Characters of Irreducible G-Modules and Cohomology of G/P for the Lie Supergroup G=Q(n)

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to Yuri Ivanovich Manin
with admiration

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Abstract. We prove a character formula for any finite-dimensional irreducible representation V of the "queer" Lie superalgebra $\mathfrak{g} = q(n)$. If expresses $\mathrm{ch}V$ in terms of the multiplicities of the irreducible $\mathfrak{g}$-subquotiens of the cohomology groups of certain dominant $\mathfrak{g}$-bundles on the Π-symmetric projective spaces (i.e. on the homogeneous superspaces G/P whose reduced space is a projective space, where $G = Q(n)$). We also establish recurent relations for the above multiplicities and this enables us to compute explicitly $\mathrm{ch}V$ for any given V. This provides a complete solution to the Kac character problem for the Lie superalgebra $q(n)$. Finally we consider the particular cases of $q(2)$, $q(3)$ and $q(4)$ in which we compare the new character formula with the generic character formula of [P4].

Introduction

In this paper we solve the Kac character problem posed in [K2], i.e. the problem of computing the character of any irreducible finite-dimensional representation, for the Lie superalgebra $\mathfrak{g} = q(n)$. Our solution is based on the same general ideas as the second author's solution of the Kac character problem for $gl(m|n)$, [S2], but in the case of $q(n)$ the argument is almost entirely geometric. The homogeneous superspaces we consider are Manin's Π-symmetric flag supermanifolds. In 1982 Manin constructed the flag supermanifolds corresponding to all classical series of simple Lie supergroups, and in particular constructed the Π-symmetric flag superspaces corresponding to $Q(n)$ (or to the the simple Lie supergroup $PSQ(n)$), see [M1] and [M2]. The standard reference today is Manin's monograph [M3]. Immediately after constructing the flag supermanifolds (of all types corresponding to the different series of classical simple Lie superalgebras) Manin formulated the problem of finding an analogue of Borel-Weil-Bott's theorem (or theory) for this case. It was quite clear that the cohomology of the flag supermanifolds deserves by itself to be calculated, but Manin's hope was that this cohomology should also give an approach to calculating the characters of the irreducible representations.

This problem of Manin turned out to be a difficult one. Some progress was made during the 80's (see for instance [P3]) where, roughly speaking, a Borel-Weil-Bott type theorem was proved for typical representations. Later (see [PS1] and [P4]) the theory (in a general $\mathcal{D}$-module version inspired by the celebrated work of Beilinson and Bernstein) was extended to generic representations. Finite-dimensional singly atypical (but not necessarily generic) $gl(m|n)$-modules have been studied in [HJKT]. However the case of arbitrary finite-dimensional irreducible representations remained essentially intractable until the papers [S1] and [S2], where Kac's character problem was solved for $gl(m|n)$ by a mixture of algebraic and geometric techniques. The method developed there by the second author enables us to carry out Manin's program also for the Π-symmetric projective superspaces and in particular to give a geometry based complete solution of Kac's character problem for $\mathfrak{g} = q(n)$.

Let us describe the contents of the paper. The objective of section 1 is to present and explain the results. In subsections 1.1 and 1.2 we fix the notations. In subsection 1.3 we state our main results in four theorems and two corollaries. The character formula of Theorem 1 reduces the problem of calculating the character of an irreducible finite-dimensional $\mathfrak{g}$-module to calculating the multiplicities of the irreducible $\mathfrak{g}'$-subquotients (where $\mathfrak{g}' = q(n-k)$) of the cohomologies of dominant $\mathfrak{g}'$-linearized bundles on the Π-symmetric projective superspaces of $G' = Q(n-k)$ for $k = 1, \ldots, n-2$. Theorems 2 and 3 establish recurrent relations which reduce the calculation of the above multiplicities to the calculation of the multiplicities of the trivial irreducible representation in the cohomologies of a certain bundle (corresponding to the highest weight of the adjoint representation) on the Π-symmetric projective superspace. Theorem 4 calculates the latter multiplicities explicitly. Together, Theorems 1–4 provide a complete solution to the Kac character problem for $\mathfrak{g} = q(n)$. Subsection 1.4 is devoted to examples: we compute explicitly the characters of all irreducible finite-dimensional $q(2)$-, $q(3)$- and $q(4)$-modules and compare the results with the approximation given by the generic character formula of [P4].

Section 1 contains practically no proofs. Since the proofs are quite technical we have chosen to present them in a separate section. This is section 3. In section 2 we prove some auxiliary results which are needed in the proofs of the main results.

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1. Preliminaries and Statement of Main Results

1.1. Algebraic preliminaries

The ground field is $\mathbb{C}$. All vector spaces are automatically assumed to be $\mathbb{Z}_2$-graded and a subscript $_0$ or $_1$ (to any $\mathbb{Z}_2$-graded object such as vector space or sheaf) always refers to the $\mathbb{Z}_2$-grading. Π denotes the functor of parity change. The *dimension* of a vector space $V = V_0 \oplus V_1$ is by definition $k + \ell\varepsilon$, where $k = \dim V_0$, $\ell = \dim \Pi V_1$, and ε is a formal odd variable with $\varepsilon^2 = 1$. One has $\dim \Pi V = \varepsilon \cdot \dim V$. If $\dim V = k + l\varepsilon$, we set $|\dim V| := k + l$. The upper index * denotes dual space.

Throughout this paper E will denote a fixed vector space of dimension $n + n\varepsilon$, $n \geq 2$. $Q(E)$ is the Lie supergroup of endomorphisms of E which preserve a given odd isomorphism $\Pi_E : E \xrightarrow{\sim} E$ with $\Pi_E^2 = \mathrm{id}$. $q(E)$ is the Lie superalgebra of $Q(E)$, i.e. $q(E) = \mathrm{Lie}G$, where Lie denotes the Lie superalgebra functor. E is the *tautological representation of* $\mathfrak{g}$ and G.

Let $\mathfrak{h}$ be a fixed *Cartan subsuperalgebra of* $\mathfrak{g}$, i.e. a nilpotent self-normalizing Lie subsuperalgebra of $\mathfrak{g}$ (see for instance [PS2]). Then dim $\mathfrak{h} = n + n\varepsilon$ and $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0 = gl(E_0)$. The *roots* Δ *of* $\mathfrak{g}$ (see [PS2]) are nothing but the roots of $gl(E_0)$:

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\} ,$$

$\varepsilon_1, \ldots, \varepsilon_n$ being a standard basis in $\mathfrak{h}_0^*$. For each $\alpha \in \Delta$ the dimension of the root space $\mathfrak{g}^{(\alpha)}$ is $1+\varepsilon$. W denotes the Weyl group of $\mathfrak{g}_0$. (W is a symmetric group of order n.) The *weights* are by definition the elements of $\mathfrak{h}_0^*$. If $\lambda \in \mathfrak{h}_0^*$, we will usually write $\lambda = (\lambda_1, \ldots, \lambda_n)$, where the *standard coordinates* λ_i *of* λ are its coordinates with respect to ε_i, i.e. $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$. We set also $\#\lambda := \#\{i \mid \lambda_i \neq 0\}$. $\Lambda = \{\lambda \in \mathfrak{h}_0^* \mid \lambda_i - \lambda_j \in \mathbb{Z}\ \forall i, j,\ 1 \leq i, j \leq n\}$ is the set of *integral weights*. We say that a weight $\tilde{\lambda}$ is a *reduced expression* of a weight λ if $\tilde{\lambda}$ is obtained from λ by replacing a maximal number of pairs of coordinates λ_i, λ_j with $\lambda_i + \lambda_j = 0$ by pairs of the form 0,0. For instance, when $n = 5$ (1,1,1,–1,–1) is a weight and all its reduced expressions are (1,0,0,0,0), (0,1,0,0,0), and (0,0,1,0,0). If $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\alpha = \varepsilon_i - \varepsilon_j$, we say that λ is *α-typical* when $\lambda_i + \lambda_j \neq 0$ and that λ is *α-atypical* when $\lambda_i + \lambda_j = 0$. The set of all α-atypical weights will be denoted by $\mathfrak{h}_\alpha^*$; $\Lambda_\alpha := \Lambda \cap \mathfrak{h}_\alpha^*$.

Z denotes the center of the enveloping algebra $U(\mathfrak{g})$. Z is a commutative $\mathbb{C}$-algebra and there is a canonical injective algebra homomorphism, the *Harish-Chandra homomorphism*

$$HC : Z \hookrightarrow S^{\cdot}(\mathfrak{h}_0)^W,$$

see [Ser1] or [P4], and by definition, for each $\lambda \in \mathfrak{h}_0^*$, $\theta^\lambda : Z \to \mathbb{C}$ is the unique homomorphism which makes the diagram

$$\begin{array}{ccc} Z & \overset{HC}{\hookrightarrow} & S^{\cdot}(\mathfrak{h}_0)^W \\ & {\scriptstyle\theta^\lambda}\searrow \quad \swarrow{\scriptstyle\lambda^W} & \\ & \mathbb{C} & \end{array}$$

commutative, $\lambda^W : S^{\cdot}(\mathfrak{h}_0)^W \to \mathbb{C}$ being the natural homomorphism induced by λ. The image of HC has been first described by Sergeev in [Ser1]. Sergeev's result implies the following statement (for the proof see [P4]):

Proposition 1.1. $\theta^\lambda = \theta^\chi \Leftrightarrow \tilde{\lambda} \in W \cdot \tilde{\chi}$, *where* $\tilde{\lambda}$ *and* $\tilde{\chi}$ *are (arbitrarily) reduced expressions respectively of* λ *and* χ. □

If $\theta : Z \to \mathbb{C}$ is a central character (i.e. simply an algebra homomorphism) we set $\#\theta := \#\tilde{\lambda}$, $\tilde{\lambda}$ being the reduced expression of any weight λ for which $\theta = \theta^\lambda$. Proposition 1.1 implies that $\#\theta$ is well defined. We define the *parity* $\bar{\theta}$ *of* θ as $\#\theta$ (mod 2).

We fix $\mathfrak{b}$ to be the *Borel subsuperalgebra* $\mathfrak{h} \supset\!\!\!\!+ \mathfrak{n}$ ($\supset\!\!\!\!+$ denoting semi-direct sum of Lie superalgebras), where $\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}^{(\alpha)}$, $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. All other Borel subsuperalgebras of $\mathfrak{g}$ which contain $\mathfrak{h}$ are $\mathfrak{h} \supset\!\!\!\!+ (\oplus_{\alpha \in w(\Delta^+)} \mathfrak{g}^{(\alpha)})$ for $w \in W \backslash \{\mathrm{id}\}$. In particular when $w = w_m$ is the element of maximal length we obtain the Borel subsuperalgebra $\mathfrak{b}^-$ *opposite to* $\mathfrak{b}$: $\mathfrak{b}^- = \mathfrak{h} \supset\!\!\!\!+ \mathfrak{n}^-$, $\mathfrak{n}^- = \oplus_{\alpha \in \Delta^- = -\Delta^+} \mathfrak{g}^{(\alpha)}$.

As usual $\mathfrak{b}$ defines a partial order $\leq_{\mathfrak{b}}$ on $\mathfrak{h}_0^*$:

$$\varkappa \leq_{\mathfrak{b}} \mu \Leftrightarrow \varkappa + \sum_{\alpha \in \Delta^+} k_\alpha \alpha = \mu \quad \text{for some} \quad k_\alpha \in \mathbb{Z}_+ .$$

The *parabolic subsuperalgebras of* $\mathfrak{g}$ (i.e. the Lie subsuperalgebras of $\mathfrak{g}$ which contain $\mathfrak{b}$) are in bijective correspondence with the parabolic subalgebras of $\mathfrak{g}_0$. Throughout this paper $\mathfrak{p}$ will denote an arbitrary parabolic subsuperalgebra (such that $\mathfrak{p} \supset \mathfrak{b}$) and $\mathfrak{p}^k$, for $k = 1, \ldots, n-1$, will denote the maximal proper parabolic subsuperalgebras: the roots $\Delta(\mathfrak{p}^k)$ of $\mathfrak{p}^k$ are by definition $\{\varepsilon_i - \varepsilon_j, \varepsilon_p - \varepsilon_q \mid i < j,\ k > p > q \geq 1,\ n \geq p > q \geq k+1\}$. For any $\mathfrak{p} \subset \mathfrak{g}$, $W^{\mathfrak{p}}$ will denote the Weyl group of the semi-simple part of $\mathfrak{p}$ considered as a subgroup of W. Finally, it is a straightforward but important observation that for every $\mathfrak{p}$ there is a unique Lie subsupergroup $P \hookrightarrow G$ (which is by definition a *parabolic subsupergroup*) such that $\mathrm{Lie} P = \mathfrak{p}$. P^k, where $k = 1, \ldots, n-1$ and $\mathrm{Lie}\, P^k = \mathfrak{p}^k$, are the maximal proper parabolic subsupergroups of G with $B \hookrightarrow P^k$.

Note now that Π acts on the category of representations of any Lie superalgebra or Lie supergroup but a particular representation may not be Π-invariant. In this paper we will restrict ourselves to considering only Π-invariant representations, i.e. we will assume that any representation considered is Π-invariant. In the case of $\mathfrak{g}$ (respectively G, $\mathfrak{p}$, P, etc.) we will call such modules $\mathfrak{g}^\Pi$*-modules* (resp. G^Π-, $\mathfrak{p}^\Pi$-modules, etc.), and an *irreducible* $\mathfrak{g}^\Pi$*-module* (resp. G^Π*-module*, etc.) is a $\mathfrak{g}^\Pi$-module with no proper $\mathfrak{g}^\Pi$-submodule (i.e. with no proper Π-invariaint $\mathfrak{g}$-submodule).

Most of the $\mathfrak{g}^\Pi$-modules ($\mathfrak{p}^\Pi$-modules, etc.), we will consider are going to be finite-dimensional (with obvious exceptions such as Verma modules) and therefore if the contrary is not explicitly stated or is not completely clear from the context, all representations considered will be assumed finite-dimensional. The category of finite-dimensional $\mathfrak{g}^\Pi$-modules (resp. $\mathfrak{p}^\Pi$-modules, etc.) will be denoted by $(\mathfrak{g}^\Pi\text{-mod})_f$ (resp. by $(\mathfrak{p}^\Pi\text{-mod})_f$ etc). If V^1 is an irreducible $\mathfrak{g}^\Pi$-submodule of a finite-dimensional $\mathfrak{g}^\Pi$-module V, $[V : V^1]$ will denote the multiplicity of V^1 in V, i.e. the number of times V^1 occurs as a $\mathfrak{g}^\Pi$-composition factor of V.

If $\theta : Z \to \mathbb{C}$ is a central character and V is a $\mathfrak{g}^\Pi$-module, then V^θ will be the direct summand of V characterized by the property that the $\mathfrak{g}^\Pi$-composition factors of V^θ coincide with all composition factors of V on which Z acts via θ.

It is easy to see that the standard outer automorphism on $\mathfrak{g}_0$ interchanging $\mathfrak{b}_0$ with $\mathfrak{b}_0^-$ extends to an automorphism of $\mathfrak{g}$ which interchanges $\mathfrak{b}$ and $\mathfrak{b}^-$. If ω is the induced functor on the category of $\mathfrak{g}^\Pi$-modules, we set $V^\vee := \omega(V)^*$. By definition, a $\mathfrak{g}^\Pi$-module V is *contragradient* iff $V^\vee \simeq V$.

