_i - 1}{d_i} b_i\right).$$

Since the linear system $|L|$ is free, these formulas immediately imply

Proposition 1.4. (i) *If* $a_X + \sum\limits_{i=1}^{k} \frac{2d_i - 1}{d_i} b_i \geq 4$, *then the strong* K^2*-condition holds.*

(ii) *If* $a_X + \sum\limits_{i=1}^{k} \frac{2d_i - 1}{d_i} b_i \geq 2$, *then the* K^2*-condition of depth 2 holds.*

(iii) *If* $\sum\limits_{i=1}^{k} \frac{d_i - 1}{d_i} b_i \geq 2$, *then* $-K_V \notin \mathrm{Int}\, A^1_+ V$ *and, the more so,* $-K_V \notin \mathrm{Int}\, A^1_{\mathrm{mov}} V$; *if, even more, the inequality above is strict, then* $-K_V \notin A^1_+ V$.

Now let us formulate the main result.

Assume that the variety V is sufficiently general in its family.

Theorem 1.5 [32]. (i) *The variety* V *is birationally rigid, the projection* $\pi\colon V \to \mathbb{P}^1$ *is the only structure of a rationally connected fiber space on* V *and the groups of birational and biregular automorphisms coincide,* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$, *if for the integral parameters of the variety* V *one of the following six cases takes place:*

- $a_X + b_V \geq 4$,
- $a_X = 1, b_V = 2$,
- $a_X = 0, b_V = 3$,
- $a_X = 3, b_V = 0$,
- $a_X = 2, b_V = 1$,
- $(a_*) = (0, \ldots, 0, 2)$ *and* $b_V = 0$.

(ii) *For* $(a_*) = (0, \ldots, 0, 1, 1)$ *and* $b_V = 0$ *a general variety* V *is birationally superrigid. However, the* K*-condition does not hold: the linear system* $|-K_V - F|$ *is movable and determines a rational map* $\varphi\colon V \dashrightarrow \mathbb{P}^1$, *the fibers of which are rationally connected. On the variety* V *there are precisely two non-trivial structures of a rationally connected fiber space: the morphism* $\pi\colon V \to \mathbb{P}^1$ *and the map* φ. *There exists a unique, up to a fiber-wise isomorphism, Fano fiber space* $\pi^+\colon V^+ \to \mathbb{P}^1$ *with the same parameters* $(a_*) = (0, \ldots, 0, 1, 1)$ *and* $b_{V^+} = 0$ *and a birational isomorphism* $\chi\colon V \dashrightarrow V^+$, *biregular in codimension one, such that the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \varphi & \downarrow & & \downarrow & \pi^+ \\ & \mathbb{P}^1 & = & \mathbb{P}^1. & \end{array}$$

The correspondence $V \to V^+$ *is an involution of the set of Fano fiber spaces of that type, that is,* $(V^+)^+ = V$.

(iii) *For* $a_X = 0$, $b_e = 2$ *for some* $e \in \{1, \ldots, k\}$ *and* $b_i = 0$ *for* $i \neq e$ *the variety* V *is birationally superrigid. However, the* K*-condition does not hold: the linear system* $|-d_e K_V - F|$ *is a pencil of rationally connected varieties. The group of birational self-maps* $\mathrm{Bir}\, V$ *is non-trivial and for a general variety* V *generated by the birational involution* τ, *which is biregular in codimension one, and moreover,* $\tau_*|F| =$

$|-d_eK_V-F|$. *On the variety* V *there are precisely two non-trivial structures of a rationally connected fiber space: the projection* $\pi\colon V\to\mathbb{P}^1$ *and the map* $\pi\tau\colon V\dashrightarrow\mathbb{P}^1$.

Fibrations into Fano cyclic covers. Let $a_*=\{0=a_0\le a_1\le\ldots\le a_{M+1}\}$ be a non-decreasing sequence of no-negative integers, $\mathcal{E}=\bigoplus\limits_{i=0}^{M+1}\mathcal{O}_{\mathbb{P}^1}(a_i)$ a locally free sheaf on $\mathbb{P}^1$, $X=\mathbb{P}(\mathcal{E})$ the corresponding projective bundle in the sense of Grothendieck. Obviously, we have

$$\operatorname{Pic}X=\mathbb{Z}L_X\oplus\mathbb{Z}R,\quad K_X=-(M+2)L_X+(a_X-2)R,$$

where L_X is the class of the tautological sheaf, R is the class of a fiber of the morphism $\pi_X\colon X\to\mathbb{P}^1$, $a_X=a_1+\ldots+a_{M+1}$, $L_X^{M+2}=a_X$. For some $a_Q,a_W\in\mathbb{Z}_+$ let

$$Q\sim mL_X+a_QR\quad\text{and}\quad W_X\sim K(lL_X+a_WR)$$

be divisors on X, where $Q\subset X$ is a smooth subvariety, $W=W_X\cap Q$ a smooth divisor on Q. Let

$$\sigma\colon V\to Q$$

be the K-sheeted cyclic cover of the variety Q, branched over the divisor W. The projection $\pi_X\,|\,_Q$ will be denoted by the symbol π_Q, the projection $\pi_Q\circ\sigma\colon V\to\mathbb{P}^1$ by the symbol π. The fiber $\pi_Q^{-1}(t)$, $t\in\mathbb{P}^1$, will be denoted by the symbol G_t (or simply G, when it is clear, which point $t\in\mathbb{P}^1$ is meant), the fiber $\pi^{-1}(t)\subset V$ by the symbol F_t or F. Set $L_Q=L_X\,|\,_Q$ and $L=\sigma^*L_Q$, respectively. Obviously,

$$\operatorname{Pic}V=\mathbb{Z}L\oplus\mathbb{Z}F,\quad K_V=-L+(a_X+a_Q+(K-1)a_W-2)F.$$

It is easy to check the formulas $(L^M\cdot F)=mK$, $L^{M+1}=K(ma_X+a_Q)$. From here we obtain $(-K_V\cdot L^M)=K((1-m)a_Q-m(K-1)a_W+2m)$ and

$$(K_V^2\cdot L^{M-1})=K(-ma_X+(1-2m)a_Q-2m(K-1)a_W+4m).$$

For convenience we write the parameters of the cover V in the form

$$((a_1,\ldots,a_{M+1}),\ (a_Q,a_W)),$$

and moreover, among the numbers $a_1,\ldots,a_{M+1}$ we specify only non-zero values, if there are any, otherwise we write simply (0). These notations are convenient because only those covers V require an individual study which have almost all parameters equal to zero. Indeed, the explicit formulas above immediately imply the following

Proposition 4.1. (i) *The variety* V *satisfies the strong* K^2*-condition, that is, the* K^2*-condition of depth* 0, *if one of the following cases takes place*

- $a_W\ge1$,
- $a_W=0$, $a_Q\ge3$,
- $a_W=0$, $a_Q=2$, $a_X\ge1$,

- $a_W = 0$, $a_Q = 1$, $a_X \geq 3$,

- $a_W = a_Q = 0$, $a_X \geq 4$.

(ii) *If* $a_W = 0$, $a_Q = 2$, $a_X = 0$, *then the variety* V *satisfies the* K^2*-condition of depth* $\frac{2}{m}$.

(iii) *If* $a_W = 0$, $a_Q = 1$, *then the variety* V *satisfies the* K^2*-condition of depth* $\frac{1}{m}$ *for* $a_X = 2$ *and of depth* $(1 + \frac{1}{m})$ *for* $a_X = 1$.

(iv) *If* $a_W = a_Q = 0$, *then the variety* V *satisfies the* K^2*-condition of depth* 1 *for* $a_X = 3$ *and of depth* 2 *for* $a_X = 2$.

Proof: this follows immediately from the fact that for any irreducible subvariety Y the inequality $(Y \cdot L^{\dim Y}) \geq 0$ holds.

Now let us formulate the main result.

We assume that the cyclic cover V is sufficiently general in the family constructed above.

Theorem 1.6 [8]. (i) *The variety* V *is birationally superrigid, the projection* $\pi: V \to \mathbb{P}^1$ *is the only structure of a rationally connected fiber space on* V, *and the groups of birational and biregular automorphisms of the variety* V *coincide,* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$, *if the integral parameters of the variety* V *either satisfy any of the six conditions of Proposition 1.5, (i), or are of one of the following six types:* $((2),(0,0))$, $((2),(1,0))$, $((1,1),(1,0))$, $((3),(0,0))$, $((1,2),(0,0))$, $((1,1,1),(0,0))$.

(ii) *The variety* V *of the type* $((1,1),(0,0))$ *is birationally superrigid. However, the* K*-condition does not hold: the linear system* $|-K_V - F|$ *is movable and determines a birational map* $\varphi: V \dashrightarrow \mathbb{P}^1$, *the fibers of which are rationally connected. On the variety* V *there are precisely two structures of a rationally connected fiber space: the projection* π *and the map* φ. *There exists a unique (up to a fiber-wise isomorphism) fibration into Fano cyclic covers* $\pi^+: V^+ \to \mathbb{P}^1$ *of the same type* $((1,1),(0,0))$ *and a birational isomorphism* $\chi: V \dashrightarrow V^+$, *biregular in codimension one, such that the following diagram of maps commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \varphi & \downarrow & & \downarrow & \pi^+ \\ & \mathbb{P}^1 & = & \mathbb{P}^1. & \end{array}$$

The correspondence $V \to V^+$ *is an involution, that is,* $(V^+)^+ = V$.

(iii) *The variety* V *of the type* $((0),(2,0))$ *is birationally superrigid. However, the* K*-condition does not hold: the linear system* $|-mK_V - F|$ *is movable and determines a birational map, the fibers of which are rationally connected. The group of birational self-maps* $\mathrm{Bir}\, V$ *is stictly larger than the group of biregular automorphisms: it contains a non-trivial birational involution* $\tau \in \mathrm{Bir}\, V \setminus \mathrm{Aut}\, V$, *and moreover,* $\mathrm{Bir}\, V \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/K\mathbb{Z})$, *where* $\mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, \tau\}$. *On* V *there are precisely two structures of a rationally connected fiber space: the projection* π *and the rational map* $\pi\tau: V \dashrightarrow \mathbb{P}^1$, *and moreover,* $|-mK_V - F| = \tau_*|F|$.

Note the obvious parallelism of Theorems 1.5 and 1.6 (their proofs, however, are essentially different). Birational (super)rigidity is also proved for some other families of Fano fiber spaces over $\mathbb{P}^1$, see [3,36-38]. Theorems 1.5 and 1.6 describe the most "massive" families.

Varieties with a pencil of del Pezzo surfaces. That class of three-dimensional rationally connected varieties for a long time was out of reach. Birational rigidity of the overwhelming majority of these varieties was proved in [3]. That paper was followed by [26,27], where birational geometry of almost all remaining types of varieties with a pencil of del Pezzo surfaces of degree 1 and 2 was completely described. That work was summarized in the survey [4].

Let $V \xrightarrow{\pi} \mathbb{P}^1$ be a fibration into del Pezzo surfaces of degree $d \in \{1,2,3\}$, $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where the variety V is smooth and for $d = 3$ sufficiently general.

Theorem 1.7 [3]. *Assume that the fiber space $V/\mathbb{P}^1$ satisfies the K^2-condition: $K_V^2 \notin \operatorname{Int} A_+^2 V$. Then the variety V is birationally rigid (superrigid for $d = 1$), the projection $\pi\colon V \to \mathbb{P}^1$ is the only structure of a rationally connected fiber space on the variety V, and the quotient group of the group of birational self-maps by the normal subgroup of birational self-maps of the generic fiber* $\operatorname{Bir} F_\eta$ *is finite, generically trivial.*

For $d = 1$ the group $\operatorname{Bir} F_\eta = \operatorname{Aut} F_\eta$ is finite, for $d = 2$ it is generated by the subgroup $\operatorname{Aut} F_\eta$ and the involutions, associated with sections of the fiber space π, for $d = 3$ it is generated by the subgroup $\operatorname{Aut} F_\eta$ and the involutions, associated with sections and bi-sections of the projection π, described in Example 1.4 (for the original description of these groups, see [5,39]).

The techniques of the proof of Theorems 1.5, 1.6 and 1.7 is explained below in Chapter 2.

3.3. The linear method: main results. The linear method is based on the theorem on "inversion of adjunction", proved by Shokurov in dimension three in [40] and by Kollár in arbitrary dimension [41]. Let us formulate a particular case of that result, which is used in the theory of birational rigidity. A discussion and the proof of inversion of adjunction are given in §2 of Chapter 3.

Theorem 1.8 (inversion of adjunction). *Let $x \in X$ be a germ of a $\mathbb{Q}$-factorial terminal variety, D an effective $\mathbb{Q}$-divisor, the support of which contains the point x. Let $R \subset X$ be an irreducible subvariety of codimension one, $R \not\subset \operatorname{Supp} D$, and, moreover, R is a Cartier divisor. Assume that the pair (X, D) is not canonical at the point x, but canonical outside that point, that is, the point x is an isolated centre of non-canonical singularities of that pair. Then the pair $(R, D_R = D|_R)$ is not log canonical at the point x.*

The inversion of adjunction (formulated above not in the most general way, but in the form in which it will be really needed) is used for excluding maximal singularities of movale linear systems Σ on a rationally connected variety V under consideration in the following way. Let $R \subset V$ be an irreducible reduced Cartier divisor (as a rule, the variety V is smooth or has elementary singularities, so that the assumptions of Theorem 1.8 are satisfied automatically), $D \in \Sigma$ a general divisor. Sine the linear system Σ is movable, we get $R \not\subset \operatorname{Supp} D$, so that the restriction D_R is well defined. Now, if the pair $(R, \frac{1}{n} D_R)$ is log canonical at the point $x \in R$, then the pair $(V, \frac{1}{n}\Sigma)$ is canonical at the point x, that is, there are no maximal singularities of the system

Σ, the centre of which is the point x. The procedure just described reduces studying birational geometry of the variety V to studying geometry of the divisor R, which sometimes essentially simplifies the work. Note that this procedure can be repeated (provided that the assumptions of Theorem 1.8 hold), reducing investigation of the singularities of the pair $(V, \frac{1}{n}\Sigma)$ to studying singularities of the pair $(R, \frac{1}{n}D_R)$, where $R \subset V$ is an irreducible subvariety (not necessarily a divisor). The most important case is restricting onto a fiber $R = \pi^{-1}(s)$ of a fibration $\pi: V \to S$.

There are three groups of results that make an essential use of the linear method.

Fano direct products. Theorems on birational geometry of Fano direct products form the largest (up to now) group of results [33,34,32,42,43].

Let F be a Fano variety of dimension ≥ 3 with $\mathbb{Q}$-factorial terminal singularities and the Picard number $\rho(F) = 1$.

Definition 1.7. We say that the variety F satisfies the condition of *divisorial canonicity*, or the condition (C) (respectively, the condition of *divisorial log canonicity*, or the condition (L)), if for any effective divisor $D \in |-nK_F|$, $n \geq 1$, the pair

$$(F, \frac{1}{n}D) \tag{8}$$

has canonical (respectively, log canonical) singularities. If the pair (8) has canonical singularities for a general divisor $D \in \Sigma \subset |-nK_F|$ of any *movable* linear system, then we say that F satisfies the condition of *movable canonicity*, or the condition (M).

Explicitly, the condition (C) is formulated in the following way: for any birational morphism $\varphi: \widetilde{F} \to F$ and any exceptional divisor $E \subset \widetilde{F}$ the following inequality holds:

$$\nu_E(D) \leq na(E). \tag{9}$$

The inequality (9) is opposite to the Noether-Fano inequality. The condition (L) is weaker: it is required that the inequality

$$\nu_E(D) \leq n(a(E) + 1) \tag{10}$$

holds. Recall that in (9) and (10) the number $a(E)$, that is, the discrepancy of an exceptional divisor $E \subset \widetilde{F}$ with respect to the model F. The inequality (10) is opposite to the *log Noether-Fano inequality*. The condition (M) means that (9) takes place for a general divisor D of any movable system $\Sigma \subset |-nK_F|$ and any discrete valuation ν_E.

In another terminology, the condition (L) maens that the *the global log canonical threshold* of the variety F is equal to one: $\mathrm{lct}(F) = 1$. The condition (C) means that the *global canonical threshold* $\mathrm{ct}(F) = 1$. The importance of these conditions is shown by the following fact [33].

Theorem 1.9. *Assume that the primitive Fano varieties $F_1, \ldots, F_K$, $K \geq 2$, satisfy the conditions (L) and (M). Then their direct product*

$$V = F_1 \times \ldots \times F_K$$

is a birationally superrigid variety, that is, for any movable linear system Σ its virtual and actual thresholds of canonical adjunction coincide:

$$c(\Sigma) = c_{\text{virt}}(\Sigma).$$

In particular:

(i) *All structures of a rationally connected fiber space on the variety V are projections onto direct factors. More precisely, let $\beta\colon V^\sharp \to S^\sharp$ be a rationally connected fiber space and $\chi\colon V - - \to V^\sharp$ a birational map. Then there exists a set of indices $I = \{i_1, \dots, i_k\} \subset \{1, \dots, K\}$ and a birational map*

$$\alpha\colon F_I = \prod_{i \in I} F_i - - \to S^\sharp$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & V^\sharp & \\ \pi_I & \downarrow & & \downarrow & \beta \\ & F_I & \stackrel{\alpha}{- - \to} & S^\sharp, & \end{array}$$

that is, $\beta \circ \chi = \alpha \circ \pi_I$, where $\pi_I\colon \prod\limits_{i=1}^{K} F_i \to \prod\limits_{i \in I} F_i$ is the natural projection onto a direct factor. In particular, on the variety V there are no structures of a fibration into rationally connected varieties of dimension strictly smaller than $\min\{\dim F_i\}$*. In particular, V has no structures of a conic bundle or a fibration into rational surfaces.*

(ii) *The groups of birational and biregular automorphisms of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

(iii) *The variety V is non-rational.*

Theorem 1.9 is proven in [33] for smooth primitive Fano varieties, however, the proof is valid word for word in a more general case, described above. Obviously, the condition (C) is stronger than (L) and (M). To apply Theorem 1.9, it is sufficient to check that a Fano variety from a given family satisfies the condition (C) (or the both conditions (L) and (M)).

For generic Fano hypersurfaces $F_d \subset \mathbb{P}^d$, $d \geq 6$, and generic Fano double space of index one $F_2 \to \mathbb{P}^d$, $d \geq 3$, the condition (C) is shown in [33]. For generic Fano double hypersurfaces $F \to Q_m \subset \mathbb{P}^d$, $d \geq 7$, branched over $W = W^*_{2l} \cap Q_m$, where Q_m and W^*_{2l} are generic hypersurfaces of degrees m and $2l$, respectively, $m + l = d$, the condition (C) is shown in [34]. For generic weighted Fano hypersurfaces of dimension three the condition (L) is checked in [42].

The proof of Theorem 1.9 and of the divisorial canonicity of Fano hypersurfaces $F_d \subset \mathbb{P}^d$ is given in §1 of Chapter 3.

Fano fibrations over $\mathbb{P}^1$. Application of the linear method simplifies the proof of birational rigidity of fiber spaces $V/\mathbb{P}^1$, see [34,43]. Assume that $\Sigma \subset |-nK_V + lF|$

is a movable linear system, where $l \in \mathbb{Z}_+$, and a fiber $F^* = \pi^{-1}(t_*)$ of the projection $\pi: V \to \mathbb{P}^1$ satisfies the condition (L). Then by the inversion of adjunction, the centre of any maximal singularity of the system Σ can not be contained in the fiber F^*: it either covers the base $\mathbb{P}^1$, or is contained in another fiber. In particular, if all fibers satisfy the conditions (L) and (M), then the system Σ has no maximal singularities at all, which implies that

$$c_{\text{virt}}(\Sigma) = c(\Sigma, V) = n.$$

Arguments of that type can be used for exclusion of certain particular types of maximal singularities as well [7].

Fano varieties of index two. This result comes out of the framework of the present survey as it deals with *non-rigid* varieties (in fact, it is the first full fledged example of a complete description of birational geometry of non-rigid varieties of arbitrary dimension). However, it stands next to the previous results and makes an essential use of the linear method. For that reason, we formulate it below and briefly discuss the scheme of its proof in §3 of Chapter 3.

Let $M \geq 5$ and $W = W_{2(M-1)} \subset \mathbb{P}^M$ be a smooth hypersurface of degree $2(M-1)$. Consider the double cover

$$\sigma: V \to \mathbb{P}^M,$$

branched over W. The variety V is a Fano variety of index two: $\operatorname{Pic} V = \mathbb{Z}H$, where H is the ample generator, $K_V = -2H$, the class H is the σ-pull back of the hyperplane in $\mathbb{P}$. On the variety V there are the following natural structures of a rationally connected fiber space: let $\alpha_P: \mathbb{P} \dashrightarrow \mathbb{P}^1$ be the linear projection from an arbitrary linear subspace P of codimension two, then the map

$$\pi_P = \alpha_P \circ \sigma: V \dashrightarrow \mathbb{P}^1$$

fibers V into $(M-1)$-dimensional Fano varieties of index 1. Assume that the variety V is sufficiently general.

Theorem 1.10 [35]. *Let $M \geq 5$ and $\chi: V \dashrightarrow Y$ be a birational map onto the total space of a rationally connected fiber space $\lambda: Y \to S$. Then $S = \mathbb{P}^1$ and for some isomorphism $\beta: \mathbb{P}^1 \to S$ and some subspace $P \subset \mathbb{P}$ of codimension two we have $\lambda \circ \chi = \beta \circ \pi_P$, that is, the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & Y & \\ \pi_P & \downarrow & & \downarrow & \lambda \\ & \mathbb{P}^1 & \stackrel{\beta}{\to} & S. & \end{array}$$

Corollary 1.4. (i) *On the variety V there are no structures of a rationally connected fiber space with the base of dimension ≥ 2. In particular, on V there are no structures of a conic bundle and del Pezzo fibration, and the variety V itself is non-rational.*

(ii) *Assume that there is a birational map* $\chi\colon V \dashrightarrow Y$, *where* Y *is a Fano variety of index* $r \geq 2$ *with factorial terminal singularities, such that* $\operatorname{Pic} Y = \mathbb{Z}H_Y$, *where* $K_Y = -rH_Y$, *where the linear system* $|H_Y|$ *is non-empty and free. Then* $r = 2$ *and the map* χ *is a biregular isomorphism.*

(iii) *The groups of birational and biregular automorphisms of the variety* V *coincide:* $\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}$.

