The Hecke algebra on the cusp forms of $\Gamma_0(p_0)$ with nebentypus

by

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Next we study the dependance of a() and b(,) on $\mathcal{L}$. We assume given to ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on G, such that the associated bundles $\tilde{L}_1$ and $\tilde{L}_2$ descend to $\mathcal{M}_1$ resp. $\mathcal{M}_2$ on A. If $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$, then the associated $\tilde{L} = \tilde{L}_1 \otimes \tilde{L}_2$ descends to $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$. Our construction gives quadratic functions a, a_1 and a_2 on Y, and bilinear b, b_1 and b_2 on Y×X. We assume that $b_1 = b_2$. Later we shall see that this is always true, but for the moment note that this holds at least for $\mathcal{L}_2 = [-1]^*(\mathcal{L}_1)$.

6.1. <u>Lemma</u> Under these hypotheses we have $a = a_1 \cdot a_2$, $b = b_1 = b_2$. In invariant formulation $\iota = \iota_1 = \iota_2$, and the Y-action on $\tilde{L}_\eta \approx \tilde{L}_{1,\eta} \otimes \tilde{L}_{2,\eta}$ is the product of the actions on $\tilde{L}_{1,\eta}$ and $\tilde{L}_{2,\eta}$.

We may assume that the various trivialisations s and t are compatible. For the proof note first that $\phi = \phi_1 + \phi_2$, and that the isomorphisms U_y are compatible with the identification $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$. Now choose non zero sections $s_1 \in \Gamma(G, \mathcal{L}_1)$ and $s_2 \in \Gamma(G, \mathcal{L}_2)$, and let $s = s_1 \otimes s_2 \in \Gamma(G, \mathcal{L})$. Then $\sigma^L{}_\chi(s) = \sum_{\alpha+\beta=\chi} \sigma^{L_1}{}_\alpha(s_1) \otimes \sigma^{L_2}{}_\beta(s_2)$. We look what happens to both sides if we replace χ by $\chi + \phi(y)$: In the sum we can change α to $\alpha + \phi_1(y)$ and β to $\beta + \phi_2(y)$, which (up to a suitable U_y) changes each summand by a factor $a_1(y) \cdot a_2(y) \cdot b_1(y,\alpha) \cdot b_2(y,\beta) = a_1(y) \cdot a_2(y) \cdot b_1(y,\chi)$. Similarly on the left we obtain a factor $a(y) \cdot b(y,\chi)$, and if $\sigma_\chi(s) \neq 0$ it follows that these two factors must be equal. It first follows that b(y,) and b_1(y,) coincide on the subgroup $\tilde{X}$ of X generated by the differences $\{\chi_1 - \chi_2\}$, for elements $\chi \in X$ with $\sigma_\chi(s_1 \otimes s_2) \neq 0$ for some choice of s_1 and s_2. This subgroup contains $\phi(Y)$, and we conclude that $a = a_1 \cdot a_2$.
To go on we first remark that b does not depend on the choice of $\mathcal{M}$ (with $\pi^*(\mathcal{M}) \approx \tilde{L}$), as two such choices differ by a character of T, which amounts to shifting the labeling of the $\mathcal{M}_\chi$ by this character. The same holds if we replace $\mathcal{L}$ by a translate $T_g^*(\mathcal{L})$, for an element $g \in G(R)$, as $\tilde{L}$ and $\mathcal{M}$ are replaced by their corresponding translates. Now consider the set of pairs $(g,h) \in G(R) \times G(R)$ such that $\lambda_1(g) + \lambda_2(h) = 0$ (in G'(R)). For any such pair $T_g^*(\mathcal{L}_1) \otimes T_h^*(\mathcal{L}_2) \approx \mathcal{L}_1 \otimes \mathcal{L}_2 \approx \mathcal{L}$, and for a suitable choice of (g,h) we have $\tilde{X} = X$: This follows from the fact that the pairs (g,h) as above are Zariski-dense in the connected component of the identity of the kernel of $\lambda_1 + \lambda_2$: $G_\eta \times G_\eta \to G_\eta$, and that $\Gamma(G_\eta, \mathcal{L}_\eta)$ is spanned by the images of $\Gamma(G_\eta, T_g^*(\mathcal{L}_1)) \otimes \Gamma(G_\eta, T_h^*(\mathcal{L}_2))$ for elements (g,h) in the identity-component of that kernel (chapter I, prop. 5.3).
All in all we now conclude that $b = b_1$, and the lemma has been shown.

For any $n \geq 1$ we derive a description of $\Gamma(G_\eta, \mathcal{L}_\eta^{\otimes n})$ in terms of Fourier-expansions. We apply this as follows: Suppose that for two pairs

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1. Introduction.

Let p_0 be a prime, $p_0 > 3$ and $\Gamma_0(p_0)$, $\Gamma_1(p_0)$, as usual, the congruence subgroups of $\Gamma = PSL_2(\mathbb{Z})$.

$$\Gamma_0(p_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod p_0 \right\}$$

$$\Gamma_1(p_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p_0) \,\middle|\, d \equiv 1 \bmod p_0 \right\}$$

Denote

$$\Delta = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, gcd(a,b,c,d) = 1, det(r) \not\equiv 0 \bmod p_0 \right\}$$

$$\Delta_0 = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \,\middle|\, c \equiv 0 \bmod p_0 \right\}$$

$$\Delta_1 = \left\{ r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0 \,\middle|\, d \equiv 1 \bmod p_0 \right\}$$

Let η be a Dirichlet character mod p_0. We define a character η of Δ_0 by

$$\eta(r) = \eta(d) \qquad \text{for } r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0$$

Let $S_k(\Gamma_0(p_0), \eta)$ be the cusp forms of $\Gamma_0(p_0)$ with the nebentypus η and $S_k(\Gamma_1(p_0))$ the cusp forms with respect to $\Gamma_1(p_0)$. It is well known that

$$S_k(\Gamma_1(p_0)) = \bigoplus_{\eta} S_k(\Gamma_0(p_0), \eta).$$

In [Wa] we have investigated the operation of the Hecke algebra on the cusp forms $S_k(\Gamma_0(p_0), (\frac{\cdot}{p_0}))$, where $(\frac{\cdot}{p_0})$ is the Legendre-symbol. Applying the Eichler-Shimura isomorphism the study of the Hecke algebra on the cusp forms is equivalent to that on the cohomology, see [Hi] or [Wa] for more details and background. By using the Shapiro-Lemma to the cohomology groups we have got a basis of the cohomology. The operation of the Hecke algebra on this basis can be determined explicitly. The characteristic polynomials of T_l on $S_k(\Gamma_0(p_0), (\frac{\cdot}{p_0}))$ for small l, p_0, k are calculeted in [Wa] . In the present paper we generalize the results in [Wa] and study the Hecke algebra on the cusp forms $S_k(\Gamma_0(p_0), \eta)$ for any Dirichlet character η mod p_0. With a little modification the basis obtained in [Wa] can also be used to compute the operation of the Hecke algebra on $S_k(\Gamma_0(p_0), \eta)$.

The characteristic polynomials of T_2, T_3, T_5, T_7 and T_{11} are computed for small p_0, n and all Dirichlet characters η.

2. The dimension of the cusp forms $S_k(\Gamma_0(p_0),\eta)$.

We recall first some results in [Wa] . Denote for $n > 0$

$$M_n = \left\{ \sum_{i=0}^{n} a_i x^i y^{n-i} \;\middle|\; a_i \in \mathbb{Q} \right\}$$

with a Δ-operation via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^i y^{n-i} = (ax+cy)^i (bx+dy)^{n-i}$$

Let $\eta : \Gamma_0(p_0)/\Gamma_1(p_0) \cong (\mathbb{Z}/p_0)^* \to \mathbb{C}^*$ be a Dirichlet character mod p_0. Since $(\mathbb{Z}/p_0\mathbb{Z})^*$ is a cyclic group, there is a primitive root ω with $(\mathbb{Z}/p_0\mathbb{Z})^* =< \omega >$. In particular, it implies $\omega^{\frac{p_0-1}{2}} = -1$. We assume throughout this paper $\eta(-1) = (-1)^k$, otherwise the cups forms $S_k(\Gamma_0(p_0),\eta) = \{0\}$. Denote $z := \eta(\omega)$, one has then $z^{\frac{p_0-1}{2}} = (-1)^k$. We extend η to Δ_0 such that η acts trivial on Δ_1, i.e. η is a character from Δ_0/Δ_1 to $\mathbb{C}^*$. Denote by $\mathbb{Q}_\eta$ the $\mathbb{Q}[\eta]$ -module of rank 1 with a Δ_0-operation given by

$$s_0.1 = \eta(s_0)\cdot 1 \qquad \forall s_0 \in \Delta_0,$$

where $\mathbb{Q}[\eta]$ is the ring generated by $\mathbb{Q}$ and the values of η. $M_{n,\eta} = M_n \otimes \mathbb{Q}_\eta$ is then a Δ_0-module. We consider now the cohomology group $H^*(\Gamma_0(p_0), M_{n,\eta})$. The Eichler-Shimura isomorphism says that the following sequence

$$0 \to S_{n+2}(\Gamma_0(p_0),\eta) \oplus \overline{S_{n+2}(\Gamma_0(p_0),\eta)} \to H^1(\Gamma_0(p_0), M_{n,\eta}\otimes\mathbb{C}) \to$$

$$\to \bigoplus_{s \text{ a cusp}} H^1(\Gamma_0(p_0)_s, M_{n,\eta}\otimes\mathbb{C}) \to 0$$

is exact, where s is a cusp with respect to $\Gamma_0(p_0)$ and $\Gamma_0(p_0)_s := \{ r \in \Gamma_0(p_0) \,|\, r.s = s \} =< T_s >$ is a cyclic infinite group. It is well known that $\Gamma_0(p_0)$ has two cusps 0, ∞. The dimension of $H^1(\Gamma_0(p_0)_s, M_{n,\eta}\otimes\mathbb{C}) \cong M_{n,\eta}/(1-T_s)M_{n,\eta}$ is 1. In particular, we obtain

$$dim(H^1(\Gamma_0(p_0), M_{n,\eta}\otimes\mathbb{C})) = 2dim(S_{n+2}(\Gamma_0(p_0)(p_0),\eta)) + 2.$$

Denote by $W_{n,\eta}$ the induced module of $M_{n,\eta}$ on $\Gamma = PSL_2(\mathbb{Z})$.

$$W_{n,\eta} = Ind^{\Gamma}_{\Gamma_0(p_0)} M_{n,\eta} = \{ f : \Gamma \to M_{n,\eta} \,|\, f(r_0 r) = r_0.f(r),\ r_0 \in \Gamma_0(p_0) \}.$$

The operation of Γ on $W_{n,\eta}$ is defined by

$$(a.f)(r) := f(ra), \qquad a, r \in \Gamma$$

We extend now this operation to an operation of Δ. For $a \in \Delta$, $r \in \Gamma$, there exist $a' \in \Delta_0$, $r' \in \Gamma$, such that $ra = a'r'$. We define then

$$(a.f)(r) := a'.f(r').$$

It is obvious that this definition coincides with the above definition. By the Shapiro-Lemma(cf. [Br] or [AS]) there is a canonical isomorphism between

$$H^1(\Gamma_0(p_0), M_{n,\eta}) \cong H^1(\Gamma, W_{n,\eta})$$

as modules under the Hecke algebra.

