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FREDHOLM CONDITIONS FOR INVARIANT OPERATORS: FINITE ABELIAN GROUPS AND BOUNDARY VALUE PROBLEMS

ALEXANDRE BALDARE, RÉMI CÔME, MATTHIAS LESCH, AND VICTOR NISTOR

We dedicate this paper to Professor Dan Voiculescu on the occasion of his 70th birthday

Abstract. We answer the question of when an invariant pseudodifferential operator is Fredholm on a fixed, given isotypical component. More precisely, let Γ be a compact group acting on a smooth, compact, manifold M without boundary and let $P \in \psi^m(M; E_0, E_1)$ be a Γ-invariant, classical, pseudodifferential operator acting between sections of two Γ-equivariant vector bundles E_0 and E_1. Let α be an irreducible representation of the group Γ. Then P induces by restriction a map $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ between the α-isotypical components of the corresponding Sobolev spaces of sections. We study in this paper conditions on the map $\pi_\alpha(P)$ to be Fredholm. It turns out that the discrete and non-discrete cases are quite different. Additionally, the discrete abelian case, which provides some of the most interesting applications, presents some special features and is much easier than the general case. Moreover, some results are true only in the abelian case. We thus concentrate in this paper on the case when Γ is *finite abelian*. We prove then that the restriction $\pi_\alpha(P)$ is Fredholm if, and only if, P is "α-elliptic," a condition defined in terms of the principal symbol of P (Definition 1.1). If P is elliptic, then P is also α-elliptic, but the converse is not true in general. However, if Γ acts freely on a dense open subset of M, then P is α-elliptic for the given fixed α if, and only if, it is elliptic. The proofs are based on the study of the structure of the algebra $\psi^m(M; E)^\Gamma$ of classical, Γ-invariant pseudodifferential operators acting on sections of the vector bundle $E \to M$ and of the structure of its restrictions to the isotypical components of Γ. These structures are described in terms of the isotropy groups of the action of the group Γ on $E \to M$.

Contents

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1. Introduction

Fredholm operators have many applications to Mathematical Physics, to Partial Differential Equations (linear and non-linear), to Geometry, and to other areas of mathematics. They have their origin in the work of several mathematicians on spectral theory and on integral equations at the end of the nineteenths century. Fredholm operators are ubiquitous in applications since on a compact manifold, a classical pseudodifferential operator is Fredholm (acting between suitable Sobolev spaces) if, and only if, it is elliptic.

In this paper, we obtain *an analogous result for the restriction of an invariant, classical pseudodifferential operator to a fixed isotypical component for the action of a finite abelian group* Γ*.* Of course, a Γ-invariant operator is Fredholm if, and only if, its restrictions to *all* isotypical components are Fredholm. Our result, focuses on one *fixed* given isotypical component. Namely, the restriction to the isotypical component corresponding to an irreducible representation α of Γ is Fredholm if, and only if, the operator is α-elliptic (Definition 1.1 and Theorem 1.2 below). The reasons we assume our group to be abelian and discrete are, first, that the main result is no longer correct as stated in the non-discrete case and, second, that the general discrete case is quite different (and much more difficult) than the abelian discrete case. Moreover, some useful intermediate results are true only in the abelian case and this is the case needed for applications to boundary value problems.

Although in the formulation of our main result we do not use C^*-algebras, for its proof, we have found it convenient to use them. Recently, there were quite a few papers using C^*-algebras to obtain Fredholm conditions, see, for instance, [17, 19, 30, 32] among many others. Often groupoids were also used [13, 20, 36, 44]. Fredholm conditions play an important role in Quantum Mechanics in the study of the essential spectrum of N-body Hamiltonians [6, 24, 23, 27, 29]. A powerful related technique is that of "limit operators" [28, 34, 35, 43]. Some of the most recent papers using similar ideas include [4, 12, 13, 14, 37, 38, 39, 51], to which we refer for further references. Besides C^*-algebras, pseudodifferential operators were also often used to obtain Fredholm conditions, see [18, 31, 26] and the references therein.

Let us now explain our main result in more detail.

1.1. **Γ-principal symbol and α-ellipticity.** In general, for any compact group G, we let $\widehat{G}$ denote the set of equivalence classes of irreducible G-modules (or representations), as usual. It is a finite set if G is finite. If $T : V \to W$ is a G-equivariant map of G-modules and $\alpha \in \widehat{G}$, we let

$$\pi_\alpha(T) : V_\alpha \to W_\alpha \tag{1}$$

denote by the induced map obtained by restricting and corestricting T to the corresponding isotypical component. Our main result, Theorem 1.2 is stated in terms of the Γ-principal symbol and α-ellipticity, two concepts which we now introduce.

Let Γ be a *finite abelian* group acting on a smooth, compact, boundaryless manifold M and let $P \in \psi^m(M; E_0, E_1)$ be a Γ-invariant classical pseudodifferential operator acting between sections of two Γ-equivariant vector bundles E_0 and E_1. Let $\alpha \in \widehat{\Gamma}$ and consider as above

$$\pi_\alpha(P) \,:\, H^s(M; E_0)_\alpha \,\to\, H^{s-m}(M; E_1)_\alpha\,, \tag{2}$$

acting between the α-isotypical components of the corresponding Sobolev spaces of sections. The main question that we answer in this paper is to determine when $\pi_\alpha(P)$ is Fredholm in terms of its principal symbol

$$\sigma_m(P) \in \Gamma(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1))\,, \tag{3}$$

regarded as a homogeneous function on the cotangent bundle of M. Note that for Γ abelian, $\widehat{\Gamma}$ consists of characters, that is, of group morphisms $\Gamma \to \mathbb{C}^*$. The reason for restricting to the case Γ finite abelian is that it presents some special features, although some, but not all, of our results extend to the case Γ finite (possibly non-abelian) and even to the case Γ compact. The case Γ abelian provides some of the most important applications and is much easier than the general case, so, for the sake of the clarity and brevity of the presentation, we will assume in our main result that Γ is abelian. Moreover, some of our results are not true in the non-abelian case, in general, and the main result (Theorem 1.2) is not true for non-discrete groups (Corollary 2.5 of [2]).

For simplicity, we will consider only classical pseudodifferential operators in this article [33, 48, 50]. Recall that a classical pseudodifferential operator P is called *elliptic* if its principal symbol is invertible (away from the zero section). If P is elliptic, then it is Fredholm, and hence $\pi_\alpha(P)$ is also Fredholm. The converse is not true, however, in general. Indeed, we introduce, for any irreducible representation α of Γ, an "α-principal symbol" $\sigma_m^\alpha(P)$ (Definition 1.1) and prove that $\pi_\alpha(P)$ is Fredholm if, and only if, its α-principal symbol is invertible (in which case we call P *α-elliptic*, see Theorem 1.2 below for the precise statement). As we have just noticed, in general, the invertibility of the α-principal symbol does not imply ellipticity. To state these results in more detail, we need to introduce some notation and terminology.

The Γ-invariance of P implies that its principal symbol is also Γ invariant:

$$\sigma_m(P) \in \Gamma(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma\,.$$

To study the space of morphisms $\Gamma(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma$ in which the principal symbol $\sigma_m(P)$ lives, let

$$\Gamma_\xi \,:=\, \{\gamma \in \Gamma \mid \gamma\xi = \xi\} \tag{4}$$

denote the isotropy group of a $\xi \in T_x^*M$ in Γ, $x \in M$, as usual. The isotropy Γ_x of $x \in M$ is defined similarly. Then the groups $\Gamma_\xi \subset \Gamma_x$ act on E_{0x} and on E_{1x}, the fibers of $E_0, E_1 \to M$ at x. If $q \in \Gamma(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma$, then $q(\xi) \in \mathrm{Hom}(E_{0x}, E_{1x})^{\Gamma_\xi}$. As we will see below, there is no loss of generality for our main result to assume that $E_0 = E_1 = E$, in which case $\mathrm{Hom}(E_0, E_1) = \mathrm{End}(E)$.

Let us consider the space

$$X_{M,E,\Gamma} := \{(\xi,\rho) \mid \xi \in T^*M \smallsetminus \{0\},\ \rho \in \widehat{\Gamma}_\xi,\ \text{and}\ \mathrm{Hom}(\rho, E_x)^{\Gamma_\xi} \neq 0\}\,. \tag{5}$$

Let also $\rho \in \widehat{\Gamma}_\xi$ be an irreducible representation of Γ_ξ, then

$$\hat{q}(\xi,\rho) := \pi_\rho(q(\xi)) \in \mathrm{End}(E_{x\rho})^{\Gamma_\xi} \tag{6}$$

denotes the restriction of $q(\xi)$ to the isotypical component corresponding to ρ, with π_ρ defined in Equation (1). Thus $\hat{q}$ is a function on $X_{M,E,\Gamma}$. Applying this construction to $\sigma_m(P) \in \Gamma(T^*M \smallsetminus \{0\}; \mathrm{End}(E))^\Gamma$, we obtain the function

$$\sigma_m^\Gamma(P) := \widehat{\sigma_m(P)} : X_{M,E,\Gamma} \to \bigcup_{(x,\rho) \in X_{M,E,\Gamma}} \mathrm{End}(E_{x\rho})^{\Gamma_\xi}\,. \tag{7}$$

That is,

$$\sigma_m^\Gamma(P)(\xi,\rho) := \pi_\rho(\sigma_m(P)(\xi)) \in \mathrm{End}(E_{x\rho})^{\Gamma_\xi}\,, \quad \xi \in T_x^*M\,. \tag{8}$$

The characterization of Fredholm operators can be reduced to each component of the manifold. We shall therefore assume for the rest of this Introduction and beginning with Subsection 4.2 that our manifold M is connected. This simplifies also the statements and the proof of our results. Let then Γ_0 be a minimal isotropy group for the connected manifold M (see Subsection 2.4.3).

The α-principal symbol $\sigma_m^\alpha(P)$ of P, $\alpha \in \widehat{\Gamma}$, is defined in terms of $\sigma_m^\Gamma(P)$, but in order to define it, we need an additional crucial ingredient that takes α into account. For $\alpha \in \widehat{\Gamma}$ and $\rho \in \widehat{\Gamma}_\xi$, we will say that α and ρ are Γ_0-*disjoint* if $\mathrm{Hom}_{\Gamma_0}(\rho,\alpha) = 0$, otherwise, we will say that they are Γ_0-*associated*. Let

$$X_{M,E,\Gamma}^\alpha := \{(\zeta,\rho) \in X_{M,E,\Gamma} \mid \rho \text{ and } \alpha \text{ are } \Gamma_0\text{-associated}\ \}\,. \tag{9}$$

Let us assume for the rest of this introduction that Γ is *abelian*. Then we have that $\alpha \in \widehat{\Gamma}$ and $\rho \in \widehat{\Gamma}_\xi$ are Γ_0-associated if, and only if, their restrictions to Γ_0 coincide, that is, if $\alpha|_{\Gamma_0} = \rho|_{\Gamma_0}$. We can now define the *α-principal symbol* $\sigma_m^\alpha(P)$ of P.