Proof of the corollary. The claim (i) and the equality $r = 2$ in (ii) are obvious (any linear subsystem of the projective dimension $\leq r-1$ in the complete linear system $|H_Y|$ defines a structure of a rationally connected fiber space on Y). Furthermore, the χ-pull back of a general divisor in the system $|H_Y|$ is a divisor in the linear system $|H|$ by Theorem 1.10, which completes the proof of the claim (ii). The part (iii) obviously follows from (ii). The proof is complete.

Chapter 2. Fano fiber spaces over the projective line

§1. Sufficient conditions of birational rigidity

1.1. Formulation of the sufficient conditions. Start of the proof. Let $\pi\colon V \to \mathbb{P}^1$ be a standard Fano fiber space, that is, V is a smooth variety, $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where F is the class of a fiber of the projection π. Assume in addition that the condition

$$A^2 V = \mathbb{Z}K_V^2 \oplus \mathbb{Z}H_F, \tag{11}$$

holds, where $H_F = (-K_V \cdot F)$ is the ample antocanonical section of the fiber, and that every fiber $F = F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$, is a Fano variety with at most non-degenerate quadratic singularities, and moreover,

$$A^1 F = \operatorname{Pic} F = \mathbb{Z}K_F \quad \text{and} \quad A^2 F = \mathbb{Z}H_F^2,$$

where $K_F = -H_F$ and H_F is considered as an element of the group $A^1 F$. These conditions are satisfied for almost all families of standard Fano fiber spaces that are by now successfully studied. The exception is made by fiber spaces with fibers of dimension 2,3 and 4, where the low dimension makes it possible to employ additional arguments. The extremal case, fibrations into cubic surfaces, is considered below in §3.

Now let us formulate additional conditions, from which birational (super)rigidity of Fano fiber spaces is derived. These conditions should be understood as some conditions of general position for the fiber space $V/\mathbb{P}^1$; more precisely, generic (in their family) fiber spaces $V/\mathbb{P}^1$ satisfy these conditions. By the *degree* of an irreducible subvariety $Y \subset V$, covering the base $\mathbb{P}^1$, $\pi(Y) = \mathbb{P}^1$ (we refer to such varieties as *horizontal*), we mean the number

$$\deg Y = (Y \cdot F \cdot (-K_V)^{\dim Y - 1}).$$

Definition 2.1. The fiber space $V/\mathbb{P}^1$ satisfies

- the condition (v), if for every irreducible vertical subvariety Y of codimension two (that is, $Y \subset F_t$ is a prime divisor, $t = \pi(Y)$) and every smooth point $o \in F_t$ the inequality
$$\frac{\operatorname{mult}_o Y}{\deg Y} \le \frac{2}{\deg V} \tag{12}$$
holds;

- the condition (f), if for every irreducible vertical subvariety Y of codimension three (that is, $\operatorname{codim}_F Y = 2$, $F = F_t \supset Y$) and every smooth point of the fiber $o \in F$, the inequality
$$\frac{\operatorname{mult}_o Y}{\deg Y} \le \frac{4}{\deg V} \tag{13}$$
holds.

- the condition (vs), if for any vertical subvariety $Y\subset F_t$ of codimension 2 (with respect to V, that is, for a prime divisor on F_t), a singular point $o\in F_t$ and an infinitely near point $x\in\widetilde{F}_t$, where $\varphi\colon\widetilde{F}_t\to F_t$ is a blow up of the point o, $\varphi(x)=o$, $\widetilde{Y}\subset\widetilde{F}_t$ the strict transform of the subvariety Y on $\widetilde{F}_t$, the following estimates hold:

$$\frac{\mathrm{mult}_o Y}{\deg Y}\leq\frac{4}{\deg V},\quad \frac{\mathrm{mult}_x \widetilde{Y}}{\deg Y}\leq\frac{2}{\deg V};$$

- the condition (h), if for any horizontal subvariety Y of codimension 2 and a point $o\in Y$ the estimate

$$\frac{\mathrm{mult}_o Y}{\deg Y}\leq\frac{4}{\deg V}.$$

  holds.

As in the previous survey [1], for the convenience of notations we write down in the sequel the ratio of the multiplicity to the degree by the single symbol

$$\frac{\mathrm{mult}_o}{\deg}Y=\frac{\mathrm{mult}_o Y}{\deg Y}.$$

The sufficient conditions of birational (super)rigidity, which can be applied to the majority of standard Fano fiber spaces $V/\mathbb{P}^1$, are collected in the following claim.

Theorem 2.1. (i) *Assume that the standard Fano fiber space $V/\mathbb{P}^1$ satisfies the (strong) K^2-condition and the conditions (v), (vs) and (h). Then $V/\mathbb{P}^1$* is a birationally superrigid Fano fiber space.

(ii) *Assume that the standard Fano fiber space $V/\mathbb{P}^1$ satisfies the K^2-condition of depth 2, the conditions (v), (vs) and at least one of the conditions (f) or (fs) at any point $o\in V$. Then for any movable linear system $\Sigma\subset|-nK_V+lF|$ with $l\in\mathbb{Z}_+$ its virtual and actual thresholds of canonical adjunction coincide:*

$$c_{\mathrm{virt}}(\Sigma)=c(\Sigma)=n.$$

In particular, if $V/\mathbb{P}^1$ satisfies the K-condition, then this fiber space is birationally superrigid.

(iii) *Assume that the standard Fano fiber space $V/\mathbb{P}^1$ satisfies the K^2-condition of depth 2 and conditions (v) and (f). If the centre of any maximal singularity of the movable linear system $\Sigma\subset|-nK_V+lF|$ with $l\in\mathbb{Z}_+$ is not a singular point of a fiber, then the virtual and actual thresholds of canonical adjunction coincide:*

$$c_{\mathrm{virt}}(\Sigma)=c(\Sigma)=n.$$

In particular, if the latter assumption holds for any movable linear system on V and the fiber space $V/\mathbb{P}^1$ satisfies the K-condition, then this fiber space is birationally superrigid.

The claim (i) is proven in [36] (see also [31]). The claim (ii) is essentially stronger (the K^2-condition of depth 2 is assumed, which makes it possible to study many fiber spaces that do not satisfy the strong K^2-condition) and requires more refined arguments [31]. The claim (iii) is a simplified version of (ii): the maximal singularities lying over the singular points of the fibers are not considered (it is assumed that they are excluded by some other method).

In the remaining part of this section we give

Proof of the claim (iii). We follow [8,32]. Proof of the claim (i) is simpler, whereas dealing with the maximal singularities lying over quadratic points of the fibers (the only step that needs to be added to the proof of part (iii), in order to get the claim (ii)) is completely similar to the arguments given below.

So let us fix a movable linear system $\Sigma \subset |-nK_V + lF|$ with $l \in \mathbb{Z}_+$. Obviously, $c(\Sigma) = n$, where $n = 0$ if and only if Σ is composed from the pencil of fibers $|F|$. Assume that the inequality $c_{\rm virt}(\Sigma) < n$ holds, in particular, $n \geq 1$. This means that there exists a birational morphism $\varphi\colon \widetilde{V} \to V$ of smooth varieties such that $c(\widetilde{\Sigma}, \widetilde{V}) < n$, where $\widetilde{\Sigma}$ is the strict transform of Σ on $\widetilde{V}$. This implies immediately that the pair $(V, \frac{1}{n}\Sigma)$ is not canonical, that is, the system Σ has a maximal singularity $E \subset \widetilde{V}$. Consider its centre $B = \varphi(E) \subset V$, an irreducible subvariety of codimension at least two.

Let (see [1, Chapter 2])

$$\begin{array}{rccc} \varphi_{i,i-1}\colon & V_i & \to & V_{i-1} \\ & \bigcup & & \bigcup \\ & E_i & \to & B_{i-1} \end{array} \tag{14}$$

be a sequence of blow ups with irreducible centres $B_{i-1} \subset V_{i-1}$, which is uniquely determined by the following conditions:

1) $V_0 = V$, $B_0 = B$, $i = 1, \dots, N$;

2) $B_j = \text{centre}(E, V_j) \subset V_j$, $E_{j+1} = \varphi^{-1}_{j+1,j}(B_j)$;

3) the valuation ν_{E_N} coincides with ν_E.

In other words, the birational map $V_N \dashrightarrow \widetilde{V}$ is biregular at the generic point of the divisor E_N and transforms E_N into E. Recall that the varieties $V_1, \dots, V_N$, generally speaking, can be singular, since the centres B_j of blow ups are not necessarily smooth. However, V_j is smooth at the generic point of B_j. By the symbol Σ^j we denote the strict transform of the linear system Σ on V_j. Set

$$\nu_j = \text{mult}_{B_{j-1}} \Sigma^{j-1}, \quad \delta_j = \text{codim}\, B_{j-1} - 1.$$

On the set of exceptional divisors $\{E_1, \dots, E_N\}$ we define in the usual way the structure of an oriented graph: E_i and E_j are joined by an oriented edge (an arrow), if and only if $i > j$ and

$$B_{i-1} \subset E_j^{i-1},$$

notation: $i \to j$. As usual, set for $i > j$

$$p_{ij} = \sharp\{\text{the paths from } E_i \text{ to } E_j\} \geq 1,$$

$p_{ii} = 1$ by definition. Set $p_i = p_{Ni}$. The Noether-Fano inequality takes the traditional form:

$$\sum_{i=1}^{N} p_i \nu_i > n \sum_{i=1}^{N} p_i \delta_i.$$

Proposition 2.1. *The centre B of the maximal singularity E on V is contained in some fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$.*

Proof. Assume the converse: $\pi(B) = \mathbb{P}^1$. Let $F \subset V$ be a fiber of general position. It is easy to see that the restriction $\Sigma_F = \Sigma\,|\,_F$ of the linear system Σ onto F is a movable linear system $\Sigma_F \subset |nH_F|$, and moreover the pair $(F, \frac{1}{n}\Sigma_F)$ is not canonical, that is, Σ_F has a maximal singularity. The easiest way to define this maximal singularity is as follows: let $\widetilde{F} \subset \widetilde{V}$ be the strict transform, E_F an irreducible component of the closed set $\widetilde{F} \cap E$. Otherwise, one can restrict the sequence of blow ups (14) onto the fiber F. The discrepancy remains the same: $a(E_F, F) = a(E, V)$. The centre of the singularity E_F is $\varphi(E_F)$, an irreducible component of the closed set $B \cap F$.

By assumption, $\operatorname{Pic} F = \mathbb{Z}K_F$, $A^2F = \mathbb{Z}K_F^2$ and for every irreducible subvariety $Y \subset F$ of codimension two at every point the inequality (13) holds. These two assumptions imply the every pair $(F, \frac{1}{n}\Sigma_F)$ is canonical, where $\Sigma_F \subset |nH_F|$ is a movable linear system (see [1, Chapter 2]). This contradiction completes the proof of the proposition.

Thus $B \subset F = F_t$ is contained in a fiber.

Lemma 2.1. *The following inequality holds:* $\operatorname{codim}_F B \geq 2$.

Proof. Assume the converse: $B \subset F$ is a prime divisor. Let $D \in \Sigma$ be a general divisor, D_F its restriction onto F. By the Noether-Fano inequality $\operatorname{mult}_B D > n$, so that $D_F = \alpha B + D^\sharp$, where $\alpha > n$ and $D^\sharp$ is an effective divisor on F. However, $D_F \sim nH_F$, which immediately gives a contradiction. Q.E.D. for the lemma.

Let $\mathcal{M} = \{T_1, \ldots, T_k\}$ be the set of all maximal singularities of the linear system Σ on $\widetilde{V}$. As we have proved above, the centre $B_E = \operatorname{centre}(\nu_E)$ of any maximal singularity $E \in \mathcal{M}$ is contained in some fiber F_t. The set $\mathcal{M}$ is finite, so that there are at most finitely many points $t \in \mathbb{P}^1$, the fibers F_t over which contain a centre of a maximal singularity. Set $\mathcal{M}_t = \{E \in \mathcal{M} | B_E \subset F_t\}$,

$$e(E) = \nu_E(\Sigma) - na(E, V) > 0$$

for $E \in \mathcal{M}$. Recall that $\Sigma \subset |-nK_V + lF|$, $l \in \mathbb{Z}_+$.

Proposition 2.2. *The following inequality holds:*

$$\sum_{t \in \mathbb{P}^1} \max_{\{E \in \mathcal{M}_t\}} \frac{e(E)}{\nu_E(F_t)} > l \tag{15}$$

Proof. Let $\widetilde{D} \in \widetilde{\Sigma}$ be a general divisor, that is, the strict transform of a divisor $D \in \Sigma$ of general position on $\widetilde{V}$. By assumption, the linear system $|\widetilde{D} + n\widetilde{K}|$ is empty (where $\widetilde{K}$ is the canonical class of the variety $\widetilde{V}$). Therefore the linear system

$$|lF - \sum_{E\in\mathcal{M}} e(E)E|$$

is empty, either. On the other hand, for $E \in \mathcal{M}_t$ by construction the divisor $F_t - \nu_E(F_t)E$ is effective, so that the divisor

$$\sum_{t\in\mathbb{P}_1}[\left(\max_{\{E\in\mathcal{M}_t\}} \frac{e(E)}{\nu_E(F_T)}\right) F_t - \sum_{E\in\mathcal{M}_t} e(E)E]$$

is also effective. From this, we immediately obtain the inequality (15).

Q.E.D. for Proposition 2.2.

1.2. The structure of the sequence of blow ups. Consider the self-intersection of the linear system Σ, that is, the effective algebraic cycle $Z = (D_1 \circ D_2)$, where $D_1, D_2 \in \Sigma$ are general divisors. Let $Z = Z^v + Z^h$ be the decomposition of the cycle Z into the vertical (Z^v) and horizontal (Z^h) parts. The cycle Z^v can be further decomposed as

$$Z^v = \sum_{t\in\mathbb{P}^1} Z^v_t, \quad \mathrm{Supp}\, Z^v_t \subset F_t.$$

Let $E \in \mathcal{M}_t$ be a maximal singularity over a point $t \in \mathbb{P}^1$. To prove Theorem 2.1 (iii), we apply to the effective cycle $Z^v_t + Z^h$ *the technique of counting multiplicities* [1, §2.2]. First of all, let us study the structure of the sequence of blow ups, resolving the singularity E. This means, to break the sequence of blow ups into segments that determine multiplicities of the cycles Z^v_t, Z^h and $(Z^h \circ F_t)$ (the intersection of a horizontal cycle with a fiber is always well defined).

So long as we discuss a fixed singularity E, we omit the symbols t and E for simplicity of notations, for instance, we write F instead of F_t, Z^v instead of Z^v_t, e instead of $e(E)$ and so on.

So let us consider the sequence of blow ups (14), associated with E. As usual, the upper index j means the operation of taking the strict transform on V_j, for instance, $F^j \subset V_j$ is the strict transform of the fiber F and so on. Set

$$N_f = \max\{i \,|\, B_{i-1} \subset F^{i-1}\}.$$

Since $\varphi_{i,i-1}(B_i) = B_{i-1}$ for any $i = 1, \dots N-1$, the codimensions $\mathrm{codim}\, B_i$ do not increase. Set

$$L = \max\{i \,|\, \mathrm{codim}\, B_{i-1} \geq 3\} \leq N.$$

We introduce the following notations: for $i \in \{1, \dots, L\}$

$$m^h_i = \mathrm{mult}_{B_{i-1}} (Z^h)^{i-1}, \quad m^v_i = \mathrm{mult}_{B_{i-1}} (Z^v)^{i-1},$$

$m^{h(v)}_i \leq m^{h(v)}_{i-1}$ for $i = 2, \dots, L$. Note that by the assumption the fiber F can be assumed to be smooth at the generic point of B and therefore the strict transform F^i

is smooth at the generic point of each variety B_i, if $B_i \subset F^i$. Thus for $i \in \{1, \ldots, N_f\}$ we get $\mathrm{mult}_{B_{i-1}} F^{i-1} = 1$. For $i > N_f$, obviously, $\mathrm{mult}_{B_{i-1}} F^{i-1} = 0$. The more so, $m_i^v = 0$ for $i > N_f$ (if $N_f < L$). Now the technique of counting multiplicities [1, §2.2] combined with the relation

$$\sum_{i=1}^{K} p_i \nu_i = n \sum_{i=1}^{K} p_i \delta_i + e,$$

$e > 0$, gives the inequality

$$\sum_{i=1}^{L} p_i m_i^h + \sum_{i=1}^{\min\{N_f, L\}} p_i m_i^v \geq \sum_{i=1}^{N} p_i \nu_i^2 \geq \frac{(n \sum_{i=1}^{N} p_i \delta_i + e)^2}{\sum_{i=1}^{N} p_i} \tag{16}$$

where p_i is the number of paths in the graph Γ of the resolution of the maximal singularity E going from the vertex E_N to E_i.

Unfortunately, the estimate (16) is not strong enough for our purposes (it would have been sufficient under the assumption that the standard K^2-condition holds, $K_V^2 \notin \mathrm{Int}\, A_+^2 V$, but we assume the weaker K^2-condition of depth 2). A more refined study of the resolution of the singularity E is needed. Set

$$\Sigma_l = \sum_{i=1}^{L} p_i, \quad \Sigma_u = \sum_{i=L+1}^{N} p_i, \quad \Sigma_f = \sum_{i=1}^{\min\{N_f, L\}} p_i.$$

Note that $\nu_E(F) = \sum_{i=1}^{N_f} p_i \geq \Sigma_f$. Obviously, $m_i^h \leq m_h = m_1^h = \mathrm{mult}_B Z^h$. Set also $d_h = \deg Z^h, \quad d_v = \deg Z_t^v$.

Now let us break the set of blow ups into a few subsets. First of all, let us separate the blow ups of subvarieties B_{i-1} of codimension three. Set

$$J_s = \{i \,|\, 1 \leq i \leq K, \quad \mathrm{codim}\, B_{i-1} \geq 4\},$$

$$J_m = \{i \,|\, 1 \leq i \leq K, \quad \mathrm{codim}\, B_{i-1} = 3\},$$

$$J_u = \{i \,|\, L+1 \leq i \leq K\}, \quad J_l = J_s \cup J_m.$$

In its turn, we break the set J_m into two disjoint subsets, $J_m = J_m^+ \coprod J_m^-$, where

$$J_m^+ = \{i \in J_m \,|\, B_{i-1} \subset F^{i-1}\},$$

$J_m^- = J_m \setminus J_m^+ = \{i \in J_m \,|\, B_{i-1} \not\subset F^{i-1}\}$. It can well turn out that J_m^+ or J_m^- (or the whole set J_m) is empty. Set, furthermore,

$$\Sigma_s = \sum_{i \in J_s} p_i, \quad \Sigma_m^{\pm} = \sum_{i \in J_m^{\pm}} p_i, \quad \Sigma_m = \Sigma_m^+ + \Sigma_m^-,$$

the symbol Σ_u retains its meaning. Obviously, $\Sigma_l = \Sigma_s + \Sigma_m$.

Now the inequality (16) can be rewritten as

$$\sum_{i\in J_l} p_i m_i^h + \sum_{i\in J_s\cup J_m^+} p_i m_i^v \geq \frac{((3\Sigma_s + 2\Sigma_m + \Sigma_u)n + e)^2}{\Sigma_s + \Sigma_m + \Sigma_u}. \tag{17}$$

The next step is estimating the horizontal multiplicities m_i^h.

Proposition 2.3. *The following inequality holds:*

$$\sum_{i\in J_s\cup J_m^+} p_i m_i^h \leq \Sigma_s \operatorname{mult}_B(Z^h \circ F). \tag{18}$$

1.3. Multiplicities of the horizontal cycles. Let us prove Proposition 2.3. The arguments below hold with obvious simplifications in the case when $J_m^+ = \emptyset$. So we assume that $J_m^+ \neq \emptyset$, so that, in particular, $J_s \subset \{1,\dots,N_f\}$.

First let us consider the following general situation. Let $Y \subset V$ be an irreducible horizontal subvariety of codimension two, $Y^i \subset V_i$ its strict transform,

$$m_Y(i) = \operatorname{mult}_{B_{i-1}} Y^{i-1} \tag{19}$$

the corresponding multiplicity. Set $Y_F = (Y \circ F)$. It is an effective cycle of codimension two in the fiber F. Let $Y_F^i \subset V_i$ be its strict transform,

$$m_{Y,F}(i) = \operatorname{mult}_{B_{i-1}} Y_F^{i-1}. \tag{20}$$

Since the support of the cycle Y_F is contained in the fiber F, the numbers $m_{Y,F}(i)$ vanish for $i \in J_m^-$.

Lemma 2.2. *The following estimate holds:*

$$\sum_{i\in J_s\cup J_m^+} p_i m_Y(i) \leq \sum_{i\in J_s} p_i m_{Y,F}(i). \tag{21}$$

Before starting the proof, recall some facts which follow immediately from the elementary intersection theory [44]. Note that here we intersect a divisor and a subvariety of arbitrary codimension, unlike [1, Chapter 2], where the case of two divisors was considered. Let X be an arbitrary smooth variety, $B \subset X$, $B \not\subset \operatorname{Sing} X$ an irreducible subvariety of codimension ≥ 2, $\sigma_B\colon X(B) \to X$ its blow up, $E(B) = \sigma_B^{-1}(B)$ the exceptional divisor. Let

$$Z = \sum m_i Z_i, \quad Z_i \subset E(B),$$

be a cycle of dimension k, $k \geq \dim B$. Define the *degree* of the cycle Z, setting

$$\deg Z = \sum_i m_i \deg\left(Z_i \bigcap \sigma_B^{-1}(b)\right),$$

where $b \in B$ is a point of general position, $\sigma_B^{-1}(b) \cong \mathbb{P}^{\mathrm{codim}\, B-1}$ and the degree in the right-hand side is the usual degree in the projective space.