We consider first the structur of Γ-module $W_{n,\eta}$. Let

$$a_i = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix}, \quad i = 0, 1, ..., p_0 - 1, \quad a_{p_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\{a_i\}$ is then a set of representatives of Γ with respect to $\Gamma_0(p_0)$:

$$\Gamma = \bigcup_{i=0}^{p_0} \Gamma_0(p_0) a_i$$

An element $f \in W_{n,\eta}$ is then determined by the values $f(a_0), f(a_1), ..., f(a_{p_0})$ by using the condition $f(r_0 r) = r_0 f(r)$. The dimension of $W_{n,\eta}$ is

$$(p_0 + 1) \cdot dim(M_{n,\eta}) = (p_0 + 1)(n + 1)$$

In other words, $W_{n,\eta}$ is generated by the elements $(w_0, w_1, ..., w_{p_0})$ with $w_i \in M_{n,\eta}$. The structur of cohomology $H^1(\Gamma, W_{n,\eta})$ is well known (cf. [Wa]):

$$H^1(\Gamma, W_{n,\eta}) \cong W_{n,\eta}/(W_{n,\eta}^S + W_{n,\eta}^Q)$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $T = SQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $W_{n,\eta}^S := \{ w \in W_{n,\eta} \mid S.w = w \}$. The dimension of the cohomology is

$$dim(H^1(\Gamma, W_{n,\eta})) = dim(W_{n,\eta}) - dim(W_{n,\eta}^S) - dim(W_{n,\eta}^Q).$$

Over $\mathbb{Q}$ we have always a decomposition $W_{n,\eta} = W_{n,\eta}^S \oplus W_{n,\eta}^Q \oplus V$ for some V. Since the group Γ is generated by S, Q with the relations $S^2 = 1$, $Q^3 = 1$ (cf. [Se]), the cohomology

$$\begin{aligned} H^1(\Gamma, W_{n,\eta}) &= \frac{\{ (\phi(S), \phi(Q)) \mid \phi(S) \in (1-S)W_{n,\eta}, \, \phi(Q) \in (1-Q)W_{n,\eta} \}}{\{ ((1-S)u, (1-Q)u) \mid u \in W_{n,\eta} \}} \\ &\cong \frac{\{ \phi(Q) \mid \phi(S) = 0, \, \phi(Q) \in (1-Q)W_{n,\eta} \}}{\{ (1-Q)u \mid u \in W_{n,\eta}^S \}} \\ &\cong \{ (1-Q)v \mid v \in V \} \end{aligned}$$

i.e., every class $\phi \in H^1(\Gamma, W_{n,\eta})$ has the form

$$\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1-Q)u, \; u \in V \end{cases}$$

We begin with the description of $W^S_{n,\eta}$. It is easy to show that

$$\begin{cases} a_0 S = a_{p_0} \\ a_i S = S_i a_j \quad i \cdot j \equiv -1 \bmod p_0, \quad S_i = \begin{pmatrix} -j & -1 \\ 1+ij & i \end{pmatrix} \in \Gamma_0(p_0) \\ a_{p_0} S = a_0 \end{cases}$$

and by the definition we obtain

$$\begin{cases} (S.f)(a_0) = f(a_{p_0}) \\ (S.f)(a_i) = S_i.f(a_j),\ i = 1, ..., p_0 - 1 \\ (S.f)(a_{p_0}) = f(a_0) \end{cases}$$

In particular, $W^S_{n,\eta}$ has the expression:

$$W^S_{n,\eta} = \{ (w_0, ..., w_{p_0}) \in W_{n,\eta} \mid w_0 = w_{p_0},\ w_i = S_i.w_j \}$$

Similarly we can show that

$$W^T_{n,\eta} = \left\{ (w_0, ..., w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid w_0 = w_1 = \cdots = w_{p_0-1} = \begin{pmatrix} 1 & 0 \\ -p_0 & 1 \end{pmatrix} w_0, w_{p_0} = T w_{p_0} \right\}$$

$$W^Q_{n,\eta} = \{ (w_0, ..., w_{p_0}) \in M_{n,\eta} \times \cdots \times M_{n,\eta} \mid T w_{p_0} = w_0, w_1 = w_{p_0}, S_i w_{j+1} = w_i \}$$

Now we compute the dimensions of $W^S_{n,\eta}, W^Q_{n,\eta}$.

Let ν_2 (resp. ν_3) the number of $\Gamma_0(p_0)$-inequivalent elliptic points of the order 2 (resp. 3).

$$\nu_2 = 0 \text{ or } 2 \equiv p_0 + 1 \bmod 4$$

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It is obvious that

$$\nu_2 = 2 \iff p_0 \equiv 1 \bmod 4 \iff \text{ there is a } i_0 \text{ with } i_0^2 \equiv -1 \bmod p_0$$

In that case $i_0 = \omega^{\frac{p_0-1}{4}}$ and $\eta(i_0) = z^{\frac{p_0-1}{4}}$. Similarly it is easy to show that

$$\nu_3 = 2 \iff p \equiv 1 \bmod 3 \iff i_0^3 \equiv -1 \bmod p_0$$

It follows then $i_0 = \omega^{\frac{p_0-1}{6}}$ and $\eta(i_0) = z^{\frac{p_0-1}{6}}$.

We show now

Lemma: *We have*

$$dim(W^S_{n,\eta}) = 2[\frac{p_0+1}{4}]\,(n+1) + 2d_S$$

$$dim(W^Q_{n,\eta}) = [\frac{p_0+1}{3}]\,(n+1) + 2d_Q$$

where

$$d_S = \begin{cases} 0 & p_0 \equiv 3 \ mod\ 4 \\ dim(Ker(1-S_{i_0})) & p_0 \equiv 1 \ mod\ 4 \end{cases}$$

$$d_Q = \begin{cases} 0 & p_0 \equiv 2 \ mod\ 3 \\ dim(Ker(1-S_{i_0})) & p_0 \equiv 1 \ mod\ 3 \end{cases}$$

In particular,

$$dim(H^1(\Gamma, W_{n,\eta})) = (p_0 + 1 - 2[\frac{p_0+1}{4}] \ - [\frac{p_0+1}{3}]\ )(n+1) - 2d_S - 2d_Q$$

$$dim(S_{n+2}(\Gamma_0(p_0), \eta)) = \frac{1}{2}(p_0 + 1 - 2[\frac{p_0+1}{4}] \ - [\frac{p_0+1}{3}]\ )(n+1) - d_S - d_Q - 1$$

Proof: For $f = (w_0, ..., w_{p_0}) \in W^S_{n,\eta}$ we have $w_i = S_i w_j$ and $S_j = S_i^{-1}$. If $j \neq i$, then w_j is determined by w_i . The number of such pair (i,j) is $2[\frac{p_0+1}{4}]$. If $j = i$, that means $p_0 \equiv 1$ mod 4 , one has $w_i \in Ker(1-S_i)$. Since there are two i with $i^2 \equiv -1 \mod p_0$, the dimension of $W^S_{n,\eta}$ has the expression

$$dim(W^S_{n,\eta}) = 2[\frac{p_0+1}{4}]\ (n+1) + 2d_S.$$

The other cases can be proved in the same manner.

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In the case that $p_0 \equiv 1$ mod 4, there is a i_0 with $i_0^2 \equiv -1 \mod p_0$ and

$$S_{i_0} = \begin{pmatrix} -i_0 & -1 \\ 1+i_0^2 & i_0 \end{pmatrix} = P \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix} P^{-1}$$

for some regular matrix P, where ζ_4 is the primivite root of the order 4, i.e. $\zeta_4 = e^{\frac{2\pi i}{4}}$. Let $m \otimes 1 \in M_{n,\eta}$, then $S_{i_0}.(m \otimes 1) = \eta(i_0)(S_{i_0} m \otimes 1)$. Therefore the dimension of the $Ker(1-S_{i_0})$ in $M_{n,\eta}$ is equal to the dimension of the $Ker(1-\eta(i_0)\begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix})$ in M_n. Since $\eta(i_0)\begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix} x^j y^{n-j} = \eta(i_0)\zeta_4^{2j-n} x^j y^{n-j}$, we have

$$d_S = \sharp \left\{ j \mid 0 \leq j \leq n,\ \eta(i_0)\zeta_4^{2j-n} = 1 \right\}$$

we know that $\eta(i_0) = z^{\frac{p_0-1}{4}}$ and $z^{\frac{p_0-1}{2}} = (-1)^n$, i.e. $\eta(i_0)^2 = (-1)^n$. If n is odd, one has $\eta(i_0)^2 = -1$ and $\eta(i_0) = \zeta_4$ or ζ_4^3. It follows then

$$d_S = \sharp \left\{ j \mid 0 \leq j \leq n,\ 1 + 2j - n = 0 \mod 4 \right\} = \frac{n+1}{2}$$

Similarly we can determine the dimensions for other cases.