Definition 1.1. The *α-principal symbol* $\sigma_m^\alpha(P)$ of P is the restriction of $\sigma_m^\Gamma(P)$ to $X_{M,E,\Gamma}^\alpha$:

$$\sigma_m^\alpha(P) := \sigma_m^\Gamma(P)|_{X_{M,E,\Gamma}^\alpha}\,.$$

We shall say that $P \in \psi^m(M;E)$ is *α-elliptic* if its α-principal symbol $\sigma_m^\alpha(P)$ is invertible everywhere on its domain of definition. This definition extends right away to operators in $\psi^m(M;E_0,E_1)$.

1.2. **Statement of the main result.** Thus P is α-elliptic if, and only if, $\sigma_m^\Gamma(P)$ is invertible on $X_{M,E,\Gamma}^\alpha$. We are ready now to state our main result.

Theorem 1.2. *Let Γ be a finite* abelian *group acting on a smooth, compact manifold M and let $P \in \psi^m(M;E_0,E_1)$ be a classical pseudodifferential operator acting between sections of two Γ-equivariant bundles $E_0, E_1 \to M$, $m \in \mathbb{R}$, and $\alpha \in \widehat{\Gamma}$. We have that $\pi_\alpha(P) : H^s(M;E_0)_\alpha \to H^{s-m}(M;E_1)_\alpha$ is Fredholm if, and only if P is α-elliptic.*

If Γ acts without fixed points on a *dense* open subset of M, then $\Gamma_0 = 1$, and hence $X_{M,E,\Gamma} = X^{\alpha}_{M,E,\Gamma}$ for all $\alpha \in \widehat{\Gamma}$. Hence, in this case, P is α-elliptic if, and only if, it is elliptic. The ellipticity of P can thus be checked in this case simply by looking at a single isotypical component. We stress, however, that if Γ is not discrete, this statement, as well as the statement of our main result (Theorem 1.2 above), are not true anymore. However, our main result, as well as many intermediate results hold for general finite groups and some even for compact Lie groups. The extension of our main result to general finite groups is work in progress [5], but the proof seems to be much more involved.

A motivation for our result comes from index theory. Let us assume that P is Γ-invariant and elliptic. Atiyah and Singer have determined, for any $\gamma \in \Gamma$, the value at γ of the character of $\mathrm{ind}_\Gamma(P) \in R(G)$, that is they have computed $ch_\gamma(\mathrm{ind}_\Gamma(P)) \in \mathbb{C}$ in terms of data at the fixed points of γ on M [3]. (Here $R(G) := \mathbb{Z}^{\widehat{G}}$ is the representation ring of G.)

Brüning [9, 10] considered the "isotypical heat trace" $\mathrm{tr}(p_\alpha e^{-t\Delta})$, which is nothing but the heat trace of $\pi_\alpha(\Delta)$, and its short time asymptotic expansion. Clearly, carrying out Brüning's programme in full would lead to a heat equation proof of an index theorem for the α isotypical component of Dirac type operators. However, the technical obstacles for this approach are enormous.

1.3. **Contents of the paper.** Let us quickly describe here the contents of our paper. We start in Section 2 with some preliminaries. We thus recall some facts about groups, most notably Frobenius reciprocity (for finite groups) and the definitions of induced representations, of minimal isotropy groups (for connected M) and of the principal orbit bundle. We also review some notions concerning the primitive ideal spectrum of C^*-algebras, as well as basic facts concerning (equivariant) pseudodifferential operators.

In Section 3, we compute the image of the algebra $\overline{\psi^{-1}}(M; E)$ of regularizing operators via π_α. We do this by proving some general results on the structure of C^*-algebras with an inner action of our group Γ. When the action of the group Γ is inner, the results and their proofs become simpler.

Let $A_M := \mathcal{C}_0(M; \mathrm{End}(E))$. The main difficulties arise in Section 4. There, we identify the primitive spectrum of the C^*-algebra A_M^Γ of Γ-invariant symbols with the set $X_{M,E,\Gamma}/\Gamma$ described above. Some care is taken to describe the corresponding topology on $X_{M,E,\Gamma}/\Gamma$. We then consider the projection from A_M^Γ to the Calkin algebra of $L^2(M; E)_\alpha$ and show that the closed subset of $\mathrm{Prim}\, A_M^\Gamma$ associated to its kernel is $X^{\alpha}_{M,E,\Gamma}/\Gamma$. These results are used in Section 5 to prove the main result of the paper, Theorem 1.2. We also discuss an application to mixed boundary value problems and explain why our result is not true when the group Γ is not discrete,

The last named author (V.N.) thanks Max Planck Institute for support while this research was performed. Since this paper is dedicated to Professor Dan Voiculescu, the last named author would like to mention that his papers, most notably [41, 42, 52], and [53] have played an important role in this author's formation as a mathematician in his early years. Moreover, we are happy to dedicate to Voiculescu this paper in which we prove a result that does not explicitly use C^*-algebras, but whose proof uses in an essential way the theory of these algebras. This was the spirit of interdisciplinarity that Voiculescu was promoting while he was a member of

INCREST, an institute that is now called the Institute of the Roumanian Academy (IMAR).

2. Preliminaries

We begin by setting up the terminology and the notation used in this paper. We also recall some basic results that are needed in the sequel.

Throughout the paper, Γ will be a compact group acting on a locally compact space M. For the most part, M will be a smooth Riemannian manifold and Γ will be a *compact Lie group* acting smoothly and isometrically on M. The final result holds only for discrete (thus finite) groups and M compact, but many intermediate results hold in greater generality, so we have tried to state the results in the greatest generality possible when this did not involve too much extra work. In particular, we shall start with a compact group Γ acting on a possibly non-compact Riemannian manifold[1] M. Eventually, we shall assume that M is compact and that Γ is discrete, hence finite. Moreover, since the case Γ abelian is simpler and presents some special features, for the final result we will assume that Γ is abelian.

2.1. **Group representations.** We follow the standard conventions, see [7, 47], to which we refer for further references and unexplained concepts.

2.1.1. *Group actions on sets.* Let us assume now that Γ is a group that acts on a set M. If $x \in M$, then Γx denotes the Γ *orbit* of x and Γ_x denotes the *isotropy* group (of M) at x, that is

$$\Gamma_x := \{\gamma \in \Gamma \mid gx = x\} \subset \Gamma\,. \tag{10}$$

We shall write $H \sim H'$ if the subgroups H and H' are conjugated in Γ. If $H \subset \Gamma$ is a subgroup, we shall denote by $M_{(H)}$ the set of elements of M whose isotropy Γ_x is conjugated to H (in Γ), i.e. $H \sim \Gamma_x$.

2.1.2. *Representations and isotypical components.* Let V be a locally convex space and $\mathcal{L}(V)$ denote the set of *continuous* linear maps $V \to V$. Let Γ now be a *compact topological group* and $\rho : \Gamma \to \mathcal{L}(\mathcal{H}_\rho)$ be a *strongly continuous* representation in a *complete, locally convex topological* vector space $\mathcal{H}_\rho$, in the sense that, for each $\xi \in \mathcal{H}_\rho$, the map $\Gamma \ni \gamma \to \rho(\gamma)\xi \in \mathcal{H}_\rho$ is continuous. We shall say then that $\mathcal{H}_\rho$ is a Γ-module and we shall often drop ρ from the notation, thus $\mathcal{H} = \mathcal{H}_\rho$ and $\gamma\xi := \rho(\gamma)\xi$. If Γ is a discrete group (as it will be the case for our final results), the continuity conditions will, of course, be automatically satisfied.

For any two Γ-modules $\mathcal{H}$ and $\mathcal{H}_1$, we shall denote by

$$\mathrm{Hom}_\Gamma(\mathcal{H}, \mathcal{H}_1) \,=\, \mathrm{Hom}(\mathcal{H}, \mathcal{H}_1)^\Gamma \,=\, \mathcal{L}(\mathcal{H}, \mathcal{H}_1)^\Gamma$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}_1$ that commute with the action of Γ, that is, $T(\gamma\xi) = \gamma T(\xi)$ for all $\xi \in \mathcal{H}$ and $\gamma \in \Gamma$.

Finally, we denote by $\hat{\Gamma}$ the set of equivalence classes of irreducible unitary Γ-modules. We shall need the following terminology.

Definition 2.1. Let $H \subset \Gamma_1$ and $H \subset \Gamma_2$ be compact groups. Two irreducible representations $\alpha_i \in \widehat{\Gamma}_i$, $i = 1, 2$, are called *H-associated* if $\mathcal{L}(\alpha_1, \alpha_2)^H \neq 0$. Otherwise, we shall say that they are *H-disjoint*.

[1]The metric on M is, however, for convenience only.

If Γ_i, $i = 1, 2$, are both abelian, then the irreducible representations α_i are characters, that is, morphisms $\alpha_i : \Gamma_i \to \mathbb{C}^*$, and we have that they are associated if, and only if, $\alpha_1|_H = \alpha_2|_H$.

2.1.3. *Isotypical component.* Let $\mathcal{H}$ be a Γ-module and $\alpha \in \hat{\Gamma}$. Then p_α will denote the Γ-invariant projection onto the α-isotypical component $\mathcal{H}_\alpha$ of $\mathcal{H}$, defined as the largest (closed) Γ submodule of $\mathcal{H}$ that is isomorphic to a multiple of α. In other words, $\mathcal{H}_\alpha$ is the sum of all Γ-submodules of $\mathcal{H}$ that are isomorphic to α. Equivalently, since Γ is compact, we have $\mathcal{H}_\alpha \simeq \alpha \otimes \mathrm{Hom}_\Gamma(\alpha, \mathcal{H})$; moreover

$$\mathcal{H}_\alpha \neq 0 \ \Leftrightarrow \ \mathrm{Hom}_\Gamma(\alpha, \mathcal{H}) \neq 0 \ \Leftrightarrow \ \mathrm{Hom}_\Gamma(\mathcal{H}, \alpha) \neq 0. \tag{11}$$

If $T \in \mathcal{L}(\mathcal{H})$ commutes with the action of Γ (i.e. it is a Γ-module morphism), then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we denote by

$$\pi_\alpha : \mathcal{L}(\mathcal{H})^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha), \quad \pi_\alpha(T) := T|_{\mathcal{H}_\alpha}, \tag{12}$$

the associated morphism and operator, as in Equation (1) of the Introduction. The morphism π_α will play a crucial role in what follows.

2.1.4. *Convolution algebras.* The algebra $\mathcal{C}(\Gamma)$ of continuous functions on Γ, endowed with the convolution product, will act on any Γ-module $\mathcal{H}$. If, moreover, $\mathcal{H}$ is a Hilbert space and Γ acts by unitary operators, we shall say then that $\mathcal{H}$ is a *unitary* Γ*-module.* We endow Γ with a fixed Haar measure and let $C_r^*(\Gamma)$ be the completion of $\mathcal{C}(\Gamma)$ acting on $L^2(\Gamma)$, which is a unitary Γ-module. Then $C_r^*(\Gamma)$ will act on any unitary Γ-module $\mathcal{H}$, since Γ is compact (and hence amenable: $C_r^*(\Gamma) = C^*(\Gamma)$).