Note that $\deg Z_i = 0$ if and only if $\sigma_B(Z_i)$ is a proper closed subset of the subvariety B.

Now let D be a prime Weil divisor on X, $Y \subset X$ an irreducible subvariety of dimension $l \leq \dim X - 1$. Assume that $Y \not\subset D$ and that $\dim B \leq l-1$. The strict transforms of the divisor D and the subvariety Y on $X(B)$ denote by the symbols D^B and Y^B, respectively.

Lemma 2.3. (i) *Assume that* $\dim B \leq l-2$. *Then*

$$D^B \circ Y^B = (D \circ Y)^B + Z,$$

where $\circ$ *means the operation of taking the algebraic cycle of the scheme-theoretic intersection,* $\mathrm{Supp}\, Z \subset E(B)$ *and*

$$\mathrm{mult}_B(D \circ Y) = \mathrm{mult}_B D \cdot \mathrm{mult}_B Y + \deg Z.$$

(ii) *Assume that* $\dim B = l-1$. *Then*

$$D^B \circ Y^B = Z + Z_1,$$

where $\mathrm{Supp}\, Z \subset E(B)$, $\mathrm{Supp}\, \sigma_B(Z_1)$ *does not contain* B *and*

$$D \circ Y = [(\mathrm{mult}_B D)(\mathrm{mult}_B Y) + \deg Z]\, B + (\sigma_B)_* Z_1.$$

Proof is easy to obtain by the standard intersection theory [44].

1.4. The technique of counting multiplicities. Let us construct a sequence of effective cycles of codimension three on the varieties V_i, setting

$$\begin{array}{rcl} Y \circ F & = & Z_0 \ (= Y_F), \\ Y^1 \circ F^1 & = & Z_0^1 + Z_1, \\ \vdots & & \\ Y^i \circ F^i & = & (Y^{i-1} \circ F^{i-1})^i + Z_i, \\ \vdots & & \end{array}$$

$i \in J_s$, where $\mathrm{Supp}\, Z_i \subset E_i$. Thus for every $i \in J_s$ we get:

$$Y^i \circ F^i = Y_F^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For any $j > i$, $j \in J_s$ set $m_{i,j} = \mathrm{mult}_{B_{j-1}}(Z_i^{j-1})$ (the multiplicity of an irreducible subvariety along a smaller subvariety is understood in the usual sense; for an arbitrary cycle we extend the multiplicity by linearity).

Now set $d_i = \deg Z_i$. We obtain the following system of equalities:

$$\begin{array}{rcl} m_Y(1) + d_1 & = & m_{Y,F}(1), \\ m_Y(2) + d_2 & = & m_{Y,F}(2) + m_{1,2}, \\ \vdots & & \\ m_Y(i) + d_i & = & m_{Y,F}(i) + m_{1,i} + \ldots + m_{i-1,i} \\ \vdots & & \end{array}$$

$i \in J_s$. Setting $S = \max\{i \in J_s\}$, consider the last equality in this sequence:

$$m_Y(S) + d_S = m_{Y,F}(S) + m_{1,S} + \ldots + m_{S-1,S}.$$

If $J_m^+ \neq \emptyset$, then by part (ii) of Lemma 2.3 we get

$$d_S \geq \sum_{i \in J_m^+} m_Y(i) \deg(\varphi_{i-1,S})_* B_{i-1} \geq \sum_{i \in J_m^+} m_Y(i).$$

Slightly modifying Definition 2.6 in [1], we say that a function $a\colon J_s \to \mathbb{R}_+$ is *compatible with the graph structure*, if

$$a(i) \geq \sum_{\substack{j \to i, \\ j \in J_s}} a(j)$$

for any $i \in J_s$. (Compared to the above-mentioned definition, only the domain of the function is changed.)

In fact, we will use only one function compatible with the graph structure, namely $a(i) = p_i$.

Proposition 2.4. *Let $a(\cdot)$ be a function, compatible with the graph structure. Then the following inequality holds:*

$$\sum_{i \in J_s} a(i) m_{Y,F}(i) \geq \sum_{i \in J_s} a(i) m_Y(i) + a(S) \sum_{i \in J_m^+} m_Y(i). \qquad (22)$$

Proof is given word for word in the same way as for the case of two divisors ([1, §2.2]): multiply the i-th equality by $a(i)$ and put them all together. In the right-hand side for any $i \geq 1$ we get the expression

$$\sum_{j \geq i+1} a(j) m_{i,j}.$$

In the left-hand side for any $i \geq 1$ we get the summand $a(i)d_i$.

Furthermore, by Lemma 2.3 in [1], if $m_{i,j} > 0$, then $j \to i$.

The next standard step is to compare the multiplicities $m_{i,j}$ with the degrees.

Lemma 2.4. *For any $i < j \in J_s$ we get $m_{i,j} \leq d_i$.*

Proof. If $m_{i,j}=0$, then there is nothing to prove. Otherwise, $j\to i$ and we have to show that

$$\mathrm{mult}_{B_{j-1}} Z_i^{j-1} \leq \deg Z_i.$$

Taking into account that the maps $\varphi_{a,b}\colon B_a \to B_b$ are surjective, it is sufficient to prove the inequality

$$\mathrm{mult}_{[B_{j-1}\cap\varphi_{i,i-1}^{-1}(t)^{j-1}]}[Z_i \cap \varphi_{i,i-1}^{-1}(t)]^{j-1} \leq \deg[Z_i \cap \varphi_{i,i-1}^{-1}(t)], \tag{23}$$

where $t\in B_{i-1}$ is a point of general position. Taking into account that $\varphi_{i,i-1}^{-1}(t)$ is the projective space $\mathbb{P}^{\mathrm{codim}\, B_{i-1}-1}$, we get that in the right-hand side in (23) we get the usual degree of a hypersurface in the projective space, whereas the set $[Z_i\cap\varphi_{i,i-1}^{-1}(t)]^{j-1}$ is obtained from this hypersurface by a finite sequence of blow ups $\varphi_{s,s-1}$, $s=i+1,\dots,j-1$, restricted onto $\varphi_{i,i-1}^{-1}(t)$. Taking into account that the multiplicities do not increase under the blow ups, we reduce the claim to the obvious case of a hypersurface in the projective space. Q.E.D. for the lemma.

As a result, we get the following estimate:

$$\sum_{j\geq i+1} a(j)m_{i,j} = \sum_{\substack{j\geq i+1\\ m_{i,j}\neq 0}} a(j)m_{i,j} \leq d_i \sum_{j\to i} a(j) \leq a(i)d_i.$$

By what was said above, we can delete in the right-hand side all the summands $m_{i,*}$, $i\geq 1$, and in the left-hand side all the summands d_i, $i\geq 1$, replacing the equality sign $=$ by the inequality sign $\leq$. Q.E.D.

Setting in the inequality (22) $a(i)=p_i$ and taking into account that for $j\geq S$ we have $p_j\leq p_S$, we complete the proof of Lemma 2.2.

Now let us complete the proof of Proposition 2.3.

Obviously, the inequality (21) remains true, if Y is an effective horizontal cycle of codimension two on V, that is, each irreducible component of the cycle Y is a horizontal subvariety. The formulae (19,20) extend by linearity to the set of all effective horizontal cycles, whereas the left-hand side and right-hand side of the inequality (21) are linear in $m_Y(\cdot)$, $m_{Y,F}(\cdot)$, respectively.

Now set $Y=Z^h$ and take into account that $m_{Y,F}(i)\leq \mathrm{mult}_B(Z^h\circ F)$ for $i\geq 1$. This proves Proposition 2.3.

1.5. The supermaximal singularity. We apply the estimates obtained above to a maximal singularity $E\in\mathcal{M}$, satisfying, apart from the Noether-Fano inequality, a certain additional condition formulated below in Proposition 2.5. Such singularities are said to be *supermaximal*. Since by assumption the K^2-condition of depth 2 holds, for the horizontal part of the self-intersection of the linear system Σ we get

$$Z^h \sim n^2K_V^2 + \alpha H_F,$$

where the coefficient $\alpha\in\mathbb{Z}$ satisfies the inequality $\alpha\geq -2n^2$. Therefore, for the vertical component we get

$$Z^v \sim (2nl-\alpha)H_F,$$

whence

$$\deg Z^v = \sum_{t\in\mathbb{P}^1} \deg Z^v_t \le (2nl + 2n^2)\deg V. \tag{24}$$

Proposition 2.5. *For some point $t \in \mathbb{P}^1$ there is a maximal singularity $E \in \mathcal{M}_t \ne \emptyset$, satisfying the estimate*

$$e(E) > \frac{\nu_E(F_t)}{2}\left(\frac{\deg Z^v_t}{n\deg V} - 2n\right) \tag{25}$$

Proof of Proposition 2.5. Compare the inequalities (15) and (24). Replacing the number l in the right-hand side of the inequality (24) by the left-hand side of the inequality (15), we get

$$\sum_{t\in\mathbb{P}^1}\left[\deg Z^v_t - 2n\deg V \max_{\{E\in\mathcal{M}_t\}} \frac{e(E)}{\nu_E(F_t)}\right] < 2n^2\deg V,$$

whence our proposition follows immediately.

Remark 2.1. If there are several maximal singularities, the centres of which lie in the fibers over distinct points $t_1,\dots,t_k$, then Proposition 2.5 gets stronger: there is a maximal singularity $E \in \mathcal{M}_t$, $t \in \{t_1,\dots,t_k\}$, satisfying the estimate

$$e(E) > \frac{\nu_E(F_t)}{2}\left(\frac{\deg Z^v_t}{n\deg V} - \frac{2n}{k}\right).$$

Thus we consider the worst possible case, setting $k = 1$.

Let $o \in B$ be a point of general position. Since by assumption $o \in F$ is a smooth point of the fiber, the conditions (f) and (v) hold. From the inequality (13) we immediately get the estimate

$$\sum_{i\in J_s\cup J^+_m} p_i m^h_i \le 4n^2\Sigma_s.$$

Since $m^h_i \le m^h_1 \le 4n^2$, we get the inequality

$$\sum_{i\in J_l} p_i m^h_i \le 4n^2(\Sigma_s + \Sigma^-_m). \tag{26}$$

This is the very estimate for the singularities of the horizontal component Z^h that we need.

Now consider the vertical component Z^v. By the condition (v) the inequality

$$m^v_i \le m^v_1 \le \frac{2}{\deg V} d_v \tag{27}$$

holds. On the other hand, the generalized K^2-condition of depth 2 implies the estimate

$$\frac{d_v}{\deg V} < \frac{2en}{\nu_E(F)} + 2n^2. \tag{28}$$

Combining (27) and (28), we obtain the inequality

$$\sum_{i\in J_s\cup J_m^+} p_i m_i^v < 2n\left(\frac{2e}{\nu_E(F)}+2n\right)(\Sigma_s+\Sigma_m^+).$$

Taking into account that by definition $\nu_E(F)=\sum\limits_{i=1}^{k} p_i\mu_i \geq \Sigma_s+\Sigma_m^+$, we get finally:

$$\sum_{i\in J_s\cup J_m^+} p_i m_i^v < 4ne+4n^2(\Sigma_s+\Sigma_m^+) \tag{29}$$

Now the inequalities (16), (26) and (29) imply the following estimate:

$$(4n^2(\Sigma_s+\Sigma_m^-)+4ne+4n^2(\Sigma_s+\Sigma_m^+))(\Sigma_s+\Sigma_m+\Sigma_u) >$$

$$> ((3\Sigma_s+2\Sigma_m+\Sigma_u)n+e)^2.$$

Taking into account that $\Sigma_m=\Sigma_m^+ + \Sigma_m^-$, after some easy arithmetic we get the inequality

$$(n(\Sigma_s-\Sigma_u)+e)^2<0.$$

A contradiction.

Q.E.D. for the claim (iii) of Theorem 2.1.

§2. Varieties with a pencil of Fano complete intersections

The aim of this section is to explain the key steps of the proof of Theorems 1.5 and 1.6. The proof is based on the techniques of Theorem 2.1, so that our work is reduced to checking the conditions (f), (v) and excluding the infinitely near maximal singularities lying over a singular point of a fiber. It is not hard to check the K^2-condition of depth 2 and the K-condition (see Propositions 1.4, 1.5) and we omit that step.

2.1. Fibrations into Fano complete intersections. Let us prove Theorem 1.5. Since the fiber space $V/\mathbb{P}^1$ is sufficiently general, we may assume that every fiber F at every point satisfies the regularity condition formulated in Definition 3.4 of the survey [1], that is, the set of polynomials

$$\{q_{i,j}\,|\,1\leq i\leq k, 1\leq j\leq d_i, (i,j)\neq(k,d_k)\} \tag{30}$$

makes a regular sequence. Now [32] or [1, Chapter 3, §2] give the condition (f). Let us prove the condition (v). Let $Y\subset F=F_t$ be a prime divisor, $o\in Y$ a point. Take a general hyperplane $H\subset\mathbb{P}$, tangent to F at the point o, that is, $H\supset T_oF$. Set $T=H\cap F$. By generality, $Y\neq T$, so that $Y_T=(Y\circ T)$ is a well defined effective cycle of codimension two on F, and moreover,

$$\frac{\mathrm{mult}_o}{\deg}Y_T \geq 2\frac{\mathrm{mult}_o}{\deg}Y.$$

Now the condition (f) implies (v).

Now to prove Theorem 1.5, it remains to check that the centre of a maximal singularity of the system $\Sigma \subset |-nK_V+lF|$ (or of the pair $(V, \frac{1}{n}\Sigma)$) can not be a singular point of a fiber. There are finitely many such points on the variety V, so that we may assume that certain additional conditions of general position are satisfied. Let us formulate these conditions.

As we did in §2 of [1, Chapter 3], take a system $(z_1,\dots,z_{M+k})$ of affine coordinates on $\mathbb{P}=\mathbb{P}^{M+k}$ with the origin at the point $o\in V$, which is a singular point of the fiber $F\ni o$, $F\subset\mathbb{P}$. By assumption, $o\in F$ is a non-degenerate quadratic singularity. Assume in addition, that the system of homogeneous equations $\{q_{i,j}=0\,|\,(i,j)\neq(k,d_k)\}$ defines a closed set of dimension two in $\mathbb{C}^{M+k}$ (respectively, a curve in $\mathbb{P}^{M+k-1}$), such that the linear span of each of its irreducible components is the linear space

$$T=\{q_{1,1}=q_{2,1}=\ldots=q_{k,1}=0\}.$$

Note that if $x\in F$ is a singularity, then the linear forms $q_{i,1}$, $i=1,\dots,k$, are linear dependent. The regularity of the point $x\in F$ means that, deleting from the set (30) exactly one linear form, say $q_{1,e}$, we obtain a regular sequence, that is, the system of equations

$$\{q_{i,j}=0\mid(i,j)\notin\{(1,e),(k,d_k)\}\} \tag{31}$$

defines a two-dimensional set in $\mathbb{C}^{M+k}$ (respectively, a curve in $\mathbb{P}^{M+k-1}$). In particular, $\operatorname{codim}T=k-1$ and the tangent cone $T_xF\subset T$ is a non-degenerate quadric. Moreover, it follows from the regularity condition that, replacing in the set (30) the linear form $q_{1,e}$ by an arbitrary linear form $l(z_1,\dots,z_{M+k-1})$, such that $l\,|_T\not\equiv 0$, we obtain a regular sequence, since neither component of the closed set (31) is contained in the hyperplane $l=0$.

Now assume that the singular point of the fiber $o\in F$ is the centre of a maximal singularity. Let $\lambda\colon F^+\to F$ be the blow up of the point o, $\lambda^{-1}(o)=E^+\subset F^+$ the exceptional divisor. The blow up λ can be looked at as the restriction of the blow up $\lambda_{\mathbb{P}}\colon\mathbb{P}^+\to\mathbb{P}$ of the point o on $\mathbb{P}$, so that $E^+\subset E$ is a non-singular quadric of dimension $M-1$, where $E=\lambda_{\mathbb{P}}^{-1}(o)\cong\mathbb{P}^{M+k-1}$ is the exceptional divisor.

Proposition 2.5. *There exists a hyperplane section B of the quadric $E^+\subset E$, satisfying the inequality*

$$\operatorname{mult}_B(\lambda^*\Sigma_F)>2n.$$

Proof is given below in §2 of Chapter 3, where we collect all facts related to the connectedness principle of Shokurov and Kollár.

Let $D\in\Sigma_F=\Sigma\,|_F$ be an effective divisor on F, $D\in|nH_F|$. For the strict transform $D^+\subset F^+$ we have $\lambda^*D=D^++(\frac{1}{2}\operatorname{mult}_o D)E^+$. Taking into account that $\operatorname{mult}_B E^+=1$, we obtain from Proposition 2.5 the inequality

$$\operatorname{mult}_o D+2\operatorname{mult}_B D^+>4n.$$

Let $H\subset\mathbb{P}$ be a general hyperplane, containing the point o and cutting out B, that is, $H^+\cap E^+=(H^+\cap E)\cap E^+=B$, where $H^+\subset\mathbb{P}^+$ is the strict transform.

Set $T = H \cap F$. The variety T is a complete intersection of type $(d_1, \dots, d_k)$ in $H = \mathbb{P}^{M+k-1}$ with an isolated quadratic singularity at the point o. The effective divisor $D_T = (D \circ T)$ on T satisfies the inequality

$$\mathrm{mult}_o D_T > 4n. \tag{32}$$

Obviously, $D_T \in |nH_T|$, where H_T is the hyperplane section of $T \subset \mathbb{P}^{M+k-1}$. By linearity, one may assume the divisor D_T to be prime, that is, an irreducible subvariety of codimension one.

Now we obtain a contradiction, repeating the arguments of [1, Chapter 3, §2] word for word: intersecting D_T with the hypertangent divisors, we construct a curve $C \subset T$, satisfying the inequality $\mathrm{mult}_o C > \deg C$, which is, of course, impossible. It remains to check that the technique of hypertangent divisors applies to our case. In [1, Chapter 3, §2] the following two facts were used:

1) the regularity condition for the complete intersection F at the point under consideration,

2) the irreducibility of the intersection $F \cap T_oF$ (it was derived from the relation $k < \frac{1}{2} \dim F$ by the Lefschetz theorem).

The arguments of [1, Chapter 3, §2] work in our case, if the conditions 1) and 2) hold.

As for the regularity condition, it is satisfied in our case due to the stronger regularity condition imposed on the singular point, formulated above. More precisely, the hyperplane section $T = H \cap F$ satisfies the usual regularity condition for any hyperplane $H \not\supset T_oF$.

Let us consider the condition 2). Recall that in this case $\mathrm{codim}\, T_oF = k-1$ (the point $o \in F$ is singular). Instead of the condition 2) we need the following fact: the intersection

$$T \cap T_oF = H \cap F \cap T_oF$$

is irreducible (and by the regularity condition has automatically the multiplicity exactly 2^k at the point o). This is true again by the Lefschetz theorem due to the inequality $k < \frac{1}{2} \dim F$, since $o \in F$ is a non-degenerate double point. This completes the proof of Theorem 1.5.

2.2. Fibrations into Fano cyclic covers. Let us prove Theorem 1.6. The conditions (f) and (v) for sufficiently general Fano cyclic covers are checked in the same way as for the complete intersections above, taking into account the additional hypertangent divisors, described in [1, Chapter 3, §2]. For the details, see [8]. Now it is enough to show that the centre of a maximal singularity can not be a singular point of a fiber $o \in F$.

Let $\sigma\colon F \to G \subset \mathbb{P} = \mathbb{P}^{M+1}$ be a realization of the fiber as a K-sheeted cyclic cover. For a generic variety V the singular point $o \in F$ can belong to strictly one of the two types:

— when the hypersurface G has a non-degenerate quadratic singularity at the point $p = \sigma(o)$ (and the point p does not lie on the branch divisor),

— when the hypersurface G is non-singular at the point $p = \sigma(o)$, but the branch divisor $W \cap G$, where $W = W_{Kl} \subset \mathbb{P}$ is a hypersurface of degree Kl, has a quadratic singularity at the point p.

A singularity $o \in F$ of the first type is excluded as a possible centre of a maximal singularity in precisely the same way as in Sec. 2.1 (again, taking into account the additional hypertangent divisors for the K-sheeted cover). Referring to [8] for the details, let us consider a singularity $o \in F$ of the second type. In this case, to exclude the maximal singularity, we need another method.

First of all, let us formulate the regularity condition for a singularity of the second type. Introducing a new coordinate u of weight l, we realize the fiber F as a complete intersection of the type $m \cdot Kl$ in the weighted projective space

$$\mathbb{P}(\underbrace{1, \ldots, 1}_{M+2}, l).$$

Namely, F is given by the system of equations

$$\left\{ \begin{array}{l} f(x_0, \ldots, x_{M+1}) = 0 \\ u^K = g(x_0, \ldots, x_{M+1}), \end{array} \right. \tag{33}$$

where $f(x_*)$ and $g(x_*)$ are homogeneous polynomials of degrees m and Kl, respectively. Recall that the integers m, l and K satisfy the relation $m+(K-1)l = M+1$.