Lemma: *Let ω a primitive root in $(\mathbb{Z}/p_0\mathbb{Z})^*$, $\eta(\omega) = z$. Then*

1. If $p_0 \equiv 1 \mod 4$, one has

$$d_S = \begin{cases} 2[\frac{n}{4}] + 1 & n \text{ even }, z^{\frac{p_0-1}{4}} = 1 \\ 2[\frac{n+2}{4}] & n \text{ even }, z^{\frac{p_0-1}{4}} = -1 \\ \frac{n+1}{2} & n \text{ odd }, z^{\frac{p_0-1}{2}} = -1 \end{cases}$$

2. If $p_0 \equiv 1 \mod 3$, one has

$$d_Q = \begin{cases} 2[\frac{n}{6}] + 1 & n \text{ even }, z^{\frac{p_0-1}{6}} = 1 \\ 2[\frac{n+2}{6}] & n \text{ even }, z^{\frac{p_0-1}{3}} + z^{\frac{p_0-1}{6}} + 1 = 0 \\ 2[\frac{n+3}{6}] & n \text{ odd }, z^{\frac{p_0-1}{6}} = -1 \\ 2[\frac{n+1}{6}] & n \text{ odd }, z^{\frac{p_0-1}{3}} - z^{\frac{p_0-1}{6}} + 1 = 0 \end{cases}$$

3. The basis set.

Defining by α_i (resp. β_i) the permutation of $\{0, 1, ..., p_0\}$ induced by the operation of S (resp. Q) on $\{a_0, a_1, ..., a_{p_0}\}$. We have (cf. §2)

$$\alpha_i \cdot i \equiv -1 \mod p_0, \quad 0 < i < p_0$$

$$\beta_i = \alpha_i + 1 \quad 1 < i < p_0$$

Definition: *For $i, j, k \in \{1, 2, ..., p_0 - 1\}$*

a. The pair (i, j) is called a α-pair if $j = \alpha_i, i = \alpha_j$, or equivalently, $i \cdot j \equiv -1 \mod p_0$;

b. The triple (i, j, k) is called a β-triple if $j = \beta_i, k = \beta_j, i = \beta_k$, or equivalently, $i \cdot j \cdot k \equiv -1 \mod p_0$;

c. Let B a subset of $\{1, 2, ..., p_0 - 1\}$. We define by $< B >$ the subset of $\{1, 2, ..., p_0 - 1\}$ determined by the following conditions:

i. $B \subset < B >$;

ii. if (i, j) is an α-pair and $j \in < B >$ then $i \in < B >$;

iii. if (i, j, k) is a β-triple and $j, k \in < B >$ then $i \in < B >$;

d. A subset B of $\{1, 2, ..., p_0 - 1\}$ is called a basis set if it satisfies:

i. $< B > = \{1, 2, ..., p_0 - 1\}$;

ii. $\forall i \in B$, $< B \setminus \{i\} > \neq \{1, 2, ..., p_0 - 1\}$.

The number of the α-pair is $2[\frac{p_0+1}{4}] - 1$ and the number of the β-triple is $[\frac{p_0+1}{3}] - 1$. Therefore

$$\sharp B = (p_0 - 1) - (2[\frac{p_0 + 1}{4}] - 1) - ([\frac{p_0 + 1}{3}] - 1) = p_0 + 1 - 2[\frac{p_0 + 1}{4}] - [\frac{p_0 + 1}{3}]$$

The important result in [Wa] is

Lemma: *There is a basis set B with the property: if $a \in B$ then $p_0 - a \in B$.*

Example: $p_0 = 13$. In that case $\nu_2 = 2$, $\nu_3 = 2$. The permutationen of $\{a_0, a_1, ..., a_{p_0}\}$ induced by the operation of S and Q are:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
S	13	12	6	4	3	5	2	11	8	10	9	7	1	0
Q	13	0	7	5	4	6	3	12	9	11	10	8	2	1

We can take the basis set $B = \{5, 8, 4, 9\}$.

4. The basis of $H^1(\Gamma, W_{n,\eta})_\pm$.

Let $\Gamma_\infty =< T >$ be the stabilizer of the cusp ∞ in Γ. We have an exact sequence:

$$0 \to H^0(\Gamma_\infty, W_{n,\eta}) \to H^1_c(\Gamma, W_{n,\eta}) \to H^1(\Gamma, W_{n,\eta}) \to H^1(\Gamma_\infty, W_{n,\eta}) \to \cdots$$

where $H^1_c(\cdot,\cdot)$ is the cohomology with the compact support, referring to [Ha] Chap.1 for details and background. Let $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We define for $\phi \in Z^1(\Gamma, W_{n,\eta})$ a cocycle

$$(\epsilon\phi)(r) := \epsilon\phi(\epsilon^{-1}r\epsilon) \qquad \forall r \in \Gamma$$

It induces an automorphism of the order 2 on the cohomologies. Hence we obtain two exact sequences:

$$0 \to H^0(\Gamma_\infty, W_{n,\eta})_+ \to H^1_c(\Gamma, W_{n,\eta})_+ \to H^1(\Gamma, W_{n,\eta})_+ \to H^1(\Gamma_\infty, W_{n,\eta})_+ \to \cdots$$

and

$$0 \to H^0(\Gamma_\infty, W_{n,\eta})_- \to H^1_c(\Gamma, W_{n,\eta})_- \to H^1(\Gamma, W_{n,\eta})_- \to H^1(\Gamma_\infty, W_{n,\eta})_- \to \cdots$$

where $H^1(\Gamma, W_{n,\eta})_\pm := \{\phi \in H^1(\Gamma, W_{n,\eta}) \mid \epsilon.\phi = \pm\phi\}$. The operation of ϵ on $W_{n,\eta}$ is clear:

$$\begin{cases} a_0\epsilon = \epsilon a_0 \\ a_i\epsilon = \begin{pmatrix} 1 & 0 \\ -p_0 & -1 \end{pmatrix} a_{p_0-i}, \quad i = 1, 2.,,, p_0 - 1 \\ a_{p_0}\epsilon = \epsilon a_{p_0} \end{cases}$$

we have for $u \in W_{n,\eta}$

$$\begin{cases} (\epsilon u)(a_0) = \epsilon.u(a_0) \\ (\epsilon u)(a_i) = E.u(a_{p_0-i}) \\ (\epsilon u)(a_{p_0}) = \epsilon.u(a_{p_0}) \end{cases}$$

where $E := \begin{pmatrix} -1 & 0 \\ p_0 & 1 \end{pmatrix}$.

In [Wa] we have shown that for n even the classes

$$\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1 - Q)u \end{cases}$$

where $u \in W_{n,\eta}$ with

$$1. \begin{cases} u(a_0) = y^n \\ u(a_j) = 0,\ j > 0 \end{cases}$$

or

$$2. \begin{cases} u(a_i) \in W_{n,\eta}\ (\text{resp. } W_{n,\eta}/W_{n,\eta}^{S_i}),\ i \in B,\ i < p_0/2,\ i^2 \not\equiv -1, i^3 \not\equiv -1\ (\text{resp. } i^2 \equiv -1 \text{ or } i^3 \equiv -1) \\ u(a_{p_0-i}) = \begin{pmatrix} -1 & 0 \\ p_0 & 1 \end{pmatrix} u(a_i) \\ u(a_j) = 0, \quad j \neq i, p_0 - i \end{cases}$$

is a basis of $H^1(\Gamma, W_{n,\eta})_-$. For n odd we have a basis of $H^1(\Gamma, W_{n,\eta})_+$:

$$\begin{cases} \phi(S) = 0 \\ \phi(Q) = (1-Q)u \end{cases}$$

with

$$\begin{cases} u(a_i) \in W_{n,\eta}\ (\text{resp. } W_{n,\eta}/W_{n,\eta}^{S_i}),\ i \in B,\ i < p_0/2,\ i^2 \not\equiv -1, i^3 \not\equiv -1\ (\text{resp. } i^2 \equiv -1 \text{ or } i^3 \equiv -1) \\ u(a_{p_0-i}) = -\begin{pmatrix} -1 & 0 \\ p_0 & 1 \end{pmatrix} u(a_i) \\ u(a_j) = 0, \quad j \neq i, p_0 - i \end{cases} .$$

5. The Hecke operator T_l on $H^1(\Gamma, W_{n,\eta})$.

To get startet, we recall the definition of the Hecke operator T_l auf (on) $H^1(\Gamma, W_{n,\eta})$, where l is a prime, $l \neq p_0$. Let

$$b_i = \begin{pmatrix} 1 & i \\ 0 & l \end{pmatrix},\ i = 0, 1, ..., l-1 \text{ and } b_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix},$$

they are a complete set of representatives of $\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma$ with respect to Γ:

$$\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \bigcup_{i=0}^{l} \Gamma b_i$$

For each $r \in \Gamma$ there is $s_i \in \Gamma$ such that $b_i r = s_i b_j$ for some j. We define for a cocycle $f \in Z^1(\Gamma, W_{n,\eta})$

$$(T_l f)(r) := \sum_{i=0}^{l} b_i' f(s_i)$$

where $b_i' := det(b_i) b_i^{-1}$.