Any irreducible $\mathfrak{h}^\Pi$-module v is determined by a weight $\lambda \in \mathfrak{h}_0^*$ via which $\mathfrak{h}_0$ acts on v, so we will denote the family of irreducible $\mathfrak{h}^\Pi$-modules by v_λ. It is easy to verify that

$$\dim v_\lambda = 2^{[\#\lambda/2]}(1+\varepsilon)$$

(cf. [P2]). If V is a $\mathfrak{g}^\Pi$-module, its generalized weight spaces $V^{(\lambda)}$ are automatically $\mathfrak{h}$-modules. We set $\mathrm{supp}V = \{\lambda \in \mathfrak{h}_0^* \mid V^{(\lambda)} \neq 0\}$. The *formal character* $\mathrm{ch}V$ *of* V is by definition the expression

$$\sum_{\mu \in \mathrm{supp}V} \dim V^{(\mu)} \cdot e^\mu .$$

A *highest weight* $\mathfrak{g}^\Pi$-*module with highest weight* λ is any (possibly infinite-dimensional) Π-invariant $\mathfrak{g}$-quotient of the *Verma module*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} v_\lambda .$$

$V(\lambda)$ will denote the unique irreducible (as a $\mathfrak{g}^\Pi$-module) quotient of $M(\lambda)$, and we will assume that $\dim V(\lambda) < \infty$. Z acts via θ^λ on $M(\lambda)$ and thus on any highest weight $\mathfrak{g}^\Pi$-module with highest weight λ. The category $\mathbf{O}$ is by definition the category of all (in general infinite-dimensional) $\mathfrak{g}^\Pi$-modules which admit a finite $\mathfrak{g}^\Pi$-filtration whose factors are highest weight modules. For any fixed central character θ, $\mathbf{O}^\theta$ is the full subcategory of $\mathbf{O}$ consisting of $\mathfrak{g}^\Pi$-modules on whose composition factors Z acts via θ.

If $\mathfrak{p}$ is a parabolic subsuperalgebra, we set

$$M_\mathfrak{p}(\lambda) := U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} v_\lambda$$

and denote by $V_\mathfrak{p}(\lambda)$ the unique irreducible (as $\mathfrak{p}^\Pi$-module) quotient of $M_\mathfrak{p}(\lambda)$. $V_0(\lambda)$ will denote the irreducible $\mathfrak{g}_0^\Pi$-module with highest weight λ and $V_{\mathfrak{p}_0}(\lambda)$ will denote the irreducible $\mathfrak{p}_0^\Pi$-module with highest weight λ.

Λ^+ is by definition the set of *dominant* integral weights, i.e.

$$\Lambda^+ := \{\lambda \in \Lambda \mid \dim V(\lambda) < \infty\}$$

and it is proved in [P2] that in coordinate form there is the following description of Λ^+:

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+$$
$$\Updownarrow$$
$$\lambda \in \Lambda,\ \lambda_i \geq \lambda_j \text{ for } i < j, \quad \text{and} \quad \lambda_i = \lambda_j \quad \text{implies} \quad \lambda_i = \lambda_j = 0.$$

By Λ_0^+ we will denote the $\mathfrak{b}_0$-dominant weights in Λ, i.e. $\Lambda_0^+ = \{\lambda = (\lambda_1, \dots \lambda_n) \in \Lambda \mid \lambda_i \geq \lambda_j \text{ for } i < j\}$; obviously $\Lambda^+ \subset \Lambda_0^+$. For each $\alpha \in \Delta$ we set also $\Lambda_\alpha^+ = \Lambda^+ \cap \mathfrak{h}_\alpha^*$, and for each parabolic subsuperalgebra we set $\Lambda_\mathfrak{p}^+ := \{\lambda \in \Lambda \mid \dim V_\mathfrak{p}(\lambda) < \infty\}$. If $\mathfrak{s}$ is a Lie subsuperalgebra of $\mathfrak{p}$ or of $\mathfrak{g}$, we say that λ is *$\mathfrak{s}$-typical in* $\mathfrak{p}$, or respectively *in* $\mathfrak{g}$, if λ is α-typical for all roots α of $\mathfrak{p}$ (resp. of $\mathfrak{g}$) which are not roots of $\mathfrak{s}$.

A complex of $\mathfrak{g}^\Pi$-modules

$$\dots \xrightarrow{d^1} M^1 \xrightarrow{d^0} M^0 \to 0$$

is a *resolution of* $V(\lambda)$ iff $M^0/\mathrm{im}d^0 \simeq V(\lambda)$, $\ker d^i = \mathrm{im} d^{i+1}$ for all $i \geq 0$. A resolution of $V(\lambda)$ is a *Bernstein-Gelfand-Gelfand resolution* (or *BGG-resolution* for short) of $V(\lambda)$ iff $M^0 \simeq M(\lambda)$, and for each i, M^i is an object of $\mathbf{O}^{\theta^\lambda}$ and M^i is free as a $U(\mathfrak{n}^-)$-module. One can show easily that for BGG-resolution each M^i admits a filtration whose quotients are Verma modules. It is also straightforward to prove (following the same lines as in the proof of a similar statement in section 2.1 of [PS1]) that for every λ (not necessarily for $\lambda \in \Lambda^+$) $V(\lambda)$ admits a BGG-resolution.

1.2. Geometric preliminaries

For any parabolic subsupergroup P of G the quotient G/P in the category of superschemes exists and coincides with one of Manin's flag superspaces of Π-symmetric flags. More precisely, in the notation of [M3], $G/P = F\Pi_{\mathrm{Spec}\,\mathbf{C}}(a_1|a_1, \dots, a_k|a_k, E)$. In our notation the type of the flags is $a_1 + a_1\varepsilon, \dots, a_k + a_k\varepsilon$, which is nothing but the type of the flag in E whose stabilizer in G_{red}[1] is P_{red}. The reduced submanifold $(G/P)_{\mathrm{red}} = G_{\mathrm{red}}/P_{\mathrm{red}}$ is the usual flag variety $F_{\mathrm{Spec}\,\mathbf{C}}(a_1, \dots, a_k, E_0)$. $\mathcal{O}_{G/P}$ denotes the structure sheaf of G/P. It is endowed with a canonical finite G_{red}-subsheaf (and $\mathcal{O}_{G/P}$-module) filtration whose adjoint factors are the symmetric powers of the conormal bundle $\mathcal{N}^*_{(G/P)_{\mathrm{red}}/G/P}$ of $(G/P)_{\mathrm{red}}$ in G/P. The corresponding graded sheaf of algebras is $S^{\cdot}(\mathcal{N}^*_{(G/P)_{\mathrm{red}}/G/P})$. (Note that the supersymmetric algebra $S^{\cdot}(\mathcal{N}^*_{(G/P)_{\mathrm{red}}/G/P}$ is simply a Grassmann algebra because the rank of $\mathcal{N}_{(G/P_{\mathrm{red}})/G/P}$ as $\mathcal{O}_{(G/P)_{\mathrm{red}}}$-module is purely odd). Furthermore $\mathcal{N}^*_{(G/P_{\mathrm{red}})/G/P}$ is

[1]By the subscript $_{\mathrm{red}}$ we indicate reduction modulo nilpotents on any supermanifold.

nothing but the cotangent bundle of $(G/P)_{\mathrm{red}}$ with changed parity. For $n > 2$ the supermanifold G/P never splits, i.e. for any P G/P is not isomorphic to the split supermanifold $((G/P)_{\mathrm{red}}, S^{\cdot}(\mathcal{N}^{*}_{(G/P)_{\mathrm{red}}/G/P})$. This follows immediately from a result of I.Skornyakov (unpublished) which claims that for $n > 2$ the Picard group of G/P (which by definition is the group of equivalence classes of even invertible $\mathcal{O}_{G/P}$-modules on G/P) is trivial, i.e. its only element is the class of $\mathcal{O}_{G/P}$.

If v is any $\mathfrak{h}$-semisimple finite-dimensional $\mathfrak{p}^{\Pi}$-module, we denote by $\mathcal{O}_{G/P}(v)$ the $\mathfrak{g}$-linearized $\mathcal{O}_{G/P}$-module "induced" from $\mathfrak{p}$ in the standard way. This means in particular that the $\mathfrak{p}_0$-module structure on the geometric fibre of $\mathcal{O}_{G/P}(v)_{\mathrm{red}} = \mathcal{O}_{(G/P)_{\mathrm{red}}} \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_{G/P}(v)$ over the closed point P_{red} of $(G/P)_{\mathrm{red}}$ is nothing but v considered as a $\mathfrak{p}_0$-module. The cohomology $H^{\cdot}(G/P, \mathcal{O}_{G/P}(v))$, which by definition is the usual sheaf cohomology of the sheaf $\mathcal{O}_{G/P}(v)$ on the topological space $(G/P)_{\mathrm{red}}$, is endowed with a canonical $\mathfrak{g}$-module structure. When $P = B$ we will sometimes write $\mathcal{O}_{G/B}(\lambda)$, where $\mathcal{O}_{G/B}(\lambda)$ is by definition the $\mathfrak{g}$-linearized $\mathcal{O}_{G/B}$-module induced from the irreducible $\mathfrak{b}^{\Pi}$-module of highest weight λ. We define also $\mathcal{O}_{G/P}(v)^{\vee}$ as $\mathcal{O}_{G/P}(v^{\vee})$, where $v^{\vee} := \omega_{\mathfrak{p}}(v)^{*}$, $\omega_{\mathfrak{p}}$ being the functor on the category $(\mathfrak{p}^{\Pi}\text{-mod})_f$ induced by the standard outer authomorphism on $\mathfrak{h} + \mathfrak{p}_{ss}$.

By Frobenius duality (in its version which applies to induced $\mathfrak{g}^{\Pi}$-linearized $\mathcal{O}_{G/P}$-modules),

$$\mathrm{Hom}_{\mathfrak{p}^{\Pi}}(V, v) = \mathrm{Hom}_{\mathfrak{g}^{\Pi}}(V, H^0(G/P, \mathcal{O}_{G/P}(v))) \tag{1.1}$$

for any $\mathfrak{g}^{\Pi}$-module V. This implies that if v is an irreducible $\mathfrak{p}^{\Pi}$-module, $H^0(\mathcal{O}_{G/P}(v))$ is an indecomposable $\mathfrak{p}^{\Pi}$-module equipped with a canonical surjection of $\mathfrak{p}^{\Pi}$-modules

$$H^0(G/P, \mathcal{O}_{G/P}(v)) \to v \tag{1.2}$$

(the latter being nothing but restriction of global sections to the geometric fibre at the closed point P_{red} of $(G/P)_{\mathrm{red}}$). If $V^{\mathfrak{p}}(v)$ is the unique irreducible $\mathfrak{g}^{\Pi}$-module which admits a surjection of $\mathfrak{p}^{\Pi}$-modules

$$V^{\mathfrak{p}}(v) \to v \ ,$$

(1.1) implies that there is a canonical injection of $\mathfrak{g}^{\Pi}$-modules

$$V^{\mathfrak{p}}(v) \to H^0(G/P, \mathcal{O}_{G/P}(v)) \tag{1.3}$$

such that the following natural diagram commutes:

$$\begin{array}{ccc} V^{\mathfrak{p}}(v) & \longrightarrow & H^0(G/P, \mathcal{O}_{G/P}(v)) \\ & \searrow \quad \swarrow & \\ & v & \end{array} .$$

If $P = B$, (1.2) means that $H^0(G/B, \mathcal{O}_{G/B}(\lambda))$ is $\mathfrak{b}$-lowest weight $\mathfrak{g}^\Pi$-module with lowest weight λ, and (1.3) means that the irreducible $\mathfrak{g}^\Pi$-module $V^\mathfrak{b}(\lambda)$ with $\mathfrak{b}$-lowest weight λ is a canonical submodule of $H^0(G/B, \mathcal{O}_{G/B}(\lambda))$.

1.3. The main results

Our objective in this paper is to establish the four theorems and two corollaries stated in this subsection.

Let us start by introducing some more notation. For any parabolic subsupergroup P of G and any two weights $\lambda, \mu \in \Lambda^+$, we set

$$(1.4) \qquad m^i_P(\lambda, \mu) := [\hat{H}^i(G/P, \mathcal{O}_{G/P}(V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda)))) : V(\mu)] \,,$$

where w_m (respectively $w^\mathfrak{p}_m$) is the element of maximal length in W (resp. $W^\mathfrak{p}$) and

$$(1.5) \quad \hat{H}^i(G/P, \mathcal{O}_{G/P}(V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda)))) := \begin{cases} H^i(G/P, \mathcal{O}_{G/P}(V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda)))) & \text{for } i > 0 \\ H^0(G/P, \mathcal{O}_{G/P}(V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda))))/V(\lambda) & \text{for } i = 0 \,. \end{cases}$$

(Note that $V(\lambda)$ admits a canonical surjection of $\mathfrak{p}$-modules $V(\lambda) \to V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda))$ and therefore there is a canonical injection of $\mathfrak{g}$-modules

$$V(\lambda) \hookrightarrow H^0(G/P, \mathcal{O}_{G/P}(V_\mathfrak{p}(w^\mathfrak{p}_m \circ w_m(\lambda)))) \ .)$$

In the special case when $P = P^1$ we will omit the subscript $_{P^1}$ and will write simply $m^i(\lambda, \mu)$ instead of $m^i_{P^1}(\lambda, \mu)$. If now k, ℓ are two non-negative integers such that $k+\ell < n$, let $G^{k,\ell}$ be the Lie subsupergroup of G with Lie superalgebra

$$\mathfrak{g}^{k,\ell} := \mathfrak{h} \ni (\bigoplus_{\alpha \in \Delta^{k,\ell}} \mathfrak{g}^{(\alpha)}) \,,$$

where $\Delta^{k,\ell} = \{\varepsilon_j - \varepsilon_p \mid k < j, p \leq n - \ell\}$. If furthermore $(P^{k,\ell})^1$ is the parabolic Lie subsupergroup of $G^{k,\ell}$ whose Lie superalgebra is $(\mathfrak{p}^{k,\ell})^1 = \mathfrak{h} \oplus (\bigoplus_{\alpha \in (\Delta^{k,\ell})^1} \mathfrak{g}^{(\alpha)})$, where $(\Delta^{k,\ell})^1 = \{\varepsilon_j - \varepsilon_p, \varepsilon_q - \varepsilon_\ell \mid k < j < p \leq n - \ell,\ k+1 < s < q \leq n - \ell\}$, we can define $m^i_{k,\ell}(\lambda, \mu)$ by formula (1.4) with G replaced by $G^{k,\ell}$, P replaced by $(P^{k,\ell})^1$, $\mathfrak{p}$ replaced by $(\mathfrak{p}^{k,\ell})^1$, and $w^\mathfrak{p}_m$ (respectively w_m) replaced by the element of maximal length in the Weyl group of the semisimple part of $(\mathfrak{p}^{k,\ell})^1$ (resp. the Weyl group of the semisimple part of $\mathfrak{g}^{k,\ell}$). Obviously $m^i(\lambda, \mu) = m^i_{0,0}(\lambda, \mu)$.