Let $F \sim (f, g)$ be our variety, $o \in F$ an arbitrary point, $p = \sigma(o) \in G = \{f = 0\}$ its image on $\mathbb{P}$. Choose a system of affine coordinates $z_1, \ldots, z_{M+1}$ with the origin at the point p. Without loss of generality we may assume that $z_i = x_i/x_0$. Set $y = u/x_0^l$. Now the standard affine set $\mathbb{A}^{M+2}_{(z_1,\ldots,z_{M+1},y)}$ is a chart for $\mathbb{P}(1, \ldots, 1, l)$. Abusing our notations, we use for the non-homogeneous polynomials, corresponding to f and g, the same symbols:

$$f = q_1 + \ldots + q_m, \quad g = w_0 + w_1 + \ldots + w_{Kl},$$

where q_i, w_j are homogeneous components of degree i, j in the variables z_*, respectively, so that in the affine chart $\mathbb{A}^{M+2}_{(z_*,y)}$, introduced above, the variety F is given by the pair of equations $f = 0$, $y^K = g$ (replacing the system (33)). If the point $o \in F$ does not lie on the ramification divisor of the morphism σ, then we always assume that $w_0 = 1$. If the point $p \in G$ is non-singular, then without loss of generality we assume that $q_1 \equiv z_{M+1}$. In the latter case we set

$$\bar{q}_i = q_i \,|\,_{\{z_{M+1}=0\}} = q_i(z_1, \ldots, z_M, 0)$$

and $\bar{w}_j = w_j \,|\,_{z_{M+1}=0} = w_j(z_1, \ldots, z_M, 0)$ for $i, j \geq 2$.

Now let us formulate the regularity condition for the singular point $o \in F$. Here $w_0 = 0$, $q_1 = z_{M+1}$, $w_1 = \lambda z_{M+1}$, where $\lambda \in \mathbb{C}$ is a constant, that is, the point $p \in G$ is non-singular. We require the quadratic form $\bar{w}_2(z_1, \ldots, z_M)$ to have the maximal rank M and the sequence

$$\bar{q}_2, \ldots, \bar{q}_m, \bar{w}_2, \ldots, \bar{w}_K$$

to be regular in $\mathcal{O}_{o,\mathbb{C}^M}$, and moreover, the system of homogeneous equations

$$\bar{q}_2 = \ldots = \bar{q}_m = \bar{w}_2 = \ldots = \bar{w}_K = 0 \tag{34}$$

should define a closed algebraic set in $\mathbb{C}^M$, neither component of which is contained in a hyperplane.

Obviously, we may assume that $\lambda \in \{0, 1\}$. Now, either $q_1 \equiv w_1 \equiv z_{M+1}$, or $q_1 \equiv z_{M+1}$ and $w_1 \equiv 0$. The germ of the variety $o \in F$ is analytically the germ of the hypersurface $y^K = \bar{w}_2(z_1, \ldots, z_M) + \ldots$, where $\bar{w}_i, \bar{q}_i$ are the restrictions of the polynomials w_i, q_i onto the hyperplane $z_{M+1} = 0$, in the space $\mathbb{C}^{M+1}_{(z_1,\ldots,z_M,y)}$. Let $\varphi: F^+ \to F$ be the blow up of the point o, $E \subset F^+$ the exceptional divisor. It follows from what was said that E realizes naturally as a quadratic hypersurface, $E \subset \mathbb{P}^M_{(z_1:\ldots:z_M:y)}$. Let $\varphi_G: G^+ \to G$ be the blow up of the point $p \in G$, $E_G = \varphi_G^{-1}(p) \subset G^+$ the exceptional divisor, $E_G \cong \mathbb{P}^{M-1}_{(z_1:\ldots:z_M)}$. It is easy to see that the morphism σ extends to a rational map $\sigma^+: F^+ \dashrightarrow G^+$, whereas the restriction

$$\sigma_E = \sigma^+|_E: E \dashrightarrow E_G$$

is the projection of the quadratic cone $E \subset \mathbb{P}^M$, given in $\mathbb{P}^M_{(z_1:\ldots:z_M:y)}$ by the equation $\bar{w}_2(z_1, \ldots, z_M) = 0$, from its vertex $o^+ = (0 : \ldots : 0 : 1)$, onto the smooth quadric $E_+ \subset E_G$, given in $E_G = \mathbb{P}^{M-1}$ by the very same equation $\bar{w}_2 = 0$. Therefore, σ^+ contracts generators of the cone E to points.

By the regularity condition, the system of homogeneous equations (34) cuts out on E_+ (and thus on E) a closed algebraic set, neither component of which is contained in a hyperplane.

Assume that there exist an effective divisor $R \in |nH_F|$ and a hyperplane section B of the quadric cone $E \subset \mathbb{P}^M$, satisfying the inequality

$$\nu + \mu > 2n, \tag{35}$$

where $R^+ \in |nH_F - \nu E|$, that is, $\nu = \frac{1}{2}\,\mathrm{mult}_o R$, and $\mu = \mathrm{mult}_B R^+$, $R^+ \subset F^+$ is the strict transform of the divisor R. By linearity of the inequality (35) one may assume the divisor R to be prime.

Lemma 2.5. *The following estimate holds:* $\nu \leq \frac{3}{2}n$.

Proof. Assume the converse: $\nu > \frac{3}{2}n$. Let

$$D_i = \sigma^*(f_i\,|_G), \quad f_i = q_1 + \ldots + q_i, \quad i = 1, \ldots, m-1,$$

and

$$\Delta_i = \sigma^*(g_i\,|_G), \quad g_i = w_1 + \ldots + w_i, \quad i = 2, \ldots, K-1,$$

be hypertangent divisors on F, $\mathcal{D} = \{D_1, \ldots, \Delta_{K-1}\}$, $\sharp\mathcal{D} = m + K - 3$. By the regularity condition we get

$$\mathrm{codim}_o \left(\bigcap_{D \in \mathcal{D}} D \right) = \sharp\mathcal{D}.$$

Lemma 2.6. *The prime divisors D_1 and R are distinct: $D_1 \neq R$.*

Proof. If $\nu \leq 2n$, then $\mu \geq 1$, that is, $R^+ \supset B$. By the regularity condition, $D_1^+ \not\supset B$. Therefore, $D_1^+ \neq R^+$.

If $\nu > 2n$, then, taking into account that $D_1^+ \in |H_F - 2E|$, we obtain again that $D_1 \neq R$. Q.E.D. for the lemma.

By the lemma the effective cycle $(R \circ D_1)$ of codimension two is well defined. The inequality

$$\frac{\mathrm{mult}_o}{\deg}(R \circ D_1) > \frac{6}{mK}$$

holds, so that there is an irreducible subvariety $Y \subset F$ of codimension two (an irreducible component of the cycle $(R \circ D_1)$), satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{6}{mK}.$$

Applying to the subvariety Y [1, Proposition 3.2], we obtain the opposite inequality

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{2}{3} \cdot \ldots \cdot \frac{m-1}{m} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{K-1}{K} = \frac{6}{mK}.$$

The contradiction just obtained shows, in addition, that the maximum of the ratio $\mathrm{mult}_o / \deg$ is attained at the divisor $\sigma^* D_1$ and equal to $4/mK$. Q.E.D. for the lemma.

Corollary 2.1. *The following inequality holds: $\mu > \frac{1}{2} n$.*

Thus the hyperplane section B is really present in the divisor $R^+ \subset F^+$. Now it is more convenient to use the technique of hypertangent linear systems (see [1, §3.1]). Set

$$\Lambda_i^G = \left| \left(\sum_{j=1}^{i} f_j s_{i-j} + \sum_{j=2}^{i} g_j r_{i-j} \right) \Bigg|_G = 0 \right|,$$

where s_k, r_k run through the set of all homogeneous polynomials in $z_1, \ldots, z_{M+1}$ of degree k. Furthermore, let

$$\Lambda_i = \sigma^* \Lambda_i^G \quad \text{and} \quad \Lambda_i^+$$

be the pull back of the system Λ_i^G on F and its strict transform on F^+, respectively. We get $\Lambda_i^+ \subset |iH_F - (i+1)E|$. Finally, let

$$\Lambda_i^E = \Lambda_i^+ \, |_E$$

be the projectivized tangent system of the linear system Λ_i at the point o.

Let H_E be the class of a hyperplane section of the cone E. We get $\Lambda_i^E \subset |(i+1)H_E|$. In the coordinate form

$$\Lambda_i^E = \sigma_E^* \left| \sum_{j=1}^{i} \bar{q}_{j+1} \bar{s}_{i-j} + \sum_{j=2}^{i} \bar{w}_{j+1} \bar{r}_{i-j} = 0 \right|,$$

where $\bar{s}_k = s_k|_{\{z_{M+1}=0\}}, \bar{r}_k = r_k|_{\{z_{M+1}=0\}}$. From this explicit presentations and the regularity condition we get at once that

$$\operatorname{codim}_E \operatorname{Bs}\Lambda_i^E = \min\{i, m-1\} + \min\{i, K-1\} - 1,$$

and moreover neither component of the closed set $\operatorname{Bs}\Lambda_i^E$ is contained in a hyperplane. Note that for $i \geq \max\{m, K\} - 1$ we get $\operatorname{Bs}\Lambda_i = \operatorname{Bs}\Lambda_{i+1}$: it is precisely the closed set

$$\bar{q}_2 = \ldots = \bar{q}_m = \bar{w}_3 = \ldots = \bar{w}_K = 0.$$

Let

$$\mathcal{L} = (D_1, \ldots, D_{m-1}, L_2, \ldots, L_{K-1}) \in \prod_{i=1}^{m-1} \Lambda_i \times \prod_{i=2}^{K-1} \Lambda_i$$

be a general set of hypertangent divisors,

$$\mathcal{L}^+ = (D_1^+, \ldots, L_{K-1}^+) \quad \text{and} \quad \mathcal{L}^E = (D_1^E, \ldots, L_{K-1}^E) = \mathcal{L}^+|_E$$

its strict transform on F^+ and restriction onto the quadric cone E, respectively.

Set $R_E = R^+|_E$ to be the projectivized tangent cone of the divisor R at the point o. By the regularity condition the closed algebraic set

$$R_E \cap \left(\bigcap_{i=2}^{m-1} D_i^E\right) \cap \left(\bigcap_{i=2}^{K-1} L_i^E\right) \tag{36}$$

is of codimension precisely $m+K-2$ with respect to E (note that in (36) the divisor D_1^E is omitted).

Therefore, the effective cycle

$$Y_E = (R_E \circ D_2^E \circ \ldots \circ D_{m-1}^E \circ L_2^E \circ \ldots \circ L_{K-1}^E)$$

of codimension $m + K - 2$ on E is well defined. Its H_E-degree is

$$\deg Y_E = \frac{1}{4} m! K! \deg R_E = \frac{1}{2} m! K! \nu.$$

Furthermore, by the regularity condition the closed set

$$R \cap \left(\bigcap_{i=2}^{m-1} D_i\right) \cap \left(\bigcap_{i=2}^{K-1} L_i\right)$$

is of codimension $m+K-2$ with respect to F *in a neighborhood of the point* o. Thus we get the uniquely determined effective algebraic cycle Y of codimension $m+K-2$ on F, each irreducible component of which contains the point o, and which coincides with the effective cycle

$$(R \circ D_2 \circ \ldots \circ D_{m-1} \circ L_2 \circ \ldots \circ L_{K-1})_U$$

on a suitable Zariski open set $U \subset F$, containing the point o. For a general set $\mathcal{L}$ we get: $Y_E = (Y^+ \circ E)$ is the projectivized tangent cone to Y at the point o. However, generally speaking, for the H_F-degree of the cycle Y we get only the inequality

$$\deg Y \leq (m-1)!(K-1)! \deg R = nm!K!,$$

since constructing the cycle Y, at the intermediate steps (the divisor R is successively intersected with D_2, L_2, D_3, L_3 and so on) we remove the components that do not contain the point o. Note that, generally speaking, certain irreducible components of the cycles Y and Y_E can be contained in D_1 and D_1^E, respectively. Let us separate those components:

$$Y = Z + Y^\sharp, \quad Y_E = Z_E + Y_E^\sharp,$$

where $\operatorname{Supp} Z \subset D_1$ and Z is the maximal subcycle of the effective cycle Y with this property (that is, neither irreducible component of the cycle $Y^\sharp$ is contained in D_1), $Z_E = (Z^+ \circ E)$, $Y_E^\sharp = ((Y^\sharp)^+ \circ E) = Y_E - Z_E$. Obviously, $\operatorname{Supp} Z_E \subset D_1^E$, but irreducible components of the cycle $Y_E^\sharp$, generally speaking, *can* be contained in D_1^E. The following fact is of key importance.

Lemma 2.7. *The algebraic cycle*

$$Y_E^\sharp - \mu(B \circ D_2^E \circ \ldots \circ D_{m-1}^E \circ L_2^E \circ \ldots \circ L_{K-1}^E)$$

is effective. In particular, the following inequality holds:

$$\deg Y_E^\sharp \geq \frac{1}{2} m!K!\mu.$$

Proof. The first claim holds because by construction the algebraic cycle $R_E - \mu B$ is effective, so that the cycle $Y_E - \mu B_{\mathcal{L}}$ is effective, either, where $B_{\mathcal{L}} = (B \circ D_2^E \circ \ldots \circ L_{K-1}^E)$. The support of the cycle $B_{\mathcal{L}}$ is a closed set $\operatorname{Supp} B_{\mathcal{L}}$ of pure codimension $m + K - 3$ with respect to E. For any irreducible component $\Delta \subset \operatorname{Supp} B_{\mathcal{L}}$ we get $\Delta \not\subset D_1^E$.

Indeed, assume the converse: $\Delta \subset D_1^E$. Then $\Delta \subset B$ and

$$\Delta \subset \left(\bigcap_{i=1}^{m-1} D_i^E \right) \cap \left(\bigcap_{i=2}^{K-1} L_i^E \right),$$

which contradicts the regularity condition.

Therefore, each irreducible component Δ of the cycle $B_{\mathcal{L}}$ cannot be a component of the cycle Z_E and so appears in the cycle $Y_E^\sharp$ only. This proves the first claim of the lemma. The second claim follows from the first one in a trivial way. Q.E.D. for the lemma.

Since the irreducble components of the cycle $Y^\sharp$ are not contained in the divisor D_1, the effective cycle $(Y^\sharp \circ D_1)$ is well defined, and moreover, the inequality

$$\frac{\operatorname{mult}_o}{\deg}(Y^\sharp \circ D_1) \geq 2 \frac{\operatorname{mult}_o}{\deg} Y^\sharp$$

holds.

Since for any irreducible subvariety $\Delta\subset F$ we have the estimate $(\mathrm{mult}_o/\deg)\Delta\leq 1$, we get

$$\deg Y^\sharp\geq 2\,\mathrm{mult}_o Y^\sharp=2\deg Y^\sharp_E\geq m!K!\mu.$$

On the other hand, we have the inequality $\deg Z\geq \mathrm{mult}_o Z=\deg Z_E$ (which is true for any effective cycle Z).

Combining these estimates, we get

$$m!K!n\geq \deg Y=\deg Z+\deg Y^\sharp\geq$$
$$\geq \deg Z_E+2\deg Y^\sharp_E=\deg Y_E+\deg Y^\sharp_E\geq$$
$$\geq \frac{1}{2}m!K!\nu+\frac{1}{2}m!K!\mu=\frac{1}{2}m!K!(\nu+\mu).$$

Therefore, $\nu+\mu\leq 2n$. Contradiction.

We have proved that a singular point of a fiber $o\in F$, lying on the ramification divisor of the morphism σ, cannot be the centre of a maximal singularity of a movable linear system.

This completes the proof of Theorem 1.6. Q.E.D.

2.3. Fiber-wise birational modifications. Let us prove Theorem 1.1. Let $V\in\mathcal{V}(d)$, $F=V\cap\{p\}\times\mathbb{P}^M$ be the fiber over the marked point. Fix a local parameter t on the curve C in a neighborhood of the point p. The hypersurface $V\subset X$ in a neighborhood of the fiber F is given by the equation

$$f=f^{(0)}+tf^{(1)}+\ldots+t^jf^{(j)}+\ldots,$$

where $f^{(j)}$ are homogeneous polynomials of degree j in the homogeneous coordinates $(x_*)=(x_0:\ldots:x_M)$ on $\mathbb{P}^M$. It is well known [6], that $\dim\mathrm{Sing}\,V\geq\dim\mathrm{Sing}\,F-1$, so that the smoothness of the hypersurface V implies that, firstly, the hypersurface $F=\{f^{(0)}=0\}\subset\mathbb{P}^M$ has at most zero-dimensional singularities and, secondly, for every point $x\in\mathrm{Sing}\,F$ we have $f^{(1)}(x)\neq 0$.

Take $V_1,V_2\in\mathcal{V}(d)$ and let $\chi^*\colon V_1^*\to V_2^*$ be a fiber-wise isomorphism outside the marked point $p\in C$. Since the fibers over points of general position $y\in C$ are smooth hypersurfaces of degree $d\geq 2$, over a point $y\in C^*$ the isomorphism χ^*_y is induced by an automorphism of the ambient projective space $\xi_y\in\mathrm{Aut}\,\mathbb{P}$. Therefore, $\chi^*=\xi^*|_{V_1}$, where $\xi^*_y=\xi_y$ is an algebraic curve $\xi^*\colon C^*\to\mathrm{Aut}\,\mathbb{P}$ of projective automorphisms. Let $\mathbb{P}=\mathbb{P}(L)$, where $L\cong\mathbb{C}^{M+1}$ is a linear space. The curve ξ^* lifts to a curve $\xi\colon C\to\mathrm{End}\,L$, $\xi(C^*)\subset\mathrm{Aut}\,L$. If $\xi(p)\in\mathrm{Aut}\,L$, then χ^* extends to a fiber-wise (biregular) isomorphism $\chi=\xi|_{V_1}$, and the varieties V_1 and V_2 are fiber-wise isomorphic. Assume that this is not the case: $\det\xi(p)=0$.

Let $\sum_{i=0}^\infty t^i\xi^{(i)}$ be the Taylor series of the curve ξ. We may assume that $\xi^{(0)}\neq 0$. The next claim is a well known fact of elementary linear algebra.

Lemma 2.8. *There exist curves of endomorphisms $\beta,\gamma\colon C\to\mathrm{End}\,L$ and a basis $(e_0,\ldots,e_M)$ of the space L such that $\beta(p),\gamma(p)\in\mathrm{Aut}\,L$ and in this basis the curve $\beta\xi\gamma^{-1}\colon C\to\mathrm{End}\,L$ is of diagonal form:*

$$\beta\xi\gamma^{-1}\colon e_i\mapsto t^{w(e_i)}e_i, \qquad (37)$$

where $w(e_i)\in\mathbb{Z}_+$.

Now replace V_1 by $\gamma(V_1)$ and V_2 by $\beta(V_2)$. We can simply assume that the fiber-wise birational correspondence ξ is of the form (37) from the start. Let us show that if $m=\max\{w(e_i)\}\geq 1$, then this is impossible.

Let $\{a_0=0<a_1<\ldots<a_k\}=\{w(e_i), i=0,\ldots,M\}\subset\mathbb{Z}_+$ be the set of weights of the diagonal transformation (37), $k\leq M$, $m=a_k$ the maximal weight. Consider the system of homogeneous coordinates $(x_0:\ldots:x_M)$, dual to the basis (e_*). Define the weight of monomials in x_*, setting

$$w(x_0^{n_0}x_1^{n_1}\ldots x_M^{n_M})=\sum_{i=0}^{M}n_iw(e_i).$$

Set $\mathcal{A}_i=\{x_j|w(e_j)=a_i\}\subset\mathcal{A}=\{x_0,\ldots,x_M\}$ to be the set of coordinates of weight a_i. We pay special attention to the sets $\mathcal{A}_*=\mathcal{A}_0$ and $\mathcal{A}^*=\mathcal{A}_k$ of coordinates of the minimal and maximal weights, respectively.

Now let $f=f^{(0)}(x)+tf^{(1)}+\ldots$ be the local over the base C equation of the hypersurface $V_2\subset C\times\mathbb{P}$, $f^{(i)}$ are homogeneous polynomials of degree $d\geq 3$ in the coordinates x_*. The series

$$f_\xi=\sum_{l=0}^{\infty}t^lf_\xi^{(l)}(x)=\sum_{l=0}^{\infty}t^lf^{(l)}(t^{w(x_0)}x_0,\ldots,t^{w(x_M)}x_M)$$

vanishes on V_1, and outside the marked fiber F_1, that is, for $t\neq 0$, gives an equation of V_1. Let $b\in\mathbb{Z}_+$ be the maximal power of the parameter t, dividing f_ξ. Then

$$t^{-b}f_\xi=g=\sum_{l=0}^{\infty}t^lg^{(l)}(x_0,\ldots,x_M)$$

gives an equation of the hypersurface V_1 in the marked fiber X_p, too.

Lemma 2.9. *For any $l\in\mathbb{Z}_+$ the polynomial $f^{(l)}$ is a linear combination of monomials of weight $\geq b-l$, and the polynomial $g^{(l)}$ is a linear combination of monomials of weight $\leq b+l$.*

Proof. Assume that the monomial x^I comes into the polynomial $f^{(l)}$ with a non-zero coefficient. Then it generates the component $t^{l+w(x^I)}x^I$ of the series f_ξ and, moreover, this component is generated *only* by this monomial in $f^{(l)}$. Therefore $l+w(x^I)\geq b$, as we claimed. Assume that the monomial x^I comes into $g^{(l)}$ with a non-zero coefficient. It comes from the monomial $t^{l+b}x^I$ of the series f_ξ, which, in its turn, can come only from the monomial x^I in the polynomial f^α, where $\alpha+w(x^I)=l+b$. Q.E.D. for the lemma.

Let

$$\begin{aligned}P_*&=\{x_j=0|w(x_j)\geq 1\}=\mathbb{P}\langle e_j|w(x_j)=0\rangle,\\ P^*&=\{x_j=0|w(x_j)\leq m-1\}=\mathbb{P}\langle e_j|w(x_j)=m\rangle\end{aligned}$$

be the subspaces of the minimal and maximal weights, respectively.