Let $\phi \in H^1(\Gamma, W_{n,\eta})$ be a class with $\phi(S) = 0$, $\phi(Q) = (1-Q)u$ for some $u \in W_{n,\eta}$. A simple calculation shows

$$\begin{cases} b_0 S = S b_l \\ b_i S = s_i b_j \quad 0 < i < l,\ i \cdot j \equiv -1 \mod l, \text{ or } i \cdot j = -1 + lm, \quad s_i = \begin{pmatrix} i & -m \\ l & -j \end{pmatrix} \\ b_l S = S b_0 \end{cases}$$

with $s_j = s_i^{-1}$ and

$$\begin{cases} b_0 Q = ST^l b_l \\ b_1 Q = s_1 T b_0 \\ b_l Q = S b_1 \\ b_i Q = s_i b_{j+1} \quad 1 < i < l,\ i \cdot j \equiv -1 \mod l \end{cases}$$

We consider the Hecke operator T_l. If $j \neq i$, then

$$\begin{aligned} b_i'\phi(s_i) + b_j'\phi(s_j) &= b_i'\phi(s_i) + b_j'\phi(s_i^{-1}) = b_i'\phi(s_i) - b_j' s_i^{-1}\phi(s_i) \\ &= b_i'\phi(s_i) - Sb_i'\phi(s_i) = (1-S)b_i'\phi(s_i). \end{aligned}$$

If $j = i$, i.e., $b_i S = s_i b_i$, or $s_i^2 = 1$. We obtain

$$\begin{aligned} 2b_i'\phi(s_i) &= b_i'\phi(s_i) + b_i'\phi(s_i) = b_i'\phi(s_i) + b_i'\phi(s_i^{-1}) \\ &= b_i'\phi(s_i) - b_i' s_i^{-1}\phi(s_i) = b_i'\phi(s_i) - Sb_i'\phi(s_i) \\ &= (1-S)b_i'\phi(s_i). \end{aligned}$$

or $b_i'\phi(s_i) = (1-S)\frac{1}{2}b_i'\phi(s_i)$. Therefore we know

$$T_l\phi(S) = \sum_i b_i'\phi(s_i) = (1-S)(\sum_{i<j} b_i'\phi(s_i) + \sum_{i=j}\frac{1}{2}b_i'\phi(s_i)) =: (1-S)m_S.$$

We study now $(T_l\phi)(Q)$. First we note that

$$b_0'\phi(ST^l) + b_l'\phi(S) + b_1'\phi(s_1 T) = (1-Q)b_0'\phi(ST^l).$$

For $1 < i,j,k < l$, $i \neq j, i \cdot j \cdot k \equiv -1 \mod l$, it is easy to see that

$$b_i'\phi(s_i) + b_j'\phi(s_j) + b_k'\phi(s_k) = (1-Q)(b_i'\phi(s_i) + (1+Q)b_j'\phi(s_j)).$$

If $j = i$, i.e. $b_i Q = s_i b_i$, one has $s_i^3 = 1$. It is obvious

$$b_i'\phi(s_i) = (1-Q)\frac{1}{3}(2b_i'\phi(s_i) + Qb_i'\phi(s_i))$$

Hence we obtain

$$\begin{aligned} T_l\phi(Q) &= (1-Q)(\phi(ST^l) + \sum_{1<i<l, i<j, i<k} (b_i'\phi(s_i) + (1+Q)b_j'\phi(s_j)) + \frac{1}{3}\sum_{1<i<l, i=j}(2b_i'\phi(s_i) + Qb_i'\phi(s_i))) \\ &=: (1-Q)m_Q. \end{aligned}$$

Finally we know that the cohomology class $T_l\phi$ is cohomolog to

$$T_l\phi \quad \sim \quad \begin{cases} (T_l\phi)(S) = 0 \\ (T_2\phi)(Q) = (1-Q)(m_Q - m_S) \end{cases}$$

To determine the operation of the Hecke operator T_l we must calculate the linear coefficients of the representations $m_Q - m_S = \sum c_i u_i$ where $\{u_i\}$ is the basis given in §4. With a modern computer we can determine the coefficients c_i explicitly.

Example: $l = 2, p_0 = 5, n = 5$. Let η be a Dirichlet character mod p_0 with $\eta(\omega) = z$, $z^{\frac{p_0-1}{2}} = (-1)^n$, i.e. $z^2 + 1 = 0$.

For $l = 2$ we have

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, b_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A simple calculation shows that

$$\begin{cases} b_0 S = S b_2 \\ b_1 S = S Q^{-1} S Q S b_1 \\ b_2 S = S b_0 \end{cases} \quad \begin{cases} b_0 T = b_1 \\ b_1 T = T b_0 \\ b_2 T = T^2 b_2 \end{cases} \quad \begin{cases} b_0 Q = Q S Q b_2 \\ b_1 Q = S Q^{-1} S Q^{-1} b_0 \\ b_2 Q = S b_1 \end{cases}$$

By the above remark we get for a class $\phi \in H^1(\Gamma, W_{n,\eta})$

$$T_2\phi \quad \sim \quad \begin{cases} (T_2\phi)(S) = 0 \\ (T_2\phi)(Q) = (1 - Q)(b_0' - b_2' Q^{-1})\phi(Q) \end{cases}$$

For $p_0 = 5$ we choose a basis set $B = \{2, 3\}$. The basis of $H^1(\Gamma, W_{n,\eta})_-$ is then $(y^n, 0, 0, 0, 0, 0)$ and $(0, 0, w_2, Ew_2, 0, 0)$, $w_2 \in M_{n,\eta}/M_{n,\eta}^{S_2}$. For $n = 5$ the numerical calculation shows that $M_{n,\eta}/M_{n,\eta}^{S_2}$ has a basis

$$\begin{aligned} v_1 &= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5 \\ v_2 &= x^4y - 8x^3y^2 + 24x^2y^3 - 32xy^4 + 16y^5 \\ v_3 &= x^3y^2 - 6x^2y^3 + 12xy^4 - 8y^5. \end{aligned}$$

Let $v_0 = y^5$, the then $\{v_0, v_1, v_2, v_3\}$ is a basis of $H^1(\Gamma, W_{n,\eta})_-$ and the Hecke operator T_2 is given by

$$T_2(v_0, v_1, v_2, v_3) = (v_0, v_1, v_2, v_3) \begin{pmatrix} 64z + 1 & 0 & 0 & 0 \\ * & z + 64 & 10(24 - 17z) & 40(10 + 9z) \\ * & 0 & 39z + 7 & 2(33z + 23) \\ * & 0 & 7 - 15z & 2(1 + 17z) \end{pmatrix}$$

The characteristic polynomial of T_2 on $H^1(\Gamma, W_{n,\eta})_-$ is

$$(x - 1 - 64z)(x - z - 64)(x^2 + (5z + 5)x - 88z)$$

The factors $(x - 1 - 64z)(x - z - 64)$ come from the operation of T_2 on the boundary cohomology. Therefor the characteristic polynomial of T_2 on $S_7(\Gamma_0(p_0), \eta)$ is

$$\chi_2(x) = x^2 + (5z + 5)x - 88z$$

The numerical computations of T_2, T_3, T_5, T_7 and T_{11} for small p_0 and n are given by the table 1.

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==

PO = 5   PRIMITIVE ROOT= 2

==

N = 2

THE VALUE ETA(ROOT)=Z SATISFIES: $Z - 1 = 0$

$T(2)= X + 4$

$T(3)= X - 2$

$T(7)= X - 6$

$T(11)= X - 32$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$T(2)= 1$

$T(3)= 1$

$T(7)= 1$

$T(11)= 1$

N = 3

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + 1 = 0$

$T(2)= X + Z + 1$

$T(3)= X - 6*Z + 6$

$T(7)= X + 26*Z + 26$

$T(11)= X + 8$

N = 4

THE VALUE ETA(ROOT)=Z SATISFIES: $Z - 1 = 0$

$T(2)= X - 2$

$T(3)= X + 4$

$T(7)= X - 192$

$T(11)= X + 148$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$T(2)= X^2 + 44$

$T(3)= X^2 + 396$

$T(7)= X^2 + 3564$

$T(11)= X^2 - 504*X + 63504$

N = 5

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + 1 = 0$

$T(2)= X^2 + 5*X*Z + 5*X - 88*Z$

$T(3)= X^2 + 15*X*Z - 15*X - 12*Z$

$T(7)= X^2 - 275*X*Z - 275*X + 25652*Z$

$T(11)= X^2 + 526*X - 1061456$

N = 6

THE VALUE ETA(ROOT)=Z SATISFIES: Z - 1 = 0

$T(2)= (X + 14) * (X^2 - 20*X + 24)$

$T(3)= (X + 48) * (X^2 - 20*X - 4764)$

$T(7)= (X + 1644) * (X^2 + 100*X - 235836)$

$T(11)= (X - 172) * (X^2 - 4544*X - 6998016)$

THE VALUE ETA(ROOT)=Z SATISFIES: Z + 1 = 0

$T(2)= X^2 + 116$

$T(3)= X^2 + 1044$

$T(7)= X^2 + 176436$

$T(11)= (X + 6828)^2$

N = 7

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + 1 = 0$

$T(2)= X^3 + X^2*Z + X^2 - 392*X*Z - 1592*Z + 1592$

$T(3)= X^3 - 36*X^2*Z + 36*X^2 + 13062*X*Z + 278532*Z + 278532$

$T(7)= X^3 + 1176*X^2*Z + 1176*X^2 - 5242902*X*Z + 1794519748*Z - 1794519748$

$T(11)= X^3 - 11596*X^2 - 149978428*X - 135134348768$

```
=================================================================
P0 = 7   PRIMITIVE ROOT= 3
=================================================================
N = 1
     ------------------------------------------------------------
     THE VALUE ETA(ROOT)=Z SATISFIES: Z + 1 = 0

     T(2)= X + 3

     T(3)= X

     T(5)= X

     T(11)= X + 6
     ------------------------------------------------------------
                                      2
     THE VALUE ETA(ROOT)=Z SATISFIES: Z  - Z + 1 = 0

               1
     T(2)= X - ---*Z + 1
               2

     T(3)= X + Z + 1

               5       7
     T(5)= X + ---*Z - ---
               2       2

     T(11)= X + 35*Z - 22
-----------------------------------------------------------------
N = 2
     ------------------------------------------------------------
     THE VALUE ETA(ROOT)=Z SATISFIES: Z - 1 = 0

     T(2)= X + 1

     T(3)= X + 2

     T(5)= X - 16

     T(11)= X + 8
     ------------------------------------------------------------
                                      2
     THE VALUE ETA(ROOT)=Z SATISFIES: Z  + Z + 1 = 0

            2   5
     T(2)= X  - ---*X*Z - Z - 1
                2

            2
     T(3)= X  + 4*X*Z + 4*X - 21*Z

            2   29        7            105
     T(5)= X  - ----*X*Z - ---*X - 28*Z - -----
                2          2              2

             2
     T(11)= X  - 172*X*Z - 137*X + 660*Z - 175

-----------------------------------------------------------------
N = 3
     ------------------------------------------------------------
     THE VALUE ETA(ROOT)=Z SATISFIES: Z + 1 = 0
```

```
    T(2)= X - 1

    T(3)= X

    T(5)= X

    T(11)= X + 206
    ------------------------------------------------------------
                                           2
    THE VALUE ETA(ROOT)=Z SATISFIES: Z  - Z + 1 = 0

          3    3  2         2
    T(2)= X  + ---*X *Z + 5*X  - 8*X*Z + 28*X - 90*Z + 45
               2

          3   2      2
    T(3)= X  + X *Z + X  - 81*X*Z - 378*Z + 189

          3   95  2      121  2                          30429
    T(5)= X  + ----*X *Z - -----*X  + 1569*X*Z - 1224*X - -------*Z + 4347
               2            2                              2

           3        2          2
    T(11)= X  + 7394*X *Z - 6200*X  + 368481*X*Z + 69252*X + 6587298*Z - 41112684
----------------------------------------------------------------
N = 4
    ------------------------------------------------------------
    THE VALUE ETA(ROOT)=Z SATISFIES: Z - 1 = 0

          3   2
    T(2)= X  + X  - 84*X + 60

          3      2
    T(3)= X  + 20*X  - 420*X - 7056

          3      2
    T(5)= X  + 74*X  - 336*X - 75264

           3      2
    T(11)= X  - 628*X  - 88032*X + 41737728
    ------------------------------------------------------------
                                           2
    THE VALUE ETA(ROOT)=Z SATISFIES: Z  + Z + 1 = 0

          2
    T(2)= X  - 2*X*Z + 36*Z + 36

          2
    T(3)= X  - 8*X*Z - 8*X - 21*Z

          2
    T(5)= X  + 38*X*Z + 3339*Z + 3339

           2
    T(11)= X  + 424*X*Z + 424*X + 25371*Z
----------------------------------------------------------------
N = 5
    ------------------------------------------------------------
    THE VALUE ETA(ROOT)=Z SATISFIES: Z + 1 = 0

          3     2
    T(2)= X  + 7*X  - 80*X - 576

             2
```