Fix now $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+$. If $\lambda_k \neq 0$, $\lambda_{k+1} = \cdots = \lambda_{\ell-1} = 0$, $\lambda_\ell \neq 0$, set $\mathfrak{p}(\lambda) := \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^k \cap \mathfrak{p}^{\ell-1} \cap \cdots \cap \mathfrak{p}^{n-1}$. If $\lambda = 0$ put $\mathfrak{p}(\lambda) = \mathfrak{g}$. $P(\lambda)$ is the Lie subsupergroup of G with $\mathrm{Lie}\, P(\lambda) = \mathfrak{p}(\lambda)$. Put

$$\sigma(\lambda) := \mathrm{Ech}\mathcal{O}_{G/P(\lambda)}(v_\lambda) \ ,$$

where Ech stands for the Euler character of a $\mathfrak{g}$-linearized cheaf, i.e the alternating sum of characters of its cohomology groups. One verifies immediately (using Borel-Weil-Bott's theorem for $(G/P(\lambda))_{\mathrm{red}}$, cf. [P3]) that

$$\sigma(\lambda) = \frac{\dim v_\lambda}{D} \cdot \sum_{w \in W} \mathrm{sgn} w \cdot w \left(e^{\lambda + \rho_0} \cdot \prod_{\alpha \in \Delta^+ \setminus (\Delta^+ \cap \Delta(\mathfrak{p}(\lambda)))} (1 + e^{-\alpha}) \right) \ ,$$

where $\rho_0 := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and D is the Weyl denominator for $\mathfrak{g}_0 = gl(E_0)$, i.e. $D = \sum_{w \in W} \mathrm{sgn} w \cdot e^{w(\rho_0)}$.

Let $\mathcal{H}$ be the infinite-dimensional vector space over $\mathbb{C}$ with basis e^λ where λ runs over Λ^+. $\mathcal{H}_\lambda$ is the finite-dimensional subspace of $\mathcal{H}$ spanned by e^ν for all $\nu \in \Lambda^+$ with $\nu \leq_{\mathfrak{b}} \lambda$. Consider the linear operator

$$\mathbf{a} : \mathcal{H} \to \mathcal{H}$$

$$\mathbf{a}(e^\nu) := \sum_{\mu \leq_{\mathfrak{b}} \nu} a_{\nu,\mu} \cdot e^\mu,$$

where for any $\mu, \nu \in \Lambda^+$ $a_{\nu,\mu}$ are the integers

$$a_{\nu,\mu} := \sum_{j \geq 0} (-1)^j [H^j(G/P(\nu),\ \mathcal{O}_{G/P(\nu)}(V_{\mathfrak{p}(\nu)}(w_m^{\mathfrak{p}(\nu)} \circ w_m(\nu)))) : V(\mu)] \ .$$

Furthermore, for any $k = 1, \dots, n-1$ we define the linear operators

$$\mathbf{a}_k : \mathcal{H} \to \mathcal{H}$$

$$\mathbf{a}_k(e^\nu) = \begin{cases} \sum_{\mu \in \Lambda^+} \left(\sum_{j \geq 0} (-1)^j m^j_{k-1,0}(\nu, \mu) \cdot e^\mu \right) & \text{if } \nu_k \neq 0 \\ 0 & \text{if } \nu_k = 0. \end{cases}$$

Theorem 1.

(a) *The restriction of* $\mathbf{a}$ *to* $\mathcal{H}_\lambda$ *is an isomorphism*

$$\mathbf{a}^\lambda := \mathbf{a}_{|\mathcal{H}_\lambda} : \mathcal{H}_\lambda \xrightarrow{\sim} \mathcal{H}_\lambda$$

for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+$, *and*

$$\mathrm{ch}V(\lambda) = \sum_{\mu \leq_{\mathfrak{b}} \lambda} b^{\lambda}_{\lambda,\mu} \cdot \sigma(\mu) \ , \tag{1.6}$$

where $b^{\lambda}_{\nu,\mu}$ *are the matrix coefficients of the inverse operator* $\mathbf{b}^{\lambda} := (\mathbf{a}^{\lambda})^{-1}$ *in the basis* e^{ν}, $\nu \leq_{\mathfrak{b}} \lambda$ *of* $\mathcal{H}_{\lambda}$. *When* $\lambda_1 = 0$ *or* $\lambda_n = 0$ *one has simply*

$$\mathrm{ch}V(\lambda) = \sigma(\lambda) \ .$$

(b) *For any* $k = 1, \ldots, n-1$, $\mathcal{H}_{\lambda}$ *is* $\mathbf{a}_k$*-invariant, and*

$$\mathbf{a}^{\lambda} = (\mathrm{id} + \mathbf{a}^{\lambda}_1) \circ \ldots \circ (\mathrm{id} + \mathbf{a}^{\lambda}_{n-1}) \ ,$$

where $\mathbf{a}^{\lambda}_k : \mathcal{H}_{\lambda} \to \mathcal{H}_{\lambda}$ *is the restriction of* $\mathbf{a}_k$ *to* $\mathcal{H}_{\lambda}$.

The next three theorems establish recurrent relations which make it possible to compute explicitly $m^{j}_{k-1,0}(\nu, \mu)$ for all $\nu \leq_{\mathfrak{b}} \lambda$, and therefore (by Theorem 1(b)) also $a_{\nu,\mu}$.

Theorem 2. *If* $P \hookleftarrow B$ *is any parabolic subsupergroup of* G *and* $\lambda \in \Lambda^+$ *is* $\mathfrak{p}$*-typical in* $\mathfrak{g}$ *(*$\mathfrak{p} := \mathrm{Lie}\ P$*), then the canonical injection*

$$V(\lambda) \hookrightarrow H^0(G/P, \mathcal{O}_{G/P}(V_{\mathfrak{p}}(w^{\mathfrak{p}}_m \circ w_m(\lambda))))$$

is an isomorphism and

$$H^i(G/P, \mathcal{O}_{G/P}(V_{\mathfrak{p}}(w^{\mathfrak{p}}_m \circ w_m(\lambda)))) = 0 \quad \text{for } i > 0.$$

Corollary 1. *If* $\lambda \in \Lambda^+$ *is* $\mathfrak{p}$*-typical in* $\mathfrak{g}$ *(*$\mathfrak{p}$ *being an arbitrary parabolic subsuperalgebra of* $\mathfrak{g}$*), then*

$$m^{i}_{P}(\lambda, \mu) = 0 \quad \text{for all } i \geq 0 \text{ and all } \ \mu \in \Lambda^+ \ . \tag{1.7}$$

Theorem 2 implies also the equivalence of a certain category of $\mathfrak{g}^{\Pi}$-modules to a category of $\mathfrak{p}^{\Pi}$-modules. Assuming that $\lambda \in \Lambda^+$ is $\mathfrak{p}$-typical in $\mathfrak{g}$, we denote by $(\mathfrak{g}^{\Pi}\text{-mod})^{(\theta^{\lambda},\mathfrak{p})}_f$ the subcategory of $(\mathfrak{g}^{\Pi}\text{-mod})_f$ consisting of $\mathfrak{g}^{\Pi}$-modules, all composition factors of which are $\mathfrak{p}$-typical (i.e. their highest weights are $\mathfrak{p}$-typical in $\mathfrak{g}$) and afford the central character θ^{λ}. By $\theta^{\lambda}_{\mathfrak{p}}$ we denote the homomorphism of the center of $U(\mathfrak{h}_0 + \mathfrak{p}_{ss})$ ($\mathfrak{p}_{ss}$ being the semi-simple part of $\mathfrak{p}$) via which this center acts on $V_{\mathfrak{p}}(\lambda)$, and $(\mathfrak{p}^{\Pi}\text{-mod})^{(\theta^{\lambda}_{\mathfrak{p}})}_f$ is by definition the category of finite-dimensional $\mathfrak{p}^{\Pi}$-modules, all composition factors of which, considered as $U(\mathfrak{h}_0 + \mathfrak{p}_{ss})$-modules, afford the central character $\theta^{\lambda}_{\mathfrak{p}}$.

Corollary 2. *If $\lambda \in \Lambda^+$ is $\mathfrak{p}$-typical in $\mathfrak{g}$, then the categories $(\mathfrak{g}^\Pi\text{-mod})_f^{(\theta^\lambda,\mathfrak{p})}$ and $(\mathfrak{p}^\Pi\text{-mod})_f^{(\theta^\lambda_\mathfrak{p})}$ are canonically equivalent.*

Theorem 3. *Let $P := P^1$ and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+$ be $\mathfrak{p}$-atypical in $\mathfrak{g}$ and $\lambda_1 > 1$. Set $\alpha = \varepsilon_1 - \varepsilon_k$, $k > 1$ being the unique index such that $\lambda_1 + \lambda_k = 0$.*

(a) *If $\lambda_1 > \lambda_2 + 1$ and $\lambda_k < \lambda_{k-1} - 1$, then*

$$m^0(\lambda, \lambda - \alpha) = 1,\ m^i(\lambda, \lambda - \alpha) = 0 \text{ for } i > 0, \tag{1.8}$$

$$m^i(\lambda, \mu) = m^{i+1}(\lambda - \alpha, \mu) \text{ for any } \mu \in \Lambda^+,\ \mu \neq \lambda - \alpha, \text{ and any } i > 0, \tag{1.9}$$

$$m^0(\lambda, \mu) = m^1(\lambda - \alpha, \mu) \text{ for any } \mu \in \Lambda^+ \text{ with } \#\mu \geq \#\lambda,\ \mu \neq \lambda - \alpha, \tag{1.10}$$

$$\begin{cases} m^0(\lambda, \mu) = m^1(\lambda - \alpha, \mu) + \overline{m^0(\lambda - \alpha, \mu)} \text{ for any } \mu \in \Lambda^+ \\ \qquad \text{with } \#\mu < \#\lambda,\ \mu \neq \lambda - \alpha, \end{cases} \tag{1.11}$$

where $\bar{x} := x(\text{mod } 2)$ for $x \in \mathbb{Z}_+$.

(b) *If $\lambda_1 = \lambda_2 + 1$ but $\lambda_k < \lambda_{k-1} - 1$, then*

$$m^i(\lambda, \mu) = m^i_{1,0}(\lambda - \alpha, \mu) \text{ for any } \mu \in \Lambda^+, \tag{1.12}$$

and if $\lambda_1 > \lambda_2 + 1$ but $\lambda_k = \lambda_{k-1} - 1$, then

$$m^i(\lambda, \mu) = m^i_{0,n-k+1}(\lambda - \alpha, \mu) \text{ for any } \mu \in \Lambda^+. \tag{1.13}$$

(c) *If $\lambda_1 = \lambda_2 + 1$ and $\lambda_k = \lambda_{k-1} - 1$, then*

$$m^0(\lambda, \mu) = 0,\ m^i(\lambda, \mu) = m^{i-1}_{1,n-k+1}(\lambda - \alpha, \mu) \text{ for any } \mu \in \Lambda^+ \text{ and all } i > 0. \tag{1.14}$$

Theorem 4. *Let $P = P^1, P^{n-1}$.*

(a) *If $\lambda = 0$ $(= (0, 0, \ldots, 0))$, then*

$$\begin{aligned} &m^i_P(\lambda, \mu) = 0 \text{ for all } \mu \in \Lambda^+,\ \mu \neq 0, \text{and all } i \geq 0, \\ &m^0_P(\lambda, 0) = 0, \\ &m^i_P(\lambda, 0) = 1 \text{ for } i = 1, \ldots, n-1. \end{aligned} \tag{1.15}$$

(b) *If $\lambda = (1, 0, \ldots, 0, -1)$, then*

$$\begin{aligned} &m^i_P(\lambda, \mu) = 0 \text{ for all } \mu \in \Lambda^+,\ \mu \neq 0, \text{ and all } i \geq 0, \\ &m^0_P(\lambda, 0) = 1, \\ &m^i_P(\lambda, 0) = 0 \text{ for } i = 1, 2, \ldots, n-3, n-1, \\ &m^{n-2}_P(\lambda, 0) = 1. \end{aligned} \tag{1.16}$$

(c) *If $n = 2$ and $\lambda = (\frac{1}{2}, -\frac{1}{2})$, then $m^i_P(\lambda, \mu) = 0$ for all $\mu \in \Lambda^+$ and all $i \geq 0$.*

Theorems 2, 3 and 4 provide us with the following procedure for computing $m^i_{\ell,0}(\lambda,\mu)$ for any $\lambda \in \Lambda^+$, any $\mu \in \Lambda^+$ and all i and ℓ, $i \geq 0$, $0 \leq \ell \leq n-2$. When $\ell = 0$ one checks first whether λ is $\mathfrak{p}^1$-typical in $\mathfrak{g}$. If yes, then by Corollary 1, $m^i(\lambda,\mu) = 0$ for all $\mu \in \Lambda^+$ and all i. If no, Theorems 3 and 4 apply. For $\lambda_1 = 0, \frac{1}{2}, 1$, Corollary 2 and Theorem 4 give the answer. Indeed, if $\lambda_1 = 0$, then $\lambda = (0,\dots,0,\lambda_t,\dots,\lambda_n)$ and by Corollary 2, $m^i(\lambda,\mu) = m^i_{0,n-t+1}(\lambda,\mu)$ for any $\mu \in \Lambda^+$. If $\lambda_1 = \frac{1}{2}$, then $\lambda = (\frac{1}{2}, -\frac{1}{2}, \lambda_3,\dots,\lambda_n)$, and if $\lambda_1 = 1$, then $\lambda = (1,0,\dots,0,-1,\lambda_s,\dots,\lambda_n)$. Corollary 2 gives respectively $m^i(\lambda,\mu) = m^i_{0,n-2}(\lambda,\mu)$ and $m^i(\lambda,\mu) = m^i_{0,n-s+1}(\lambda,\mu)$, and both latter multiplicities are computed by Theorem 4. When $\lambda_1 > 1$ one of the cases in Theorem 3 applies. If it is (a), one expresses $m^i(\lambda,\mu)$ in terms of $m^j(\lambda-\alpha,\mu)$. If it is (b) or (c) one expresses $m^i(\lambda,\mu)$ in terms of $m^i_{1,0}(\lambda-\alpha,\mu)$, $m^i_{0,1}(\lambda-\alpha,\mu)$, or $m^{i-1}_{1,1}(\lambda-\alpha,\mu)$. Continuing this process one reduces ultimately the computation of $m^i(\lambda,\mu)$ to Theorem 4. (Note that in this computation one applies Theorems 3 and 4 only to the supergroups $(P^{k,\ell})^1$. However, as formulated, Theorem 2 applies to any P, Theorem 3 applies to P^1 only and Theorem 4 applies to P^1 and P^{n-1}. The reader will straightforwardly find a symmetric version of Theorem 3 applying to P^{n-1} (one has to assume that $\alpha = \varepsilon_k - \varepsilon_n$ and to define the corresponding analogs of $m^i_{p,q}(\lambda,\mu)$). In this way Theorems 3 and 4 should be viewed as an inductive procedure for calculating the cohomology of the Π-symmetric projective superspaces G/P^1 and G/P^{n-1} with coefficients in any irreducible dominant $\mathfrak{g}$-linearized locally free sheaf.) The multiplicities $m^i_{\ell,0}(\lambda,\mu)$ for $\ell = 1,\dots,n-2$ are computed by a completely similar process.