For any algebra A, we shall denote by $Z(A)$ the *center* of A, that is, the set of elements $z \in A$ that commute with all other elements $a \in A$. An element $z \in Z(A)$ will be called *central* (in A). For instance, if $\alpha \in \widehat{\Gamma}$, then its character defines a central projection $p_\alpha \in Z(\mathcal{C}(\Gamma)) \subset Z(C^*(\Gamma))$. More explicitely, for any $\gamma \in \Gamma$ we have that

$$p_\alpha(\gamma) = \chi_\alpha(\gamma) := \mathrm{tr}(\alpha(\gamma)).$$

Given a representation ρ of Γ on $\mathcal{H}$, the image of p_α is then

$$\rho(p_\alpha) = \int_\Gamma \chi_\alpha(g)\rho(g)dg,$$

where integration is against the Haar measure. We are interested in this projection since $\mathcal{H}_\alpha = p_\alpha \mathcal{H}$.

2.2. **Induction and Frobenius reciprocity.** We now review some basic definitions and results for induced representations. We will use induction *for finite groups only,* so we assume in this discussion of the Frobenius reciprocity (i.e. in this subsection) that Γ is finite.

2.2.1. *Definition of the induced module.* Since we are assuming in this subsection that Γ is finite, we have that $C^*(\Gamma) = \mathcal{C}(\Gamma) = \mathbb{C}[\Gamma]$, the group algebra of Γ. We will use the standard notation $V^{(I)} := \{f : I \to V\}$, valid for I finite. If $H \subset \Gamma$ is a subgroup (hence also finite) and V is a H-module, we let

$$\mathrm{Ind}_H^\Gamma(V) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V \simeq \{\, \xi : \Gamma \to V \mid f(gh^{-1}) = hf(g) \,\} \simeq V^{(\Gamma/H)} \tag{13}$$

be the *induced representation* from V. The last isomorphism is obtained by choosing a set of representatives of the right cosets Γ/H. The action of Γ on $\mathrm{Ind}_H^\Gamma(V)$ is by left multiplication on $\mathbb{C}[\Gamma]$, and the indicated isomorphism is an isomorphism of Γ-modules. The Γ-module $\mathrm{Ind}_H^\Gamma(V)$ depends functorially on V.

Remark 2.2. If V is an algebra and the group H acts on V by algebra homomorphisms, then the isomorphism $\mathrm{Ind}_H^\Gamma(V) \simeq \{\xi : \Gamma \to V \mid f(gh^{-1}) = hf(g)\}$ shows that $\mathrm{Ind}_H^\Gamma(V)$ is an algebra for the pointwise product. If V_1 is a left V-module (with a structure compatible with the action of Γ), then $\mathrm{Ind}_H^\Gamma(V_1)$ is a $\mathrm{Ind}_H^\Gamma(V)$ module, again with the pointwise multiplication. The induction is moreover compatible with morphisms of modules and algebras (change of scalars), again by the function representation of the induced representation. In particular, if $\phi : V \to W$ is a H-morphism of algebras, if V_1 and W_1 are modules over these algebras, and $\psi : V_1 \to W_1$ is a H-module morphism such that $\psi(ab) = \phi(a)\psi(b)$, then, if *mult* denotes the multiplication map, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{ind}_H^\Gamma(V) \otimes \mathrm{ind}_H^\Gamma(V_1) & \xrightarrow{\phi\otimes\psi} & \mathrm{ind}_H^\Gamma(W) \otimes \mathrm{ind}_H^\Gamma(W_1) \\ {\scriptstyle mult}\downarrow & & \downarrow{\scriptstyle mult} \\ \mathrm{ind}_H^\Gamma(V_1) & \xrightarrow{\psi} & \mathrm{ind}_H^\Gamma(W_1)\,. \end{array} \tag{14}$$

(All maps, including the multiplications, are assumed to be compatible with the action of Γ.)

2.2.2. *Explicit isomorphisms.* We will use the following form of the *Frobenius reciprocity*: the map

$$\begin{aligned} \Phi = \Phi_{H,V}^{\Gamma,\mathcal{H}} : \mathrm{Hom}_H(\mathcal{H}, V) \; &\to \; \mathrm{Hom}_\Gamma(\mathcal{H}, \mathrm{Ind}_H^\Gamma(V))\,, \\ \Phi(f)(\xi) \; &:= \; \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} f(g^{-1}\xi)\,, \end{aligned} \tag{15}$$

is an isomorphism ($\xi \in \mathcal{H}$, $f \in \mathrm{Hom}_H(\mathcal{H}, V)$). This version of the Frobenius reciprocity is not valid in general, but is valid for finite groups [7, 47]. Often one writes $\mathrm{Hom}_H(\mathrm{Res}_{\mathrm{H}}^\Gamma(\mathcal{H}), V)$ instead of $\mathrm{Hom}_H(\mathcal{H}, V)$. Let α be an irreducible representation of Γ, $H \subset \Gamma$ be a subgroup, and β an irreducible representation of H. Frobenius reciprocity gives, in particular, that the multiplicity of α in $\mathrm{ind}_H^\Gamma(\beta)$ is the same as the multiplicity of β in the restriction of α to H. In particular, α is contained in $\mathrm{ind}_H^\Gamma(\beta)$ if, and only if, β is contained in the restriction of α to H i.e. α and β are *H-associated*. Furthemore, by taking $\mathcal{H}$ to be the trivial Γ-module $\mathbb{C}$, we obtain an isomorphism

$$\begin{aligned} \Phi : V^H \; &= \; \mathrm{Hom}_H(\mathbb{C}, V) \simeq \mathrm{Hom}_\Gamma(\mathbb{C}, \mathrm{Ind}_H^\Gamma(V)) \; = \; \mathrm{Ind}_H^\Gamma(V)^\Gamma\,, \\ \Phi(\xi) \; &:= \; \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} \xi \; = \; \sum_{x\in\Gamma/H} x \otimes \xi\,. \end{aligned} \tag{16}$$

The chosen normalization in the definition of Φ is such that it is an isomorphism of algebras if V is an algebra.

2.2.3. *The abelian case.* If Γ is abelian and V is an irreducible H-module, then the action of H on V is via scalars: $h \cdot v = \chi_V(h)v$, for some group morphism (i.e. character) $\chi_V : H \to \mathbb{C}^*$. The induced module $\mathrm{Ind}_H^\Gamma(V)$ splits then as the direct sum of the irreducible Γ modules β that are H-associated to V. If χ is a character of Γ, we shall denote by V_χ the H-module equal to $\mathbb{C}$ as a vector space, with the action of $h \in H$ given by $h \cdot v = \chi(h)v$.

Lemma 2.3. *Assume that Γ is a finite abelian group and that H is a subgroup of Γ. Let V be an irreducible H-module corresponding to the character $\chi_V : H \to \mathbb{C}^*$. Then*

$$\mathrm{Ind}_H^\Gamma(V) \simeq \bigoplus_{\substack{\chi \in \hat{\Gamma}, \\ \chi|_H = \chi_V}} \mathrm{Ind}_H^\Gamma(V)_\chi.$$

Moreover, by writing $\mathrm{Ind}_H^\Gamma(V) \simeq \mathbb{C}[\Gamma/H] \otimes V$ as vector spaces, we obtain an action of $\widehat{\Gamma/H}$ on $\mathrm{Ind}_H^\Gamma(V)$ by the formula

$$\rho \cdot (\gamma H \otimes v) := \rho(\gamma H)\gamma H \otimes v, \quad \gamma \in \Gamma,\ v \in V, \quad \text{and} \quad \rho \in \widehat{\Gamma/H}\,.$$

This action maps $\mathrm{Ind}_H^\Gamma(V)_\chi$ to $\mathrm{Ind}_H^\Gamma(V)_{\rho\chi}$.

Proof. Given $\chi \in \widehat{\Gamma}$, we have by the Frobenius isomorphism that χ is contained in $\mathrm{Ind}_H^\Gamma(V)$ if, and only if, $\chi|_H = \chi_V$. If A is a finite abelian group, we have the (non-canonical) isomorphism $\widehat{A} \simeq A$ of groups. There are $|\widehat{\Gamma/H}| = |\Gamma/H|$-many characters χ of Γ with the property $\chi|_H = \chi_V$. It follows, by counting dimensions, that they all appear with multiplicity one in $\mathrm{Ind}_H^\Gamma(V)$. The statement about the action of $\widehat{\Gamma/H}$ is proved by a direct computation. This proof is complete. □

2.2.4. *Inducing endomorphism modules.* Let now α be an irreducible Γ-module, let β_j, $j = 1, \ldots, N$, be non-isomorphic irreducible H-modules (with H a subgroup of Γ, as above), and

$$\beta := \oplus_{j=1}^N \beta_j^{k_j}\,. \tag{17}$$

If H is abelian, then each β_j is one dimensional, and hence H acts by scalars on each $\beta_j^{k_j}$.

We want to study the algebra $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma$ acting on $\mathrm{Ind}_H^\Gamma(\beta)$ and on its isotypical component $p_\alpha(\mathrm{Ind}_H^\Gamma(\beta)) = \mathrm{Ind}_H^\Gamma(\beta)_\alpha$ (see 2.1.3 for the definition of the projection p_α). We have, by the Frobenius isomorphism and by the form of the H-module β, that

$$\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq \oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H \simeq \oplus_{j=1}^N M_{k_j}(\mathbb{C}), \tag{18}$$

which is a semi-simple algebra. Moreover, we have that $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma = \Phi(\mathrm{End}(\beta)^H)$, where Φ is the map of Equation (16). From the properties of the induction functor Ind_H^Γ, we also have that $\mathrm{Ind}_H^\Gamma(\beta) = \oplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$.

Lemma 2.4. *Let $\beta := \oplus_{j=1}^N \beta_j^{k_j}$ be as in Equation (17), let*

$$T = (T_j) \in \mathrm{End}(\beta)^H \simeq \oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H\,,$$

with $T_j \in \mathrm{End}(\beta_j^{k_j})^H$, and let $\xi_j \in \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$. We let

$$\xi := (\xi_j) \in \oplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j}) \simeq \mathrm{Ind}_H^\Gamma(\beta)\,.$$

Then $\Phi(T)(\xi) = (\Phi(T_j)\xi_j)_{j=1,\ldots,N}$.

In other words, the Frobenius isomorphism Φ of Equation (16) is compatible with direct sums and with the action of morphisms on modules.

Proof. This follows from the naturality of the product, the isomorphism

$$\operatorname{Ind}_H^\Gamma(\oplus_{j=1}^N \operatorname{End}(\beta_j^{k_j}))^\Gamma \xrightarrow{\sim} \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta))^\Gamma \,, \tag{19}$$

and Remark 2.2 (especially Equation (14)). □

Put differently, the simple factor of the algebra $\operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta))^\Gamma$ corresponding to $\operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma$ acts only on the jth component of $\oplus_{i=1}^N \operatorname{Ind}_H^\Gamma(\beta_i^{k_i}) = \operatorname{Ind}_H^\Gamma(\beta)$. The following proposition will play a crucial role in what follows.