$T(3)= X*(X + 2040)$

$T(5)= X*(X^2 + 2040)$

$T(11)= X^3 - 3710*X^2 + 4193452*X - 1498724712$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 - Z + 1 = 0$

$T(2)= X^3 - 4*X^2*Z - 98*X*Z + 98*X - 24$

$T(3)= X^3 + X^2*Z + X^2 - 1113*X*Z - 13818*Z + 6909$

$T(5)= X^3 + 55*X^2*Z - 110*X^2 + 28875*X*Z - 28875*X - 2756250*Z + 1378125$

$T(11)= X^3 + 403*X^2*Z - 403*X^2 + 2487695*X*Z - 437463057$

N = 6

THE VALUE ETA(ROOT)=Z SATISFIES: $Z - 1 = 0$

$T(2)= (X + 6) * (X^2 + 3*X - 214)$

$T(3)= (X + 42) * (X^2 - 94*X + 1344)$

$T(5)= (X + 84) * (X^2 - 330*X + 5600)$

$T(11)= (X + 5568) * (X^2 - 2844*X - 887776)$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + Z + 1 = 0$

$T(2)= X^4 + 6*X^3*Z + 412*X^2*Z + 412*X^2 - 1704*X + 9312*Z$

$T(3)= X^4 + 28*X^3*Z + 28*X^3 - 4986*X^2*Z + 43092*X - 5359473*Z - 5359473$

$T(5)= X^4 - 252*X^3*Z + 90490*X^2*Z + 90490*X^2 + 19632900*X + 11594625*Z$

$T(11)= X^4 - 3972*X^3*Z - 3972*X^3 - 17626474*X^2*Z - 19048929420*X - 7298108160225*Z - 7298108160225$

N = 7

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$T(2)= (X^2 - 16*X - 120)^2 *(X + 31)$

$T(3)= X*(X^4 + 17184*X^2 + 40430880)$

$$T(5)= X\cdot(X^4 + 1809120\cdot X^2 + 736852788000)$$

$$T(11)= (X^2 + 11084\cdot X + 29862948) \cdot (X^2 - 13154)$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 - Z + 1 = 0$

$$T(2)= X^5 + \frac{1}{2}\cdot X^4\cdot Z + 7\cdot X^4 - 572\cdot X^3\cdot Z + 600\cdot X^3 - 4102\cdot X^2\cdot Z + 2643\cdot X^2 - 16136\cdot X\cdot Z + 4144\cdot X - 49224\cdot Z - 49224$$

$$T(3)= X^5 - 5\cdot X^4\cdot Z - 5\cdot X^4 - 14916\cdot X^3\cdot Z + 1080048\cdot X^2\cdot Z - 540024\cdot X^2 - 10664325\cdot X\cdot Z + 10664325\cdot X - 103078899\cdot Z + 206157798$$

$$T(5)= X^5 + \frac{315}{2}\cdot X^4\cdot Z - \frac{1005}{2}\cdot X^4 + 804774\cdot X^3\cdot Z - 857274\cdot X^3 - 141737655\cdot X^2\cdot Z + 106941390\cdot X^2 - 197426947725\cdot X\cdot Z + 25039192500\cdot X - 17453615484375\cdot Z - \frac{24435061678125}{2}$$

$$T(11)= X^5 - \frac{9087283}{2}\cdot X^4\cdot Z + 5735310\cdot X^4 + 10755380048\cdot X^3\cdot Z - 2125936604\cdot X^3 + 620149473541312\cdot X^2\cdot Z + 2365321548729164\cdot X^2 + 688332443374312529\cdot X\cdot Z - 852381297869410651\cdot X + 35911221494916534827656 2\cdot Z - \frac{14918448765878664244745 1}{2}$$

===

PO = 11   PRIMITIVE ROOT= 2

===

N = 1

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^5 + 1 = 0$

$$T(2)= X + Z^4 - Z^2 + Z + 1$$

$$T(3)= X - Z^4 - Z^3 + 2\cdot Z^2 - 2\cdot Z + 1$$

$$T(5)= X + Z^4 - Z^3 - 3\cdot Z^2 - Z + 1$$

$$T(7)= X + 4\cdot Z^4 + 4\cdot Z^3 + 2\cdot Z^2 - 2$$

N = 2

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^5 - 1 = 0$

$$T(2)= X^2 - X\cdot Z^4 - 2\cdot X\cdot Z^3 - X\cdot Z^2 + X\cdot Z + X + 2\cdot Z^2 - 6\cdot Z + 2$$

$$T(3)= X^2 - 2\cdot X\cdot Z^4 + X\cdot Z^3 + X\cdot Z^2 + X\cdot Z + X - 8\cdot Z^4 - 15\cdot Z^3 - 8\cdot Z^2 - 8\cdot Z - 8$$

$$T(5)= X^2 + X\cdot Z^4 - 4\cdot X\cdot Z^3 + 4\cdot X\cdot Z^2 - 4\cdot X\cdot Z + X - 111\cdot Z^4 - 13\cdot Z^3 - 27\cdot Z^2 - 27\cdot Z - 13$$

$$T(7)= X^{2} - 6*X*Z^{4} - 6*X*Z^{3} + 3*X*Z^{2} - 14*X*Z + 3*X - 28*Z^{4} + 110*Z^{3} - 112*Z^{2} + 110*Z - 28$$

N = 3

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^{5} + 1 = 0$

$$T(2)= X^{3} - 3*X^{2}*Z^{4} + 3*X^{2}*Z^{2} + X^{2}*Z + X^{2} + 4*X*Z^{2} - 22*X*Z + 4*X - 38*Z^{3} - 18*Z^{2} - 18*Z - 38$$

$$T(3)= X^{3} + 5*X^{2}*Z^{4} - X^{2}*Z^{3} - 4*X^{2}*Z^{2} + 4*X^{2}*Z + X^{2} + 51*X*Z^{4} + 123*X*Z^{3} + 51*X*Z^{2} + 6*X*Z - 6*X + 315*Z^{4} + 252*Z^{3} - 189*Z^{2} + 252*Z + 315$$

$$T(5)= X^{3} + X^{2}*Z^{4} + 9*X^{2}*Z^{3} + 3*X^{2}*Z^{2} + 9*X^{2}*Z + X^{2} - 715*X*Z^{4} + 300*X*Z^{3} - 381*X*Z^{2} + 381*X*Z - 300*X + 8309*Z^{4} - 8309*Z^{3} + 10689*Z^{2} - 9093*Z + 10689$$

$$T(7)= X^{3} - 40*X^{2}*Z^{4} - 40*X^{2}*Z^{3} - 16*X^{2}*Z^{2} + 16*X^{2} - 715*X*Z^{4} - 1738*X*Z^{3} + 954*X*Z^{2} - 1738*X*Z - 715*X + 30254*Z^{4} - 30254*Z^{2} + 14378*Z + 14378$$

N = 4

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^{5} - 1 = 0$

$$T(2)= X^{4} - X^{3}*Z^{4} + 4*X^{3}*Z^{3} - X^{3}*Z^{2} + X^{3}*Z + X^{3} - 2*X^{2}*Z^{4} - 2*X^{2}*Z^{3} - 4*X^{2}*Z^{2} - 78*X^{2}*Z - 4*X^{2} - 196*X*Z^{4} + 48*X*Z^{3} - 36*X*Z^{2} - 36*X*Z + 48*X - 184*Z^{4} + 344*Z^{3} + 432*Z^{2} + 344*Z - 184$$

$$T(3)= X^{4} + X^{3}*Z^{4} + X^{3}*Z^{3} - 11*X^{3}*Z^{2} - 11*X^{3}*Z + X^{3} + 34*X^{2}*Z^{4} - 429*X^{2}*Z^{3} + 34*X^{2}*Z^{2} + 34*X^{2}*Z + 34*X^{2} + 2367*X*Z^{4} + 771*X*Z^{3} - 1809*X*Z^{2} + 771*X*Z + 2367*X - 20457*Z^{4} - 20457*Z^{3} + 13131*Z^{2} + 32832*Z + 13131$$

$$T(5)= X^{4} + X^{3}*Z^{4} + 11*X^{3}*Z^{3} - 29*X^{3}*Z^{2} + 11*X^{3}*Z + X^{3} - 4881*X^{2}*Z^{4} - 1594*X^{2}*Z^{3} + 1621*X^{2}*Z^{2} + 1621*X^{2}*Z - 1594*X^{2} - 73311*X*Z^{4} - 73311*X*Z^{3} + 16329*X*Z^{2} + 69869*X*Z + 16329*X + 1566134*Z^{4} + 4787070*Z^{3} + 1566134*Z^{2} - 3589606*Z - 3589606$$

$$T(7)= X^{4} - 38*X^{3}*Z^{4} - 38*X^{3}*Z^{3} - 58*X^{3}*Z^{2} + 98*X^{3}*Z - 58*X^{3} - 6245*X^{2}*Z^{4} - 3176*X^{2}*Z^{3} - 26266*X^{2}*Z^{2} - 3176*X^{2}*Z - 6245*X^{2} + 2837816*X*Z^{4} - 1563704*X*Z^{3} + 2837816*X*Z^{2} + 864000*X*Z + 864000*X - 65806216*Z^{4} + 79260992*Z^{3} - 73260124*Z^{2} - 73260124*Z + 79260992$$