Theorems 1–4 give a solution of the Kac character problem for $\mathfrak{g}$. Given $\lambda \in \Lambda^+$ one calculates the matrix of the operator $\mathbf{a}^\lambda$ by using Theorem 1 (b) and Theorems 2,3 and 4, and then obtains $\mathrm{ch}V(\lambda)$ by Theorem 1 (a).

If λ is generic, i.e. if $\lambda_i \gg \lambda_{i+1}$ for $i = 1,\dots,n-1$ ($\gg$ means "much greater"), then it is true that

$$\mathrm{ch}V(\lambda) = \frac{\dim v_\lambda}{D} \cdot \sum_{w \in W} \mathrm{sgn}w \cdot w \left(e^{\lambda+\rho_0} \cdot \prod_{\substack{\alpha \in \Lambda^+ \\ \lambda \text{ is } \alpha\text{-typical}}} (1+e^{-\alpha}) \right). \tag{1.17}$$ [2]

Formula (1.17) is the *generic character formula.* If λ is *typical,* i.e. λ is α-typical for all $\alpha \in \Delta^+$, or if more generally $\lambda_k = \lambda_\ell = 0$ whenever λ is $\varepsilon_k - \varepsilon_\ell$-atypical, then the right-hand side of (1.17) simply coincides with $\sigma(e^\lambda)$.

[2]The precise meaning of this statement is that there exist positive constants k_i such that (1.17) holds whenever $\lambda_i - \lambda_{i+1} > k_i$ for $i = 1,\dots,n-1$.

1.4. Comments and examples

The generic character formula was known previously in three cases: for $\lambda \in \Lambda^+$ with $\lambda_1 = 0$ or $\lambda_n = 0$, for a typical $\lambda \in \Lambda^+$, and for a generic λ. In the first case it is due to Sergeev, [Ser2]. In both other cases it has been established by the first author respectively in [P2] and [P4]. The typical character formula of [P2] is a particular case of (1.17) and it extends the pioneering work of V.Kac, [K1], [K2], to the case of $q(n)$. Formula (1.17) was first proved in [P4] and was inspired by the work of Bernstein and Leites on $gl(1 + n\varepsilon)$, [BL].

The rest of this section is devoted to examples. We consider in detail the cases when $n = 2$, 3 and 4 and identify all weights $\lambda \in \Lambda^+$ for which the right-hand side of (1.6) does not coincide with the right-hand side of (1.17). We then compute the respective differences in all cases. For $n \leq 4$ this replaces the somewhat vague genericity condition by an explicit description of all $\lambda \in \Lambda^+$ for which (1.17) is true. We denote by $\mathrm{Gen}(\lambda)$ the right-side of (1.17) for any $\lambda \in \Lambda^+$.

If $n = 2$ the generic formula is valid for all $\lambda \in \Lambda^+$. This follows from [P2] and can also be easily verified directly. If $n = 3$, Theorem 2 implies that the generic formula applies to all weights $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^+$ with $\lambda_1 + \lambda_3 \neq 0$. Let $\lambda_1 + \lambda_3 = 0$ and $\lambda_2 = a \neq 0$. If $a > 1$, (1.12) gives

$$\mathrm{ch}V(a+1, a, -a-1) + \mathrm{ch}V(a, a-1, 1-a) = \mathrm{Ech}\,\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-a-1, a+1, a)).$$

Furthermore,

$$\mathrm{Ech}\,\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-a-1, a+1, a)) = \mathrm{Gen}(a+1, a, -a-1) + \widetilde{\mathrm{Gen}}(a, a, -a)$$

where

$$\widetilde{\mathrm{Gen}}(a, a, -a) := \frac{2(1+\varepsilon)}{D} \cdot \sum_{w \in W} \mathrm{sgn}\, w \cdot w \left(e^{(a,a,-a)+\rho_0} \cdot \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \varepsilon_1 - \varepsilon_3}} (1 + e^{-\alpha}) \right) .$$

Since

$$\mathrm{ch}V(a, a-1, 1-a) = \mathrm{Gen}(a, a-1, 1-a)$$

by Theorem 2, we obtain

$$\mathrm{ch}\, V(a+1, a, -a-1) = \mathrm{Gen}(a+1, a, -a-1) + \widetilde{\mathrm{Gen}}(a, a, -a) - \mathrm{Gen}(a, a-1, 1-a).$$

Finally, a direct calculation shows that

$$\mathrm{Gen}(a, a-1, 1-a) = \widetilde{\mathrm{Gen}}(a, a, -a),$$

and this gives

$$\mathrm{ch}\, V(a+1, a, -a-1) = \mathrm{Gen}(a+1, a, -a+1)$$

for $a \geq 1$.

For $a = \frac{1}{2}$ (1.12) implies

$$\mathrm{ch} V(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{3}{2}) = \mathrm{Ech}\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2})),$$

and clearly

$$\mathrm{Ech}\mathcal{O}_{G/P^1}(-\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2})) = \mathrm{Gen}(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{3}{2}) + \widetilde{\mathrm{Gen}}(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}).$$

One verifies directly that $\widetilde{\mathrm{Gen}}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) = 0$, and in this way

$$\mathrm{ch} V(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{3}{2}) = \mathrm{Gen}(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{3}{2}).$$

Therefore (1.17) holds for any $\lambda \in \Lambda^+$ with $\lambda_1 + \lambda_3 = 0$, $\lambda_2 \geq \frac{1}{2}$. One checks in the same way that (1.17) holds for any $\lambda \in \Lambda^+$ with $\lambda_1 + \lambda_3 = 0$ and $\lambda_2 \leq -\frac{1}{2}$, and therefore it remains to consider the case when $\lambda_2 = 0$ and $\lambda_1 + \lambda_3 = 0$. For $\lambda_1 = 1$ a trivial calculation shows that

$$\mathrm{ch} V(1, 0, -1) = \mathrm{Gen}(1, 0, -1) - 2\mathrm{ch} V(0, 0, 0)\ . \tag{1.18}$$

($V(1, 0, -1)$ is nothing but the direct sum $psq(3) \oplus \Pi psq(3)$, where $psq(3)$ is the simple subquotient of the Lie superalgebra $q(3)$ ($psq(3) = sq(3)/\mathbb{C}$ and $q(3)/sq(3) = \Pi\mathbb{C}$.) For $\lambda_2 = 2$ Theorems 3 and 4 give

$$\begin{aligned} \mathrm{ch} V(2, 0, -2) + \mathrm{ch} V(1, 0, -1) + 2\mathrm{ch} V(0, 0, 0) &= \mathrm{Ech}\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-2, 2, 0)) \\ &= \mathrm{Gen}(2, 0, -2) + \mathrm{Gen}(1, 0, -1) \end{aligned} \tag{1.19}$$

and therefore (1.18) implies

$$\mathrm{ch} V(2, 0, -2) = \mathrm{Gen}(2, 0, -2).$$

For $\lambda_2 = k > 2$, according to Theorems 3 and 4

$$\mathrm{ch} V(k, 0, -k) + \mathrm{ch} V(k-1, 0, 1-k) = \mathrm{Gen}(k, 0, -k) + \mathrm{Gen}(k-1, 0, 1-k),$$

and thus induction on k gives immediately

$$\mathrm{ch}V(k,0,-k) = \mathrm{Gen}(k,0,-k)$$

for $k \geq 2$.

Finally it is obvious that

$$\mathrm{ch}V(0,0,0) = \mathrm{Gen}(0,0,0)$$

($V(0,0,0) = \mathbb{C} \oplus \Pi\mathbb{C} = \mathbb{C}^{1+\epsilon}$), and thus we conclude that for $n = 3$ $\mathrm{ch}V(\lambda) = \mathrm{Gen}(\lambda)$ for any $\lambda \in \Lambda^+$ except $\lambda \in (1,0,-1)$, $\mathrm{ch}V(\lambda)$ being given in the latter case by formula (1.18).

The case of $n = 4$ can be analyzed by the same methods. First of all one notes that (1.17) holds for any typical $\lambda \in \Lambda^+$ as well as for any $\lambda \in \Lambda^+$ which is $\mathfrak{p}^1 \cap \mathfrak{p}^2$-typical or $\mathfrak{p}^2 \cap \mathfrak{p}^3$-typical on $\mathfrak{g}$. This follows from Theorem 2. If λ is $\mathfrak{p}^1$-typical in $\mathfrak{p}$ or $\mathfrak{p}^3$-typical, Theorem 2 and our consideration of the case $n = 3$ imply that (1.7) holds unless the weight λ is one of the following:

$$\lambda = (k,1,0,-1) \text{ for } k > 1, \tag{1.20}$$

$$\lambda = (1,0,-1,\ell) \text{ for } \ell < -1. \tag{1.21}$$

Theorem 2 applies to these cases too and gives that

$$\mathrm{ch}V(\lambda) = \frac{1}{D} \cdot \sum_{w \in W} \mathrm{sgn}w \cdot w(\mathrm{ch}V_{\mathfrak{p}}(\lambda) \cdot e^{\rho_0} \cdot \prod_{\alpha \in \Delta^+,\ \alpha \notin \Delta(\mathfrak{p})} (1 + e^{-\alpha}))\ ,$$

where $\mathfrak{p} = \mathfrak{p}^1$ for (1.21) and $\mathfrak{p}^{n-1}$ for (1.22). But $\mathrm{ch}V_{\mathfrak{p}}(\lambda)$ is given in each case an obvious modification of formula (1.20), and a trivial calculation shows that

$$\mathrm{ch}V(\lambda) = \mathrm{Gen}(\lambda) - 2\mathrm{ch}V(k,0,0,0) \quad \text{in the case of (1.20)} \tag{1.22}$$

and

$$\mathrm{ch}V(\lambda) = \mathrm{Gen}(\lambda) - 2\mathrm{ch}V(0,0,0,\ell) \quad \text{in the case of (1.21).} \tag{1.23}$$

Therefore it remains to consider the case when $\lambda_1 + \lambda_4 = 0$. It has two subcases: $\lambda_2 + \lambda_3 \neq 0$ and $\lambda_2 + \lambda_3 = 0$. Assume first that $\lambda_2 + \lambda_3 \neq 0$. Then $\lambda = (a,b,c,-a)$, $b + c \neq 0$. Let $a = b + 1$, i.e. let $\lambda = (b+1,b,c,-b-1)$. If $b > 2$, using Theorems 3 and 4 one verifies that

$$\mathrm{ch}V(b+1,b,c,-b-1) + \mathrm{ch}V(b,b-1,c,1-b) = \mathrm{Gen}(b+1,b,c,-b-1) + \widetilde{\mathrm{Gen}}(b,b,c,-b),$$

where

$$\widetilde{\mathrm{Gen}}(b,b,c,-b) = \frac{\dim v_{(b,b,c,-b)}}{D} \cdot \sum_{w\in W} \mathrm{sgn} w \cdot w(e^{(b,b,c,-b)+\rho_0} \cdot \prod_{\substack{\alpha\in\Delta^+ \\ \alpha\neq\varepsilon_1-\varepsilon_4}} (1+e^{-\alpha})) \ .$$

But $\mathrm{ch}V(b,b-1,c,1-b) = \mathrm{Gen}(b,b-1,c,1-b)$, and a direct checking shows that

$$\mathrm{Gen}(b,b-1,c,1-b) = \widetilde{\mathrm{Gen}}(b,b,c,-b),$$

which gives

$$\mathrm{ch}V(b+1,b,c,b-1) = \mathrm{Gen}(b+1,b,c,b-1)$$

for $b > 2$. For $b = 2$ one considers the three possibilities $c = 1, 0, -1$, and (using in particular formula (1.19)) verifies that (1.17) holds in any of these cases. Let now $b = 1$. Then $c = 0$, i.e.

(1.25) $$\lambda = (2,1,0,-2).$$

Here a straightforward calculation based on the relations (1.12)–(1.15) gives

(1.26) $$\mathrm{ch}V(2,1,0,-2) = \mathrm{Gen}(2,1,0,-2) - 2\mathrm{ch}V(1,0,0,0) \ .$$

In this way we have now fully analyzed the case when $\lambda = (b+1,b,c,-b-1)$, $b > 0$, $b+c \neq 0$. Using induction on k one verifies now that if $\lambda = (b+k,b,c,-b-k) \in \Lambda^+$ for $k \geq 2$, then

$$\mathrm{ch}V(b+k,b,c,-b-k) = \mathrm{Gen}(b+k,b,c,-b,-b-k)$$

for all $b > 0$ and all c.

The case when $\lambda = (b+k,b,c,-b-k) \in \Lambda^+$ with $c < 0$, $b+c \neq 0$ is completely analogous and the result is that (1.17) holds except when

(1.27) $$\lambda = (2,0,-1,-2),$$

and that in the latter case

(1.28) $$\mathrm{ch}V(2,0,-1,-2) = \mathrm{Gen}(2,0,-1,-2) - 2\mathrm{ch}V(0,0,0,-1) \ .$$

It remains to consider the case

(1.29) $$\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \text{ with } \lambda_1+\lambda_4 = 0,\ \lambda_2+\lambda_3 = 0 \ .$$

Let us first single out the following weights:

$$\lambda = (0,0,0,0), \tag{1.30}$$

$$\lambda = (1,0,0,-1), \tag{1.31}$$

$$\lambda = (\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}), \tag{1.32}$$

$$\lambda = (2,0,0,-2), \tag{1.33}$$

$$\lambda = (2,1,-1,-2), \tag{1.34}$$

$$\lambda = (k+1, k, -k, -k-1) \quad \text{for } k > 1. \tag{1.35}$$

In the cases (1.30)–(1.34) one verifies directly that:

$$\mathrm{ch}V(0,0,0,0) = \mathrm{Gen}(0,0,0,0)\ , \tag{1.36}$$

$$\mathrm{ch}V(1,0,0,-1) = \mathrm{Gen}(1,0,0,-1) + 2\mathrm{ch}V(1,0,0,-1)\ , \tag{1.37}$$

$$\mathrm{ch}V(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}) = \mathrm{Gen}(\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2})\ , \tag{1.38}$$

$$\mathrm{ch}V(2,0,0,-2) = \mathrm{Gen}(2,0,0,-2) - 2\mathrm{ch}V(0,0,0,0)\ , \tag{1.39}$$

$$\mathrm{ch}V(2,1,-1,-2) = \mathrm{Gen}(2,1,-1,-2) - 2\mathrm{ch}V(1,0,0,-1)\ . \tag{1.40}$$

In the case of (1.35) Theorem 3 implies

$$\mathrm{ch}V(k+1,k,-k,-k-1) - \mathrm{ch}V(k,k-1,1-k,-k) = \mathrm{Ech}\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-k-1,k+1,k,-k))\ .$$

Furthermore

$$\mathrm{Ech}\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-k-1,k+1,k,-k)) = \mathrm{Gen}(k+1,k,-k,-k-1) + \widetilde{\mathrm{Gen}}(k,k,-k,-k) \tag{1.41}$$

for

$$\widetilde{\mathrm{Gen}}(k,k,-k,-k) = \frac{4(1+\varepsilon)}{D} \cdot \sum_{w \in W} \mathrm{sgn} w \cdot w(e^{(k,k,-k,-k)+\rho_0} \cdot \prod_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \varepsilon_1 - \varepsilon_4, \varepsilon_2 - \varepsilon_3}} (1 + e^{-\alpha}))\ ,$$

and a straightforward calculation gives

$$\widetilde{\mathrm{Gen}}(k,k,-k,-k) = -8\mathrm{ch}V_0(k,k-1,1-k,-k) \tag{1.42}$$

where for any $\mu \in \Lambda_0^+$ $V_0(\mu)$ denotes the irreducible $\mathfrak{g}_0^\Pi$-module with highest weight μ. Using (1.40) and (1.41) one verifies by induction on k that for $k > 1$

$$\begin{aligned} \mathrm{ch}V(k+1,k,-k,-k-1) &= \mathrm{Gen}(k+1,k,-k,-k-1) - \mathrm{ch}V(k,k-1,1-k,-k) \\ &= \mathrm{ch}V_0(k+1,k,-k,-k-1)\ . \end{aligned} \tag{1.43}$$

Finally, if λ is as in (1.29) but not as in (1.30)–(1.35), the reader will show by induction on $\lambda_1 - \lambda_2$ that (1.17) holds.