Lemma 2.10. *If $b \geq m+1$, then $P_* \subset \operatorname{Sing} F_2$. If $m(d-1) \geq b+1$, then $P^* \subset \operatorname{Sing} F_1$.*

Proof. Assume that $b \geq m+1$. The fiber $F_2 \subset \mathbb{P}$ over the marked point is given by the equation $f^{(0)} = 0$. By assumption, $f^{(0)}$ is a linear combination of the monomials of weight $\geq m+1$. If the monomial x^I comes into $f^{(0)}$ with a non-zero coefficient, then x^I is divided by a quadratic monomial in the variables $\mathcal{A} \setminus \mathcal{A}_*$ (otherwise $w(x^I) \leq m$). Therefore all first partial derivatives of the polynomial $f^{(0)}$ vanish on P_*. Therefore $P \subset \operatorname{Sing} F_2$.

Similarly, if $b \leq m(d-1)-1$, then any monomial x^I in $g^{(0)}$ are divided by a quadratic monomial in $\mathcal{A} \setminus \mathcal{A}^*$, otherwise we get $w(x^I) \geq m(d-1)$, which contradicts the assumption and Lemma 2.9. Q.E.D. for Lemma 2.10.

Note that for $d \geq 3$ the inequalities $b \leq m$ and $b \geq m(d-1)$ can not hold simultaneously. Therefore, at least of the two inequalities of Lemma 2.10 holds. Let $b \geq m+1$. Since V_2 is non-singular, P_* is a point. Let $\mathcal{A}_* = \{x_0\}$, so that $P_* = (1, 0, \dots, 0)$. Again from the fact that V_2 is non-singular, we get $f^{(1)}(1, 0, \dots, 0) \neq 0$. Therefore, the monomial x_0^d comes into $f^{(1)}$ with a non-zero coefficient. By Lemma 1.4 we get $b \leq 1$. Therefore $m = 0$, a contradiction.

In the case $b \leq m(d-1)-1$ the arguments are symmetric: V_1 is non-singular, P^* is a point $(0, \dots, 0, 1)$, $\mathcal{A}^* = \{x_M\}$ and $g^{(1)}(0, \dots, 0, 1) \neq 0$, so that $md \leq b+1$, whence we get again $m = 0$, a contradiction.

Therefore, there are no non-trivial weights and ξ is a fiber-wise biregular isomorphism. So $\chi = \xi|_{V_1}$ is a fiber-wise isomorphism as well. Proof of Theorem 1.1 is complete.

§3. Varieties with a pencil of cubic surfaces

In this section we sketch the proof of Theorem 1.7 for $d = 3$, that is, for varieties with a pencil of cubic surfaces.

3.1. Maximal singularities. Existence of a line. Let $\pi\colon V \to \mathbb{P}^1$ be a fibration into cubic surfaces which is a standard Fano fiber space, $\Sigma \subset |-nK_V + lF|$ a movable linear system. Since by assumption $V/\mathbb{P}^1$ satisfies the K^2-condition, the K-condition holds as well, that is, $l \in \mathbb{Z}_+$. Now we argue in the same way as in §1, with the only difference that the system Σ can have maximal singularities, covering the base $\mathbb{P}^1$. More precisely, assume that the inequality $c_{\mathrm{virt}}(\Sigma) < c(\Sigma, V) = n$ holds. Then we have

Proposition 2.6. *The linear system Σ has a maximal singularity. Moreover, either there is a maximal singularity, covering the base $\mathbb{P}^1$, or on some model $\varphi\colon V^+ \to V$ there is a finite set of exceptional divisors*

$$\mathcal{M} = \{E \subset V^+ \mid e(E) = \operatorname{ord}_E \varphi^*\Sigma - na(E) > 0\},$$

such that the following inequality holds:

$$\sum_{t \in \mathbb{P}^1} \left(\max_{\{E \in \mathcal{M} \mid \varphi(E) \in F_t\}} \frac{e(E)}{\operatorname{ord}_E \varphi^* F_t} \right) > l.$$

The standard **proof** (see §1) is omitted.

Now if $C \subset V$ is a horisontal curve (that is, $\pi(C) = \mathbb{P}^1$), which is the centre of a maximal singularity, then $\operatorname{mult}_C \Sigma > n$, so that C is a section or bi-section of the projection π. Untwisting such curves by fiber-wise involutions, described in Example 1.4 of Chapter 1, we come to the situation when the system Σ has no maximal singularities, covering the base. Now, arguing as in §1, we get the existence of a supermaximal singularity $E \in \mathcal{M}$, satisfying the inequality

$$6ne(E) > \operatorname{ord}_E \varphi^* F_t \cdot \deg Z_t^v,$$

where $\varphi(E) = x \in F_t = \pi^{-1}(t)$ and Z_t^v is the vertical component of the self-intersection $Z = (D_1 \circ D_2)$ of the linear system Σ, contained in the fiber F_t. From this moment, the singularity E is fixed, the fiber F_t is denoted by F, instead of Z_t^v we write Z^v etc.

Proposition 2.7. *Through the point $x \in F$ there is at least one line $L \subset F \subset \mathbb{P}^3$.*

Proof. Assume the converse. Then the point x is a smooth point of the cubic surface $F \subset \mathbf{P}^3$. Moreover, the curve $R = T_x F \cap F$ is irreducible, its degree is equal to 3 and its multiplicity at the point x is equal to 2 exactly. If $C \subset F$ is any other curve, then

$$\deg C = (C \cdot R) \geq (C \cdot R)_x \geq 2 \operatorname{mult}_x C.$$

Thus for any curve $Q \subset F$ we get the inequality

$$\operatorname{mult}_x Q \leq \frac{2}{3} \deg Q.$$

Therefore,

$$\operatorname{mult}_x Z^v \leq \frac{2}{3} \deg Z^v < \frac{4ne}{\nu_E(F)}.$$

Now the computations of §1 (simplified) give a contradiction. Q.E.D. for the proposition.

The main difficulty in the proof of Theorem 1.7 for $d = 3$ is precisely the existence of lines, the input of which into the self-intersection Z of the system Σ can be too large, so that the computations of §1 do not give a contradiction. In the present survey we consider the case when there is precisely one line through the point x on the cubic surface F. The remaining cases are considered in [3].

If the line $L \ni x$ is unique, then the point x is smooth on F and $T_x F \cap F = L + Q$, where $Q \subset F$ is a smooth conic. The arguments given above show that for any curve $C \subset F$, $C \neq L$, the following inequality takes place:

$$\operatorname{mult}_x C \leq \frac{1}{2} \deg C.$$

Write down $Z^v = C + kL$, where C is an effective 1-cycle, not containing L. Now

$$k + \frac{1}{2} \deg C \geq \frac{4ne}{\nu_E(F)}, \quad k + \deg C < \frac{6ne}{\nu_E(F)}.$$

This implies that

$$\deg C < \frac{4ne}{\nu_E(F)}.$$

In particular, $(C \cdot L) \le \deg C < \dfrac{4ne}{\nu_E(F)}$.

3.2. The main construction: the staircase, associated with the line L. We assume that the line L does not contain singular points of the fiber F (if there are such points). The general case is done in [3].

An infinite series of blow ups

$$\begin{array}{cccc} \sigma_i: & V^{(i)} & \to & V^{(i-1)} \\ & \bigcup & & \bigcup \\ & E^{(i)} & \to & L_{i-1}, \end{array}$$

$i \ge 1$, starting from $V^{(0)} = V$, where L_{i-1} is the centre of the i-th blow up, and $E^{(i)} = \sigma_i^{-1}(L_{i-1})$ is its exceptional divisor, $L_0 = L$, is said to be a *staircase, associated with the line* L, or, simply, an *L-staircase*, if the following conditions are satisfied:

L_i is a curve for all $i \in \mathbb{Z}_+$, $E^{(i)}$ is a ruled surface of the type $\mathbb{F}_1$ over L_{i-1} and $L_i \subset E^{(i)}$ is the exceptional section (i.e. the (-1)-curve).

Obviously, by this definition the staircase is unique. Just below we show that it exists. Its starting segment, consisting of the blow ups σ_i for $1 \le i \le M$, is said to be a (finite) staircase of the length M.

It is convenient to prove the existence of the staircase together with some of its properties.

For convenience of notations set $E^{(0)}$ to be the fiber F of the morphism π, which contains L. The operation of taking the proper inverse image on the i-th step (i.e. on $V^{(i)}$) is denoted by adding the bracketed upper index i. For instance, the proper inverse image of the surface $E^{(i)}$ on $V^{(j)}$ for $j \ge i$ is written down as $E^{(i,j)}$. Set also:

s_i to be the class of L_i in $A^2(V^{(i)})$, $s_0 = f$;

$f_i \in A^2(V^{(i)})$ to be the class of the fiber of the ruled surface $E^{(i)}$ over a point $\in L_{i-1}$.

Abusing our notations, we sometimes treat s_i and f_i as numerical classes of curves on the ruled surface $E^{(i)}$:

$$A^1 E^{(i)} = \operatorname{Pic} E^{(i)} = \mathbb{Z}s_i \oplus \mathbb{Z}f_i,$$

so that, in particular, the formulas like $(s_i \cdot s_i) = -1$, $(s_i \cdot f_i) = 1$ etc. make sense.

In these notations we have the following

Proposition 2.8. (i) *For $i \ge 2$ the effective 1-cycle $(E^{(i-1,i)} \circ E^{(i)})$ is just the irreducible curve $E^{(i-1,i)} \cap E^{(i)}$. Its numerical class is equal to $(s_i + f_i)$. In particular, this curve does not intersect $L_i \sim s_i$.*

(ii) *The following equalities hold: $(E^{(i)})^3 = 1$, $(E^{(i)} \cdot L_i) = 0$. Taking into account the isomorphism $L_i \cong \mathbb{P}^1$, we can write down*

$$\mathcal{N}_{L_i/V^{(i)}} \cong \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(-1).$$

In this presentation the first component is uniquely determined. It corresponds to the exceptional section $L_{i+1} \subset E^{(i+1)} = \mathbb{P}(\mathcal{N}_{L_i/V^{(i)}})$. For the second component we can take the one-dimensional subbundle, corresponding exactly to the curve $E^{(i,i+1)} \cap E^{(i+1)}$.

(iii) *The classes s_i and f_i satisfy the relations*

$$\sigma^* s_{i-1} = s_i,$$

$$\sigma_* f_i = 0$$

for $i \geq 1$.

Assuming that $L \cap \operatorname{Sing} F = \emptyset$, we give a simultaneous **proof** of the existence of the staircase and of Proposition 2.8.

Let us consider the first step of the staircase, that is, the morphism $\sigma_1 : V^{(1)} \to V^{(0)} = V$, blowing up the line $L_0 = L \subset F$. We get the exact sequence

$$0 \to \mathcal{N}_{L/F} \to \mathcal{N}_{L/V} \to \mathcal{O}_V(F)\,|_L \to 0,$$

which can be rewritten down in the following way:

$$0 \to \mathcal{O}_L(-1) \to \mathcal{N}_{L/V} \to \mathcal{O}_L \to 0.$$

Consequently, $E^{(1)}$ is a ruled surface of the type $\mathbb{F}_1$, $(E^{(1)})^3 = 1$, whence $(E^{(1)} \cdot E^{(1)}) \sim (-s_1 - f_1)$ and $(E^{(0,1)} \cdot L_1) = ((F - E^{(1)}) \cdot s_1) = 0$. Thus all the requirements (i)-(iii) of the definition of the staircase are satisfied for the first blow up.

We proceed by induction on $i \geq 1$. We get the exact sequence

$$0 \to \mathcal{N}_{L_i/E^{(i)}} \to \mathcal{N}_{L_i/V^{(i)}} \to \mathcal{O}_{V^{(i)}}(E^{(i)})\big|_{L_i} \to 0.$$

Taking into account the facts which were already proved, this sequence can be rewritten down as follows:

$$0 \to \mathcal{O}_{L_i}(-1) \to \mathcal{N}_{L_i/V^{(i)}} \to \mathcal{O}_{L_i} \to 0.$$

Again this implies that $E^{(i+1)} = \mathbb{P}(\mathcal{N}_{L_i/V^{(i)}})$ is a ruled surface of the type $\mathbb{F}_1$ and $(E^{(i+1)})^3 = 1$, so that

$$E^{(i+1)}\,|_{E^{(i+1)}} \sim (-s_{i+1} - f_{i+1}).$$

Thus $(E^{(i+1)} \cdot L_{i+1}) = 0$, (i) and (iii) are satisfied in an obvious way. The proof is complete. Q.E.D.

Remark 2.2. (i) Since $E^{(i-1,i)}$ does not intersect L_i (for $i \geq 1$ in the non-special and for $i \geq 2$ in the special case), we get

$$E^{(i-1,i)} = E^{(i-1,i+1)} = \ldots = E^{(i-1,j)} = \ldots$$

for any $j \geq i$. In particular, if $C \subset E^{(i-1)}$ is a curve, which is not the exceptional section L_{i-1}, then its proper inverse images on all the varieties $V^{(j)}$, $j \geq i$, are the same:

$$C^{(i)} = C^{(i+1)} = \ldots = C^{(j)}.$$

(ii) Abusing our notations, we call an irreducible curve $C \subset E^{(i)}$, $i \geq 1$, a *horizontal* one, if $\sigma_i(C) = L_{i-1}$, and a *vertical* one, if $\sigma_i(C)$ is a point on L_{i-1}. Respectively, we define horizontal and vertical 1-cycles with the support in $E^{(i)}$. The *degree* of a horizontal curve C is equal to $\deg C = \deg \sigma_i|_C = (C \cdot f_i)$, the *degree* of a vertical curve C is equal to $\deg C = (C \cdot L_i) = 1$. We define the degree of a horizontal and a vertical 1-cycle with the support in $E^{(i)}$ as its intersection with f_i and L_i, respectively. In particular, the degree of a vertical 1-cycle is just the number of lines (fibers) in it. Note that if an effective horizontal 1-cycle C does not contain the exceptional section L_i as a component, then its class in $A^1(E^{(i)})$ or $A^2(V^{(i)})$ is equal to $\alpha s_i + \beta f_i$, where $\alpha \geq 1$ and $\beta \geq \alpha$.

(iii) Obviously, the graph of the sequence of the blow ups σ_i is a chain. In particular,

$$K_{V^{(M)}} = \sigma^*_{M,0} K_V + \sum_{i=1}^{M} \sigma^*_{M,i} E^{(i)}$$

(where $\sigma_{i,j}$, as always, stands for the composition $\sigma_{j+1} \circ \ldots \circ \sigma_i$) and the canonical multiplicity of the valuation $\nu_{E^{(i)}}$ is equal to i. In the non-special case

$$\sigma^*_{M,0} F = F^{(M)} + \sum_{i=1}^{M} E^{(i,M)},$$

Proposition 2.9. *There exists a finite L-staircase of the length $M \geq 1$ such that for $i = 0, \ldots, M-1$ the centre of the valuation ν_E on $V^{(i)}$ is a point $x_i \in L_i$, $x_0 = x$, whereas the centre of the valuation ν_E on $V^{(M)}$ is either:*

A) a point $x_M \notin L_M$, $x_M \notin E^{(M-1,M)}$;

B) the line $B = \sigma_M^{-1}(x_{M-1})$, that is, a fiber of the ruled surface $E^{(M)}$;

C) the point $x_M = E^{(M-1,M)} \cap \sigma_M^{-1}(x_{M-1})$.

Proof. If the centre of the valuation ν_E is contained in $E^{(i)}$, then $i = a(E^{(i)}, V) \leq a(\nu_E, V)$. Therefore, there exists $M \geq 1$ such that the centre of the valuation ν_E on $V^{(M)}$ is not a point on L_M. The remaining part of the proof is an obvious listing of possible cases. Q.E.D. for the proposition.

Let us fix the constructed staircase of the length M.

3.3. End of the proof: the technique of counting multiplicities. We only exclude the case A): the cases B) and C) are excluded in a similar way, but the formulas are somewhat more complicated. See [3] for all details. Let $\Sigma^{(i)}$ be the strict transform of the system Σ on $V^{(i)}$. Set $\lambda_i = \mathrm{mult}_{L_{i-1}} \Sigma^{(i-1)}$, $n \geq \lambda_1 \geq \ldots$. The surface $E^{(M)} \subset V^{(M)}$ is for convenience denoted by the symbol E^*. Let

$$\begin{array}{cccc} \varphi_{i,i-1}: & V_i & \to & V_{i-1} \\ & \bigcup & & \bigcup \\ & E_i & \to & B_{i-1}, \end{array}$$

$i = 1, \ldots, K$, $V_0 = V^{(M)}$, be the resolution of the valuation ν_E, considered as a discrete valuation on the variety $V^{(M)}$. We introduce the following notations:

$$\nu_i = \mathrm{mult}_{B_{i-1}} \Sigma^{i-1}$$

is the multipplicity of the strict transform of the system Σ on V_{i-1} along the centre of the blow up;

$p_i = p(E_K, E_i)$ is the number of paths in the oriented graph of the valuation $\nu = \nu_{E_K}$ from E_K to E_i (here ν is considered as a discrete valuation on the variety $V_0 = V^{(M)}$!);

$N^* = \max\{i \,|\, 1 \leq i \leq K, B_{i-1} \subset E^{i-1}\}$;

$L = \max\{i \,|\, 1 \leq i \leq K, B_{i-1} -$ is a point$\}$ (so that for $j \leq L$ B_{j-1} is a point and for $j \geq L+1$ B_{j-1} is a curve); finally, set

$$N = \min\{N^*, L\}, \quad \Sigma_0 = \sum_{i=1}^{L} p_i, \quad \Sigma_1 = \sum_{i=L+1}^{K} p_i, \quad \Sigma^* = \sum_{i=1}^{N^*} p_i, \quad \Sigma_* = \sum_{i=1}^{N} p_i.$$

Obviously, in these notations we get $\nu_E(E^*) = \varepsilon = \Sigma^*$ and $\nu_E(F) = \varepsilon$. Furthermore,

$$\nu_E(\Sigma) = \nu_{E^*}(\Sigma)\nu_E(E^*) + \nu_E(\Sigma^{(M)}).$$

Now the Noether-Fano inequality takes the form

$$\sum_{i=1}^{K} p_i \nu_i = \varepsilon \sum_{i=1}^{M} (n - \lambda_i) + n \sum_{i=1}^{K} p_i \delta_i + e.$$

As always, let $D_i^{(M)}$, $i = 1, 2$, be the proper inverse images of general divisors from the system Σ. Let $Z^{(M)} = (D_1^{(M)} \circ D_2^{(M)})$ be the effective 1-cycle of their scheme-theoretic intersection. Set

$$m_i = \mathrm{mult}_{B_{i-1}} (Z^{(M)})^{i-1}$$

for $i \leq L$. In accordance with the technique of counting multiplicities, we obtain the estimate

$$\sum_{i=1}^{L} p_i m_i \geq \frac{(2\Sigma_0 n + \Sigma_1 n + \varepsilon \sum\limits_{i=1}^{M} (n - \lambda_i) + e)^2}{\Sigma_0 + \Sigma_1}.$$

Estimating the minimum of this quadratic form under the restrictions specified above and taking into account the Noether-Fano inequality, we get

$$\mathrm{mult}_B \, Z^{(M)} \geq \frac{\left(\Sigma n + \varepsilon \sum\limits_{i=1}^{M} (n - \lambda_i) + e \right)^2}{\sum\limits_{i=1}^{K} p_i^2}.$$

3.4. The cycle $Z^{(M)}$ in terms of the staircase. Now to complete the proof of our theorem we must get some estimates of the upper bounds of the left hand parts of the principal inequality, which was obtained above. The computations to be performed are rather tiresome. However, they are quite clear geometrically. Coming back to our main construction – that is, the staircase,– let us introduce some new terminology and notations, connected with the linear system Σ. First of all, set

$$z_i = (D_1^{(i)} \cdot D_2^{(i)}) \in A^2 V^{(i)}$$

to be the class of the effective 1-cycle $Z^{(i)} = (D_1^{(i)} \circ D_2^{(i)})$. On the "zeroth" step of our staircase we have the decomposition

$$Z = Z^v + Z^h.$$

Let us trace down the changes which the 1-cycle $Z^{(k)}$ undergoes when k comes from $i-1$ to i. Naturally, instead of the components of the cycle $Z^{(i-1)}$, which are different from L_{i-1}, their proper inverse images come into the cycle $Z^{(i)}$. Instead of the curve L_{i-1}, which is present in $Z^{(i-1)}$ with some multiplicity k_{i-1}, the cycle $Z^{(i)}$ contains an effective sub-cycle with the support in the exceptional divisor $E^{(i)}$. Let us break this sub-cycle into three parts:

1) $C_h^{(i)}$ includes all the curves, which are horizontal with respect to the morphism $\sigma_i \colon E^{(i)} \to L_{i-1}$, and different from the exceptional section L_i,

2) $C_v^{(i)}$ includes all the vertical curves, that is, the fibers of σ_i over points of the curve L_{i-1},

3) the exceptional section L_i with a certain multiplicity $k_i \in \mathbb{Z}_+$.

To make our notations uniform, set $C_h^{(0)}$ to be the part of the cycle Z^v, which includes all the curves different from L. Set also

$$d_{h,v}^{(i)} = \deg C_{h,v}^{(i)}$$

(see Remark 2.2, (ii) of the previous section). Now we get the following presentation of the cycles $Z^{(i)}$:

$$Z^{(0)} = Z^h + Z^v = Z^h + C_h^{(0)} + k_0 L,$$
$$Z^{(1)} = (Z^h)^{(1)} + C_h^{(0,1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1,$$
$$\dots,$$
$$Z^{(i)} = (Z^h)^{(i)} + C_h^{(0,1)} + C_h^{(1,2)} + C_v^{(1,2)} + \ldots +$$
$$+ C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i.$$

3.5. Computation of the class z_M and the end of the proof. Obviously, the class of the cycle $C_v^{(i)}$ in $A^1 V^{(i)}$ is equal to $d_v^{(i)} f_i$, and the class of the cycle $C_h^{(i)}$ is equal to $d_h^{(i)} s_i + \beta_i f_i$, where the coefficients satisfy the important inequality

$$\beta_i \geq d_h^{(i)}$$

(see Remark 2.2, (ii)). Furthermore, the class of the cycle $C_v^{(i,i+1)}$ is equal to $d_v^{(i)}(f_i - f_{i+1})$ and the class of the cycle $C_h^{(i,i+1)}$ is equal to

$$d_h^{(i)} s_i + \beta_i f_i - (\beta_i - d_h^{(i)}) f_{i+1}.$$

Setting $\alpha_i = \big((Z^h)^{(i-1)} \cdot L_{i-1}\big)$, we can write down $z_i^h = z_{i-1}^h - \alpha_i f_i$, where z_i^h is the numerical class of the horizontal cycle $(Z^h)^{(i)}$.