N = 5

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^{5} + 1 = 0$

$T(2)= X^5 + 5X^4Z^4 - 5X^4Z^2 + X^4Z + X^4 + 40X^3Z^4 - 40X^3Z^3 - 4X^3Z^2 - 198X^3Z - 4X^3 + 586X^2Z^3 - 34X^2Z^2 - 34X^2Z$
$+ 586X^2 + 4176XZ^4 - 1000XZ^3 + 6328XZ^2 - 1000XZ + 4176X - 4224Z^4 + 1408Z^3 + 1408Z^2 - 4224Z$

$T(3)= X^5 - 16X^4Z^4 - X^4Z^3 - 10X^4Z^2 + 10X^4Z + X^4 - 321X^3Z^4 + 1878X^3Z^3 - 321X^3Z^2 + 57X^3Z - 57X^3 + 17850X^2Z^4 -$
$6333X^2Z^3 + 16590X^2Z^2 - 6333X^2Z + 17850X^2 + 347670XZ^4 - 347670XZ^3 + 282726XZ^2 - 654129XZ + 282726X -$
$4457691Z^4 + 4938930Z^3 - 4938930Z^2 + 4457691Z - 3977568$

$T(5)= X^5 + X^4Z^4 - 56X^4Z^3 + 72X^4Z^2 - 56X^4Z + X^4 - 37560X^3Z^4 - 12705X^3Z^3 + 9130X^3Z^2 - 9130X^3Z + 12705X^3 +$
$2018725X^2Z^4 - 2018725X^2Z^3 - 1603025X^2Z^2 + 1608800X^2Z - 1603025X^2 - 237823875XZ^4 - 253047125XZ^3 -$
$237823875XZ^2 - 79581125XZ + 79581125X + 15203688750Z^4 + 6943711250Z^3 - 6943711250Z^2 - 15203688750Z$
$- 17840901250$

$T(7)= X^5 + 204X^4Z^4 + 204X^4Z^3 - 73X^4Z^2 + 73X^4 + 6931X^3Z^4 - 88422X^3Z^3 + 202294X^3Z^2 - 88422X^3Z + 6931X^3 -$
$20670259X^2Z^4 + 20670259X^2Z^2 - 19226843X^2Z - 19226843X^2 + 11220928720XZ^4 - 8368901800XZ^3 + 5289369840XZ^2 -$
$5289369840XZ + 8368901800X - 107155727360Z^4 + 2935502900Z^3 + 2935502900Z^2 - 107155727360Z$

N = 6

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^5 - 1 = 0$

$T(2)= X^6 + 7X^5Z^4 - 8X^5Z^3 + 7X^5Z^2 + X^5Z + X^5 - 22X^4Z^4 - 22X^4Z^3 - 48X^4Z^2 - 462X^4Z - 48X^4 + 2644X^3Z^4 - 2912 \cdot$
$X^3Z^3 - 572X^3Z^2 - 572X^3Z - 2912X^3 + 5288X^2Z^4 + 9152X^2Z^3 + 48536X^2Z^2 + 9152X^2Z + 5288X^2 + 265648XZ^4 +$
$15328XZ^3 + 15328XZ^2 + 265648XZ - 154160X - 690624Z^4 - 209792Z^3 - 690624Z^2 - 342848Z - 342848$

$T(3)= X^6 + 13X^5Z^4 + X^5Z^3 + 13X^5Z^2 + 13X^5Z + X^5 + 949X^4Z^4 - 8304X^4Z^3 + 949X^4Z^2 - 104X^4Z - 104X^4 - 51714X^3Z^4$
$+ 20442X^3Z^3 - 126594X^3Z^2 + 20442X^3Z - 51714X^3 + 279099X^2Z^4 + 279099X^2Z^3 - 2638440X^2Z^2 + 15929379X^2Z$
$- 2638440X^2 + 9693081XZ^4 + 4978989XZ^3 + 4978989XZ^2 + 9693081XZ + 165287709X - 5744321226Z^4 + 2548480644Z^3$
$- 1476668313Z^2 - 1476668313Z + 2548480644$

$T(5)= X^6 + X^5Z^4 + X^5Z^3 - 71X^5Z^2 + X^5Z + X^5 - 265476X^4Z^4 + 18522X^4Z^3 - 8032X^4Z^2 - 8032X^4Z + 18522X^4 - 2282090 \cdot$

```
  3 4            3 3             3 2            3              3                2 4               2 3
X *Z  - 2282090*X *Z  - 4747850*X *Z  + 2935330*X *Z - 4747850*X  - 2879059975*X *Z  + 14329103075*X *Z  -

           2 2             2               2                 4                3                2
2879059975*X *Z  - 269759050*X *Z - 269759050*X  + 213305166625*X*Z  - 66112449875*X*Z  - 66112449875*X*Z  +

                                                          4                3                 2
213305166625*X*Z + 672520376125*X - 22224093428750*Z  + 73178894471250*Z  - 72126624106250*Z  + 73178894471250*Z

- 22224093428750

      6       5 4       5 3       5 2       5        5          4 4          4 3           4 2            4
T(7)= X  + 674*X *Z  + 674*X *Z  + 198*X *Z  - 686*X *Z + 198*X  + 343502*X *Z  - 500880*X *Z  - 1623044*X *Z  - 500880*X *Z

          4             3 4              3 3             3 2              3               3
 + 343502*X  + 155679548*X *Z  + 1630785968*X *Z  +155679548*X *Z  - 1611342156*X *Z - 1611342156*X  +

             2 4                2 3              2 2                2                   2
482001856094*X *Z  + 288953809449*X *Z  + 31075147408*X *Z  + 31075147408*X *Z + 288953809449*X  - 157624211135368*X*

 4                   3                   2
Z  + 495394762240520*X*Z  + 495394762240520*X*Z  - 157624211135368*X*Z - 589310709588832*X +

                  4                    3                    2
102641162227378192*Z  + 102641162227378192*Z  - 394508695361513348*Z  - 128696324560430264*Z - 394508695361513348
-----------------------------------------------------------------
N = 7
      -----------------------------------------------------------

                                  5
THE VALUE ETA(ROOT)=Z SATISFIES: Z  + 1 = 0

      7     6 4     6 2    6      6      5 4      5 3     5 2        5        5         4 3        4 2
T(2)= X  - 3*X *Z  + 3*X *Z  + X *Z + X  + 48*X *Z  - 48*X *Z  + 4*X *Z  - 1270*X *Z + 4*X  - 2966*X *Z  - 450*X *Z  - 450*

 4          4          3 4         3 3          3 2          3           3            2 4            2 3
X *Z - 2966*X  + 22496*X *Z  + 15824*X *Z  + 423216*X *Z  + 15824*X *Z + 22496*X  + 853216*X *Z  - 156256*X *Z  -

      2 2          2            4               3              2                                  4
156256*X *Z  + 853216*X *Z + 433536*X*Z  - 33380096*X*Z  + 433536*X*Z  - 3036416*X*Z + 3036416*X + 52801408*Z  +

         3           2
52801408*Z  - 18406784*Z  + 18406784

      7      6 4    6 3      6 2      6      6        5 4         5 3         5 2        5          5
T(3)= X  + 35*X *Z  - X *Z  + 56*X *Z  - 56*X *Z + X  + 3087*X * Z  + 22704*X *Z  + 3087*X *Z  - 4176*X *Z + 4176*X  -

       4 4          4 3          4 2          4           4            3 4             3 3              3
918597*X *Z  - 150537*X *Z  - 476238*X *Z  - 150537*X *Z - 918597*X  - 36227493*X *Z  + 36227493*X *Z  - 54071550*X *

 2             3            3              2 4              2 3              2 2             2
Z  - 141830631*X *Z - 54071550*X  - 2026167336*X *Z  - 4416728706*X *Z  + 4416728706*X *Z  + 2026167336*X *Z +

          2                4                3              2
1392692985*X  - 145595816010*X*Z  - 244411605630*X*Z  - 8525300625*X*Z  + 8525300625*X*Z + 244411605630*X +

             4                3                2
5261707743255*Z  + 4252898483550*Z  + 5261707743255*Z  + 849193780230*Z - 849193780230

      7   6 4      6 3       6 2      6     6         5 4         5 3          5 2          5
T(5)= X  + X *Z  + 279*X *Z  - 147*X *Z  + 279*X *Z + X  - 1311130*X *Z  - 57560*X *Z  - 160206*X *Z  + 160206*X *Z +

      5          4 4           4 3             4 2             4             4              3 4
57560*X  + 6664174*X *Z  - 6664174*X *Z  + 446321414*X *Z  - 74483598*X *Z + 446321414*X  - 10541917075*X *Z  -
```

$$363348448055 X^{3} Z^{3} - 10541917075 X^{3} Z^{2} - 72756546810 X^{3} Z + 72756546810 X^{3} - 144912642015175 X^{2} Z^{4} + 30795252790475 X^{2}$$

$$Z^{3} - 30795252790475 X^{2} Z^{2} + 144912642015175 X^{2} Z - 16515791466375 X^{2} + 12738304565676000 X Z^{4} - 5361974464430000 X Z^{3}$$

$$+ 3517194529318000 X Z^{2} -$$

$$5361974464430000 X Z + 12738304565676000 X - 1018534932916807500 Z^{4} - 203413537834152500 Z^{3} -$$

$$283029056684047500 Z^{2} + 283029056684047500 Z + 203413537834152500$$

$$T(7)= X^{7} - 80 X^{6} Z^{4} - 80 X^{6} Z^{3} + 1094 X^{6} Z^{2} - 1094 X^{6} + 2235954 X^{5} Z^{4} + 2172870 X^{5} Z^{3} + 22201536 X^{5} Z^{2} + 2172870 X^{5} Z +$$

$$2235954 X^{5} + 23118712736 X^{4} Z^{4} - 23118712736 X^{4} Z^{2} + 6710658780 X^{4} Z + 6710658780 X^{4} + 121188412500526 X^{3} Z^{4} +$$

$$51745515606505 X^{3} Z^{3} + 42520604145362 X^{3} Z^{2} - 42520604145362 X^{3} Z - 51745515606505 X^{3} - 73533730812752868$$

$$X^{2} Z^{4} + 52335160723169226 X^{2} Z^{3} + 52335160723169226 X^{2} Z^{2} - 73533730812752868 X^{2} Z + 176260331390591844508 X Z^{4} -$$

$$176260331390591844508 X Z^{3} - 202579045540153555532 X Z^{2} - 187181476541811422048 X Z - 202579045540153555532 X +$$

$$154441792548428150685920 Z^{4} + 134683030297070242251880 Z^{3} - 134683030297070242251880 Z - 154441792548428150685920$$