The conclusion is that for $n = 4$ (1.17) holds except in the cases (1.20), (1.21), (1.25), (1.27), (1.31), (1.33), (1.34), (1.35). The corresponding "correction terms" to (1.17) are given respectively in the formulas (1.22), (1.23), (1.26), (1.28), (1.37), (1.39), (1.40), (1.43).

2. Auxiliary Results

2.1. A lemma on central characters

Lemma 2.1. *Let $\lambda, \mu \in \Lambda_0^+$, $\mu \leq_{\mathfrak{b}} \lambda$, and $\theta^\lambda = \theta^\mu$. Then $\lambda = \mu + \sum_{i=1}^{s} \alpha_i$ for some sequence $\alpha_1, \ldots, \alpha_s$, $\alpha_j \in \Delta^+$, such that $\mu + \sum_{j=1}^{i} \alpha_j \in \Lambda_0^+$ and $\mu + \sum_{j=1}^{i} \alpha_j$ is α_{i+1}-atypical for any $i = 1, \ldots, s-1$.*

PROOF. Let $\lambda - \mu = \sum_t b_t \beta_t$, $\beta_t \in \Delta^+$ being simple roots. We will prove the statement by induction on n and on $|\lambda - \mu| := \sum_t b_t$. The induction assumption with respect to n and Lemma 1.1 in [P4] enable us to assume that $\lambda_k \neq \mu_k$ for $k = 1, \ldots, n$. Furthermore, let the reduced expressions of λ and μ be respectively $\tilde{\lambda} = a_1 \varepsilon_{i_1} + \cdots + a_k \varepsilon_{i_k}$, $\tilde{\mu} = a_1 \varepsilon_{j_1} + \cdots + a_k \varepsilon_{j_k}$ ($\tilde{\mu} \in W \cdot \tilde{\lambda}$ since $\theta^\lambda = \theta^\mu$, see Proposition 1.1). Consider first the case when $j_1 \neq 1$. Let ℓ be the maximal index such that $\mu_1 + \mu_\ell = 0$. Setting $\mu' := \mu + \varepsilon_1 - \varepsilon_\ell$ we see that the pair λ, μ' satisfies the conditions of the Lemma with $|\lambda - \mu'| < |\lambda - \mu|$, and therefore the induction assumption implies our claim. Let $j_1 = 1$. Then $\mu_1 = a_1$, $\lambda_1 > \mu_1$, and thus $i_1 > 1$. Let r be the maximal index such that $\lambda_r > a_1$. Clearly $r \leq i_1 - 1$. Let r' be the minimal index for which $\lambda_r + \lambda_{r'} = 0$. Set $\lambda' := \lambda + \varepsilon_{r'} - \varepsilon_r$. The pair λ', μ satisfies the conditions of the Lemma and $|\lambda' - \mu| < |\lambda - \mu|$, i.e. the induction assumption gives the result in this case too. □

2.2. On Verma module homomorphisms

Proposition 2.1. *Let $\lambda \in \mathfrak{h}_\alpha^*$ for some $\alpha \in \Delta^+$. Then*

$$\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda - \alpha), M(\lambda)) \neq 0 \ .$$

PROOF. Consider first the particular case when in addition $\lambda \in \Lambda_\alpha^+$, λ is generic, and $\lambda \notin \mathfrak{h}_\beta^*$ for $\beta \neq \alpha$. Under these assumptions Corollary 2.2 in [P4] implies the existence of an exact sequence of $\mathfrak{g}^\Pi$-modules

$$0 \to V(\lambda - \alpha) \to H^0_{G/B}(\lambda) \to V(\lambda) \to 0, \tag{2.1}$$

where $H^0_{G/B}(\lambda) := H^0(G/B, \mathcal{O}_{G/B}(-\lambda))^*$. Since $H^0_{G/B}(\lambda)$ is a highest weight g^Π-module with highest weight λ ($H^0_{G/B}(G/B, \mathcal{O}_{G/B}(-\lambda))$ being a lowest weight module with lowest weight $-\lambda$), (2.1) implies that $[M(\lambda) : V(\lambda - \alpha)] \neq 0$. Therefore also $[N : V(\lambda - \alpha))] \neq 0$, where N is the kernel of the canonical surjection $M(\lambda) \to V(\lambda)$. We claim that $\lambda - \alpha$ is an extremal weight for N, i.e. a maximal element in supp$N$ with respect to $\leq_{\mathfrak{b}}$. Note first that if $\theta^\lambda = \theta^\nu$ for some $\nu \in \Lambda^+$, then ν cannot satisfy both inequalities $\lambda >_{\mathfrak{b}} \nu$, $\nu >_{\mathfrak{b}} \lambda - \alpha$. The latter is a combinatorial observation which the reader will verify immediately using Proposition 1.1. Since now $\lambda >_{\mathfrak{b}} \nu$ for any $\nu \in \text{supp}N$, $\nu \neq \lambda$, ν cannot satisfy $\nu >_{\mathfrak{b}} \lambda - \alpha$, i.e. $\lambda - \alpha$ is indeed an extremal weight of N. Therefore any $v \in N^{(\lambda-\alpha)}$ is an $\mathfrak{n}$-singular vector, which implies that $\text{Hom}_{\mathfrak{g}^\Pi}(M(\lambda - \alpha), M(\lambda)) \neq 0$ for any generic $\lambda \in \Lambda^+_\alpha$ with $\lambda \notin \mathfrak{h}^*_\beta$ for all $\beta \neq \alpha$.

To complete the proof it suffices to notice that the set $\{\lambda \in \mathfrak{h}^*_\alpha \mid \text{Hom}_{\mathfrak{g}^\Pi}(M(\lambda - \alpha), M(\lambda)) \neq 0\}$ is a Zariski closed set in $\mathfrak{h}^*_\alpha$. Indeed, it is an elementary exercise to show that a Zariski closed subset in $\mathfrak{h}^*_\alpha$ which contains all generic weights in $\mathfrak{h}^*_\alpha \cap \Lambda^+$ coincides itself with $\mathfrak{h}^*_\alpha$. Therefore $\text{Hom}_{\mathfrak{g}^\Pi}(M(\lambda - \alpha), M(\lambda)) \neq 0$ for any $\lambda \in \mathfrak{h}^*_\alpha$ and the proof is complete. □

2.3. Tensor product functors

Lemma 2.2. *Let $\lambda \in \mathfrak{h}^*_0$, $X(\lambda)$ be a highest weight $\mathfrak{g}^\Pi$-module with highest weight λ, and V be any finite-dimensional $\mathfrak{g}^\Pi$-module. Then there is a $\mathfrak{g}$-filtration*

$$0 = F^0 \subset F^1 \subset \cdots \subset F^k = X(\lambda) \otimes V$$

such that $F^i/F^{i-1} \simeq X(\mu^i)$ for $1 \leq i \leq k$, where $X(\mu^i)$ is a highest weight $\mathfrak{g}$-module with highest weight $\mu^i \in \lambda + \text{supp}V$, and such that $\mu^i \leq_{\mathfrak{b}} \mu^j$ for $i > j$. Moreover, $m(X(\lambda) \otimes V, \mu) \leq [v_\lambda \otimes V^{(\mu-\lambda)} : v_\mu]$ where $m(X(\lambda) \otimes V, \mu) := \#\{i \mid \mu = \mu^i\}$.

The proof is standard (see [BGG]) and we omit it. □

Lemma 2.3. *Let $\lambda \in \mathfrak{h}^*$. Any $\mathfrak{n}$-singular $\mathfrak{h}$-submodule in $V(\lambda) \otimes E$ or $V(\lambda) \otimes E^*$ (i.e. any $\mathfrak{h}$-submodule consisting of $\mathfrak{n}$-singular vectors) is isomorphic to a submodule of $v_\lambda \otimes v_{\varepsilon_i}$, or respectively $v_\lambda \otimes v_{-\varepsilon_i}$, for some i, $1 \leq i \leq n$.*

The proof is completely similar to the proof of Lemma 5.3 in [S2]. □

We introduce now the functors

$$T^+_\theta : \mathbf{O} \rightsquigarrow \mathbf{O}^\theta , \qquad T^+_\theta(V) := (V \otimes E)^\theta ,$$
$$T^-_\theta : \mathbf{O} \rightsquigarrow \mathbf{O}^\theta , \qquad T^-_\theta(V) := (V \otimes E^*)^\theta ,$$

$\theta : Z \to \mathbb{C}$ being any central character. If θ' is another central character such that $\bar{\theta}' = \bar{\theta}$, then for any object V of $\mathbf{O}^{\theta'}$

$$T^+_\theta(V) \simeq (V\hat{\otimes}E)^\theta \oplus (V\hat{\otimes}E)^\theta ,$$

where $V\hat{\otimes}E$ is the eigenspace of eigenvalue 1 of the map $\Pi \otimes \Pi_E \ : \ V \otimes E \simeq V \otimes E$.[3] Similarly

$$T^-_\theta(V) \simeq (V\hat{\otimes}E^*)^\theta \oplus (V\hat{\otimes}E^*)^\theta ,$$

(If $\bar{\theta}' \neq \bar{\theta}$, $(V\hat{\otimes}E)^\theta$ and $(V\hat{\otimes}E^*)^\theta$ are only $\mathfrak{g}$-modules but not necessarily $\mathfrak{g}^\Pi$-modules). Therefore for $\bar{\theta} = \bar{\theta}'$ we have also the functors

$$\hat{T}^+_{\theta',\theta} : \mathbf{O}^{\theta'} \rightsquigarrow \mathbf{O}^\theta, \qquad \hat{T}^+_{\theta',\theta}(V) := (V\hat{\otimes}E)^\theta ,$$
$$\hat{T}^-_{\theta',\theta} : \mathbf{O}^{\theta'} \rightsquigarrow \mathbf{O}^\theta, \qquad \hat{T}^-_{\theta',\theta}(V) := (V\hat{\otimes}E^*)^\theta .$$

The following three lemmas are straightforward and the the reader will easily prove all of them or find analogous proofs in [BGG].

Lemma 2.4.

(a) *The functors $T^\pm_\theta$ and $T^\pm_{\theta',\theta}$ are exact.*

(b) *(T^+_θ, T^-_θ) and $(\hat{T}^\pm_{\theta',\theta}, \hat{T}^\mp_{\theta,\theta'})$ are pairs of adjoint functors.*

□

Lemma 2.5. *Let $\bar{\theta} = \bar{\theta}^\lambda$ for some $\lambda \in \mathfrak{h}_0^*$. Then $\hat{T}^\pm_{\theta^\lambda,\theta}(M(\lambda))$ has a finite $\mathfrak{g}^\Pi$-filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^{p^\pm} = \hat{T}^\pm_{\theta^\lambda,\theta}(M(\lambda))$ such that $F^i/F^{i-1} \simeq M(\mu^i)$ for $1 \leq i \leq p^\pm$, where $\mu^i = \lambda \pm \varepsilon_{j_i}$ for some j_i and $j_i \neq j_k$ for $i \neq k$ (and of course $\theta^{\lambda \pm \varepsilon_{j_i}} = \theta$). If $\bar{\theta} \neq \bar{\theta}^\lambda$, the same is true for $T^\pm_\theta(M(\lambda))$.*

□

Lemma 2.6. *For any central character θ and any object V of $\mathbf{O}$,*

$$T^\pm_\theta(V)^\vee \simeq T^\pm_\theta(V^\vee) .$$

If $\bar{\theta}' = \bar{\theta}$ and V is an object of $\mathbf{O}^{\theta'}$, then also

$$\hat{T}^\pm_{\theta',\theta}(V)^\vee \simeq \hat{T}^\pm_{\theta',\theta}(V^\vee) .$$

□

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[3] In section 3 we will also consider $V\hat{\otimes}E$ and $V\hat{\otimes}E^*$ for a $\mathfrak{p}^\Pi$- module V.

Lemma 2.7. *Let $\lambda, \lambda - \alpha \in \Lambda_\alpha^+$ for some $\alpha \in \Delta^+$. Then if*

$$\cdots \to M^2 \to M^1 \to M(\lambda) \to 0$$

is a BGG-resolution of $V(\lambda)$, $M(\lambda - \alpha)$ is necessarily present among the quotients of any $\mathfrak{g}^\Pi$-filtration of M^1 all quotients of which are Verma modules.

PROOF. Since $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda-\alpha), M(\lambda)) \neq 0$ (Proposition 2.1) but $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda - \alpha), V(\lambda)) = 0$, we have $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda-\alpha), M^1) \neq 0$. Furthermore, for any $\mathfrak{g}^\Pi$-filtration of M^1 whose quotients are Verma modules, $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda-\alpha), M(\nu)) \neq 0$ for some quotient $M(\nu)$. We claim that necessarily $\nu = \lambda - \alpha$. Indeed, assume the contrary. If $\alpha = \varepsilon_i - \varepsilon_j$, $i < j$, set $\lambda' = \lambda - \varepsilon_i$ and $\nu' = \nu - \varepsilon_i$. Then, since $\theta^{\lambda'} = \theta^{\lambda - \varepsilon_k}$ implies $k = i$, Lemma 2.5 gives

$$M(\lambda') \hookrightarrow T^-_{\theta^{\lambda'}}(M(\lambda - \alpha)) \hookrightarrow M(\lambda') \oplus M(\lambda') \ .$$

Furthermore, a straightforward combinatorial argument based on the inequalities $\lambda \geq_{\mathfrak{b}} \nu \geq_{\mathfrak{b}} \lambda - \alpha$ shows that $\theta^{\nu + \varepsilon_k - \varepsilon_i} = \theta^\nu$ implies $k = i$, and therefore (again by Lemma 2.5)

$$M(\nu) \hookrightarrow T^+_{\theta^\lambda}(M(\nu')) \hookrightarrow M(\nu) \oplus M(\nu) \ .$$

Now, by Lemma 2.4 (b),

$$\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda-\alpha), T^+_{\theta^\lambda}(M(\nu'))) \simeq \mathrm{Hom}_{\mathfrak{g}^\Pi}(T^-_{\theta^{\lambda'}}(M(\lambda-\alpha)), M(\nu')))$$

and, since $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda-\alpha), M(\nu)) \neq 0$, we obtain $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda'), M(\nu')) \neq 0$. But $\mathrm{Hom}_{\mathfrak{g}^\Pi}(M(\lambda'), M(\nu')) = 0$ simply because $\lambda' \not\leq_{\mathfrak{b}} \nu'$. This contradiction proves that $\nu = \lambda - \alpha$. □

If $\lambda \in \Lambda^+$, we set for any central character θ

$$\Omega^\pm_\theta(\lambda) := \{\lambda \pm \varepsilon_i \mid \lambda \pm \varepsilon_i \in \Lambda^+, \ \theta^{\lambda \pm \varepsilon_i} = \theta\} \ .$$

(In section 3 below we will also need

$$\Omega^\pm_{\theta_\mathfrak{p}}(\lambda) := \{\lambda \pm \varepsilon_i \mid \lambda \pm \varepsilon_i \in \Lambda^+_\mathfrak{p}, \ \theta_\mathfrak{p}^{\lambda \pm \varepsilon_i} = \theta_\mathfrak{p}\}$$

for $\lambda \in \Lambda^+_\mathfrak{p}$ and a central character $\theta_\mathfrak{p}$ of $\mathfrak{h} + \mathfrak{p}_{ss}$).