Proposition 2.5. *Let $\beta := \oplus_{j=1}^N \beta_j^{k_j}$ be as in Equation* (17)*. Let $J \subset \{1, 2, \ldots, N\}$ be the set of indices j such that $\alpha|_H = \beta_j$ (i.e. α and β_j are H-associated). Then the morphism*

$$\pi_\alpha : \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta))^\Gamma \simeq \oplus_{j=1}^N \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma \to \operatorname{End}(p_\alpha \operatorname{Ind}_H^\Gamma(\beta))^\Gamma$$

is such that we have natural isomorphisms

$$\ker(\pi_\alpha) \simeq \oplus_{j\notin J} \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma \quad \textit{and} \quad \operatorname{Im}(\pi_\alpha) \simeq \oplus_{j\in J} \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma \,.$$

Proof. By Lemma 2.4, we can assume that $N = 1$. By Lemma 2.3, $\operatorname{Ind}_H^\Gamma(\beta_1^{k_1}) \simeq \oplus_{\chi\in\widehat{\Gamma}} V_\chi$, with $\chi|_H = \beta_1$. We obtain

$$p_\alpha \operatorname{Ind}_H^\Gamma(\beta_1^{k_1}) \simeq \begin{cases} \alpha^{k_1} & \text{if } \alpha|_H = \beta_1 \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Thus, if $\alpha|_H \neq \beta_1$, $J = \emptyset$, $\pi_\alpha = 0$, and hence

$$\begin{aligned} \ker(\pi_\alpha) &= \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_1^{k_1}))^\Gamma = \oplus_{j\notin J} \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma \quad \text{and} \\ \operatorname{Im}(\pi_\alpha) &= 0 = \oplus_{j\in J} \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_j^{k_j}))^\Gamma \,, \end{aligned}$$

as claimed.

Let us assume now that that $\alpha|_H = \beta_1$. The morphism π_α can then be written as the composition

$$\begin{aligned} \pi_\alpha : \operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta_1^{k_1}))^\Gamma &\simeq \operatorname{End}(\beta_1^{k_1})^H \simeq M_{k_1}(\mathbb{C}) \xrightarrow{\Psi_\alpha} M_{k_1}(\mathbb{C}) \\ &\simeq \operatorname{End}(\alpha^{k_1})^\Gamma \simeq \operatorname{End}(p_\alpha \operatorname{Ind}_H^\Gamma(\beta_1^{k_1}))^\Gamma \,. \end{aligned}$$

To prove our proposition in this case ($\chi|_H = \beta_1$, and hence in general, by Lemma 2.4), it thus suffices to show that $\Psi_\alpha = id$, which equivalent to the fact that $\Psi_\alpha \neq 0$ for $\alpha|_H = \beta_1$.

To prove this (that $\Psi_\alpha \neq 0$ for $\alpha|_H = \beta_1$), let us begin by noticing that the morphism $\operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta)) \to \operatorname{End}(\operatorname{Ind}_H^\Gamma(\beta))$ is injective. Hence $\operatorname{Ind}_H^\Gamma(\operatorname{End}(\beta))^\Gamma \to \operatorname{End}(\operatorname{Ind}_H^\Gamma(\beta))^\Gamma$ is injective as well. This means that not all of the maps Ψ_ρ, with $\rho|_H = \beta_1$, can be zero, by Lemma 2.3. On the other hand, it can be checked by a direct calculation that the action of $\widehat{\Gamma/H}$ on $\operatorname{Ind}_H^\Gamma(\beta)$ of the same lemma permutes the morphisms Ψ_ρ. It follows either that they are all zero or that they are all non-zero. As their direct sum is non-zero, it follows that all $\Psi_\rho \neq 0$, $\rho|_H = \beta_1$. □

2.3. **The primitive ideal spectrum.** Let us begin by recalling a few basic facts about C^*-algebras [21]. Recall that a two-sided ideal $I \subset A$ of a C^*-algebra A is called *primitive* if it is the kernel of an irreducible, non-zero $*$-representation of A (so A is *not* considered to be a primitive ideal of itself). We will denote by $\mathrm{Prim}(A)$ the *primitive ideal spectrum* of A, which, we recall, is defined as the set of primitive ideals of A. For instance, if $A = \mathcal{C}_0(X)$, the space of continuous functions $X \to \mathbb{C}$ vanishing at infinity on the locally compact space, then we have a natural identification (homeomorphism)

$$\mathrm{Prim}(\mathcal{C}_0(X)) \simeq X \,, \tag{21}$$

which is an identification that lies at the heart of non-commutative geometry [15, 16]. All the C^*-algebras considered in this paper are type I. The relevance of this fact, for us, is that, if A is type I, then $\mathrm{Prim}(A)$ identifies with the set of isomorphism classes of irreducible representations of A. Any C^*-algebra with only finite dimensional representations is a type I algebra [21]. All the algebras considered in this paper (except the algebras of compact operators on various Hilbert spaces), have this property.

Remark 2.6. The following example will be used several times. Let H be a finite group and $\beta = \oplus_{j=1}^N \beta_j^{k_j}$ be a finite dimensional H-module with β_j non-isomorphic simple H-modules. Since $\mathrm{Hom}_H(\beta_j^{k_j}, \beta_j^{k_j}) \simeq M_{k_j}(\mathbb{C})$ and $\mathrm{Hom}(\beta_i^{k_i}, \beta_j^{k_j})^H = 0$ for $i \neq j$ by the assumption that the simple H-modules β_i and β_j are non-isomorphic, we obtain

$$\begin{aligned}\mathcal{L}(\beta)^H \,=\, \mathrm{End}_H(\beta) \,\simeq\, \mathrm{Hom}_H(\oplus_i \beta_i^{k_i}, \oplus_j \beta_j^{k_j}) \,\simeq\, \oplus_{i,j} \mathrm{Hom}_H(\beta_i^{k_i}, \beta_j^{k_j}) \\ \simeq\, \oplus_j \mathrm{End}_H(\beta_j^{k_j}) \,\simeq\, \oplus_j M_{k_j}(\mathbb{C})\,.\end{aligned}$$

The algebra $\mathcal{L}(\beta)^H = \mathrm{End}_H(\beta)$ is thus a C^*-algebra with only finite dimensional representations and we have natural bijections

$$\mathrm{Prim}(\mathrm{End}_H(\beta)) \,\leftrightarrow\, \{\beta_1, \beta_2, \ldots, \beta_N\} \,\leftrightarrow\, \{1, 2, \ldots, N\}\,.$$

The algebra $\mathcal{L}(\beta)^H$ is thus a semi-simple complex algebra with simple factors $\mathrm{End}_H(\beta_j^{k_j}) \simeq M_{k_j}(\mathbb{C})$, $j = 1, 2, \ldots, N$.

We shall need the connection between ideals and the topology of the primitive ideal spectrum, which we now recall. Let J be a closed, two-sided ideal of A. Then

$$\begin{aligned}\mathrm{Prim}(J) \,&=\, \{I \in \mathrm{Prim}(A) |\, J \not\subset I\} \ \text{ and} \\ \mathrm{Prim}(A/J) \,&\cong\, \{I \in \mathrm{Prim}(A) |\, J \subset I\} \,=\, \mathrm{Prim}(J)^c\,.\end{aligned} \tag{22}$$

The second bijection sends an ideal $I \supset J$ to $\pi_{A,J}^{-1}(I)$, where $\pi_{A,J} : A \to A/J$ is the canonical bijection. In the following, we shall identify the two sets using this bijection.

In analogy with the Zariski topology, $\mathrm{Prim}(A)$ is endowed with the Jacobson (or hull-kernel) topology, which is the topology whose open sets are those of the form $\mathrm{Prim}(J)$. Conversely, given an open subset $U \subset \mathrm{Prim}(A)$, then $U = \mathrm{Prim}(J_U)$, where $J_U := \cap_{\mathfrak{P} \in \mathrm{Prim}(A) \smallsetminus U} \mathfrak{P}$. For us, it will be convenient to regard these properties as a *one-to-one* correspondence between the closed, two-sided ideals of A and the closed subsets $\mathrm{Prim}(A/J) = \mathrm{Prim}(A) \smallsetminus \mathrm{Prim}(J)$ of $\mathrm{Prim}(A)$. If $I \subset A$ is an ideal such that $a \in A$ and $aI = 0$ implies $a = 0$, then I is called an *essential ideal* of A and its associated open set $\mathrm{Prim}(J)$ is dense in $\mathrm{Prim}(A)$.

Let Z be a commutative C^*-algebra and $\phi : Z \to M(A)$ be a *-morphism to the multiplier algebra of A [1, 11]. Assume that $\phi(Z)$ commutes with A and $\phi(Z)A = A$. Then Schur's lemma gives that there exists a natural continuous map $\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$, which we shall call also the "central character map" (associated to ϕ).

2.4. **Lie group actions on manifolds.** *We will assume from now on that* Γ *is a compact Lie group, that* M *is a Riemannian manifold, and that* Γ *acts smoothly and isometrically on* M.

2.4.1. *Slices and tubes.* Let us fix $x \in M$ arbitrarily. Then its orbit Γx is a smooth, compact submanifold of M. Let N_x be the orthogonal of $T_x(\Gamma x)$ in $T_x M$. (If Γ is discrete, as in our main results, then, of course, $N_x = T_x M$.) Then the isotropy group Γ_x acts linearly and isometrically on N_x. For $r > 0$, let $U_x := (N_x)_r$ denote the set of vectors of length $< r$ in N_x. It is known then that, for $r > 0$ small enough, the exponential map gives a Γ-equivariant isometric diffeomorphism

$$W_x \simeq \Gamma \times_{\Gamma_x} U_x := \{(\gamma, y) \in \Gamma \times U_x \mid (\gamma h, y) \equiv (\gamma, hy),\, h \in \Gamma_x\}\,, \tag{23}$$

where W_x is a Γ-invariant neighborhood of x in M. More precisely, W_x is the set of $y \in M$ at distance $< r$ to the orbit Γx, if $r > 0$ is small enough. The set W_x is called a *tube* around x (or Γx) and the set U_x is called the *slice* at x. The range of r depends, of course, on x, namely it must satisfy $0 < r < r_x$, for some $r_x > 0$. We assume that, for each $x \in M$, such an $r \in (0, r_x)$ has been chosen and we will keep fixed the notation for the slice U_x and for the tube W_x associated to x and to the fixed choice of r.

2.4.2. *Equivariant vector bundles.* Let $E \to M$ be a Γ-equivariant smooth vector bundle. All our vector bundles will be assumed to be finite dimensional. Let us fix $x \in M$ and consider the tube $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ around x of Equation (23). We use this diffeomorphism to identify U_x with a subset of M, in which case, we can also assume the restriction of E to the slice U_x to be trivial. More precisely,

$$\begin{aligned} E|_{U_x} &\simeq U_x \times \beta \ \text{ and} \\ E|_{W_x} &\simeq \Gamma \times_{\Gamma_x} (U_x \times \beta)\,, \end{aligned} \tag{24}$$

for some Γ_x-module β, the second isomorphism being Γ-equivariant.