==

PO = 13   PRIMITIVE ROOT= 2

==

N = 1

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^{2} + 1 = 0$

$$T(2)= X^{2} + 2 X Z + 2 X - 3 Z$$

$$T(3)= X^{2} + 2 X - 9$$

$$T(5)= X^{2} - 4 X Z - 4 X + 3 Z$$

$$T(7)= X^{2} - 6 X Z + 6 X - 13 Z$$

$$T(11)= X^{2} - 2 X Z + 2 X + 78 Z$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^{4} - Z^{2} + 1 = 0$

$$T(2)= X (X + Z^{3} - Z^{2} + 1)$$

$$T(3)= X^{2} + X Z^{2} + 2 Z^{3} - 3 Z^{2} - Z + 3$$

$T(5) = X^2 + 2 X Z^3 + X Z^2 - 2 X Z + 3 X + 19 Z^3 - Z^2 - 8 Z + 17$

$T(7) = X^2 + \frac{13}{2} X Z^3 - 3 X Z^2 - \frac{11}{2} X Z + 27 Z^3 - 29 Z^2 - Z + 19$

$T(11) = X^2 + 9 X Z^3 - 10 X Z^2 + \frac{23}{2} X Z + 4 X - 169 Z^3 + 92 Z^2 + 212 Z + 131$

N = 2

THE VALUE ETA(ROOT)=Z SATISFIES: $Z - 1 = 0$

$T(2) = X^3 + 4 X^2 - 9 X - 20$

$T(3) = X^3 + 2 X^2 - 67 X - 224$

$T(5) = X^3 + 10 X^2 + 19 X - 14$

$T(7) = X^3 + 22 X^2 - 377 X - 6422$

$T(11) = X^3 - 54 X^2 - 1092 X + 25688$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$T(2) = X^2 + 9$

$T(3) = (X + 1)^2$

$T(5) = X^2 + 81$

$T(7) = X^2 + 225$

$T(11) = X^2 + 2304$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + Z + 1 = 0$

$T(2) = X (X^3 - X^2 Z - X^2 - 18 X Z - 8)$

$T(3) = X^4 + 4 X^3 Z + 4 X^3 - 43 X^2 Z - 2 X - 192 Z - 192$

$T(5) = X^4 + 6 X^3 Z - 12 X^3 - 12 X^2 Z - 285 X^2 - 1830 X Z + 3900 X + 5100 Z - 8500$

$T(7) = X^4 - \frac{55}{2} X^3 Z - \frac{35}{2} X^3 + 523 X^2 Z - \frac{179}{2} X^2 + 6160 X Z + 7760 X - 7800 Z + 18200$

$$T(11)= X^4 + \frac{45}{2} X^3 Z + 175 X^3 - 4278 X^2 Z - \frac{4575}{2} X^2 - 217770 X Z + 25262 X - 1060800 Z - 5374720$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 - Z + 1 = 0$

$$T(2)= X(X^2 + X Z + X - 6 Z)$$

$$T(3)= X^3 + 2 X^2 Z - 2 X^2 + 29 X Z + 42$$

$$T(5)= X^3 - 8 X^2 Z + 17 X^2 - 182 X Z + 178 X + 144 Z + 240$$

$$T(7)= X^3 + \frac{5}{2} X^2 Z + \frac{3}{2} X^2 + 382 X Z - \frac{829}{2} X + 3588 Z - 2808$$

$$T(11)= X^3 + \frac{85}{2} X^2 Z - 185 X^2 - 887 X Z - \frac{2275}{2} X + 47580 Z + 11700$$

N = 3

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + 1 = 0$

$$T(2)= X^3 + X^2 Z + X^2 - 23 X Z - 29 Z + 29$$

$$T(3)= X^3 + 2 X^2 - 105 X - 144$$

$$T(5)= X^3 + 7 X^2 Z + 7 X^2 - 173 X Z - 1199 Z + 1199$$

$$T(7)= X^3 + 24 X^2 Z - 24 X^2 + 4803 X Z - 77506 Z - 77506$$

$$T(11)= X^3 - 16 X^2 Z + 16 X^2 + 16982 X Z + 209924 Z + 209924$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^4 - Z^2 + 1 = 0$

$$T(2)= X(X^4 + X^3 Z + X^3 + X^2 Z^2 - 47 X^2 Z + X^2 - 20 X Z^3 - 10 X Z^2 - 10 X Z - 20 X + 54 Z^3 + 234 Z^2 + 54 Z)$$

$$T(3)= X^5 + X^4 Z^2 - 34 X^3 Z^3 - 216 X^3 Z^2 + 17 X^3 Z + 216 X^3 - 564 X^2 Z^3 + 1128 X^2 Z^2 - 117 X^2 - 3735 X Z^3 - 6075 X Z^2 - 3735 X Z - 36288 Z^3 - 25596 Z^2 + 18144 Z + 25596$$

$$T(5)= X^5 + 77 X^4 Z^3 - 30 X^4 Z^2 + 12 X^4 Z + 13 X^4 - 277 X^3 Z^3 + 2794 X^3 Z^2 + 2466 X^3 Z - 1964 X^3 + 6528 X^2 Z^3 + 1942 X^2 Z^2 - 17624 X^2 Z - 55608 X^2 + 750096 X Z^3 + 1140768 X Z^2 - 1394352 X Z - 1249254 X - 20429820 Z^3 - 11454048 Z^2 + 4085964 Z + 7636032$$

$$T(7) = X^5 + \frac{329}{2} X^4 Z^3 - 21 X^4 Z^2 - \frac{467}{2} X^4 Z + 14 X^4 + 6952 X^3 Z^3 - 7719 X^3 Z^2 - 22 X^3 Z + 4098 X^3 + 260426 X^2 Z^3 - 564745 X^2 Z^2 - \frac{419905}{2} X^2 Z + 882619 X^2 + 8577708 X Z^3 - 5920312 X Z^2 + 6699201 X Z - 7999763 X + 236905864 Z^3 - 180030656 Z^2 - 104481936 Z + 168855752$$

$$T(11) = X^5 + 6116 X^4 Z^3 + 150 X^4 Z^2 + \frac{719}{2} X^4 Z - 50 X^4 + 665525 X^3 Z^3 - 576180 X^3 Z^2 - 912263 X^3 Z - 369780 X^3 - \frac{60886241}{2} X^2 Z^3 + 36348214 X^2 Z^2 + \frac{27561599}{2} X^2 Z - 34698354 X^2 - 5420592278 X Z^3 - 1336947379 X Z^2 - 2290858307 X Z - 6100828903 X - 280860301332 Z^3 - 79362293400 Z^2 + 260775435024 Z + 142834837860$$

N = 4

THE VALUE ETA(ROOT)=Z SATISFIES: $Z - 1 = 0$

$$T(2) = X^5 - 2X^4 - 117X^3 + 10X^2 + 2052X + 888$$

$$T(3) = X^5 + 20X^4 - 418X^3 - 4784X^2 + 51297X + 94428$$

$$T(5) = X^5 - 14X^4 - 12842X^3 + 255988X^2 + 29351241X - 466661574$$

$$T(7) = X^5 + 96X^4 - 19466X^3 - 1267856X^2 + 23160969X - 55434704$$

$$T(11) = X^5 - 180X^4 - 282120X^3 + 7234400X^2 + 14506526736X + 212623859136$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$$T(2) = X^6 + 161X^4 + 5856X^2 + 18864$$

$$T(3) = (X^3 - 8X^2 - 549X + 4068)^2$$

$$T(5) = X^6 + 8018X^4 + 13754433X^2 + 2485690416$$

$$T(7) = X^6 + 82950X^4 + 1662348177X^2 + 423560602764$$

$$T(11) = X^6 + 405548X^4 + 36683341824X^2 + 7521473396736$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + Z + 1 = 0$

$$T(2) = X^4 + 5X^3 Z + 5X^3 - 60X^2 Z + 232X + 72Z + 72$$

$$T(3)= X^4 - 8X^3Z - 8X^3 - 469X^2Z - 920X - 7680Z - 7680$$

$$T(5)= X^4 + 10X^3 - 2807X^2 + 8740X + 357156$$

$$T(7)= X^4 + 68X^3Z + 8113X^2Z + 8113X^2 - 76456X + 8507824Z$$

$$T(11)= X^4 + 480X^3Z + 480X^3 - 258621X^2Z + 165775832X + 20230428816Z + 20230428816$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 - Z + 1 = 0$

$$T(2)= X^5 + X^4Z + X^4 - 110X^3Z - 16X^2Z + 8X^2 + 1752XZ - 1752X + 1296Z - 2592$$

$$T(3)= X^5 - 10X^4Z + 10X^4 + 665X^3Z + 3882X^2 + 44676XZ - 44676X + 191592Z$$

$$T(5)= X^5 - 98X^4Z + 49X^4 + 6407X^3 - 510766X^2Z + 255383X^2 + 13150056X - 475145568Z + 237572784$$

$$T(7)= X^5 - 92X^4Z + 184X^4 + 21579X^3Z - 21579X^3 + 1733904X^2Z - 866952X^2 - 206167104XZ + 2967045120Z + 2967045120$$

$$T(11)= X^5 + 80X^4Z + 80X^4 - 247535X^3Z + 7647760X^2Z - 3823880X^2 + 9166417728XZ - 9166417728X - 302334888960Z + 604669777920$$