Proposition 2.2. *Let $\bar{\theta} = \bar{\theta}^\lambda$ for some $\lambda \in \Lambda^+$. Then the following statements hold.*

(a) $\#\Omega_{\bar{\theta}}^{\pm}(\lambda) \leq 2$.

(b) *If $\Omega_{\bar{\theta}}^{\pm}(\lambda) = \emptyset$, then $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) = 0$.*

(c) *If $\Omega_{\bar{\theta}}^{\pm}(\lambda) = \{\mu\}$, then $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \simeq V(\mu)$ or $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) = 0$.*

(d) *If $\Omega_{\bar{\theta}}^{\pm}(\lambda) = \{\mu, \mu'\}$ for some $\mu' <_{\mathfrak{b}} \mu$, then $\mu - \mu' \in \Delta^+$. Furthermore, if $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \neq 0$, then $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))$ is isomorphic to $V(\mu)$ or to $V(\mu')$, or there is a unique highest weight module $X^{\pm}(\mu)$ with highest weight μ for which there exists an exact sequence of $\mathfrak{g}^\Pi$-modules*

$$0 \to X^{\pm}(\mu) \to \hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \to Y^{\pm}(\mu') \to 0, \tag{2.2}$$

$Y^{\pm}(\mu)$ being a $\mathfrak{b}$-highest weight module with highest weight μ'. In this latter case $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))$ is an indecomposable $\mathfrak{g}^\Pi$-module such which has a unique irreducible submodule and a unique irreducible quotient both isomorphic to $V(\mu')$.

Proof. (a) For any two indices i, j, $\theta^{\lambda\pm\varepsilon_i} = \theta^{\lambda\pm\varepsilon_j}$ implies via Lemma 1.1 in [P4] $\lambda_i = \lambda_j$ or $\lambda_i + \lambda_j \pm 1 = 0$. Since $\lambda \in \Lambda^+$ we obtain that $\lambda_i = \lambda_j = 0$ or $\lambda_i + \lambda_j \pm 1 = 0$. Let now $\theta^{\lambda\pm\varepsilon_i} = \theta^{\lambda\pm\varepsilon_j} = \theta^{\lambda\pm\varepsilon_k}$ for $i < j < k$. Then, as one checks immediately, $\lambda_i = \lambda_j = 0$ or $\lambda_j = \lambda_k = 0$. But therefore $\bar{\theta}^{\lambda\pm\varepsilon_j} \neq \bar{\theta}^\lambda$, and this contradiction proves that $\#\Omega_{\bar{\theta}}^{\pm}(\lambda) \leq 2$.

(b) follows immediately from Lemma 2.2.

(c) By Lemma 2.2, $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \neq 0$ gives that $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))$ is a highest weight module with highest weight μ (the multiplicity inequality in Lemma 2.2 implies the irreducibility of the $\mathfrak{g}^\Pi$-module $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))^{(\mu)}$). Furthermore, $V(\lambda)^\vee \simeq V(\lambda)$ and thus $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))^\vee \simeq \hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))$ by Lemma 2.6. However a contragradient highest weight module is necessarily irreducible as a $\mathfrak{g}^\Pi$-module, i.e. $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \simeq V(\mu)$.

(d) Lemma 2.2 implies the existence of an exact sequence of $\mathfrak{g}^\Pi$-modules

$$0 \to X^{\pm}(\mu) \to \hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda)) \to Y^{\pm}(\mu')))) \to 0\ , \tag{2.3}$$

where each $\mathfrak{g}^\Pi$-module $X^{\pm}(\mu)$ and $Y^{\pm}(\mu')$ is either zero or is a highest weight module with respective highest weight μ and μ'. If $X^{\pm}(\mu) = 0$ or $Y^{\pm}(\mu') = 0$, the same argument as in the proof of (c) gives that the other module is irreducible as a $\mathfrak{g}^\Pi$-module. If $X^{\pm}(\mu), Y^{\pm}(\mu') \neq 0$, then $\mu, \mu' \in \Omega_{\bar{\theta}}^{\pm}(\lambda)$ implies $\mu - \mu' = \alpha \in \Delta^+$. Furthermore, Lemma 2.3 also gives the uniqueness of $X^{\pm}(\mu)$ because $X^{\pm}(\mu)$ is generated by the irreducible $\mathfrak{h}^\Pi$-module of $\hat{T}^{\pm}_{\theta^\lambda,\theta}(V(\lambda))$ consisting of $\mathfrak{n}$-singular vectors of weight μ.

We claim that $X^{\pm}(\mu)$ is a reducible $\mathfrak{g}^{\Pi}$-module. Assume the contrary. Then $X^{\pm}(\mu) \simeq V(\mu)$. Since $V(\lambda)^{\vee} \simeq V(\lambda)$, Lemma 2.6 implies that sequence (2.3) splits. Therefore $Y^{\pm}(\mu')^{\vee} \simeq Y^{\pm}(\mu')$, which gives $Y^{\pm}(\mu') \simeq V(\mu')$. In this way $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda)) \simeq V(\mu) \oplus V(\mu')$. Consider now a BGG-resolution of $V(\lambda)$

$$\cdots \to M^1 \to M(\lambda) \to 0 .$$

By Lemma 2.4(a), $\hat{T}^{\pm}_{\theta^{\lambda},\theta}$ is an exact functor, therefore

$$\cdots \to \hat{T}^{\pm}_{\theta^{\lambda},\theta}(M^1) \to \hat{T}^{\pm}_{\theta^{\lambda},\theta}(M(\lambda)) \to 0 \tag{2.4}$$

is a resolution of $V(\mu) \oplus V(\mu')$. Noting that our argument in the proof of (a) implies $\{\lambda \pm \varepsilon_i | \theta^{\lambda \pm \varepsilon_i} = \theta^{\lambda}\} = \{\mu, \mu'\}$, we see that Lemma 2.5 gives the existence of an exact sequence

$$0 \to M(\mu) \to \hat{T}^{\pm}_{\theta^{\lambda},\theta}(M(\lambda)) \to M(\mu') \to 0 .$$

Applying now Lemma 2.7 to the obvious subcomplex of (2.4) which is a BGG-resolution of $V(\mu)$ we obtain that $M(\mu')$ is a subquotient of $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(M^1)$. This forces the existence of a weight ν for which $M(\nu)$ is a subquotient of M^1 and $\mu' = \nu \pm \varepsilon_k$ for some k. But then $\nu \leq_{\mathfrak{b}} \lambda$ and $\theta^{\nu} = \theta^{\lambda}$, which (as one checks immediately) contradicts the equality $\nu = \mu' \mp \varepsilon_k$. This contradiction proves that $X^{\pm}(\mu)$ is indeed a reducible $\mathfrak{g}^{\Pi}$-module.

Lemma 2.3 implies now that the minimal $\mathfrak{g}^{\Pi}$-submodule of $X^{\pm}(\mu)$ is isomorphic to $V(\mu')$ and that any singular vector in $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$ belongs to $X^{\pm}(\mu)$. This gives the indecomposability of $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$ and the fact that $V(\mu')$ is the only irreducible $\mathfrak{g}^{\Pi}$-submodule of $\hat{T}^{\pm}_{\theta^{\lambda},\theta}$. Finally the contragrediency of $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$ implies that $V(\mu')$ is also the only irreducible $\mathfrak{g}^{\Pi}$-quotient of $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$. □

Corollary 2.1. *Let $\lambda \in \Lambda^+$. If $\#\theta \geq \#\theta^{\lambda}$ and $\bar{\theta} = \bar{\theta}^{\lambda}$, then $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$ is an irreducible $\mathfrak{g}^{\Pi}$-module.*

Proof. If $\hat{T}^{\pm}_{\theta^{\lambda},\theta}(V(\lambda))$ is reducible, then $\#\Omega^{\pm}_{\theta}(\lambda) = 2$ by Proposition 2.2. But this implies $\#\theta^{\lambda} > \#\theta$ which contradicts to the condition $\#\theta \geq \#\theta^{\lambda}$. □

2.4. The functors $H^i_{G/P}(\cdot)$ and the multiplicities $m^i_P(\lambda, \mu)$

In the proofs of the main results it will be convenient to consider the following twisted version of the cohomology of induced $\mathcal{O}_{G/P}$-modules:

$$H^i_{G/P} : \mathfrak{p}^{\Pi}\text{-mod}_f \rightsquigarrow \mathfrak{g}^{\Pi}\text{-mod}_f$$
$$H^i_{G/P}(V) := H^i(G/P, \mathcal{O}_{G/P}(V^*))^* .$$

Since $H^0(G/P, \mathcal{O}_{G/P}(V^*))$ is a lowest weight module, see **1.2**, $H^0_{G/P}(V_{\mathfrak{p}}(\lambda))$ is a highest weight module whenever $H^0_{G/P}(V_{\mathfrak{p}}(\lambda)) \neq 0$. Furthermore, the canonical isomorpisms $V_{\mathfrak{p}}(\lambda)^* = V_{\mathfrak{p}}(-w^{\mathfrak{p}}_m(\lambda))$ and $V(\mu)^* = V(-w_m(\mu))$ imply

$$[H^i_{G/P}(V_{\mathfrak{p}}(\lambda)) : V(\mu)] = \begin{cases} m^i_P(\lambda,\mu) & \text{for } i \neq 0 \text{ or } \mu \neq \lambda \\ 1 & \text{for } i = 0 \text{ and } \mu = \lambda \ . \end{cases}$$

Lemma 2.8. *If $m^i_P(\lambda,\mu) \neq 0$, then there exist $w \in W$ and $\beta_1, \ldots, \beta_k \in \Delta^+$, $\beta_p \neq \beta_q$, so that $\mu = w(\lambda) - \sum_p \beta_p$ and i is equal to the minimal length $\ell(w \circ W^{\mathfrak{p}})$ of elements in the coset $w \circ W^{\mathfrak{p}}$ where $\mathfrak{p} = \mathrm{Lie}P$.*

PROOF. $\mathcal{O}_{G/P}(V_{\mathfrak{p}}(\lambda)^*)$ has a $\mathfrak{g}_0$-sheaf filtration with factors $\mathcal{O}_{(G/P)_{\mathrm{red}}}(V_{\mathfrak{p}_0}(\nu)^*)$ where ν runs over a subset of the set $\lambda + \mathrm{supp}S^{\cdot}(\mathfrak{n}_1^-)$. By the Borel-Weil-Bott theorem (applied to $\mathcal{O}_{(G/P)_{\mathrm{red}}}(V_{\mathfrak{p}_0}(\nu)^*)$) $H^i_{G/P}(V_{\mathfrak{p}}(\lambda))$ has a $\mathfrak{g}_0$-module filtration with factors $V_0(w_\nu(\nu + \rho_0) - \rho_0))$, where w_ν is such that $w_\nu(\nu + \rho_0) \in \Lambda^+_0$ and $i = \ell(w_\nu \circ W^{\mathfrak{p}})$. Therefore, if $m^i_P(\lambda,\mu) \neq 0$, then $\mu = w_\nu(\nu + \rho_0) - \rho_0 = w_\nu(\lambda) + w_\nu(\sum_i \alpha_i + \rho_0) - \rho_0$ where $\alpha_i \in \Delta^-$, $\alpha_t \neq \alpha_s$. But obviously $w_\nu(\sum_i \alpha_i + \rho_0) - \rho_0 = \sum_p \beta_p$ for some $\beta_1, \ldots, \beta_k \in \Delta^+$, $\beta_p \neq \beta_q$. Setting $w := w_\nu$ we complete the proof. □

Corollary 2.2. *If $m^i_P(\lambda,\mu) \neq 0$, then $\mu \leq_{\mathfrak{b}} \lambda$. The equality $\mu = \lambda$ is possible only when $w(\lambda) = \lambda$ for some $w \in W$, $w \notin W^{\mathfrak{p}}$.* □

Corollary 2.3. *If $m^i_P(\lambda,\mu) \neq 0$, the pair λ, μ satisfies the conditions of Lemma 2.1.* □

3. Proofs of the Main Results

3.1. Proof of Theorem 1

The relations (1.7)–(1.16)[4] imply that for each $k = 1, \ldots, n-1$ $\mathbf{a}_k$ preserves $\mathcal{H}_\lambda$ and furthermore that its restriction $\mathbf{a}^\lambda_k$ to $\mathcal{H}_\lambda$ is strictly lower triangular for any linear order of the basis e^ν for $\nu \leq_{\mathfrak{b}} \lambda$ which is compatible with the partial order $\leq_{\mathfrak{b}}$. Therefore $\mathrm{id} + \mathbf{a}^\lambda_k$ is invertible for all k. Furthermore, if $\nu_1 \neq 0$, we have the tower of natural projections

$$G/P(\nu) \xrightarrow{p^{n-1}_\nu} G/\tilde{P}^{n-2} \xrightarrow{p^{n-2}_\nu} \cdots \xrightarrow{p^2_\nu} G/\tilde{P}^1 \xrightarrow{p^1_\nu} \mathrm{Spec}\mathbb{C} \ ,$$

[4]The proofs of Theorems 2–4 are completely independent of Theorem 1 so we can use Theorems 2–4 to prove Theorem 1.

where $\text{Lie}\tilde{P}^r = \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^k \cap \mathfrak{p}^{\ell-1} \cap \cdots \cap \mathfrak{p}^r$ for $\ell - 1 \leq r \leq n-1$, $\text{Lie}\tilde{P}^r = \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^k$ for $k < r < \ell - 1$, and $\text{Lie}\tilde{P}^r = \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^r$ for $1 \leq r \leq k$. Set $p_\nu := p_\nu^1 \circ \cdots \circ p_\nu^{n-1}$. The fact that

$$R^{\cdot}(p_\nu)_*(\mathcal{O}_{G/P(\nu)}(V_{\mathfrak{p}(\nu)}(w_m^{\mathfrak{p}(\nu)} \circ w_m(\nu)))) =$$
$$R^{\cdot}(p_\nu^1)_*(R^{\cdot}(p_\nu^2)_*(\ldots(R^{\cdot}(p_\nu^{n-1})_*(\mathcal{O}_{G/P(\nu)}(V_{\mathfrak{p}(\nu)}(w_m^{\mathfrak{p}(\nu)} \circ w_m(\nu)))))\ldots)))$$

implies

$$\mathbf{a}(e^\nu) = ((\text{id} + \mathbf{a}_1) \circ \cdots \circ (\text{id} + \mathbf{a}_{n-1}))(e^\nu)$$

for $\nu_1 \neq 0$. But if $\nu_1 = 0$, or $\nu_n = 0$, Theorem 2 applied to $\mathcal{O}_{G/P(\nu)}(V_{\mathfrak{p}(\nu)}(w_m^{\mathfrak{p}(\nu)} \circ w_m(\nu)))$ gives

$$\mathbf{a}(e^\nu) = e^\nu.$$

Therefore

$$\mathbf{a}^\lambda := a_{|\mathcal{H}_\lambda} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$$

is a well-defined isomorphism for any $\lambda \in \Lambda^+$, and

$$\mathbf{a}^\lambda = ((\text{id} + \mathbf{a}_1^\lambda) \circ \cdots \circ (\text{id} + \mathbf{a}_{n-1}^\lambda)) .$$