Let $F \to Y$ be a Hermitian vector bundle on a locally compact measure space. Then $L^2(Y; F)$ denotes the set of (equivalence classes) of square integrable sections of the vector bundle F and $\mathcal{C}_0(Y; F)$ denote the set of its continuous sections vanishing at infinity.

2.4.3. *The principal orbit bundle.* Let us assume from now on that M is connected, except where explicitly stated otherwise. The reason of this assumption is that it is known then [49] that there exists a *minimal isotropy* subgroup $\Gamma_0 \subset \Gamma$, in the sense that $M_{(\Gamma_0)}$ is a dense open subset of M. (Recall that $M_{(H)}$ denotes the set of points of M whose stabilizer is conjugated in Γ to H.) In particular, there exist minimal elements for inclusion for the set of isotropy groups of points in M and all minimal isotropy groups are conjugated (to a fixed subgroup, denoted Γ_0 in what follows). The notation Γ_0 will remain fixed from now on. By the definition, the set $M_{(\Gamma_0)}$ consists of the points whose stabilizer is conjugated to that minimal subgroup. The set $M_{(\Gamma_0)}$ is called the *principal orbit bundle* of M.

If $x \in M_{(\Gamma_0)}$, then Γ_x acts *trivially* on the slice U_x at x, by the minimality of Γ_0. (See 2.4.1 for notation.) In general, for $x \in M$, the isotropy of Γ_x acting on U_x will contain a subgroup conjugated to Γ_0.

If Γ is abelian, there is only one minimal isotropy group Γ_0 (recall that we are assuming M to be connected). Moreover, we can then factor the action of Γ to an action of Γ/Γ_0 on M, which has trivial minimal isotropy, that is, it is free on a dense, open subset of M.

2.5. **Pseudodifferential operators.** We continue to assume that Γ is a compact Lie group that acts smoothly and isometrically on a smooth Riemannian manifold M. We let $\psi^m(M;E)$ denote the space of order m, *classical* pseudodifferential operators on M with *compactly supported* distribution kernel. Recall that, in this article, we consider only classical pseudodifferential operators. We let $\overline{\psi^0}(M;E)$ and $\overline{\psi^{-1}}(M;E)$ denote the norm closures of $\psi^0(M;E)$ and $\psi^{-1}(M;E)$, respectively. The action of Γ then extends to an action on E and on $\psi^m(M;E)$, $\overline{\psi^0}(M;E)$, and $\overline{\psi^{-1}}(M;E)$. We will denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$. We will write $\mathcal{K}$ instead of $\mathcal{K}(\mathcal{H})$ when the Hilbert spaces $\mathcal{H}$ is clear from the context. We have

$$\overline{\psi^{-1}}(M;E) \,=\, \mathcal{K}(L^2(M;E))\,, \tag{25}$$

since we have considered only pseudodifferential operators with compactly supported distribution kernels.

Let S^*M denote the *unit cosphere bundle* of M, that is, the set of unit vectors in T^*M, as usual. We will denote, as usual, by $\mathcal{C}_0(S^*M;\mathrm{End}(E))$ the set of continuous sections of the *lift* of the vector bundle $\mathrm{End}(E) \to M$ to S^*M.

Corollary 2.7. *We have an exact sequence*

$$0 \,\to\, \mathcal{K}^\Gamma = \overline{\psi^{-1}}(M;E)^\Gamma \,\to\, \overline{\psi^0}(M;E)^\Gamma \,\xrightarrow{\sigma_0}\, \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \,\to\, 0\,.$$

Proof. Recall then that the principal symbol extends to a continuous, surjective, Γ-equivariant map

$$\sigma_0 \,=\, \sigma_0^M : \overline{\psi^0}(M;E) \,\to\, \mathcal{C}_0(S^*M;\mathrm{End}(E)) \tag{26}$$

with kernel $\overline{\psi^{-1}}(M;E) = \mathcal{K} = \mathcal{K}(L^2(M;E))$. In other words, we have the following well known exact sequence

$$0 \,\to\, \mathcal{K} \,\to\, \overline{\psi^0}(M;E) \,\xrightarrow{\sigma_0}\, \mathcal{C}_0(S^*M;\mathrm{End}(E)) \,\to\, 0.$$

The exact sequence of the corollary is obtained from the fact that the functor $\mathcal{H} \to \mathcal{H}^\Gamma$ is exact on the category of Γ–modules with continuous Γ-action, since Γ is compact. $\square$

2.6. **Reduction to order-zero operators.** We assume here that M is a compact Riemannian manifold with metric g. Let us fix a Γ-invariant metric on $E \to M$. Let ∇ be a metric preserving, Γ-invariant connection on E. In what follows $\Delta := \Delta_g^E = \nabla^*\nabla$ will denote the (positive) Laplacian on M with coefficients in E. Since M is compact, the Sobolev space $H^s(M;E)$ can be defined for any $s \in \mathbb{R}$ as the domain of $(1+\Delta)^{s/2}$ and $(1+\Delta)^{s/2} : H^s(M;E) \to L^2(M;E)$ is an isomorphism.

Lemma 2.8. *Assume that M is compact. We have that a bounded operator $P : H^s(M;E) \to H^{s-m}(M;E)$ is Fredholm if, and only if, $\widetilde{P} := (1+\Delta)^{(s-m)/2}P(1+\Delta)^{-s/2} : L^2(M;E) \to L^2(M;E)$ is Fredholm. Moreover, if P is the limit in*

$\mathcal{L}(H^s(M;E), H^{s-m}(M;E))$ *of a sequence of operators* $P_n \in \psi^m(M;E)$, *then* $\widetilde{P} \in \overline{\psi^0}(M;E)$.

Proof. The first part follows from the fact $(1+\Delta)^s : H^m(M;E) \to H^{m-2s}(M;E)$ is an isomorphism for all $m, s \in \mathbb{R}$, since M is compact. The second part follows from the well known fact that the powers of the Laplace operator are classical pseudodifferential operators [46] and the continuity of the multiplication in the operator norm. □

The case of operators in $\psi^m(M;E,F)$ acting between *two* vector bundles $E, F \to M$ can be reduced to the case of a single vector bundle by considering $\psi^m(M;E\oplus F)$. Moreover, $P \in \overline{\psi}^0(M;E,F) \subset \overline{\psi}^0(M;E\oplus F)$ will be Fredholm if, and only if,

$$\begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix} \in \overline{\psi}^0(M;E\oplus F)$$

is Fredholm (here all operators act on L^2-spaces). Therefore it is sufficient to consider only the case of order-zero operators acting on a single vector bundle.

3. The structure of regularizing operators

We continue to assume that M is a complete Riemannian manifold and that Γ is a compact Lie group acting by isometries on M. From now on, all our vector bundles will be Γ-equivariant vector bundles.

As explained in the Introduction, we want to identify the structure of the restrictions of Γ-invariant pseudodifferential operators on M to the isotypical components of $L^2(M;E)$. Let π_α be this restriction morphism to the α-isotypical component. More precisely, we want to understand the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$, for any fixed $\alpha \in \Gamma$. See Equations (1) and (12) for the definition of the restriction morphism π_α and of the projectors $p_\alpha \in C^*(\Gamma)$.

In this section, we study two basic cases: that of inner actions and that of free actions (of Γ).

3.1. **Inner actions of Γ: the abstract case.** In this subsection we deal with the case when the action of Γ is implemented by unitaries in the multiplier algebra (the case of inner actions). This allows us, in particular, to settle the case of regularizing operators. We shall need the following notion of direct sum. Recall that $M(A)$ denotes the multiplier algebra of a C^*-algebra A. Recall the following standard definition.

Definition 3.1. Let $\phi : \Gamma \to \mathrm{Aut}(A)$ be the action of a group Γ by automorphisms on a C^*-algebra A. We shall say that this action is *inner* if the morphism ϕ lifts to a morphism $\psi : \Gamma \to U(M(A))$ to the group of unitary elements of $M(A)$ such that

$$\phi_g(a) = \psi(g)a\psi(g)^{-1}, \quad a \in A, \ g \in \Gamma.$$

Remark 3.2. If $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an inner action as in Definition 3.1, then we obtain, in particular, a morphism $C^*(\Gamma) \to M(A)$. If, moreover, $\pi : A \to \mathcal{L}(\mathcal{H})$ is a $*$-representation, then it extends to a representation of $M(A)$. This induces naturally a unitary representation of Γ on $\mathcal{H}$. This representation is uniquely determined if π is non-degenerate (i.e. if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$).

If A_n, $n \geq 1$ is a sequence of C^*-algebras, we shall denote by $c_0 - \oplus_{n=1}^{\infty} A_n$ the inductive limit $\lim_{N\to\infty} \oplus_{n=1}^{N} A_n$. This definition extends immediately to countable families of C^*-algebras. Recall that $\widehat{\Gamma}$ denotes the set of isomorphism classes of irreducible unitary representations of Γ. We shall need then the following general result.

Proposition 3.3. *Let A be a C^*-algebra with a *-morphism $C^*(\Gamma) \to M(A)$ and let $p_\alpha \in C^*(\Gamma)$ denote the central projector corresponding to $\alpha \in \widehat{\Gamma}$. Let $A^\Gamma := \{a \in A \,|\, af = fa,\ f \in C^*(\Gamma)\}$. Then*

$$A^\Gamma \simeq c_0 - \oplus_{\alpha\in\hat{\Gamma}}\, p_\alpha A^\Gamma \,. \tag{27}$$

*If $I \subset A$ is a closed two-sided ideal, then $C^*(\Gamma)I \subset I$ and hence we obtain *-morphisms $C^*(\Gamma) \to M(I)$ and $C^*(\Gamma) \to M(A/I)$ such that the induced sequence*

$$0 \to p_\alpha I^\Gamma \to p_\alpha A^\Gamma \to p_\alpha (A/I)^\Gamma \to 0$$

is exact for any $\alpha \in \widehat{\Gamma}$.

Proof. Let us arrange the elements of $\widehat{\Gamma}$ in a sequence ρ_n, $n \geq 1$. Then, for any $a \in A$, $\lim_{N\to\infty} \sum_{n=1}^{N} p_{\rho_n} a = a$. Since, for any $a \in A^\Gamma$, we have $p_\alpha a = a p_\alpha$, the isomorphism $A^\Gamma \simeq c_0 - \oplus_{\alpha\in\hat{\Gamma}} A^\Gamma p_\alpha$ follows. Finally, it is known that if I is a two-sided ideal of A, then $M(A)I \subset I$. □

If $A \subset \mathcal{L}(\mathcal{H})$ is a sub-C^*-algebra of the algebra of bounded operators on a Hilbert space $\mathcal{H}$ together with a compatible Γ-module structure on $\mathcal{H}$, we let $\pi_\alpha : A^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)$ be the restriction morphism to the α-isotypical component, as before, see Equations (1) and (12).