Lemma 2.11. *The following inequality is true:*

$$\alpha_i \leq \deg Z^h = 3n^2.$$

Proof. Since $L \subset F$, and $\deg Z^h$ is equal to $(Z^h \cdot F)$, this is obvious. Q.E.D.

Proposition 2.10. *The classes z_i satisfy the chain of relations*

$$z_i = z_{i-1} - (2\lambda_i n + \lambda_i^2) f_i - \lambda_i^2 s_i.$$

Proof. We have

$$z_i = (D^{(i)})^2 = (D^{(i-1)} - \lambda_i E^{(i)})^2 =$$

$$= z_{i-1} - 2\lambda_i (D^{(i-1)} \cdot L_{i-1}) f_i - \lambda_i^2 (s_i + f_i).$$

It follows from what was proved above, that for any $j \in \mathbb{Z}_+$ $(D^{(j)} \cdot L_j) = (D \cdot L) = n$. Q.E.D.

Proposition 2.11. *For $i \geq 2$ the integers k_i, α_i, β_i and $d_{h,v}^{(i)}$ satisfy the following system of relations:*

$$d_v^{(i)} + \beta_i = \alpha_i + d_v^{(i-1)} + (\beta_{i-1} - d_h^{(i-1)}) - 2\lambda_i n - \lambda_i^2.$$

For $i = 1$ we get

$$d_v^{(1)} + \beta_1 = \alpha_1 + (C_h^{(0)} \cdot L) - 2\lambda_1 n - \lambda_1^2.$$

Proof. To obtain this proposition, it is necessary to write out explicitly the class of the cycle $Z^{(i)}$ in terms of the parameters introduced above, and to use the previous proposition. The corresponding computations are elementary.

Proposition 2.12. *For any $i \geq 1$ we get the inequality*

$$d_v^{(i)} + \beta_i \leq (C_h^{(0)} \cdot L) + \sum_{j=1}^{i} (3n^2 - 2\lambda_j n - \lambda_j^2).$$

Proof. It is necessary to apply the inequality of the previous proposition i times and to use the last lemma. Q.E.D.

Finally, let us complete the exclusion of the case A). It is clear that among all the curves, lying on the divisor

$$\bigcup_{i=0}^{M} E^{(i,M)},$$

only those can possibly contain the point x_M, which lie entirely in $E^{(M)}$ and are different from the exceptional section L_M. Consequently, we are justified in writing down

$$Z^{(M)} = (Z^h)^{(M)} + C_v^{(M)} + C_h^{(M)} + \dots,$$

where the dots stand for the sum of all the curves, which do not contain the point x_M. Set

$$W=C_v^{(M)}+C_h^{(M)},\quad m_i^v=\mathrm{mult}_{B_{i-1}}W^{i-1},\quad m_i^h=\mathrm{mult}_{B_{i-1}}(Z^h)^{(M),i-1}$$

for $i\leq L$, so that $m_i=m_i^v+m_i^h$. Obviously, the multiplicities m_i^v vanish for $N+1\leq i\leq L$. Furthermore, $m_i^{h,v}\leq m_1^{h,v}$, and similarly to Lemma 2.11 we get $m_1^h\leq 3n^2$. Finally, $m_1^v\leq d_v^{(M)}+d_h^{(M)}\leq d_v^{(M)}+\beta_M$, so that, summing up our information, we get

$$3n^2\Sigma_0+\Sigma_*\left((C_h^{(0)}\cdot L)+\sum_{i=1}^{M}(3n^2-2\lambda_i n-\lambda_i^2)\right)\geq\sum_{i=1}^{L}p_i m_i^h+\sum_{i=1}^{N}p_i m_i^v\geq$$

$$\geq\frac{\left(2\Sigma_0 n+\Sigma_1 n+\varepsilon\sum\limits_{i=1}^{M}(n-\lambda_i)+e\right)^2}{\Sigma_0+\Sigma_1}.$$

Replacing Σ_* by $\varepsilon=\Sigma^*$, we make our inequality stronger, and replacing $\varepsilon(C_h^{(0)}\cdot L)$ by $4ne$, we get a strict inequality. Subtract the left-hand side from the right-hand one and look at the expression just obtained as a quadratic form in λ_i on the domain $0\leq\lambda_i\leq n$. By symmetry, its minimum is attained somewhere on the diagonal line, that is, at $\lambda_i=\lambda$, $0\leq\lambda\leq n$. Replace all the λ_i's by this value λ. Thus we get the strict inequality

$$\Phi<0,$$

where the expression Φ by means of elementary arithmetic can be transformed as follows:

$$\begin{gathered}\Phi=(\Sigma_0^2+\Sigma_0\Sigma_1+\Sigma_1^2)n^2+M\varepsilon\Sigma_0(n-\lambda)^2-\\ -M\varepsilon\Sigma_1(n-\lambda)(n+\lambda)+\\ +M^2\varepsilon^2(n-\lambda)^2-2e\Sigma_1 n+2M\varepsilon e(n-\lambda)+e^2.\end{gathered}$$

Since $\lambda\leq n$, we can replace $(n+\lambda)$ by $2n$, preserving the strict inequality. However, it is easy to check that the last expression is the sum of the complete square

$$(\Sigma_1 n-M\varepsilon(n-\lambda)-e)^2$$

and a few non-negative components. Thus it can not be negative. Our proof is complete. Q.E.D.

Chapter 3. Varieties with many rationally connected structures

§1. Fano direct products

In this section, following [33], we give a proof of Theorem 1.9 on birational superrigidity of Fano direct products and prove the divisorial canonicity of generic Fano hypersurfaces of index one.

1.1. Maximal singularities of movable linear systems. We prove Theorem 1.9 by induction on the number of factors K. When $K = 1$, the theorem holds in a trivial way: the condition (M) means that movable linear systems on the variety $F = F_1$ have no maximal singularities. This immediately implies birational superrigidity of the variety F.

Starting from this moment, $K \geq 2$.

Assume the converse: there is a moving linear system Σ on V such that the inequality $c_{\mathrm{virt}}(\Sigma) < c(\Sigma)$ holds. By the definition of the virtual threshold of canonical adjunction it means that there exists a sequence of blow ups $\varphi\colon \widetilde{V} \to V$ such that the inequality

$$c(\widetilde{\Sigma}) < c(\Sigma) \tag{38}$$

holds, where $\widetilde{\Sigma}$ is the strict transform of the linear system Σ on $\widetilde{V}$. To prove that the variety V is birationally superrigid, we must show that the inequality (38) is impossible, that is, to obtain a contradiction.

Let $H_i = -K_{F_i}$ be the positive generator of the group $\operatorname{Pic} F_i$. Set

$$S_i = \prod_{j \neq i} F_i,$$

so that $V \cong F_i \times S_i$. Let $\rho_i\colon V \to F_i$ and $\pi_i\colon V \to S_i$ be the projections onto the factors. Abusing our notations, we write H_i instead of $\rho_i^* H_i$, so that

$$\operatorname{Pic} V = \bigoplus_{i=1}^{K} \mathbb{Z} H_i \quad \text{and} \quad K_V = -H_1 - \ldots - H_K.$$

We get $\Sigma \subset |n_1 H_1 + \ldots + n_K H_K|$, whereas $c(\Sigma) = \min\{n_1, \ldots, n_K\}$. Without loss of generality we assume that $c(\Sigma) = n_1$. By the inequality (38) we get $n_1 \geq 1$. Set $n = n_1$, $\pi = \pi_1$, $F = F_1$, $S = S_1$. We get

$$\Sigma \subset |-nK_V + \pi^* Y|,$$

where $Y = \sum\limits_{i=2}^{K} (n_i - n) H_i$ is an effective class on the base S of the fiber space π.

Now we need to modify the birational morphism φ. For an arbitrary sequence of blow ups $\mu_S\colon S^+ \to S$ we set $V^+ = F \times S^+$ and obtain the following commutative diagram:

$$\begin{array}{ccc} V^+ & \stackrel{\mu}{\to} & V \\ \pi_+ \quad \downarrow & & \downarrow \quad \pi \\ S^+ & \stackrel{\mu_S}{\to} & S, \end{array} \tag{39}$$

where π_+ is the projection and $\mu = (\mathrm{id}_F, \mu_S)$. Let $E_1, \ldots, E_N \subset \widetilde{V}$ be all exceptional divisors of the morphism φ.

Proposition 3.1. *There exists a sequence of blow ups $\mu_S \colon S^+ \to S$ such that in the notations of the diagram (39) the centre of each discrete valuation E_i, $i = 1, \dots, N$, covers either S^+ or a divisor on S^+:*

$$\operatorname{codim}[\pi_+(\operatorname{centre}(E_i, V^+))] \leq 1.$$

Proof. Let $E \subset \widetilde{V}$ be the exceptional divisor of the birational morphism $\varphi \colon \widetilde{V} \to V$, $B = \varphi(E)$ the centre of the discrete valuation E on V. Assume that $\operatorname{codim}_S \pi(B) \geq 2$. Construct a sequence of commutative diagrams

$$\begin{array}{ccccc} & V_j & \stackrel{\varepsilon_j}{\to} & V_{j-1} & \\ \pi_j & \downarrow & & \downarrow & \pi_{j-1} \\ & S_j & \stackrel{\lambda_j}{\to} & S_{j-1} & , \end{array} \tag{40}$$

where $j = 1, \dots, l$, satisfying the following conditions:

1) $V_0 = V$, $S_0 = S$, $\pi_0 = \pi$;

2) $V_j = F \times S_j$, π_j is the projection onto the factor S_j, $\varepsilon_j = (\mathrm{id}_F, \lambda_j)$ for all $j \geq 1$;

3) λ_j is the blow up of the irreducible subvariety

$$B_{j-1} = \pi_{j-1}(\operatorname{centre}(E, V_{j-1})) \subset S_{j-1},$$

where $\operatorname{codim} B_{j-1} \geq 2$.

It is obvious that the properties 1)-3) determine the sequence of diagrams (40) in a unique way.

Lemma 3.1. *The following inequality holds: $l \leq a(E, V)$.*

Proof. Let $\Delta_j \subset V_j$ be the exceptional divisor of the morphism ε_j. By construction we get $\operatorname{centre}(E, V_j) \subset \Delta_j$, so that $\nu_E(\Delta_j) \geq 1$. Now we obtain

$$a(E, V) = a(E, V_l) + \sum_{j=1}^{l} \nu_E(\Delta_j) a(\Delta_j, V) \geq l.$$

Q.E.D. for the lemma.

Therefore the sequence of diagrams (40) terminates: we may assume that $\operatorname{centre}(E, V_l)$ covers a divisor on the base S_l. From this fact (by the Hironaka theorem on the resolution of singularities) Proposition 3.1 follows immediately.

Let Σ^+ be the strict transform of the linear system Σ on V^+. Now the arguments break into two parts due to the following fact.

Proposition 3.2. *The following alternative holds:*

(i) *either the inequality $c(\Sigma^+) < c(\Sigma)$ is true,*

(ii) *or for a general divisor $D^+ \in \Sigma^+$ the pair $(V^+, \frac{1}{n} D^+)$ is not canonical, and moreover, for some $i = 1, \dots, N$ the discrete valuation E_i determines a non-canonical singularity of this pair.*

Remark 3.1. The alternative of Proposition 3.2 should be understood in the "and/or" sense: *at least one* of the two possibilities (i), (ii) takes place (or both).

Proof of Proposition 3.2. Consider the diagram of maps (39). Let $\tau\colon V^\sharp \to V^+$ be the resolution of singularities of the composite map

$$V^+ \stackrel{\mu}{\to} V \stackrel{\varphi^{-1}}{\to} \widetilde{V}.$$

Set $\psi = \varphi^{-1}\circ\mu\circ\tau\colon V^\sharp \to \widetilde{V}$. There exist an open set $U \subset V^\sharp$ and a closed set of codimension two $Y \subset \widetilde{V}$ such that

$$\psi_U = \psi|_U\colon U \to \widetilde{V}\setminus Y$$

is an isomorphism. Obviously, if $E \subset V^\sharp$ is an exceptional divisor of the morphism τ and $E \cap U \neq \emptyset$, then $E\cap U = \psi_U^{-1}(E_i)$ for some exceptional divisor E_i of the morphism φ.

Let Σ^+ and $\Sigma^\sharp$ be the strict transforms of the linear system Σ on V^+ and $V^\sharp$, respectively, $\Sigma_U = \Sigma^\sharp|_U$. If $D^\sharp \in \Sigma^\sharp$ is a general divisor, then

$$\widetilde{D} = \psi_U(D_U^\sharp) \in \widetilde{\Sigma}$$

is a general divisor of the linear system $\widetilde{\Sigma}$ (we make no difference between $\widetilde{\Sigma}$ and its restriction onto $\widetilde{V}\setminus Y$, since the set Y is of codimension two). We know that

$$\widetilde{D} + nK_{\widetilde{V}} \notin A^1_+\widetilde{V},$$

see (38). Therefore,

$$D_U^\sharp + nK_U \notin A^1_+U. \tag{41}$$

Let $\mathcal{E}$ be the set of exceptional divisors of the morphism τ with a non-empty intersection with U. By (41) we get

$$\tau^*(D^+ + nK^+)|_U - \sum_{E\in\mathcal{E}}(\nu_E(D^+) - na^+(E))E_U \notin A^1_+U,$$

where K^+ is the canonical class of V^+, $a^+(E) = a(E, V^+)$. Consequently, either

$$D^+ + nK^+ \notin A^1_+V^+,$$

and we are in the case (i) of Proposition 3.2, or there exists an exceptional divisor $E \in \mathcal{E}$, satisfying the Noether-Fano inequality $\nu_E(D^+) > n\cdot a^+(E)$, that is, the discrete valuation E realizes a non-canonical singularity of the pair $(V^+, \frac{1}{n}D^+)$. In the latter case we get part (ii) of the alternative of Proposition 3.2, since $E \in \mathcal{E}$ and thus $E = E_i$ for some $i = 1,\dots,N$ (as discrete valuations). Q.E.D. for Proposition 3.2.

1.2. Reduction to the base of the fiber space. Assume that the case (i) of the alternative of Proposition 3.2 takes place, that is, $D^+ + nK^+ \notin A^1_+V^+$. Let $z \in F$ be a point of general position. Set

$$S_z^+ = \{z\}\times S^+,\ S_z = \{z\}\times S.$$

It is clear that $K_z^+ = K^+|_{S_z^+}$ and $K_z = K_V|_{S_z}$ are the canonical classes $K_S^+ = K_{S^+}$ and K_S, respectively. Let

$$\Sigma_z = \Sigma|_{S_z} \quad \text{and} \quad \Sigma_z^+ = \Sigma^+|_{S_z^+}$$

be the restriction of the linear systems Σ, Σ^+ onto S_z and S_z^+. Take general divisors $D_z \in \Sigma_z$ and $D_z^+ \in \Sigma_z^+$. We get a movable linear system Σ_z on the variety $S = F_2 \times \ldots \times F_K$. Moreover,

$$\Sigma_z \subset |n_2H_2 + \ldots + n_KH_K|,$$

so that $c(\Sigma_z) = \min\{n_2, \ldots, n_K\} \geq n = c(\Sigma)$.

Lemma 3.2. *The following estimate holds:* $D^+ + nK^+ = \pi_+^*(D_z^+ + nK_z^+)$.

Proof. Set $\mathcal{E}_S$ to be the set of exceptional divisors of the morphism μ_S. The exceptional divisors of the morphism μ are $F \times E = \pi_+^*E$ for $E \in \mathcal{E}_S$. We get

$$K_S^+ = \mu_S^*K_S + \sum_{E\in\mathcal{E}_S} a_EE \quad \text{and} \quad K^+ = \mu^*K_V + \pi_+^*(\sum_{E\in\mathcal{E}_S} a_EE),$$

where $a_E = a(E)$ is the discrepancy of the divisor E. For some numbers $b_E \geq 0$ we get

$$D^+ = \mu^*D - \sum_{E\in\mathcal{E}_S} b_E\pi_+^*E,$$

whereas for a point $z \in F$ of general position

$$D_z^+ = \mu_S^*D_z - \sum_{E\in\mathcal{E}_S} b_EE.$$

Now taking into account that $D + nK_V = \pi^*Y$ and $D_z + nK_z = D_z + nK_S = Y$, we obtain the claim of the lemma.

Corollary 2. $D_z^+ + nK_z^+ \notin A_+^1S^+$.

Proof. Indeed, it is clear that

$$\pi_+^*A_+^1S^+ \subset A_+^1V^+.$$

Q.E.D. for the corollary.

Thus for the strict transform Σ_z^+ of the linear system Σ_z on S^+ we get the inequality $c(\Sigma_z^+) < c(\Sigma_z)$. The more so, $c_{\text{virt}}(\Sigma_z) < c(\Sigma_z)$. Therefore the variety S is not birationally superrigid. This contradicts the induction hypothesis.

1.3. Reduction to the fiber of the fiber space. End of the proof. By Proposition 3.2 and what was said above, for a general divisor $D^+ \in \Sigma^+$ the pair $(V^+, \frac{1}{n}D^+)$ is not canonical, that is, there exists a birational morphism $V^\sharp \to V^+$ and an exceptional divisor $E \subset V^\sharp$, satisfying the Noether-Fano inequality $\nu_E(\Sigma^+) > n \cdot a_E^+$, where $a_E^+ = a(E, V^+)$. Moreover, we can assume that the centre $B = \text{centre}(E, V)$ of the valuation E covers a divisor on the base or the whole base: $\text{codim}_{S^+} T \leq 1$, where $T = \pi_+(B)$.

Let $t \in T$ be a point of general position. The fiber $F_t = \pi_+^{-1}(t)$ cannot lie entirely in the base set $\mathrm{Bs}\,\Sigma^+$ of the moving linear system Σ^+, since

$$\mathrm{codim}_{V^+}\,\pi_+^{-1}(T) \leq 1.$$

Therefore, $\Sigma_t^+ = \Sigma^+|_{F_t}$ is a non-empty linear system on F, $\Sigma_t^+ \subset |nH| = |-nK_F|$ (if $T \subset S^+$ is a divisor, then Σ_t^+ can have fixed components). Let $D_t^+ \in \Sigma_t^+$ be a general divisor. By inversion of adjunction (see Theorem 1.8 and §2 of this chapter), the pair

$$(F, \frac{1}{n} D_t^+)$$

is not log canonical. We get a contradiction with the condition (L). This contradiction completes the proof of birational superrigidity of the variety V.

1.4. The structures of a rationally connected fiber space and birational self-maps. Let us prove the remaining claims of Theorem 1.9. Let $\beta\colon V^\sharp \to S^\sharp$ be a rationally connected fiber space, $\chi\colon V - - \to V^\sharp$ a birational map. Take a very ample linear system $\Sigma_S^\sharp$ on the base $S^\sharp$ and let $\Sigma^\sharp = \beta^*\Sigma_S^\sharp$ be a movable linear system on $V^\sharp$, $c(\Sigma^\sharp) = 0$. Let Σ be the strict transform of the system $\Sigma^\sharp$ on V. By our remark, $c_{\mathrm{virt}}(\Sigma) = 0$, so that by what we proved above we conclude that $c(\Sigma) = 0$. Therefore, in the presentation

$$\Sigma \subset |-n_1 H_1 - \ldots - n_K H_K|$$

we can find a coefficient $n_e = 0$. We may assume that $e = 1$. Setting $S = F_2 \times \ldots \times F_K$ and $\pi\colon V \to S$ to be the projection, we get $\Sigma \subset |\pi^* Y|$ for a non-negative class Y on S. But this means that the birational map χ of the fiber space V/S onto the fiber space $V^\sharp/S^\sharp$ is fiber-wise: there exists a rational dominant map $\gamma\colon S \dashrightarrow S^\sharp$ making the diagram

$$\begin{array}{ccccc} & V & \overset{\chi}{\dashrightarrow} & V^\sharp & \\ \pi & \downarrow & & \downarrow & \beta \\ & S & \overset{\gamma}{\dashrightarrow} & S^\sharp & \end{array}$$

commutative. For a point $z \in S^\sharp$ of general position let $F_z^\sharp = \beta^{-1}(z)$ be the corresponding fiber, $F_z^\chi \subset V$ its strict transform with respect to χ. By assumption, the variety F_z^χ is rationally connected. On the other hand,

$$F_z^\chi = \pi^{-1}(\gamma^{-1}(z)) = F \times \gamma^{-1}(z),$$

where $F = F_1$ is the fiber of π. Therefore, the fiber $\gamma^{-1}(z)$ is also rationally connected.

Thus we have reduced the problem of description of rationally connected structures on V to the same problem for S. Now the claim (i) of Theorem 1.9 is easy to obtain by induction on the number of direct factors K. For $K = 1$ it is obvious that there are no non-trivial rationally connected structures. The second part of the claim (i) (about the structures of conic bundles and fibrations into rational surfaces) is obvious since $\dim F_i \geq 3$ for all $i = 1, \ldots, K$. Non-rationality of V is now obvious.