In order to prove (1.6), note that any $\nu \in \Lambda^+$, $\nu \leq_{\mathfrak{b}} \lambda$,

$$\sum_{\mu \leq_{\mathfrak{b}} \nu} a_{\nu,\mu} \text{ch} V(\mu) = \text{Ech}\mathcal{O}_{G/P(\nu)}(V_{\mathfrak{p}(\nu)}(w_m^{\mathfrak{p}(\nu)} \circ w_m(\nu))) = \sigma(\nu) .$$

This is a system of linear equations whose matrix is the matrix of $\mathbf{a}^\lambda$ and whose right hand-side is the vector-column $(\sigma(\nu))$ for $\nu \leq_{\mathfrak{b}} \lambda$. By solving this system we obtain in particular (1.6). If $\lambda_1 = 0$ or $\lambda_n = 0$, then respectively $\nu_1 = 0$ or $\nu_n = 0$ for any $\nu \leq_{\mathfrak{b}} \lambda$, $\nu \in \Lambda^+$, and thus $\mathbf{a}_{|\mathcal{H}_\lambda} = \text{id}$, which means that

$$\text{ch}V(\lambda) = \sigma(\lambda).$$

□

3.2. Proof of Theorem 2 and Corollary 1

Let $\mathfrak{p} = \text{Lie}\ P = \mathfrak{r} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is the reductive part of $\mathfrak{p}$. Consider an element $z \in \mathfrak{h}_0$ such that $\alpha(z) = 0$ for each simple root α of $\mathfrak{b}$ which is a root of $\mathfrak{u}$ and $\alpha(z) = 1$ for each simple root of $\mathfrak{b}$ which is not a root of $\mathfrak{u}$. Clearly z is unique up to addition of elements from the center of $\mathfrak{g}_0$.

Theorem 2 is obviously equivalent to Corollary 1, so all we need to prove is that $m^i_P(\lambda,\mu) = 0$ for all $\mu \in \Lambda^+$, $i \geq 0$. Assuming to the contrary that $m^i_P(\lambda,\mu) \neq 0$ for some μ and some $i \geq 0$. Then, by Corollary 2.3, the pair λ, μ satisfies the condition of Lemma 2.1 and thus $\lambda = \mu + \sum_j \alpha_j$ as in Lemma 2.1. The fact that λ is $\mathfrak{p}$-typical implies that all α_j are roots of $\mathfrak{u}$. On the other hand, $\mu = w(\lambda) - \sum_p \beta_p$ as in Lemma 2.8. Therefore

$$\lambda - w(\lambda) = \sum_j \alpha_j - \sum_p \beta_p \ .$$

Let $i = \ell(w \cdot W^{\mathfrak{p}}) \neq 0$. Then $(\lambda - w(\lambda))(z) > 0$ since λ is $\mathfrak{p}$ typical. But $\sum_j \alpha_j(z) = 0$ and $\sum_p \beta_p(z) \geq 0$, and thus $(\sum_j \alpha_j - \sum_p \beta_p)(z) \leq 0$. This shows that necessarily $i = \ell(w \cdot W^{\mathfrak{p}}) = 0$. In this case all α_j's are roots of $\mathfrak{u}$ and

$$\lambda(z) = \mu(z) \ . \tag{3.1}$$

We claim that (3.1) is contradictory. Indeed, as it is easy to see, the canonical surjection of $\mathfrak{p}^\Pi$-modules (1.2) induces by duality a canonical surjection of $\mathfrak{g}^\Pi$-modules

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathfrak{p}}(\lambda) \xrightarrow{\pi_\lambda} H^0_{G/P}(V_{\mathfrak{p}}(\lambda))$$

such that the natural diagram

$$\begin{array}{ccc} H^0_{G/P}(V_{\mathfrak{p}}(\lambda)) & \longrightarrow & V(\lambda) \\ \nwarrow \pi_\lambda & & \nearrow pr \\ & U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathfrak{p}}(\lambda) & \end{array} \tag{3.2}$$

is commutative. But this means that $\kappa(z) < \lambda(z)$ for any $\kappa \in \mathrm{supp}(\ker pr)$ and in particular that $\mu(z) < \lambda(z)$. This contradiction finally gives $m^i_P(\lambda,\mu) = 0$ for all $i \geq 0$. □

3.3. Proof of Corollary 2

Note first that Theorem 2 is obviously equivalent to the statement that for any $\lambda \in \Lambda^+$ which is $\mathfrak{p}$-typical in $\mathfrak{g}$,

$$\begin{aligned} &H^0_{G/P}(V_{\mathfrak{p}}(\lambda)) = V(\lambda) \ , \\ &H^i_{G/P}(V_{\mathfrak{p}}(\lambda)) = 0 \quad \text{for } i > 0 \ . \end{aligned} \tag{3.3}$$

Moreover we claim that the functor

$$H^0_{G/P} : (\mathfrak{p}^\Pi - \mathrm{mod})_f^{(\theta^\lambda_{\mathfrak{p}})} \rightsquigarrow (\mathfrak{g}^\Pi\text{-mod})_f^{(\theta^\lambda, \mathfrak{p})} \tag{3.4}$$

is the desired equivalence of categories. First of all, using Proposition 1.1 one verifies immediately that if a simple finite-dimensional $\mathfrak{p}^\Pi$-module $V_\mathfrak{p}(\lambda')$, considered as a $U(\mathfrak{h}_0 + \mathfrak{p}_{ss})$-module, affords the central character $\theta_\mathfrak{p}^\lambda$ of $V_\mathfrak{p}(\lambda)$, then $\lambda' \in \Lambda^+$, λ' is $\mathfrak{p}$-typical in $\mathfrak{g}$ and $V(\lambda')$ affords the central character θ^λ. This, together with the observation that (because of (3.3)) the $\mathfrak{g}^\Pi$-composition factors of $H^0_{G/P}(V_\mathfrak{p})$ for any object $V_\mathfrak{p}$ of $(\mathfrak{p}^\Pi\text{-mod})_f^{(\theta_\mathfrak{p}^\lambda)}$ have the same highest weights as the composition factors of $V_\mathfrak{p}$ itself, implies in particular that $H^0_{G/P}$ is indeed a well-defined functor between the above categories.

There is also a natural "localization functor"

$$L_\mathfrak{p} : (\mathfrak{g}^\Pi\text{-mod})_f^{(\theta^\lambda,\mathfrak{p})} \rightsquigarrow (\mathfrak{p}^\Pi\text{-mod})_f^{(\theta_\mathfrak{p}^\lambda)} .$$

Indeed, if V is an object of $(\mathfrak{g}^\Pi\text{-mod})_f^{(\theta^\lambda,\mathfrak{p})}$, let $V(\mathfrak{p})$ denote the intersection of all $\mathfrak{p}^\Pi$-submodules of V which generate V as a $\mathfrak{g}^\Pi$-module. Clearly $V(\mathfrak{p})$ is a canonical $\mathfrak{p}^\Pi$-submodule of V and we set $L_\mathfrak{p}(V) := V(\mathfrak{p})$.

The fact that $L_\mathfrak{p}$ and $H^0_{G/P}$ are mutually inverse functors is established by a straightforward checking which we leave to the reader. □

3.4. Proof of Theorem 3

We start with the following two lemmas.

Lemma 3.1. *If α and λ are as in Theorem 3 and $k < n$, then*

$$m^i(\lambda, \mu) = m^i_{0,n-k}(\lambda, \mu) \tag{3.5}$$

for any $\mu \in \Lambda^+$.

Proof. Let $K \hookrightarrow G$ be the parabolic subsupergroup with $\mathfrak{k} := \mathrm{Lie}K = \mathfrak{p}^k \cap \cdots \cap \mathfrak{p}^{n-1}$. Consider the double bundle

$$\begin{array}{ccc} & G/L & \\ p \swarrow & & \searrow q \\ G/P^1 & & G/K \end{array} ,$$

where L is the parabolic subsupergroup of G with $\mathfrak{l} = \mathrm{Lie}\ L = \mathfrak{p}^1 \cap \mathfrak{k}$. Notice that λ is $\mathfrak{l}$-typical in $\mathfrak{p}^1$ and therefore by (an obvious relative version of) Theorem 2, $R^i p_* \mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*) = 0$ for $i > 0$ and $R^0 p_* \mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*) = \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)^*)$. Therefore $H^j_{G/L}(V_\mathfrak{l}(\lambda)) = H^j_{G/P^1}(V_{\mathfrak{p}^1}(\lambda))$ for all $j \geq 0$.

Let us describe now the composition factors of the $\mathfrak{g}^\Pi$-linearized $\mathcal{O}_{G/K}$-modules $R^i q_* \mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*)$ for all i. Clearly for each μ the multiplicity of $\mathcal{O}_{G/K}(V_\mathfrak{k}(\mu)^*)$ in

$R^iq_*\mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*)$ equals $m^i_{0,n-k}(\lambda,\mu)$. Furthermore, Corollary 2.2 (applied to $G = K$, $P = L$) implies that any μ with $m^i_{0,n-k}(\lambda,\mu) \neq 0$ is $\mathfrak{k}$-typical in $\mathfrak{g}$. Therefore, $H^j(G/K,\ R^iq_*\mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*)) \neq 0$ for all $j > 0$ and (by Leray's spectral sequence),

$$H^i_{G/L}(V_\mathfrak{l}(\lambda)^*) = H^0(G/K,\ R^iq_*\mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*))^*\ .$$

This means that the multiplicity of $H^0(G/K,\ \mathcal{O}_{G/K}(V_\mathfrak{k}(\mu)^*))^*$ in $H^0(G/K,$ $R^iq_*\mathcal{O}_{G/L}(V_\mathfrak{l}(\lambda)^*))^*$ is $m^i_{0,n-k}(\lambda,\mu)$. But since $H^0(G/K,\ \mathcal{O}_{G/K}(V_\mathfrak{k}(\mu)^*))^*$ is nothing but $V(\mu)$ and since the multiplicity of $V(\mu)$ in $H^i_{G/L}(V_\mathfrak{l}(\lambda)^*)$ is $m^i(\lambda,\mu)$, we have established (3.5). □

Lemma 3.1 implies that it suffices to prove Theorem 3 under the assumption that $\alpha = \varepsilon_1 - \varepsilon_n$. This assumption will be valid throughout the rest of the proof.

The following lemma is a more specific version of Corollary 2.1.

Lemma 3.2. *Let $\lambda \in \Lambda^+$, $\#\theta \geq \#\theta^\lambda$, $\bar\theta = \bar\theta^\lambda$, and $\Omega^+_\theta(\lambda) = \{\lambda + \varepsilon_i\}$ (respectively $\Omega^-_\theta(\lambda) = \{\lambda - \varepsilon_i\}$) for some i. If λ is $\mathfrak{p}^{i-1}$-typical (respectively λ is $\mathfrak{p}^i$-typical) in $\mathfrak{g}$, then $\hat T^\pm_{\theta^\lambda,\theta}(V(\lambda)) = V(\lambda \pm \varepsilon_i)$.*

PROOF. Corollary 2.1 implies that $\hat T^\pm_{\theta^\lambda,\theta}(V(\lambda)) = 0$ or $\hat T^\pm_{\theta^\lambda,\theta}(V(\lambda)) \simeq V(\lambda \pm \varepsilon_i)$. Therefore all we need to prove is that $\hat T^\pm_{\theta^\lambda,\theta}(V(\lambda)) \neq 0$. We will do it for $T^+_{\theta^\lambda,\theta}(V(\lambda))$ and we will leave the case of $T^-_{\theta^\lambda,\theta}(V(\lambda))$ (which is completely similar) to the reader.

Put $P := P^{i-1}$. Since λ is $\mathfrak{p}$-typical, Theorem 2 implies $H^0_{G/P}(V_\mathfrak{p}(\lambda)) = V(\lambda)$. Therefore $H^0_{G/P}(V_\mathfrak{p}(\lambda)\hat\otimes E)^\theta \simeq T^+_{\theta^\lambda,\theta}(V(\lambda))$. Furthermore there are the following exact sequences of $\mathfrak{p}$-modules:

$$0 \to V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_1) \to V_\mathfrak{p}(\lambda)\hat\otimes E \to V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_i) \to 0\ , \tag{3.6}$$

$$0 \to V_\mathfrak{p}(\lambda+\varepsilon_i) \to V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_i) \to (V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_i))/V_\mathfrak{p}(\lambda+\varepsilon_i) \to 0\ . \tag{3.7}$$

Since $\Omega^+_\theta(\lambda) = \{\lambda+\varepsilon_i\}$, we have $\theta^\mu \neq \theta$ for all composition factors $V_\mathfrak{p}(\mu)$ of $V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_1)$ or of $(V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_i))/V_\mathfrak{p}(\lambda+\varepsilon_i)$. Therefore for any j,

$$H^j_{G/P}(V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_1))^\theta = 0$$

and

$$H^j_{G/P}(V_\mathfrak{p}(\lambda)\hat\otimes V_\mathfrak{p}(\varepsilon_i)/V_\mathfrak{p}(\lambda+\varepsilon_i))^\theta = 0\ .$$

In this way (3.6) yields

$$H^j_{G/P}(V_{\mathfrak{p}}(\lambda)\hat{\otimes}E)^\theta \simeq H^j_{G/P}(V_{\mathfrak{p}}(\lambda)\hat{\otimes}V_{\mathfrak{p}}(\varepsilon_i))^\theta$$

for any j, and (3.7) yields

$$H^j_{G/P}(V_{\mathfrak{p}}(\lambda)\hat{\otimes}V_{\mathfrak{p}}(\varepsilon_i))^\theta \simeq H^j_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i))^\theta$$

for any j. In particular

$$T^+_{\theta^\lambda,\theta}(H^0_{G/P}(V_{\mathfrak{p}}(\lambda))) = H^0_{G/P}(V_{\mathfrak{p}}(\lambda)\hat{\otimes}E)^\theta \simeq H^0_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i)) \ .$$

But $H^0_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i)) \neq 0$ because we have a canonical surjection $H^0_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i)) \to V(\lambda+\varepsilon_i)$ (induced by the canonical $\mathfrak{g}^\Pi$-injection

$$V(\lambda+\varepsilon_i)^* \hookrightarrow H^0(G/P,\ \mathcal{O}_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i)^*))\ ,$$

and this proves the Lemma. (We proved also that under the conditions of the Lemma, $H^0_{G/P}(V_{\mathfrak{p}}(\lambda+\varepsilon_i)) \simeq V(\lambda+\varepsilon_i)$.) □

Now we can turn to the actual proof of Theorem 3. The plan is as follows. We will first establish Proposition 3.1 which is a weak version of Theorem 3(a). Then we will prove Theorem 3(b) and (c), and only after that we will complete the proof of Theorem 3(a).