Proposition 3.4. *In addition to the assumptions of Proposition 3.3, let us suppose that $A \subset \mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Let $\alpha \in \widehat{\Gamma}$, as before. Then the morphism $\pi_\alpha : A^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)$ restricts to an isomorphism $p_\alpha A^\Gamma \to \pi_\alpha(A^\Gamma)$.*

Proof. Let us recall that, as explained in Remark 3.2, the group Γ will be represented on $\mathcal{H}$. The rest follows from Proposition 3.3, whose notation we shall use freely. Indeed, if $\alpha \neq \beta \in \widehat{\Gamma}$, then $\pi_\alpha(p_\beta) = 0$. On the other hand $\pi_\alpha(p_\alpha) = p_\alpha$. The result follows by combining this property with $\mathcal{H}_\alpha = p_\alpha \mathcal{H}$ and with Proposition 3.3. □

3.2. **Inner actions of Γ: pseudodifferential operators.** We now apply the results of the previous subsection to the algebra of pseudodifferential operators. In particular, this gives a rather complete picture for the case of negative order operators (recall that the closure of regularizing operators coincides with that of negative order operators). Since we are eventually interested only in the case Γ finite, we discuss only briefly the issues related to the non-discrete case (such as the continuity of the action of Γ).

A crucial first observation is that if $\gamma \in \Gamma$ and $P \in \psi^{-\infty}(M;E)$, then $\gamma P, P\gamma \in \psi^{-\infty}(M;E)$. This leads to several interesting consequences. We record this as a lemma.

Lemma 3.5. *We have $\gamma\psi^{-\infty}(M;E) = \psi^{-\infty}(M;E)\gamma = \psi^{-\infty}(M;E)$, for all $\gamma \in \Gamma$, and the induced actions (to the right and to the left) of Γ on $\psi^{-\infty}(M;E)$ are continuous and unitary in the operator norm. Consequently, we have*

$$C^*(\Gamma)\overline{\psi^{-1}}(M;E) + \overline{\psi^{-1}}(M;E)C^*(\Gamma) \subset \overline{\psi^{-1}}(M;E)\,.$$

Proof. Since smoothing operators have smooth kernels $k(x,y) \in \mathrm{Hom}(E_y, E_x)$, the action of Γ induced an action on $\psi^{-\infty}(M;E)$. Thanks to the Γ-invariance of the metric on M, we have $\|\gamma P\| = \|P\gamma\| = \|P\|$ in the L^2-operator norm, for any $P \in \psi^{-\infty}(M;E)$. The second part follows from the first part. □

We then have the following:

Proposition 3.6. *The multiplication by Γ on $\overline{\psi^{-1}}(M;E)$ defines a *-morphism $C^*(\Gamma) \to U(M(\overline{\psi^{-1}}(M;E)))$ to the multiplier algebra of $\overline{\psi^{-1}}(M;E)$.*

Proof. This follows from Lemma 3.5. □

We obtain the following corollary.

Corollary 3.7. *Let $A = \overline{\psi^{-1}}(M;E)$. If Γ acts trivially on M (so that the action of $C^*(\Gamma)$ extends to the algebra $\overline{\psi^0}(M;E)$), we also allow $A = \overline{\psi^0}(M;E)$ or $A = \mathcal{C}_0(S^*M;\mathrm{End}(E))$. We then have isomorphisms*

$$A^\Gamma \;\simeq\; c_0 - \oplus_{\alpha\in\widehat{\Gamma}}\, p_\alpha A^\Gamma\,.$$

Moreover, if Γ acts trivially on M and $\alpha \in \hat{\Gamma}$, we have an exact sequence

$$0 \to p_\alpha\overline{\psi^{-1}}(M;E)^\Gamma \to p_\alpha\overline{\psi^0}(M;E)^\Gamma \to \mathcal{C}_0(S^*M;\mathrm{End}(p_\alpha E)^\Gamma) \to 0\,.$$

Proof. The first part follows from Proposition 3.3 applied to the algebra $\overline{\psi^0}(M;E)$ and to its ideal $\mathcal{K}$. The second part follows from the exactness of the functors $V \to V^\Gamma$ and $V \to p_\alpha V$ on the category of Γ-modules. □

Let $\alpha \in \hat{\Gamma}$ and let π_α be the representation of $\overline{\psi^0}(M;E)^\Gamma$ on $L^2(M;E)_\alpha$ defined by restriction as before, Equations (1) and (12). The assumptions of Proposition 3.4 are satisfied for $A = \overline{\psi^{-1}}(M;E)$, so we obtain the following.

Corollary 3.8. *The morphism π_α restricts to an isomorphism from $p_\alpha\overline{\psi^{-1}}(M;E)^\Gamma$ to $\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$.*

We also have the following simple result, which makes the last corollary more precise. Recall that $\mathcal{K} \simeq \overline{\psi^{-1}}(M;E)$. This allows us to better describe the structure of $\overline{\psi^{-1}}(M;E)^\Gamma$.

Proposition 3.9. *The algebra $\pi_\alpha(\mathcal{K}^\Gamma)$ is the algebra of Γ–equivariant compact operators on $L^2(M;E)_\alpha$.*

Proof. Let $T \in \mathcal{K}$ commute with Γ. Then its restriction to a Γ-invariant subspace is still compact and still commutes with Γ. This shows that $\pi_\alpha(\mathcal{K}^\Gamma)$ is contained in the set K_α of Γ-invariant compact operators on $L^2(M;E)_\alpha$. Conversely, $K_\alpha \subset \mathcal{K}^\Gamma$ and π_α acts as the identity on K_α. □

3.3. **The case of free actions.** Let us now tackle the opposite case, that is when Γ acts freely on M. We shall assume that Γ is finite, for simplicity (we only need this case), and hence, in particular, the action of Γ is proper. We have then the following well-known result (see [16, 40] and the references therein).

Proposition 3.10. *Let us assume that Γ is a finite group acting freely on M and let $\alpha(\gamma) = 1$ be the trivial representation, so $\pi_\alpha = \pi_1$. Let us denote by $F := E/\Gamma \to M/\Gamma$ the resulting vector bundle and $\phi : \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \to$*

$\mathcal{C}_0(S^*M/\Gamma; \mathrm{End}(E/\Gamma))$ *the resulting isomorphism. Then we have the following morphism of exact sequences, with the vertical arrows surjective.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\psi^{-1}}(M;E)^\Gamma & \longrightarrow & \overline{\psi^{0}}(M;E)^\Gamma & \longrightarrow & \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \overline{\psi^{-1}}(M/\Gamma,F) & \longrightarrow & \overline{\psi^{-0}}(M/\Gamma,F) & \longrightarrow & \mathcal{C}_0(S^*M/\Gamma;\mathrm{End}(F)) & \longrightarrow & 0 \end{array}$$

For $G \subset M$, let $A_G := \mathcal{C}_0(S^*G; \mathrm{End}(E))$ and consider the *surjective* map

$$\begin{aligned} (28) \quad \mathcal{R}_G : A_G^\Gamma := \mathcal{C}_0(S^*G;\mathrm{End}(E))^\Gamma &\simeq \overline{\psi^0}(G;E)^\Gamma / \overline{\psi^{-1}}(G;E)^\Gamma \\ &\to \pi_\alpha(\overline{\psi^0}(G;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(G;E)^\Gamma)\,. \end{aligned}$$

Proposition 3.11. *Let Γ be a finite group acting on a smooth compact manifold M (without boundary). Assume that the action of Γ is free on a dense, open subset of M. Let $E \to M$ be an equivariant vector bundle. Then the map $\mathcal{R}_M : \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \to \pi_1(\overline{\psi^0}(M;E)^\Gamma)/\pi_1(\overline{\psi^{-1}}(G;E)^\Gamma)$ of Equation* (28) *is injective, and hence an isomorphism of algebras.*

Proof. Let $M_0 \subset M$ be an open, dense subset on which Γ acts freely. Proposition 3.10 for M replaced with M_0 shows that $\mathcal{R}_{M_0}$ is injective. Hence $\mathcal{R}_M$ is injective on $\mathcal{C}_0(S^*M_0;\mathrm{End}(E))^\Gamma$, because the restriction of $\mathcal{R}_M$ to $\mathcal{C}_0(S^*M_0;\mathrm{End}(E))^\Gamma$ is $\mathcal{R}_{M_0}$. Since the later is an essential ideal in $\mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma$, it follows that $\mathcal{R}_M$ is also injective (everywhere on $\mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma$). □

4. The principal symbol

Let us fix an irreducible representation α of Γ and consider the fundamental restriction morphism π_α of Equation (1). See also Subsection 2.1.2, especially Equation (12), for more details on the morphism π_α. We are mostly concerned with the morphism $\pi_\alpha : \overline{\psi^0}(M;E)^\Gamma \to \mathcal{L}(L^2(M;E)_\alpha)$ and, in this section, we identify the quotient

$$\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)\,.$$

Since $\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ was identified in the previous section, information on the above quotient algebra will give further insight into the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions.

In the beginning of this section, we continue to assume that M is a complete Riemannian manifold and that Γ is a compact Lie group acting by isometries on M. Since the results are different in the discrete and non-discrete case, we will assume beginning with Subsection 4.2 that Γ is finite. Moreover, for the main result, we shall assume that Γ is abelian, since the abelian case is simpler and presents some additional features. Some intermediate results are true only in the abelian case. It is also the abelian case that will be used for our application to boundary value problems.

4.1. **The primitive ideal spectrum of the symbol algebra.** We now turn to the description of the primitive ideal spectrum of the algebra $\mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma$ of symbols. For simplicity, for $O \subset M$ open, we denote again $A_O := \mathcal{C}_0(S^*O;\mathrm{End}(E))$, as in the definition of the morphism $\mathcal{R}_O$ of Equation (28). We shall be mostly concerned with the cases $O = M$ and $O = O_0 := M_{(\Gamma_0)}$. We have the following standard result.