N = 5

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + 1 = 0$

$$T(2)= X^6 - 3X^5Z - 3X^5 - 263X^4Z + 527X^3Z - 527X^3 - 16536X^2 + 17920XZ + 17920X - 9216Z$$

$$T(3)= X^6 + 2X^5 - 2670X^4 - 960X^3 + 1950561X^2 - 784602X - 382594752$$

$$T(5)= X^6 - 54X^5Z - 54X^5 - 58832X^4Z + 2317580X^3Z - 2317580X^3 - 1009817925X^2 + 19420909750XZ + 19420909750X + 5214355683750Z$$

$$T(7)= X^6 + 199X^5Z - 199X^5 + 243962X^4Z - 39230622X^3Z - 39230622X^3 - 19239992257X^2 - 1925608673651XZ + 1925608673651X - 484478063529356Z$$

$$T(11)= X^6 + 843X^5Z - 843X^5 + 4339472X^4Z - 6889604900X^3Z - 6889604900X^3 + 7436774127708X^2 + 1743719941228772XZ - 1743719941228772X - 299087179543071504Z$$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^4 - Z^2 + 1 = 0$

$$T(2)= X^6 - 4X^5Z^3 + 4X^5Z^2 + 5X^5Z + X^5 + 5X^4Z^2 - 251X^4Z + 5X^4 - 812X^3Z^3 - 138X^3Z^2 - 138X^3Z - 812X^3 - 1274X^2$$

$$*Z^{3} + 14610*X^{2}*Z^{2} - 1274*X^{2}*Z - 12176*X*Z^{3} + 30120*X*Z^{2} + 42296*X*Z - 42296*X - 139008*Z^{3} - 3456*Z^{2} + 1728$$

$$T(3)= X^{6} + X^{5}*Z^{2} + 416*X^{4}*Z^{3} - 2673*X^{4}*Z^{2} - 208*X^{4}*Z + 2673*X^{4} + 5928*X^{3}*Z^{3} - 11856*X^{3}*Z + 11109*X^{3} + 129090*X^{2}*Z^{3} - 1405980$$

$$*X^{2}*Z^{2} + 129090*X^{2}*Z + 15188004*X*Z^{3} - 6852204*X*Z^{2} - 7594002*X*Z + 6852204*X + 12430548*Z^{3} - 24861096*Z - 3055428$$

$$T(5)= X^{6} - 23*X^{5}*Z^{3} + 11*X^{5}*Z^{2} - 11*X^{5}*Z + 23*X^{5} + 47975*X^{4}*Z^{3} + 900*X^{4}*Z^{2} - 450*X^{4} + 1673125*X^{3}*Z^{3} + 444525*X^{3}*Z^{2} + 444525*$$

$$X^{3}*Z + 1673125*X^{3} - 29898625*X^{2}*Z^{3} + 59797250*X^{2}*Z - 315757125*X^{2} + 8720112500*X*Z^{3} - 7951263750*X*Z^{2} +$$

$$7951263750*X*Z - 8720112500*X + 45531243750*Z^{3} - 519842212500*Z^{2} + 259921106250$$

$$T(7)= X^{6} - 121*X^{5}*Z^{3} + 400*X^{5}*Z^{2} - 279*X^{5}*Z - 279*X^{5} + 235244*X^{4}*Z^{3} + 112293*X^{4}*Z^{2} - 235244*X^{4}*Z - 224586*X^{4} - 86116958*X^{3}*$$

$$Z^{3} - 31324255*X^{3}*Z^{2} + 31324255*X^{3}*Z + 86116958*X^{3} - 18508754416*X^{2}*Z^{3} - 17986999024*X^{2}*Z^{2} + 9254377208*X^{2}*Z +$$

$$17986999024*X^{2} + 8689208086780*X*Z^{3} + 8689208086780*X*Z^{2} - 3275402043784*X*Z - 5413806042996*X - 568353880755232*Z^{3} -$$

$$827433879388480*Z^{2} + 413716939694240$$

$$T(11)= X^{6} - 1100*X^{5}*Z^{3} - 1100*X^{5}*Z^{2} + 607*X^{5}*Z + 493*X^{5} - 1006789*X^{4}*Z^{2} + 4359800*X^{4}*Z - 1006789*X^{4} - 2809477303*X^{3}*Z^{3} -$$

$$1455227313*X^{3}*Z^{2} + 1455227313*X^{3}*Z + 2809477303*X^{3} - 2129962934896*X^{2}*Z^{3} + 4358608603764*X^{2}*Z^{2} - 2129962934896*$$

$$X^{2}*Z - 1790767233461128*X*Z^{3} + 485523630861984*X*Z^{2} + 1305243602599144*X*Z + 1305243602599144*X -$$

$$138641008301470272*Z^{3} - 789070162471800768*Z^{2} + 394535081235900384$$

N = 6

THE VALUE ETA(ROOT)=Z SATISFIES: Z - 1 = 0

$$T(2)= X^{7} - 6*X^{6} - 589*X^{5} + 3534*X^{4} + 92836*X^{3} - 468456*X^{2} - 2565760*X - 772800$$

$$T(3)= X^{7} - 52*X^{6} - 8698*X^{5} + 412216*X^{4} + 19868025*X^{3} - 711862020*X^{2} - 10023042816*X - 30177681408$$

$$T(5)= X^{7} + 390*X^{6} - 274766*X^{5} - 89721084*X^{4} + 18803876945*X^{3} + 4949782534350*X^{2} - 332386338998500*X - 43056397766235000$$

$$T(7)= X^{7} - 1056*X^{6} - 2664238*X^{5} + 1960008208*X^{4} + 2198662428217*X^{3} - 682797580314112*X^{2} - 357493958138352204*X$$

$$- 2237347633058825456$$

$$T(11)= X^{7} + 7620*X^{6} - 57575652*X^{5} - 390877797312*X^{4} + 1147296819645936*X^{3} + 5064902947464268608*X^{2} -$$

$7596720433860597787072*X - 4804139800332584086921728$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z + 1 = 0$

$T(2)= X^6 + 449*X^4 + 37224*X^2 + 205776$

$T(3)= (X^3 +28*X^2 - 2601*X - 71748)^2$

$T(5)= X^6 + 243506*X^4 + 1206410625*X^2 + 93756690000$

$T(7)= X^6 + 847206*X^4 + 231424342425*X^2 + 20471634652072500$

$T(11)= X^6 + 76413428*X^4 + 1813281980887296*X^2 + 13610733591480665702400$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 + Z + 1 = 0$

$T(2)= X^8 + 9*X^7*Z + 9*X^7 - 760*X^6*Z + 5328*X^5 - 167776*X^4*Z - 167776*X^4 - 661680*X^3*Z - 13537024*X^2 - 21204096*X*Z$

$- 21204096*X + 329482752*Z$

$T(3)= X^8 + 28*X^7*Z + 28*X^7 - 11350*X^6*Z + 224524*X^5 - 35893881*X^4*Z - 35893881*X^4 - 500392440*X^3*Z - 21077052876*X^2 -$

$340328307888*X*Z - 340328307888*X - 1189005516288*Z$

$T(5)= X^8 - 192*X^7 - 308690*X^6 + 64761480*X^5 + 22132512425*X^4 - 5345908156200*X^3 - 192752481265000*X^2 + 100779892364460000*$

$X - 5307832734751350000$

$T(7)= X^8 + 196*X^7*Z + 3829354*X^6*Z + 3829354*X^6 - 151949196*X^5 + 4118389058657*X^4*Z + 357519216541536*X^3*Z + 357519216541536$

$*X^3 - 1371206296183633312*X^2 + 2660606177663166609920*X*Z - 8966244354382355872000*Z - 8966244354382355872000$

$T(11)= X^8 - 5052*X^7*Z - 5052*X^7 - 64608570*X^6*Z - 213696447588*X^5 - 1154441639089281*X^4*Z - 1154441639089281*X^4 +$

$826179486061178160*X^3*Z - 4455829409680165990096*X^2 + 257175043836419528961024*X*Z$

$+ 257175043836419528961024*X+ 3247334958272486704198877184*Z$

THE VALUE ETA(ROOT)=Z SATISFIES: $Z^2 - Z + 1 = 0$

$T(2)= X^7 + X^6*Z + X^6 - 638*X^5*Z - 1888*X^4*Z + 944*X^4 + 108336*X^3*Z - 108336*X^3 + 142416*X^2*Z - 284832*X^2 + 3398688*X$

$-1247616*Z - 1247616$

$T(3)= X^7 + 26*X^6*Z - 26*X^6 + 9062*X^5*Z - 131496*X^4 + 24886233*X^3*Z - 24886233*X^3 - 170028990*X^2*Z - 21435733404*X$

$- 24783330888*Z + 24783330888$

$$T(5) = X^7 + 446 X^6 Z - 223 X^6 + 188999 X^5 + 95032930 X^4 Z - 47516465 X^4 + 1918851600 X^3 + 3990835680000 X^2 Z - 1995417840000 X^2 - 366159769200000 X - 11879271244800000 Z + 5939635622400000$$

$$T(7) = X^7 - 924 X^6 Z + 1848 X^6 + 2928726 X^5 Z - 2928726 X^5 + 5733261912 X^4 Z - 2866630956 X^4 - 1997882120097 X^3 Z - 2291877869815296 X^2 Z - 2291877869815296 X^2 + 308321892235345248 X - 4993473182305167770560 Z + 998694636461033541120$$

$$T(11) = X^7 - 2172 X^6 Z - 2172 X^6 - 54695366 X^5 Z + 159513023256 X^4 Z - 79756511628 X^4 + 587416572414417 X^3 Z - 587416572414417 X^3 - 727653230918975328 X^2 Z + 1455306461837950656 X^2 - 389588053028986330272 X - 40062112421493368344320 Z - 40062112421493368344320$$