Let us prove the claim (ii) of Theorem 1.9. Set $RC(V)$ to be the set of all structures of a rationally connected fiber space with a non-trivial base on V. By the claim (i) we have

$$RC(V) = \{\pi_I: V \to F_I = \prod_{i\in I} F_i \,|\, \emptyset \neq I \subset \{1,\dots,K\}\}.$$

The set $RC(V)$ has a natural structure of an ordered set: $\alpha \leq \beta$ if β factors through α. Obviously, $\pi_I \leq \pi_J$ if and only if $J \subset I$. For $I = \{1\dots,K\} \setminus \{e\}$ set $\pi_I = \pi_e$, $F_I = S_e$. It is obvious that $\pi_1 \dots, \pi_K$ are the minimal elements of $RC(V)$.

Let $\chi \in \mathrm{Bir}\, V$ be a birational self-map. The map

$$\chi^*: RC(V) \to RC(V),$$

$$\chi^*: \alpha \longmapsto \alpha \circ \chi,$$

is a bijection preserving the relation $\leq$. From here it is easy to conclude that χ^* is of the form

$$\chi^*: \pi_I \longmapsto \pi_{I^\sigma},$$

where $\sigma \in S_K$ is a permutation of K elements and for $I = \{i_1,\dots,i_k\}$ we define $I^\sigma = \{\sigma(i_1),\dots,\sigma(i_k)\}$. Furthermore, for each $I \subset \{1,\dots,K\}$ we get the diagram

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V & \\ \pi_I & \downarrow & & \downarrow & \pi_{I^\sigma} \\ & F_I & \stackrel{\chi_I}{\dashrightarrow} & F_{I^\sigma}, & \end{array}$$

where χ_I is a birational map. In particular, χ induces birational isomorphisms $\chi_e: F_e \dashrightarrow F_{\sigma(e)}$, $e = 1,\dots,K$. However, all the varieties F_e are birationally superrigid, so that all the maps χ_e are biregular isomorphisms. Thus

$$\chi = (\chi_1,\dots,\chi_K) \in \mathrm{Bir}\, V$$

is a biregular isomorphism, too: $\chi \in \mathrm{Aut}\, V$. Q.E.D. for Theorem 1.9.

1.5. An example of varieties satisfying the condition of divisorial canonicity. Let $\mathbb{P} = \mathbb{P}^M$, $M \geq 3$, be the complex projective space. Set $\mathcal{F} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(M)))$ be the space of hypersurfaces of degrees M.

Theorem 3.1. *For $M \geq 6$ there exists a non-empty Zariski open subset $\mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}$ such that any hypersurface $F \in \mathcal{F}_{\mathrm{reg}}$ is non-singular and satisfies the condition (C).*

Proof. The set $\mathcal{F}_{\mathrm{reg}}$ is defined by explicit regularity conditions which we will now formulate. Let $F = F_M \subset \mathbb{P} = \mathbb{P}^M$ be a smooth Fano hypersurface. For a point $x \in F$ fix a system of affine coordinates $z_1,\dots,z_M$ with the origin at x and let

$$f = q_1 + q_2 + \ldots + q_M$$

be the equation of the hypersurface F, $q_i = q_i(z_*)$ are homogeneous polynomials of degree $\deg q_i = i$,

$$f_i = q_1 + \ldots + q_i$$

to be the left segments of the polynomial f, $i = 1, \ldots, M$.

The condition $(R1.1)$: the sequence

$$q_1, q_2, \ldots, q_{M-1}$$

is regular in $\mathcal{O}_{x,\mathbb{P}}$, that is, the system of equations

$$q_1 = q_2 = \ldots = q_{M-1} = 0$$

defines a one-dimensional subset, a finite set of lines in $\mathbb{P}$, passing through the point x. This is the standard regularity condition, which was used in [1, Chapter 3].

The condition $(R1.2)$: the linear span of any irreducible component of the closed algebraic set

$$q_1 = q_2 = q_3 = 0$$

in $\mathbb{C}^M$ is the hyperplane $q_1 = 0$ (that is, the tangent hyperplane $T_x F$).

$(R1.3)$ The closed algebraic set

$$\overline{\{f_1 = f_2 = 0\} \cap F} = \overline{\{q_1 = q_2 = 0\} \cap F} \subset \mathbb{P} \tag{42}$$

(the bar means the closure in $\mathbb{P}$) is irreducible and any section of this set by a hyperplane $P \ni x$ is

- either also irreducible and reduced,
- or breaks into two irreducible components $B_1 + B_2$, where $B_i = F \cap S_i$ is the section of F by a plane $S_i \subset \mathbb{P}$ of codimension 3, and moreover $\mathrm{mult}_x B_i = 3$,
- or is non-reduced and is of the form $2B$, where $B = F \cap S$ is the section of F by a plane S of codimension 3, and moreover $\mathrm{mult}_x B = 3$.

Set $\mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}$ to be the set of Fano hypersurfaces, satisfying the conditions $(R1.1 - R1.3)$ at every point (in particular, every hypersurface $F \in \mathcal{F}_{\mathrm{reg}}$ is smooth). It is clear that $\mathcal{F}_{\mathrm{reg}}$ is a Zariski open subset of the projective space $\mathcal{F}$. We have

Proposition 3.3. *For $M \geq 6$ the set $\mathcal{F}_{\mathrm{reg}}$ is non-empty.*

Proof is given in [33]. For $M \geq 8$ in the condition $(R1.3)$ we may require that the section of the set (42) by any hyperplane $P \ni x$ were irreducible and reduced, a general hypersurface satisfies this stronger condition. On the other hand, for $M = 4, 5$ it is easy to show that for any hypersurface $F \in \mathcal{F}$ there is a point where the conditions $(R1.2)$ and $(R1.3)$ are not satisfied.

Let us prove that the condition (C) is satisfied for a regular Fano hypersurface $F \in \mathcal{F}_{\mathrm{reg}}$.

Let $\Delta \in |nH|$ be an effective divisor, $n \geq 1$, where $H \in \operatorname{Pic} F$ is the class of a hyperplane section, $K_F = -H$. We have to show that the pair $(F, \frac{1}{n}\Delta)$ has canonical singularities.

Assume the converse. Then for a certain sequence of blow ups $\varphi: F^+ \to F$ and an exceptional divisor $E^+ \subset F^+$ the Noether-Fano inequality

$$\nu_{E^+}(\Delta) > n \cdot a(E^+) \tag{43}$$

is satisfied. For a fixed E^+ the inequality (43) is linear in Δ, so that without loss of generality we may assume that $\Delta \subset F$ is a prime divisor, that is, an irreducible subvariety of codimension 1. From (43) it follows easily that the centre $Y = \varphi(E^+)$ of the valuation E^+ on F satisfies the inequality $\operatorname{mult}_Y \Delta > n$. On the other hand, it is well known [1, Lemma 2.1], that for any irreducible curve $C \subset F$ the inequality $\operatorname{mult}_C \Delta \leq n$ holds. Thus $Y = x$ is a point. Let $\varepsilon: \widetilde{F} \to F$ be its blow up, $E \subset \widetilde{F}$ the exceptional divisor $E \cong \mathbb{P}^{M-2}$. By Proposition 3.6, which is proven below, for some hyperplane $B \subset E$ the inequality

$$\operatorname{mult}_x \Delta + \operatorname{mult}_B \widetilde{\Delta} > 2n, \tag{44}$$

holds, where $\widetilde{\Delta} \subset \widetilde{F}$ is the strict transform of the divisor Δ.

Let $\mathbb{T} = \overline{T_x F} \subset \mathbb{P}$ be the tangent hyperplane at the point x. The divisor E can be naturally identified with the projectivization $\mathbb{P}(T_x\mathbb{T}) = \mathbb{P}(T_x F)$. There is a unique hyperplane $\mathbb{B} \subset \mathbb{T}$, $x \in \mathbb{B}$, such that $B = \mathbb{P}(T_x\mathbb{B})$ with respect to the above-mentioned identification. Let $\Lambda_{\mathbb{B}}$ be the pencil of hyperplanes in $\mathbb{P}$, containing $\mathbb{B}$, and $\Lambda_B = \Lambda_{\mathbb{B}}|_F \subset |H|$ its restriction onto F. Consider a general divisor $R \in \Lambda_B$. It is a hypersurface of degree M in $\mathbb{P}^{M-1}$, smooth at the point x. Let $\widetilde{R} \subset \widetilde{F}$ be the strict transform of the divisor R. Obviously,

$$\widetilde{R} \cap E = B.$$

Set $\Delta_R = \Delta|_R = \Delta \cap R$. It is an effective divisor on the hypersurface R.

Lemma 3.3. *The following estimate holds:*

$$\operatorname{mult}_x \Delta_R > 2n. \tag{45}$$

Proof. We have $(\widetilde{\Delta} \circ \widetilde{R}) = \widetilde{\Delta}_R + Z$, where Z is an effective divisor on E. According to the elementary rules of the intersection theory [44], $\operatorname{mult}_x \Delta_R = \operatorname{mult}_x \Delta + \deg Z$, since $\operatorname{mult}_x R = 1$. However, Z contains B with multiplicity at least $\operatorname{mult}_B \widetilde{\Delta}$. Therefore, the inequality (44) implies the estimate (45). Q.E.D. for the lemma.

Lemma 3.4. *The divisor $T_R = T_x R \cap R$ on the hypersurface R is irreducible and has multiplicity exactly 2 at the point x.*

Proof. The irreducibility is obvious (for instance, for $M \geq 6$ one can apply the Lefschetz theorem). By the condition $(R1.2)$ the quadric $\{q_2|_E = 0\}$ does not contain a hyperplane in E as a component, in particular, it does not contain the hyperplane $B \subset E$. Thus the quadratic component of the equation of the divisor T_R, that is, the polynomial $q_2|_B$, is non-zero. Q.E.D. for the lemma.

Let us continue our proof of Theorem 3.1. By Lemmas 3.3 and 3.4 we can write

$$\Delta_R = aT_R + \Delta_R^\sharp,$$

where $a \in \mathbb{Z}_+$ and the effective divisor $\Delta_R^\sharp \in |n^\sharp H_R|$ on the hypersurface R satisfies the estimate $\mathrm{mult}_x \Delta_R^\sharp > 2n^\sharp$. Moreover, $\Delta_R^\sharp$ does not contain the divisor T_R as a component. Without loss of generality we can assume the divisor $\Delta_R^\sharp$ to be irreducible and reduced.

Now consider the second hypertangent system [1, Chapter 3]

$$\Lambda_2^R = |s_0 f_2 + s_1 f_1|_R,$$

where s_i are homogeneous polynomials of degree i in the linear coordinates z_*. Its base set

$$S_R = \{q_1|_R = q_2|_R = 0\}$$

is by condition $(R1.3)$ of codimension 2 in R and either irreducible and of multiplicity 6 at the point x, or breaks into two plane sections of R, each of multiplicity 3 at the point x. In any case, for a general divisor $D \in \Lambda_2^R$ we get $\Delta_R^\sharp \not\subset \mathrm{Supp}\, D$, so that the following effective cycle of codimension two on R,

$$\Delta_D = (D \circ \Delta_R^\sharp),$$

is well defined. Since $\mathrm{mult}_x D = 3$ and $\Lambda_2^R \subset |2H_R|$, the cycle Δ_D satisfies the estimate

$$\frac{\mathrm{mult}_x}{\deg} \Delta_D > \frac{3}{M}. \tag{46}$$

We can replace the cycle Δ_D by its suitable irreducible component and thus assume it to be an irreducible subvariety of codimension 2 in R. Comparing the estimate (46) with the description of the set S_R given above, we see that $\Delta_D \not\subset S_R$. This implies that $\Delta_D \not\subset T_R$. Indeed, if this were not true, we would have got

$$f_1|_{\Delta_D} \equiv q_1|_{\Delta_D} \equiv 0. \tag{47}$$

However, $\Delta_D \subset D$, so that for some $s_0 \neq 0$, $s_1 \neq 0$ (the divisor D is chosen to be general) we have

$$(s_0 f_2 + s_1 f_1)|_{\Delta_D} \equiv 0.$$

By (47) this implies that $f_2|_{\Delta_D} \equiv (q_1 + q_2)|_{\Delta_D} \equiv 0$ (since $s_0 \neq 0$ is just a constant), so that $\Delta_D \subset S_R$. A contradiction.

Thus $\Delta_D \not\subset T_R$. Therefore the effective cycle $\Delta^+ = (\Delta_D \circ T_R)$ is well defined. It satisfies the estimate

$$\frac{\mathrm{mult}_x}{\deg} \Delta^+ > \frac{6}{M}. \tag{48}$$

The effective cycle Δ^+ as a cycle on F is of codimension 4. Now recall the following fact ([45] or [1, Chapter 3]): if the Fano hypersurface F at the point x satisfies the regularity condition ($R1.1$), then for any effective cycle Y of pure codimension $l \leq M-2$ the inequality

$$\frac{\mathrm{mult}_x}{\deg} Y \leq \frac{l+2}{M}$$

holds. Therefore, the inequality (48) for an effective cycle of codimension 4 is impossible.

The proof of Theorem 3.1 is complete.

§2. The connectedness principle and its applications

In this section, we formulate the connectedness principle of Shokurov and Kollár and consider its geometric applications.

2.1. The connectedness principle. Inversion of adjunction. Let X, Z be normal varieties or analytic spaces and $h\colon X \to Z$ a proper morphism with connected fibers and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X.

Theorem 3.2 (the connectedness principle, [6, Theorem 17.4]). *Assume that D is effective ($d_i \geq 0$) and the class $-(K_X + D)$ is h-numerically effective and h-big. Let*

$$f\colon Y \xrightarrow{h} X \xrightarrow{h} Z$$

be a resolution of singularities of the pair (X, D). Set

$$K_Y = g^*(K_X + D) + \sum e_i E_i.$$

The support of the $\mathbb{Q}$-divisor $\sum\limits_{e_i \leq -1} e_i E_i$, that is, the closed algebraic set

$$\bigcup_{e_i \leq -1} E_i,$$

is connected in a neighborhood of any fiber of the morphism f.

Proof see in [41, Chapter 17]. It has been also reproduced in the survey [15] and in [46] for a particular case (in which the arguments follow the same scheme as in [41]).

The connectedness principle has numerous applications, which we will now consider. The first application is Theorem 1.8 (inversion of adjunction).

Proof of Theorem 1.8. We use the notations of Theorem 1.8. Let $D = \sum_{i \in I} d_i D_i$ be an effective $\mathbb{Q}$-divisor, $d_i \in \mathbb{Q}_+$ for all $i \in I$. Since the pair (X, D) is canonical outside the point x, we get the inequality $d_i \leq 1$ for all $i \in I$. Replacing D by $\frac{1}{1+\varepsilon} D$ for a small $\varepsilon \in \mathbb{Q}_+$, we may assume that $d_i < 1$ for all $i \in I$.

Let $\varphi\colon \widetilde{X} \to X$ be a resolution of singularities of the pair $(X, D + R)$. Write down

$$K_{\widetilde{X}} = \varphi^*(K_X + D + R) + \sum_{j \in J} e_j E_j - \sum_{i \in I} d_i \widetilde{D}_i - \widetilde{R}, \tag{49}$$

where E_j, $j \in J$, are all exceptional divisors of the morphism φ, $\widetilde{D}_i$ and $\widetilde{R}$ are the strict transforms of the divisors D_i, R on $\widetilde{X}$, respectively. Set

$$b_j = \mathrm{ord}_{E_j} \varphi^* D, \quad a_j = a(E_j, X),$$

$j \in J$. In these notations for $j \in J$ we get $e_j = a_j - b_j - r_j$, where $r_j = \mathrm{ord}_{E_j} \varphi^* R$. Obviously,

$$\varphi^{-1}(x) = \bigcup_{j \in J^+} E_j$$

for some subset $J^+ \subset J$. Recall that by assumption R is a Cartier divisor, containing the point x, which implies (it is a key point) that for $j \in J^+$

$$r_j = \mathrm{ord}_{E_j} \varphi^* R \geq 1.$$

Furthermore, by assumption the pair (X, D) is not canonical, but canonical outside the point x. Therefore, among the indices $j \in J^+$ there is an index k such that $a_k < b_k$. For this index we have $e_k < -1$.

Now by the connectedness principle we get: there is an index $l \in J$, such that $e_l < -1$ and

$$E_l \cap \widetilde{R} \neq \emptyset.$$

Now from (49) by the adjunction formula we get

$$K_{\widetilde{R}} = (K_{\widetilde{X}} + \widetilde{R})|_{\widetilde{R}} = \varphi_R^*(K_R + D_R) + (\sum_{j \in J} e_j E_j|_{\widetilde{R}} - \sum_{i \in I} d_i \widetilde{D}_i|_{\widetilde{R}}),$$

where $\varphi_R = \varphi|_{\widetilde{R}} \colon \widetilde{R} \to R$ is the restriction of the sequence of blow ups φ onto R. By what was said, in the last bracket there is at least one prime divisor of the form $E_l|_{\widetilde{R}}$, where $l \in J^+$, the coefficient at which is strictly less than -1. Q.E.D. for Theorem 1.8.

The following version of the inversion of adjunction is useful.

Proposition 3.4. *Let $x \in X$ be a germ of a smooth variety, D an effective $\mathbb{Q}$-divisor, the pair (X, D) is not canonical, but canonical outside the point x, that is, the point x is an isolated centre of non canonical singularities of the pair. Let $R \ni x$ be a non-singular divisor where $T_x R$ is a hyperplane of general position in $T_x X$. Then the pair (R, D_R) is not log canonical, but canonical outside the point x.*

Proof. In the notations of the proof of Theorem 1.8, the index k, which realizes a non log canonical singularity of the pair (R, D_R), lies in J^+: by the assumption of general position, the divisor R does not contain any centres of singularities of the pair (X, D) outside the point x. Q.E.D. for the proposition.

Here is one more application of the connectedness principle.

Proposition 3.5. *Let $x \in X$ be a germ of a smooth variety, D an effective $\mathbb{Q}$-divisor, the pair (X, D) is not canonical at the point x, but canonical outside that point. Let $\lambda \colon X^+ \to X$ be the blow up of the point x, $E = \lambda^{-1}(x) \subset X^+$*

the exceptional divisor, D^+ and R^+ the strict transforms of the divisors D and R, respectively. Furthermore, let $\mu\colon \widetilde{X} \to X^+$ be a resolution of singularities of the pair $(X^+, D^+ + R^+)$,

$$\varphi = \lambda \circ \mu\colon \widetilde{X} \to X$$

the composite map. Now write down

$$K_{\widetilde{X}} = \varphi^*(K_X + D + R) + \sum_{j\in J} e_j E_j - \sum_{i\in I} d_i \widetilde{D}_i - \widetilde{R}, \tag{50}$$

where E_j, $j \in J$, are all exceptional divisors of the morphism φ, $\widetilde{D}_i$ and $\widetilde{R}$ are the strict transforms of the divisors D_i and R on $\widetilde{X}$, respectively.

Then the following alternative takes place:

(1) either $\operatorname{mult}_x D > \dim X$,

(2) or the set

$$\mu\Big(\bigcup_{b_j > a_j+1} E_j\Big) \subset E$$

is connected.

Proof. By the assumptions the claim follows immediately from the connectedness principle.

2.2. Further applications of the connectedness principle. First of all, let us show the following useful fact.

Proposition 3.6. *Assume that the pair (X, D) is the same as in Proposition 3.5, $\lambda\colon X^+ \to X$ is the blow up of the point x, $E = \lambda^{-1}(x) \subset X^+$ is the exceptional divisor, D^+ the strict transform of the divisor D. Then the following alternative takes place:*

1) either $\operatorname{mult}_x D > 2$,

2) or there is a hyperplane $B \subset E$, which is uniquely determined by the pair (X, D) such that the inequality

$$\operatorname{mult}_x D + \operatorname{mult}_B D^+ > 2. \tag{51}$$

holds.

Proof. Canonicity is stronger than log canonicity. Therefore one can apply inversion of adjunction (in the form of Proposition 3.4) several times, subsequently restricting the pair (X, D) onto smooth subvarieties

$$R_1 \supset R_2 \supset \ldots \supset R_k,$$

where $R_1 \subset X$ is a smooth divisor, $R_{i+1} \subset R_i$ is a smooth divisor, $x \in R_k$ and $R_k \not\subset \operatorname{Supp} D$. All the pairs

$$(R_i, D|_{R_i})$$

are not log canonical at the point x. Thus Proposition 3.4 holds for a generic smooth germ $R \ni x$ of codimension $k \leq \dim X - 1$. In particular, it holds for a general surface $S \ni x$. (This fact was for the first time used by Corti [1] in order to obtain an alternative proof of the $4n^2$-inequality, see also [34, Proposition 1.5].) Thus the pair

$$(S, D_S = D|_S)$$

has at the point x an isolated (for a general S) non log canonical singularity. Let us consider the two-dimensional case more closely. Let $x \in S$ be a germ of a smooth surface, $C \subset S$ a germ of an effective (possibly reducible) curve, $x \in C$. Consider a sequence of blow ups

$$\varphi_{i,i-1} \colon S_i \to S_{i-1},$$

$S_0 = S$, $i = 1, \ldots, N$, $\varphi_{i,i-1}$ blows up a point $x_{i-1} \in S_{i-1}$, $E_i = \varphi_{i,i-1}^{-1}(x_{i-1}) \subset S_i$ is the exceptional line. For $i > j$ set

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1} \colon S_i \to S_j,$$

$\varphi = \varphi_{N,0}$, $\widetilde{S} = S_N$. We assume that the points x_i lie one over another, that is, $x_i \in E_i$, and that $x_0 = x$, so that all the points x_i, $i \geq 1$, lie over x:

$$\varphi_{i,0}(x_i) = x \in S.$$

Let Γ be the graph with the vertices $1, \ldots, N$ and oriented edges (arrows) $i \to j$, that connect the vertices i and j if and only if $i > j$ and

$$x_{i-1} \in E_j^{i-1},$$

where for a curve $Y \subset S_j$ its strict transform on S_a, $a \geq j$, is denoted by the symbol Y^a. Assume that the point x is the centre of an isolated non log canonical singularity of the pair $(S, \frac{1}{n}C)$ for some $n \geq 1$. This means that for some exceptional divisor $E \subset \widetilde{S}$ the log Noether-Fano inequality

$$\nu_E(C) = \mathrm{ord}_E\, \varphi^* C > n(a_E + 1), \tag{52}$$

holds, where a_E is the discrepancy of E. Without loss of generality we may assume that $E = E_N$ is the last exceptional divisor.