Proposition 4.1. *The algebra $Z_M := \mathcal{C}_0(S^*M)^\Gamma = \mathcal{C}_0(S^*M/\Gamma)$ identifies with a central subalgebra of $A_M^\Gamma := \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$. Let z_ξ be the maximal ideal of Z_M associated to the orbit $\Gamma\xi$ for some $\xi \in S_x^*M$, let E_x be a fiber of E corresponding to ξ, and let $E_x \simeq \oplus_{j=1}^N \beta_j^{k_j}$ be its decomposition into Γ_ξ-isotypical components, with β_j simple, non-isomorphic Γ_ξ modules. Then $A_M^\Gamma/z_\xi A_M^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}$ is a semi-simple, finite-dimensional (complex) algebra with N simple factors $\mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j}) \simeq M_{k_j}(\mathbb{C})$, $j \in \{1, 2, \ldots, N\}$.*

Proof. We have $\mathcal{C}_0(M) \subset \mathcal{C}_0(M; \mathrm{End}(E)) \subset \mathrm{End}(\mathcal{C}_0(M; E))$, with $f \in \mathcal{C}_0(M)$ acting as a scalar on each fiber E_x. In fact, this identifies $\mathcal{C}_0(M)$ with the center $Z(\mathcal{C}_0(M; \mathrm{End}(E)))$ of $\mathcal{C}_0(M; \mathrm{End}(E))$. By considering Γ invariant functions, we obtain that $Z_M := \mathcal{C}_0(S^*M)^\Gamma = \mathcal{C}_0(S^*M/\Gamma)$ is contained in the center $Z(A_M^\Gamma)$ of $A_M^\Gamma := \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$. Let $\Gamma\xi$ denote the orbit in S^*M that we consider and let J be the (non-maximal, in general) ideal of $\mathcal{C}_0(S^*M)$ corresponding to functions vanishing on this orbit. Then J is Γ invariant and $J^\Gamma = z_\xi$. By taking the Γ invariants in the exact sequence $0 \to JA_M \to A_M \to A_M/JA_M \to 0$ and using Frobenius reciprocity for $A_M/JA_M \simeq \mathrm{Ind}_{\Gamma_\xi}^\Gamma(\mathrm{End}(E_x))$, we obtain that

$$A_M^\Gamma/z_\xi A_M^\Gamma \simeq (A_M/JA_M)^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}. \tag{29}$$

The proof is completed using Remark 2.6 for $H = \Gamma_\xi$. □

See [8, 22, 25, 45, 54] for similar results. Recall that if $\phi : Z \to M(A)$ is a central *-morphism (i.e. $\phi(z)a = a\phi(z)$, for $a \in A$ and $z \in Z$) such that $\phi(Z)A = A$, then it defines a natural "central character" map $\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$ by Schur's Lemma. The same proof yields the following.

Corollary 4.2. *There is a one-to-one correspondence between the primitive ideals of $A_M^\Gamma := \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$ and the Γ-orbits of the pairs (ξ, ρ), where $\xi \in S_x^*M$ and $\rho \in \widehat{\Gamma}_\xi$ appears in E_x (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$). The group Γ acts by joint conjugation on both ξ and ρ. The inclusion $Z_M := \mathcal{C}_0(S^*M)^\Gamma \to A_M^\Gamma$ is such that the associated canonical central character map of spectra*

$$\mathrm{Prim}(A_M^\Gamma) \to \mathrm{Prim}(Z_M) = S^*M/\Gamma$$

is continuous, finite-to-one, and maps the orbit $\Gamma(\xi, \rho)$ to the orbit $\Gamma\xi$.

Proof. Let π be an irreducible representation of $A_M^\Gamma := \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$. Then π is a multiple of a character on $Z_M := \mathcal{C}_0(S^*M)^\Gamma \subset Z(A_M^\Gamma)$, by Schur's lemma. Let this character correspond to the orbit $\Gamma\xi \in S^*M/\Gamma$, with $\xi \in S_x^*M$ and denote by z_ξ the corresponding maximal ideal of Z_M, as in the proof of the last lemma. In other words, z_ξ is the value of the central character map corresponding to the inclusion $Z_M \subset A_M^\Gamma$ applied to π. Then π factors out through an irreducible representation of $A_M^\Gamma/z_\xi A_M^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}$. Let us write $E_x \simeq \oplus_{j=1}^N \beta_j^{k_j}$ with β_j non-isomorphic simple Γ_ξ modules, as in the statement of Proposition 4.1. Then $\mathrm{End}(E_x)^{\Gamma_\xi} \simeq \oplus_{j=1}^N \mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j})$, a direct sum of simple algebras. Thus π factors through one of the simple algebras $\mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j})$ This associates to π the pair $(\xi, \rho) = (\xi, \beta_j)$, as desired. This pair is not unique, but depends on the choice of ξ. It becomes unique modulo the action of Γ, however. Conversely, given such a pair (ξ, ρ), we obtain an irreducible representation of A_M^Γ following exactly the same procedure in reverse order. The first part of the result follows.

To prove that ϕ^* is finite to one, we notice that, by construction, $\phi^*(\xi,\rho)=\xi$. Since only a finite number of (isomorphism classes of) simple Γ_ξ modules appears in E_x, the finiteness follows. $\square$

Remark 4.3. Let us denote by $X_{M,E,\Gamma}$ the set of pairs (ξ,ρ), where $\xi\in T_x^*M\setminus\{0\}$ and $\rho\in\widehat{\Gamma}_\xi$ appears in E_x (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho,E_x)\neq 0$), as in the statement of Corollary 4.2. The main result of that corollary is a natural bijection

$$X_{M,E,\Gamma}/\Gamma \simeq \mathrm{Prim}(A_M^\Gamma)\,. \tag{30}$$

This bijection can be explicitely described as follows: to an orbit $\Gamma(\xi,\rho)$ in $X_{M,E,\Gamma}$ is associated $\ker(\pi_{\xi,\rho})$ in $\mathrm{Prim}\,A_M^\Gamma$, where for any $f\in A_M^\Gamma$ we define $\pi_{\xi,\rho}(f)$ as the restriction of $f(\xi)$ to the ρ-isotypical component of E_x.

4.2. **Factoring out the minimal isotropy.** Recall that we are assuming M to be connected; in that case there is a minimal isotropy type for the action of Γ. We shall also assume from now on that Γ is abelian, for the reasons discussed in the Introduction. In particular, it is the case needed for our applications to boundary value problems and, moreover, some results are not true in the non-abelian case.

Let $\alpha\in\widehat{\Gamma}$ as before, and recall that we want to determine the structure of the quotient $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ of the restricted algebras to the α-isotypical component. To this end, recall the morphism

$$\mathcal{R}_M : A_M^\Gamma := \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \to \pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$$

of Equation (28). The main result of this subsection is to determine the kernel of this morphism.

The main reason why the abelian case is simpler than the general case is that in the abelian case all minimal isotropy subgroups of Γ acting on M coincide. The (unique) minimal isotropy subgroup of Γ acting on M will be denoted by Γ_0, as before. Recall that the set $O_0 := M_{(\Gamma_0)}$ of points $x\in M$ with isotropy $\Gamma_x=\Gamma_0$ is called the principal orbit bundle of M; it is a dense, open subset of M. For every (other) $x\in M$, we have $\Gamma_0\subset\Gamma_x$.

We obtain that the group Γ_0 acts trivially on M. Moreover, there exists a unitary group morphism (representation) $\Gamma_0\to\mathrm{End}(E)$ that implements the action of Γ_0 on $\overline{\psi^0}(M;E)$. Let $p_\beta^{(0)}\in C^*(\Gamma_0)$, $\beta\in\widehat{\Gamma}_0$, be the central projectors associated to the irreducible representations of Γ_0 (the additional exponent is to differentiate them from the projectors p_α, $\alpha\in\widehat{\Gamma}$). Corollary 3.7 then gives the exact sequence

$$0\to p_\beta^{(0)}\overline{\psi^{-1}}(M;E)^{\Gamma_0}\to p_\beta^{(0)}\overline{\psi^0}(M;E)^{\Gamma_0}\to\mathcal{C}_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^{\Gamma_0}\to 0\,.$$

Moreover,

$$\overline{\psi^0}(M;E)^{\Gamma_0} \simeq \oplus_{\beta\in\widehat{\Gamma}_0}\, p_\beta^{(0)}\overline{\psi^0}(M;E)^{\Gamma_0}\,.$$

(Here the direct sum is finite, so there is no need to include the "c_0"-specification like in Corollary 3.7.) Since the actions of Γ and Γ_0 commute, we can further take the Γ-invariants to obtain:

$$0\to p_\beta^{(0)}\overline{\psi^{-1}}(M;E)^{\Gamma}\to p_\beta^{(0)}\overline{\psi^0}(M;E)^{\Gamma}\to\mathcal{C}_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^{\Gamma}\to 0\,. \tag{31}$$

Moreover,

$$\overline{\psi^0}(M;E)^{\Gamma} \simeq \oplus_{\beta\in\widehat{\Gamma}}\, p_\beta^{(0)}\overline{\psi^0}(M;E)^{\Gamma}\,. \tag{32}$$

In particular, we have that

Lemma 4.4. *Let Γ be a finite abelian group and $E \to M$ a Γ-equivariant vector bundle over a smooth, compact, connected manifold M (thus without boundary). Let $\Gamma_0 \subset \Gamma$ be the minimal isotropy group. We have*

$$\mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma \ \simeq \ \oplus_{\beta \in \widehat{\Gamma}_0} \mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma$$

Proof. We successively have

$$\begin{aligned}\mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma \ &\simeq \ \big(\mathcal{C}_0(S^*M; \mathrm{End}(E))^{\Gamma_0}\big)^{\Gamma/\Gamma_0}\\ &\simeq \ \big(\mathcal{C}_0(S^*M; \mathrm{End}(E)^{\Gamma_0})\big)^{\Gamma/\Gamma_0} \ \simeq \ \oplus_{\beta \in \widehat{\Gamma}_0} \big(\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E)^{\Gamma_0})\big)^{\Gamma/\Gamma_0}\\ &\simeq \ \oplus_{\beta \in \widehat{\Gamma}_0} \mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma\,,\end{aligned}$$

where we have used that $\mathrm{Hom}(p_\beta^{(0)} E, p_{\beta'}^{(0)} E)^{\Gamma_0}$ for $\beta \neq \beta' \in \widehat{\Gamma}_0$. □

Let us record now the following corollary of Proposition 3.11.

Corollary 4.5. *Let Γ be a finite abelian group acting on a smooth, connected compact manifold M (without boundary). Let $E \to M$ be an equivariant vector bundle. Assume that minimal isotropy is trivial: $\Gamma_0 = 1$. Then the map $\mathcal{R}_M : \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma \to \pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(O;E)^\Gamma)$ of Equation* (28) *is injective, and hence an isomorphism of algebras.*

Proof. By replacing the action π of Γ on E with $\pi_0 := \pi\alpha^{-1}$, that is, with, $\pi_0(g)\xi := \alpha^{-1}(g)\pi(g)\xi$, we can assume that $\alpha = 1$. The action of Γ is moreover free on the dense open subset $M_{(1)} = M_{\Gamma_0}$ of M. Proposition 3.11 then allows us to conclude. □

We now turn to the main result of this section.

Theorem 4.6. *Let Γ be a finite abelian group acting on a smooth compact, connected manifold M (without boundary) and let $E \to M$ be a Γ-equivariant vector bundle. Then the kernel of the morphism*

$$\begin{aligned}\mathcal{R}_M : \oplus_{\beta \in \widehat{\Gamma}_0} \mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma \ \simeq \ \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma \ =: A_M^\Gamma&\\ \to \ \pi_\alpha(\overline{\psi^0}(O;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)&\end{aligned}$$

of Equation (28) *is $\oplus_{\beta \in \widehat{\Gamma}_0, \beta \neq \alpha'} \mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma$, where $\alpha' := \alpha|_{\Gamma_0}$. In particular, $\mathcal{R}_M(\mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma) \simeq \mathcal{C}_0(S^*M; \mathrm{End}(p_{\alpha'}^{(0)} E))^\Gamma$.*

Proof. It is enough to identify the action of $\mathcal{R}_M$ on each direct summand $\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma$ of $\mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$. We can thus study the action of the morphism $\mathcal{R}_M$ one isotypical component $\beta \in \widehat{\Gamma}_0$ at a time.