As usual, for $i > j$ let the symbol p_{ij} denote the number of paths in the graph Γ from the vertex i to the vertex j, for $i < j$ set $p_{ij} = 0$, as always $p_{ii} = 1$. In terms of the numbers p_{ij} the log Noether-Fano inequality (52) takes the traditional form

$$\sum_{i=1}^{N} p_{Ni}\mu_i > n\Big(\sum_{i=1}^{N} p_{Ni} + 1\Big), \tag{53}$$

where $\mu_i = \mathrm{mult}_{x_{i-1}} C^{i-1}$.

Proposition 3.7. *Either $\mu_1 > 2n$ (that is, the first exceptional divisor $E_1 \subset S_1$ already gives a non log canonical singularity of the pair $(S, (1/n)C)$), or $N \geq 2$ and the following inequality holds:*

$$\mu_1 + \mu_2 > 2n.$$

Proof. If $N=1$, then $\mu_1>2n$ by means of log Noether-Fano inequality. Assume that $\mu_1\le 2n$, then $N\ge 2$. Obviously, $\mu_1>n$. If $\mu_2\ge n$, then $\mu_1+\mu_2>2n$, as we claim. So assume that $\mu_2<n$. Then for each $i\in\{2,\dots,N\}$ we have $\mu_i\le\mu_2<n$ (since the point x_{i-1} lies over x_1). Therefore from the inequality (53) we get

$$p_{N1}(\mu_1-n)+\sum_{i=2}^{N}p_{Ni}(\mu_2-n)>n.$$

However,

$$p_{N1}=\sum_{j\to 1}p_{Nj}\le\sum_{i=2}^{N}p_{Ni},$$

so that the more so

$$\sum_{i=2}^{N}p_{Ni}(\mu_1+\mu_2-2n)>n.$$

Therefore $\mu_1+\mu_2>2n$. Q.E.D. for the proposition.

Let us now complete the proof of Proposition 3.6. Consider a general surface $S\ni x$. The pair (S,D_S) is not log canonical, but log canonical outside the point x. By Proposition 3.7, either $\operatorname{mult}_x D_S>2$, but in this case $\operatorname{mult}_x D>2$, so that the first of the two cases of Proposition 3.6 takes place, or the pair (S^+,D_S^+) (that is, the strict transform of the pair (S,D_S) on X^+) is not log canonical, but log canonical outside some proper closed connected subset

$$Z_S\subset E_S=E\cap S^+\cong\mathbb{P}^1.$$

Obviously, Z_S is a point $y_S\in E_S$. Since the surface S is general, there is a hyperplane $B\subset E$ such that

$$y_S=B\cap S^+.$$

By Proposition 3.7, the inequality

$$\operatorname{mult}_x D_S+\operatorname{mult}_{y_S}D_S^+>2$$

holds. This immediately implies the inequality (51) and Proposition 3.6.

2.3. Isolated hypersurface singularities. As one more application of the connectedness principle, consider a germ $x\in X$ of an isolated terminal singularity with the following properties. Let

$$\varphi\colon X^+\to X$$

be the blow up of the point x, $E=\varphi^{-1}(x)$ the exceptional divisor, which is irreducible and reduced. The varieties X, X^+ and E have $\mathbb{Q}$-factorial terminal singularities. Let $\delta=a(E,X)$ be the discrepancy of E, D an effective $\mathbb{Q}$-divisor on X, D^+ its strict transform on X^+. Define the number $\nu_E(D)$ by the formula

$$\varphi^*D=D^++\nu_E(D)E.$$

Proposition 3.8. *Assume that the pair (X,D) is not canonical at the point x, which is an isolated centre of a non-canonical singularity of this pair. Assume also that for some integer $k\geq 1$ the inequality*

$$\nu_E(D)+k\leq\delta \tag{54}$$

holds. Then the pair (X^+,D^+) is not log canonical and there is a non log canonical singularity $\widetilde{E}\subset\widetilde{X}$ of that pair (where $\widetilde{X}\to X^+$ is some model), the centre of which

$$\mathrm{centre}(\widetilde{E},X^+)\subset E$$

is of dimension $\geq k$.

Proof. Assuming $X\subset\mathbb{P}^N$ to be projectively embedded, consider a *generic* linear subspace $P\subset\mathbb{P}^N$ of codimension k, containing the point x. Let Λ_P be the linear system of hyperplanes, containing P, and Λ be the corresponding linear system of sections of the variety X. Let $\varepsilon>0$ be a sufficiently small rational number of the form $\frac{1}{K}$ and

$$\{H_I\,|\,i\in I\}\subset\Lambda$$

a set of $\sharp I=Kk$ generic divisors. Set

$$R=D+\sum_{i\in I}\varepsilon H_i,$$

and let R^+ be the strict transform of R on X^+.

Obviously, the pair (X^+,D^+) is not log canonical. The centre of any of its non log canonical singularities is contained in E. Furthermore, being non log canonical is an open property, so that, slightly decreasing the coeffients in D, we may assume that the strict version of the inequality (54) holds, that is, $\nu_E(D)+k<\delta$ (whereas other assumptions still hold).

Now consider the pair (X^+,R^+) (we still assume that $X\ni x$ is a germ, so that all constructions are local in a neighborhood of the point x). It is non log canonical, and all its non log canonical singularities are non log canonical singularities of the pair (X^+,D^+), with the exception of one additional singularity, the germ $(P\cap X)^+$ of the section of X by the plane P, that is, the base set of the system Λ. By the strict version of the inequality (54), the class $-(K_{X^+}+R^+)$ is obviously φ-nef and φ-big, so that, applying the connectedness principle (to $X=X^+$, $Z=X$, $h=\varphi$, $D=R^+$), we conclude that the union of the centres of non log canonical singularities of the pair (X^+,R^+) on X^+ is connected. Since P is generic, this is only possible if $(P\cap X)^+$ intersects some centre of a non log canonical singularity of the pair (X^+,D^+), which should be of dimension at least k. Q.E.D. for the proposition.

The fact which we have just proven will be applied to our case of a hypersurface singularity $x\in X$ with a smooth exceptional divisor.

1.2. Singularities of pairs on a smooth hypersurface. Let $X\subset\mathbb{P}^N$ be a smooth hypersurface of degree $m\in\{2,\dots,N-1\}$, $D\in|\,lH_X\,|$ an effective divisor, which is cut out on X by a hypersurface of degree $l \geq 1$. (So that H_X is the class of a hyperplane section of X.) The following fact and its proof are well known [47,48].

Proposition 3.9. *For any $n \geq l$ the pair $(X, \frac{1}{n}D)$ is log canonical.*

Proof. We may consider the case $n = l$. Assume the converse: the pair $(X, \frac{1}{n}D)$ is not log canonical. Since for any curve $C \subset X$ the inequality $\mathrm{mult}_C D \leq n$ holds (see [1, Lemma 2.1]), the centre of a non log canonical singularity of the pair $(X, \frac{1}{n}D)$ can only be a point. Let $x \in X$ be such a point. Consider now a general projection $\pi\colon \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$. Its restriction onto X is a finite morphism $\pi_X\colon X \to \mathbb{P}^{N-1}$ of degree m, which is an analytic isomorphism at the point x, and one may assume that

$$\pi_X^{-1}(\pi_X(x)) \cap \mathrm{Supp}\, D = \{x\}.$$

This implies that the germ of the pair $(X, \frac{1}{n}D)$ at the point x and the germ of the pair $(\mathbb{P}^{N-1}, \frac{1}{n}\pi(D))$ at the point $\pi(x)$ are analytically isomorphic. In particular, the point $\pi(x)$ is an isolated centre of a non log canonical singularity of the pair $(\mathbb{P}^{N-1}, \frac{1}{n}\pi(D))$. However, this is impossible.

Being non log canonical is an open property, so that for a rational number $s < n^{-1}$, sufficiently close to n^{-1}, the pair

$$(\mathbb{P}^{N-1}, s\pi(D))$$

still has the point $\pi(x)$ as an isolated centre of a non log canonical singularity. Let $P \subset \mathbb{P}^{N-1}$ be a hyperplane, not containing the point $\pi(x)$. By the inequality $smn + 1 < N$ the $\mathbb{Q}$-divisor $-(K_{\mathbb{P}^{N-1}} + s\pi(D) + P)$ is ample, so that one may apply to the pair

$$(\mathbb{P}^{N-1}, s\pi(D) + P)$$

the connectedness principle of Shokurov and Kollár (in the notations of Theorem 3.2, $X = \mathbb{P}^{N-1}$, Z is a point, for the $\mathbb{Q}$-divisor D we take $s\pi(D) + P$, the conditions of Theorem 3.2 are satisfied in a trivial way by what was said above) and obtain a contradiction: the point $\pi(x)$ is an isolated centre of a non log canonical singularity and the divisor P comes into the $\mathbb{Q}$-divisor $s\pi(D) + P$ with the coefficient one, however $\pi(x) \notin P$, so that the connectedness is violated. Q.E.D. for Proposition 3.9.

2.5. The weak local inequality for an isolated hypersurface singularity. Let $x \in X$ be a germ of isolated hypersurface terminal singularity. More precisely, if $\varphi\colon X^+ \to X$ is the blow up of the point x, $\varphi^{-1}(x) = E \subset X^+$ is the exceptional divisor, we assume that X^+ and E are smooth, whereas E is isomorphic to a smooth hypersurface of degree $\mu = \mathrm{mult}_o V$ in $\mathbb{P}^M$.

Furthermore, let $D \ni x$ be a germ of a prime divisor, $D^+ \subset X^+$ its strict transform, $D^+ \sim -\nu E$ for $\nu \in \mathbb{Z}_+$, so that the equality

$$\mathrm{mult}_o D = \mu\nu$$

holds.

Proposition 3.10. *Assume that the pair* $(X, \frac{1}{n}D)$ *is not canonical at the point* x*, which is an isolated centre of a non-canonical singularity of that pair. Then the inequality*

$$\nu > n \tag{55}$$

holds.

Proof. Assume the converse: $\nu \leq n$. Then the pair $(X^+, \frac{1}{n}D^+)$ is not canonical, and moreover, the centre of any non-canonical singularity of this pair (that is, of any maximal singularity of the divisor D^+) is contained in the exceptional divisor E. By the inversion of adjunction the pair $(E, \frac{1}{n}D^+_E)$, where $D^+_E = D^+ \,|\, _E$, is not log canonical. Let $H_E = -E \,|\, _E$ be the generator of the Picard group $\operatorname{Pic} E$, that is, the hyperplane section of E with respect to the embedding $E \subset \mathbb{P}^M$. We get

$$D^+_E \sim -\nu E \,|\, _E = \nu H_E.$$

Since $\nu \leq n$, the non log canonicity of the pair $(E, \frac{1}{n}D^+_E)$ contradicts to Proposition 3.9. Q.E.D.

As one more application of the connectedness principle, consider the following local situation. Let $x \in X$ be a germ of a quadratic singularity, $\dim X \geq 3$. Let us blow up the point x:

$$\lambda \colon X^+ \to X,$$

and denote by the symbol E the exceptional divisor $\lambda^{-1}(x)$, which we consider as a quadric hypersurface $E \subset \mathbb{P}^{\dim X}$. Let, furthermore, D be an effective $\mathbb{Q}$-Cartier divisor on the variety X, D^+ its strict transform on X^+. Assuming the exceptional quadric E to be irreducible, define the number $\beta \in \mathbb{Q}_+$ by the relation

$$D^+ \sim \lambda^* D - \beta E.$$

Proposition 3.11. *Assume that the rank of the quadric hypersurface* E *is at least 4 and the pair* (X, D) *has the point* x *as an isolated centre of a non canonical singularity, that is, it is non canonical, but canonical outside the point* x*. Then the following inequality holds:* $\beta > 1$.

Proof. If $\dim X = 3$, then by assumption the point $x \in X$ is a non-degenerate quadratic singularity, and this fact is well known [21]. (If $\beta \leq 1$, then the pair (X^+, D^+) is non canonical, so that by inversion of adjunction the pair (E, D^+_E) is not log canonical, but $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and D^+_E is an effective curve of bidegree (β, β), which is impossible [20].) If $\dim X \geq 4$, then, restricting D onto a generic hyperplane section $Y \ni x$ of the variety X with respect to some embedding $X \hookrightarrow \mathbb{P}^N$, and repeating this procedure $\dim X - 3$ times, we reduce the problem (by inversion of adjunction) to the already considered case $\dim X = 3$. Proof of the proposition is complete.

§3. The double spaces of index two

In this section we sketch the proof of Theorem 1.10 on the double spaces of index two. This brevity of exposition comes, firstly, from the fact that these varieties are not birationally rigid (in the sense of any of the definitions of birational rigidity and superrigidity that are used today) and for that reason are outside the framework of the present survey (which is on birationally rigid varieties), and, secondly, from the physical size of the proof (see [35]), so that it is not possible to give it here. We just describe the key steps of the proof.

Theorem 1.10 is based on some claim on the virtual threshold of canonical adjunction of a movable linear system on V. For an arbitrary linear subspace $P \subset \mathbb{P}$ of codimension two let V_P be the blow up of the subvariety $\sigma^{-1}(P) \subset V$ (it is irreducible because the variety V is assumed to be generic). For a movable linear system Σ on V the symbol Σ_P stands for its strict transform on V_P.

Theorem 1.10 is implied by the following technical fact.

Theorem 3.3. *Assume that for a movable linear system Σ the inequality*

$$c_{\rm virt}(\Sigma) < c(\Sigma, V) \tag{56}$$

holds. Then there exists a uniquely determined linear subspace $P \subset \mathbb{P}$ of codimension two, satisfying the inequality

$$\mathrm{mult}_{\sigma^{-1}(P)} \Sigma > c(\Sigma, V),$$

whereas for the strict transform Σ_P the equality

$$c_{\rm virt}(\Sigma) = c_{\rm virt}(\Sigma_P) = c(\Sigma_P, V_P)$$

holds.

Let us obtain Theorem 1.10 from Theorem 3.3. Let us fix a movable linear system Σ, satisfying the inequality (56). Taking, if necessary, a symmetric power of Σ, we may assume that $\Sigma \subset |2nH| = |-nK_V|$, where $n \geq 1$ is a positive integer. The system Σ (and the integer n) are fixed throughout the proof. Obviously, $c(\Sigma, V) = n$. According to Theorem 3.3, there exist a (unique) linear subspace $P \subset \mathbb{P}$ of codimension two, satisfying the estimate $\mathrm{mult}_R \Sigma > n$, where $R = \sigma^{-1}(P)$ is an irreducible variety. Singularities of the varieties R are ar most zero-dimensional. Let $\varphi: V^+ \to V$ be the blow up of the (possibly singular) subvariety $R = \sigma^{-1}(P)$, $E = \varphi^{-1}(R)$ the exceptional divisor.

Lemma 3.5. (i) *The variety V^+ is factorial and has at most finitely many isolated double points (not necessarily non-degenerate).*

(ii) *The linear projection $\pi_{\mathbb{P}}: \mathbb{P} \dashrightarrow \mathbb{P}^1$ from the plane P generates the regular projection*

$$\pi = \pi_{\mathbb{P}} \circ \sigma \circ \varphi: V^+ \to \mathbb{P}^1,$$

the general fiber of which $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$ is a non-singular Fano variety of index one, and finitely many fibers have isolated double points.

(iii) *The following equalities hold:*

$$\mathrm{Pic}\, V^+ = \mathbb{Z}H \oplus \mathbb{Z}E = \mathbb{Z}K^+ \oplus \mathbb{Z}F,$$

where $H = \varphi^* H$ for simplicity of notations, $K^+ = K_{V^+}$ is the canonical class of the variety V^+, F is the class of a fiber of the projection π, whereas

$$K^+ = -2H + E,\ F = H - E.$$

Proof. These claims follow directly from the definition of the blow up φ, the genericity of the variety V and the well known fact that an isolated hypersurface singularity of a variety of dimension ≥ 4 is factorial (see [51]).

Let Σ^+ be the strict transform of the system Σ on the blow up V^+ of the subvariety R. By Theorem 3.3, the following equality holds:

$$c_{\mathrm{virt}}(\Sigma^+) = c(\Sigma^+, V^+). \tag{57}$$

From this fact we immediately get

Proposition 3.12. *Assume that $c_{\mathrm{virt}}(\Sigma^+) = 0$. Then the system Σ^+ is composed from the pencil $|H - R|$, that is, $\Sigma^+ \subset |2nF|$.*

Proof of the proposition. Assume the converse:

$$\Sigma^+ \subset |-mK^+ + lF|,$$

where $m \geq 1$. By the part (iii) of Lemma 3.5,

$$m = 2n - \nu,\ l = 2\nu - 2n \geq 2,$$

so that for the threshold of canonical adjunction we get $c(\Sigma^+, V^+) = m$. Since $c_{\mathrm{virt}}(\Sigma^+) = 0$, by Theorem 3.3 we get $m = 0$, which is what we claimed. Q.E.D. for the proposition.

Proof of Theorem 1.10. For the linear system Σ we take the strict transform with respect to χ of any linear system of the form $\lambda^*\Lambda$, where Λ is a movable system on the base S. Applying Theorem 3.3 and Proposition 3.12, we complete the proof.

From the arguments above, one can see that the proof of Theorem 1.10 is based on the two key claims:

1) the existence (and uniqueness) of the maximal subvariety of the form $\sigma^{-1}(P)$, where $P \subset \mathbb{P}$ is a linear subspace of codimension two and

2) on the equality (57) of the thresholds of canonical adjunction.

In the proof of the claim 1) of crucial importance is the following local fact, which is known as the $8n^2$-inequality.

Let $o \in X$ be a germ of a smooth variety of dimension $\dim X \geq 4$. Let Σ be a movable linear system on X, and the effective cycle $Z = (D_1 \circ D_2)$, where $D_1, D_2 \in \Sigma$ are generic divisors, its self-intersection. Blow up the point o:

$$\varphi\colon X^+ \to X,$$

$E = \varphi^{-1}(o) \cong \mathbb{P}^{\dim X - 1}$ is the exceptional divisor. The strict transform of the system Σ and the cycle Z on X^+ we denote by the symbols Σ^+ and Z^+, respectively.

Proposition 3.13 ($8n^2$**-inequality).** *Assume that the pair* $(X, \frac{1}{n}\Sigma)$ *is not canonical, but canonical outside the point* o*, where* n *is some positive number. There exists a linear subspace* $P \subset E$ *of codimension two (with respect to* E*), such that the inequality*

$$\mathrm{mult}_o Z + \mathrm{mult}_P Z^+ > 8n^2$$

holds.

An equivalent claim, but formulated in a rather cumbersome way, was several times published by Cheltsov [50-52], however his proof is essentially faulty (see [53]). For a complete proof, see [53].

Now let us sketch a proof of the claim (57), that is, the claim 2).

Assume that the inequality

$$c_{\mathrm{virt}}(\Sigma^+) < c(\Sigma^+, V^+) = m$$

holds. Then the pair $(V^+, \frac{1}{m}\Sigma^+)$ is not canonical, so that the linear system Σ^+ has a maximal singularity, that is, for some birational morphism $\psi: \widetilde{V} \to V^+$ and irreducible exceptional divisor $E^+ \subset \widetilde{V}$ the Noether-Fano inequality holds:

$$\nu_E(\Sigma^+) > ma(E^+, V^+).$$

Lemma 3.6. *The centre of maximal singularity* E^+ *is contained in some fiber* $F_t = \pi^{-1}(t)$*, that is,* $B = \pi \circ \psi(E^+) = t \in \mathbb{P}^1$*.*

Proof. Assume the converse: $\pi \circ \psi(E^+) = \mathbb{P}^1$. Retsricting the linear system Σ^+ onto the fiber of general position $F = F_s$, we get that the pair

$$(F, \frac{1}{m}\Sigma_F)$$

is not canonical, where $\Sigma_F \subset |-mK_F|$. However, F is a smooth double space of index one and it is well known [54], that this is impossible. Q.E.D. for the lemma.

For simplicity of notations, let $F = F_t$ be the fiber, containing the centre of singularity E^+.

Proposition 3.14. *The centre* B *is a singular point of the fiber* F*.*

Proof. See [35].

Now let $\lambda: V^\sharp \to V^+$ be the blow up of the point o, $E^\sharp = \lambda^{-1}(o) \subset V^\sharp$ the exceptional divisor, which can be seen as a quadratic hypersurface in $\mathbb{P}^M$. One can show [35] that for $M \geq 6$ we may assume that for a generic hypersurface $W \subset \mathbb{P}$, arbitrary plane $P \subset \mathbb{P}$ of codimension two and any singularity $o \in V^+$ the quadric $E^\sharp$ is of rank at least 4. Define the integer $\beta \in \mathbb{Z}_+$ by the formula

$$D^\sharp \sim \lambda^* D - \beta E^\sharp,$$

where $D \in \Sigma^+$ is a generic divisor, $D^\sharp$ its strict transform on $V^\sharp$. To the generic variety V we may apply Proposition 3.11, which gives the inequality

$$\beta > m.$$

Furthermore, the divisor $\lambda_F^* D_F - \beta E_F^\sharp$ on the strict transform $F^\sharp \subset V^\sharp$ is effective (the symbols λ_F and $E_F^\sharp$ stand for the blow up of the point $o \in F$ and for the exceptional divisor $\lambda_F^{-1}(o)$, respectively). This implies the inequality

$$\mathrm{mult}_o D_F \geq 2\beta > 2m,$$

which is impossible (the H-degree of the divisor D_F as an effective algebraic cycle is $2m$). This proves the coincidence of the thresholds (57) for $M \geq 6$.

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