The relation between the central projectors of $C^*(\Gamma_0)$ and $C^*(\Gamma)$ is that $p_\beta^{(0)} = \sum_{\alpha|_{\Gamma_0} = \beta} p_\alpha$, $\beta \in \widehat{\Gamma}_0$. Of course, $p_\alpha p_{\alpha'} = 0$ if $\alpha \neq \alpha' \in \widehat{\Gamma}$. It follows that

$$\pi_\alpha(p_\beta^{(0)} P) \ = \ p_\alpha p_\beta^{(0)} P|_{L^2(M;E)_\alpha} \ = \begin{cases} p_\alpha P|_{L^2(M;E)_\alpha} & \text{if } \ \alpha|_{\Gamma_0} = \beta \\ 0 & \text{otherwise.}\end{cases} \tag{33}$$

This shows that $\mathcal{R}_M = 0$ on $\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)} E))^\Gamma$ if $\alpha|_{\Gamma_0} \neq \beta$.

On the other hand, for $\alpha|_{\Gamma_0} = \beta$, we shall show that $\mathcal{R}_M$ is an isomorphism on $\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$. By replacing the action π of Γ on E with $\pi_0 := \alpha^{-1}\pi$, that is, with, $\pi_0(g)\xi := \alpha^{-1}(g)\pi(g)\xi$, we can assume that Γ_0 acts trivially on $E_\beta = p_\beta^{(0)}E$ (we already know that Γ_0 acts trivially on M). We can then factor the action of Γ to an action of Γ/Γ_0, and thus assume that the minimal isotropy is trivial: $\Gamma_0 = 1$.

After these reductions, the orbit bundle of M is $O_0 := M_{(1)}$, and the action of Γ on O_0 is free (and proper since Γ is compact). Corollary 4.5 then shows that $\mathcal{R}_M$ is injective on $\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$. $\square$

We note that the component of $\sigma_m(P)$ in $\mathcal{C}_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$ is $\sigma_m^\alpha(P)$, $\alpha|_{\Gamma_0} = \beta$, that is, the restriction of $\sigma_0^\Gamma(P)$ to $X_{M,E,\Gamma}^\alpha$. In this regard, we notice that

$$\begin{cases} X_{M,E,\Gamma}^{\alpha} = X_{M,E,\Gamma}^{\alpha'} & \text{if } \alpha|_{\Gamma_0} = \alpha'|_{\Gamma_0} \\ X_{M,E,\Gamma}^{\alpha} \cap X_{M,E,\Gamma}^{\alpha'} = \emptyset & \text{otherwise.} \end{cases} \tag{34}$$

Let us denote $X_{M,E,\Gamma}^\beta = X_{M,E,\Gamma}^\alpha$ if $\alpha|_{\Gamma_0} = \beta$. This gives the *disjoint union* decomposition

$$X_{M,\Gamma,E} = \bigsqcup_{\beta \in \widehat{\Gamma}_0} X_{M,E,\Gamma}^\beta \,. \tag{35}$$

It would be interesting to establish an analogous relation in the nonabelian case.

5. Applications and extensions

We now prove the main result of the paper on the characterization of Fredholm operators and discuss some extensions of our results.

5.1. **Fredholm conditions.** We now turn to the proof of our main result. We assume that M is a compact smooth manifold. We have the following Γ–equivariant version of Atkinson's theorem.

Proposition 5.1. *Let V be a unitary Γ–module and P be a Γ–equivariant bounded operator on V. We have that P is Fredholm if, and only if, it is invertible modulo $\mathcal{K}(V)^\Gamma$, in which case, we can choose the parametrix (i.e. the inverse modulo the compacts) to also be Γ-invariant.*

Proof. This follows from the inclusion of C^*-algebras

$$\mathcal{L}(V)^\Gamma/\mathcal{K}(V)^\Gamma \subset \mathcal{L}(V)/\mathcal{K}(V).$$

It is a standard fact that, if $B \subset A$ is an inclusion of unital C^*-algebras, then an element $a \in B$ is invertible in A if, and only if, it is invertible in B [21, Proposition 1.3.10]. Therefore if $P \in \mathcal{L}(V)^\Gamma$, then its projection in $\mathcal{L}(V)^\Gamma/\mathcal{K}(V)^\Gamma$ is invertible if, and only if, it is invertible in the greater algebra $\mathcal{L}(V)/\mathcal{K}(V)$. By Atkinson's theorem, the latter is equivalent to P being Fredholm. $\square$

Since $\pi_\alpha(\mathcal{K}^\Gamma) = \pi_\alpha(\mathcal{K}(L^2(M;E)_\alpha))^\Gamma$ and $\overline{\psi^{-1}}(M;E) = \mathcal{K} := \mathcal{K}(L^2(M;E)_\alpha)$, we obtain the following corollary.

Corollary 5.2. *Let $P \in \overline{\psi^0}(M;E)^\Gamma$ and $\alpha \in \hat{\Gamma}$. We have that $\pi_\alpha(P)$ is Fredholm on $L^2(M;E)_\alpha$ if, and only if, $\pi_\alpha(P)$ is invertible modulo $\pi_\alpha(\mathcal{K}^\Gamma)$ in $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$.*

We are now in a position to prove the main result of this paper, Theorem 1.2.

Proof ot Theorem 1.2. Lemma 2.8 implies that we may assume $P \in \overline{\psi^0}(M; E)^\Gamma$. Corollary 5.2 then states that $\pi_\alpha(P)$ is Fredholm if, and only if, the image of its principal symbol $\sigma_0(P)$ is invertible in the quotient algebra

$$\mathcal{R}_M(A_M^\Gamma) = \pi_\alpha(\overline{\psi^0}(M; E)^\Gamma)/\pi_\alpha(\mathcal{K}^\Gamma).$$

We have shown the isomorphism $\mathrm{Prim}(A_M^\Gamma) \simeq X_{M,E,\Gamma}/\Gamma$ in Equation (30). Now Theorem 4.6 and the discussion following it identify the primitive spectrum of $\mathcal{R}_M(A_M^\Gamma)$, which is a closed subset of $\mathrm{Prim}(A_M^\Gamma)$, with the set $X_{M,E,\Gamma}^\alpha/\Gamma$. Recall that

$$X_{M,E,\Gamma}^\alpha = \{(\xi, \rho) \in T^*M \setminus \{0\} \times \widehat{\Gamma}_\xi \mid \rho_{|\Gamma_0} = \alpha_{|\Gamma_0}\},$$

as defined in the introduction.

Therefore $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi, \rho) \in X_{M,E,\Gamma}^\alpha$, i.e. if and only if P is α-elliptic. $\square$

5.2. **Boundary value problems.** In this subsection, we very briefly indicate an application to mixed boundary value problems. Let M be a smooth compact manifold with boundary and choose a tubular neighborhood $U \simeq [0, 1) \times \partial M$ of the boundary. Let M^d be the *double* of M along ∂M: as a topological space, the space M^d is the quotient of $M \times \{-1, 1\}$ by the subspace $\partial M \times \{-1, 1\}$. We shall denote $M_\pm := M \times \{\pm 1\}$. On M^d we consider the smooth structure such that

(1) the projections $p_\pm : M_\pm \to M$ are smooth maps, and
(2) the map $U^d \simeq (0, 1) \times \partial M$ is smooth.

Thus the smooth structure on M^d thus depends on our choice of tubular neighborhood. For any $x = (x', i) \in M^d$, we denote by $-x$ its symmetrical counterpart, i.e. $-x = (x', -i)$. Then the map $x \mapsto -x$ gives a natural smooth action of $\mathbb{Z}_2$ on M^d.

If $E \to M$ is a smooth vector bundle, then we define $E^d \to M^d$ as the smooth vector bundle obtained by gluing two copies of E on M_+ and M_- along ∂M. Then the $\mathbb{Z}_2$-action on M^d extends to an action on E^d, which maps an element $v \in E_x^d$ to its copy in E_{-x}^d, for any $x \in M^d$.

We generalize this construction to the case when we have a disjoint union decomposition of the boundary $\partial M = \partial_D M \cup \partial_N M$ into two disjoint, closed and open subsets. Then we double first with respect to the "Dirichlet" part of the boundary $\partial_D M$ and then with respect to the "Neumann" part of the boundary $\partial_N M$. We obtain accordingly an action of $\mathbb{Z}_2^2$ on the resulting manifold M^{dd}. We let this group act on the resulting vector bundle E^{dd} such that the action of the first component of $\mathbb{Z}_2$ is twisted (i.e. tensored) with its only non-trivial character, namely -1. We have the following standard lemma.

Lemma 5.3. *The restriction map $r_+ : \mathcal{C}^\infty(M^{dd}; E^{dd}) \to \mathcal{C}^\infty(M_+; E)$ induces a isomorphisms*

(1) $L^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2} \simeq L^2(M; E)$,
(2) $H^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2} \simeq H^2(M; E) \cap \{\, u|_{\partial_D M} = 0,\ \partial_\nu u|_{\partial_N M} = 0\,\}$.

An order-2, $\mathbb{Z}_2^2$-invariant pseudodifferential operator P on M^{dd} will map invariant sections to invariant sections; this means that we consider the case $\alpha = 1$ in Theorem 1.2. Because the action of $\mathbb{Z}_2^2$ is free on a dense subset of M^{dd}, Theorem 1.2 implies that P is Fredholm from $H^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2}$ to $L^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2}$ if, and only if, it is elliptic. This then yields Fredholm conditions for the restriction of P to M, with mixed Dirichlet/Neumann boundary conditions on $\partial_D M$ and $\partial_N M$.

5.3. **The case of non-discrete groups.** If Γ is not discrete, then it is enough for our operators to be *transversally* elliptic. Indeed, let us assume that M is a compact smooth manifold and that Γ is a compact Lie group acting on M. Denote by $\mathfrak{g}$ the Lie algebra of Γ. Recall that any $X \in \mathfrak{g}$ defines as usual the vector field X_M^* given by $X_M^*(m) = \frac{d}{dt}_{|t=0} e^{tX} \cdot m$. Let first introduce the Γ-transversal space

$$T_\Gamma^* M := \{\alpha \in T^*M \mid \alpha(X_M^*(\pi(\alpha))) = 0, \forall X \in \mathfrak{g}\}.$$

A Γ-invariant classical pseudodifferential operator P of order m is said Γ-*transversally elliptic* if its principal symbol is invertible on $T_\Gamma^* M \setminus \{0\}$. Let $P \in \psi^m(M; E_0, E_1)$ be Γ-transversally elliptic. Recall the now classical result of Atiyah and Singer [2, Corollary 2.5]

Theorem 5.4. *Assume P is Γ-transversely elliptic. Then for every irreducible representation $\alpha \in \widehat{\Gamma}$,*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha,$$

is Fredholm.

Note that this implies that Theorem 1.2 is not true anymore for Γ-transversally elliptic operators if Γ is non-discrete.

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