We start with

Proposition 3.1. *If λ is as in Theorem 3(a) (and $\alpha = \varepsilon_1 - \varepsilon_n$) then (1.8) and (1.9) hold, and furthermore*

(3.8) $m^1(\lambda-\alpha,\mu) \leq m^0(\lambda,\mu) \leq m^1(\lambda-\alpha,\mu)+m^0(\lambda-\alpha,\mu)$ *for any* $\mu \in \Lambda^+,\ \mu \neq \lambda-\alpha$.

PROOF.

Claim 1. *If λ is as in Theorem 3(a) then there is an exact sequence of $\mathfrak{g}^\Pi$-sheaves*

$$0 \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}E)^{\theta^\lambda} \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \to 0,$$

where more generally $\mathcal{O}_{G/P}(v)^\theta$ is the generalized eigenspace of eigenvalue θ in the $\mathfrak{g}^\Pi$-sheaf $\mathcal{O}_{G/P}(v)$.

Indeed, the exact sequence of $(\mathfrak{p}^1)^\Pi$-modules

$$0 \to V_{\mathfrak{p}^1}(\varepsilon_1) \to E \to V_{\mathfrak{p}^1}(\varepsilon_2) \to 0$$

gives rise to the exact sequence

$$0 \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}E)^{\theta^\lambda} \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}V_{\mathfrak{p}^1}(\varepsilon_2))^{\theta^\lambda} \to 0 .$$

(Note that $E' := V_{\mathfrak{p}^1}(\varepsilon_2)$ is tautological representation of $\mathfrak{h}+\mathfrak{p}^1_{ss}$, $\dim E' = n-1+(n-1)\varepsilon$ and $V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}V_{\mathfrak{p}^1}(\varepsilon_2)$ is defined as in **2.3** with E replaced by E'). But $\lambda-\varepsilon_1$ and $\theta^{\lambda-\alpha}_{\mathfrak{p}^1}$ satisfy the conditions of Lemma 3.2 applied to $\mathfrak{h}+\mathfrak{p}^1_{ss}$ ($\mathfrak{h}+\mathfrak{p}^1_{ss}$ is isomorphic to a trivial central extension of $q(E')$, so Lemma 3.2 is obviously valid) because $\lambda-\varepsilon_1$ is $\mathfrak{p}^1\cap\mathfrak{p}^n$-typical in $\mathfrak{p}^1$, $\#\theta^{\lambda-\varepsilon_1}_{\mathfrak{p}^1} = \#\theta^{\lambda-\alpha}_{\mathfrak{p}^1}$, and $\Omega^+_{\theta^{\lambda-\alpha}_{\mathfrak{p}^1}}(\lambda-\varepsilon_1) = \{\lambda-\alpha\}$. Therefore by Lemma 3.2,

$$\mathcal{O}_{G/P^1}((V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}V_{\mathfrak{p}^1}(\varepsilon_2))^{\theta^{\lambda-\alpha}_{\mathfrak{p}^1}}) \simeq \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) .$$

Noting that $\theta^{\lambda-\varepsilon_1+\varepsilon_r} \neq \theta^\lambda$ for $1 < r < n$, we obtain

$$\mathcal{O}_{G/P^1}((V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}V_{\mathfrak{p}^1}(\varepsilon_2))^{\theta^{\lambda-\alpha}_{\mathfrak{p}^1}}) = \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}V_{\mathfrak{p}^1}(\varepsilon_2))^{\theta^\lambda},$$

and we have established Claim 1.

Claim 1 implies immediately the existence of the following exact sequence

$$\begin{aligned} \cdots \to H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1))\hat{\otimes}E)^{\theta^\lambda} \to \\ \to H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \to H^{i-1}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to \cdots \end{aligned} \tag{3.9}$$

Since $\lambda-\varepsilon_1$ is $\mathfrak{p}^1$-typical in $\mathfrak{g}$, (3.9) yields the $\mathfrak{g}^\Pi$-isomorphisms

$$H^{i+1}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \tilde{\to} H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \quad \text{for } i>0 \tag{3.10}$$

and the exact sequence

$$\begin{aligned} 0 \to H^1_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \to H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \xrightarrow{\psi_\lambda} \\ \to (H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1))\hat{\otimes}E)^{\theta^\lambda} \to H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \to 0. \end{aligned} \tag{3.11}$$

This is sufficient to conclude that (1.8) and (1.9) hold. Indeed, (3.10) gives (1.9) directly; Corollary 2.2 and (3.10) imply $m^i(\lambda,\lambda-\alpha)=0$ for $i>0$, and, by noting that

$(H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1))\hat{\otimes}E)^{\theta^\lambda} = \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1))$, we obtain from 3.11 and Proposition 2.2(d) that $m^0(\lambda,\lambda-\alpha)=1$.

In order to establish (3.8) notice that

$$\text{im } \psi_\lambda \simeq H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda))/H^1_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha))$$

and that therefore it suffices to prove

Claim 2. *For any $\mu \in \Lambda^+$, $\mu \neq \lambda$,*

$$[\text{im } \psi_\lambda : V(\mu)] \leq [H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) : V(\mu)] .$$

As we already noted

$$H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\varepsilon_1)\hat{\otimes}E)^{\theta^\lambda} = \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1)) ,$$

and moreover the exact sequence

$$(3.12) \qquad 0 \to \text{im } \psi_\lambda \to \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1)) \xrightarrow{\phi_\lambda} H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) \to 0$$

induced by (3.11) is nothing but (2.2), where

$$X^+(\lambda=(\lambda-\varepsilon_1)+\varepsilon_1) := \text{im } \psi_\lambda$$

and

$$Y^+((\lambda-\varepsilon_1)+\varepsilon_n=\lambda-\alpha) := H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) .$$

By Proposition 2.2(d), im ψ_λ is the unique highest weight submodule in $\hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1))$ of highest weight λ. Since $\hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1))$ is contragradient, there is a unique projection

$$s_\lambda : \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1)) \to S_\lambda ,$$

where $S_\lambda \simeq (\text{im } \psi_\lambda)^\vee$.

We claim now that

$$\ker r_\lambda \subset \ker s_\lambda ,$$

where $r_\lambda : \text{im}\psi_\lambda \to V(\lambda)$ is the canonical projection. Indeed, $s_\lambda(\ker r_\lambda) \hookrightarrow S_\lambda$ but $[s_\lambda(\ker r_\lambda) : V(\lambda)] = 0$. Since S_λ has a unique $\mathfrak{g}^\Pi$-irreducible submodule isomorphic to $V(\lambda)$ (which corresponds to the unique irreducible quotient of im ψ_λ isomorphic to $V(\lambda)$

(see Proposition 2.2(d)) we have necessarily $s_\lambda(\ker r_\lambda) = 0$ or equivalently $\ker r_\lambda \subset \ker s_\lambda$. This means that the map

$$\tilde{s}_\lambda : \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda - \varepsilon_1))/\mathrm{im}\psi_\lambda \to S_\lambda/V(\lambda)$$

is a well-defined projection. Therefore (3.12) gives

$$(3.13) \qquad [S_\lambda/V(\lambda) : V(\mu)] \leq [H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda - \alpha)) : V(\mu)]$$

for any $\mu \in \Lambda^+$. But since $S_\lambda/V(\lambda) \simeq (\ker r_\lambda)^\vee$ we have

$$(3.14) \qquad [S_\lambda/V(\lambda) : V(\mu)] = [\ker r_\lambda : V(\mu)] \ .$$

(3.13) and (3.14) give now Claim 2 immediately. This completes the proof of Proposition 3.1. □

PROOF OF THEOREM 3(B).

We start with the claim that if $0 \neq \lambda \in \Lambda^+$ is $\alpha = \varepsilon_1 - \varepsilon_n$-atypical, then

$$H^j_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) = H^j_{G/P^{n-1}}(V_{\mathfrak{p}^{n-1}}(\lambda))$$

for all j. Indeed, consider the double bundle

$$\begin{array}{ccc} & G/S & \\ p^1 \swarrow & & \searrow p^{n-1} \\ G/P^1 & & G/P^{n-1} \end{array} \ ,$$

S being the parabolic subsupergroup of G with Lie $S = \mathfrak{s} := \mathfrak{p}^1 \cap \mathfrak{p}^{n-1}$. Since $\lambda = (\lambda_1, \ldots, \lambda_n = -\lambda_1)$ with $\lambda_1 > \lambda_2, \lambda_{n-1} > \lambda_n$, λ is $\mathfrak{s}$-typical in $\mathfrak{p}^1$ and $\mathfrak{p}^{n-1}$. Therefore (by an obvious relative version of Theorem 2)

$$R^0 p^1_* \mathcal{O}_{G/S}(V_\mathfrak{s}(\lambda)^*) = \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)^*) \ ,$$
$$R^0 p^{n-1}_* \mathcal{O}_{G/S}(V_\mathfrak{s}(\lambda)^*) = \mathcal{O}_{G/P^{n-1}}(V_{\mathfrak{p}^{n-1}}(\lambda)^*) \ ,$$

and (by the Leray spectral sequence)

$$H^j_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) = H^j_{G/S}(V_\mathfrak{s}(\lambda)^*) = H^j_{G/P^{n-1}}(V_{\mathfrak{p}^{n-1}}(\lambda)^*) \ .$$

In the rest of the proof of Theorem 3(b) we will assume that $\lambda_1 > \lambda_2 + 1$ and $\lambda_n = \lambda_{n-1} - 1$ since our claim implies that it is sufficient to prove Theorem 3(b) under these conditions.

Obviously $\Omega^{+}_{\theta^{\lambda}_{\mathfrak{p}^1}}(\lambda - \varepsilon_1) = \{\lambda\}$ and $\theta^{\lambda-\varepsilon_1+\varepsilon_i} \neq \theta^{\lambda}$ for $1 < i < n$, and thus Proposition 2.2(c) implies

$$\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda - \varepsilon_1)\hat{\otimes} E)^{\theta^{\lambda}} \simeq \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \ .$$

Therefore

$$\hat{T}^{+}_{\theta^{\lambda-\varepsilon_1},\theta^{\lambda}}(H^j_{G/P^1}(V_{\mathfrak{p}^1}(\lambda - \varepsilon_1))) \simeq H^j_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \ . \tag{3.15}$$

Furthermore, by Lemma 3.1,

$$m^j(\lambda - \varepsilon_1, \mu') = m^j_{0,1}(\lambda - \varepsilon_1, \mu') \ , \tag{3.16}$$

which means that $\lambda_n = \mu'_n$ when $m^j(\lambda - \varepsilon_1, \mu') \neq 0$. In this way, if $m^j(\lambda - \varepsilon_1, \mu') \neq 0$, Lemma 3.2 applies to μ' with $i = n$ and to $\theta := \theta^{\lambda} = \theta^{\mu'+\varepsilon_n}$, and thus $\hat{T}^{+}_{\theta^{\lambda-\varepsilon_1},\theta^{\lambda}}(V(\mu')) \simeq V(\mu' + \varepsilon_n)$. This, together with (3.15) and (3.16), gives $m^j(\lambda - \varepsilon_1, \mu') = m^j(\lambda, \mu' + \varepsilon_n) = m^j_{0,1}(\lambda - \varepsilon_1, \mu')$. But clearly $m^j_{0,1}(\lambda - \varepsilon_1, \mu') = m^j_{0,1}(\lambda - \alpha, \mu' + \varepsilon_n)$, and therefore

$$m^j(\lambda, \mu' + \varepsilon_n) = m^j_{0,1}(\lambda - \alpha, \mu' + \varepsilon_n)$$

whenever $m^j(\lambda - \varepsilon_1, \mu') \neq 0$. Since it is obvious that $m^j(\lambda - \varepsilon_1, \mu') \neq 0$ iff $m^j(\lambda, \mu' + \varepsilon_n) \neq 0$, setting $\mu := \mu' + \varepsilon_n$ we obtain (1.13) for $k = n$ which is all we need to prove Theorem 3(b).

PROOF OF THEOREM 3(C). It is based on several preliminary assertions.

Lemma 3.3. *Let $\nu \in \Lambda^{+}_{\mathfrak{p}^1}$ and $\nu_1 = \nu_2 > 0$. Then for any μ,*

$$\begin{aligned} m^j(\nu, \mu) &= m^{j-1}_{1,0}(\nu, \mu) \qquad \text{if } j \geq 1, \\ m^0(\nu, \mu) &= 0 \ . \end{aligned} \tag{3.17}$$

PROOF. Let $\tau : G/R \to G/P^1$ be the canonical submersion, where R is the parabolic subsupergroup with Lie $R = \mathfrak{r} := \mathfrak{p}^1 \cap \mathfrak{p}^2$. Consider the exact triangle

$$0 \to \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*) \to R^{\cdot}\tau_* \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*) \to \hat{R}^{\cdot}\tau_* \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*) \to 0 \tag{3.18}$$

in the (bounded) derived category of sheaves of $\mathcal{O}_{G/P^1}$-modules, where $\hat{R}^{\cdot}\tau_* \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)$ is defined simply as the quotient $R^{\cdot}\tau_* \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)/\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*)$. Fixing an appropriate representative for $R^{\cdot}\tau_* \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)$ in the derived category, we can consider (3.18) as an exact sequence of complexes of $\mathcal{O}_{G/P^1}$-modules ($\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*)$ being a trivial complex concentrated in degree 0). (3.18) gives rise to the long exact sequence of hypercohomology

$$(3.19)\qquad \begin{aligned}&0 \to \mathbb{H}^0(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*)) \to \mathbb{H}^0(G/P^1, R^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)) \to \\ &\mathbb{H}^0(G/P^1, \hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)) \to \mathbb{H}^1(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*)) \to \dots\end{aligned}$$

We will now analyze this sequence and show that it implies (3.17).

Note first that $\mathbb{H}^i(G/P^1, R^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)) = 0$ for all i. This is an immediate consequence of the fact that the sheaf of $\mathcal{O}_{G/R}$-modules $\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)$ is acyclic, i.e. all its cohomology groups vanish. The latter can be established by the technique of Demazure reflections developed in [P3] (see also [P4]) which gives that

$$H^j(G/R, \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)) = H^{j+1}(G/R, \mathcal{O}_{G/R}(V_{\mathfrak{r}}(w_{12}(\nu)^*))$$

(w_{12} permuting ν_1 and ν_2) for each i, and thus that

$$H^j(G/R, \mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu))^*)) = 0 \; .$$

Consider next the spectral sequence with second term $E_2^{p,q} = H^q(G/P^1, \mathcal{H}^p(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)))$ which abuts to $\mathbb{H}^{p+q}(G/P^1, \hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))$ (and where $\mathcal{H}^p(\cdot)$ denotes the p^{th} cohomology sheaf of a complex of sheaves). We have

$$\begin{aligned}\mathcal{H}^p(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))) = R^p\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*) = \mathcal{O}_{G/P^1}(H^p_{P^1/R}(V_{\mathfrak{r}}(\nu))^*) \quad \text{for} \quad p > 0, \\ R^0\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*)/\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*) = \mathcal{O}_{G/P^1}(H^0_{P^1/R}(V_{\mathfrak{r}}(\nu)^*)/V_{\mathfrak{p}^1}(\nu)^*) \; .\end{aligned}$$

It is crucial to observe that the $\mathfrak{g}^{\Pi}$-composition factors of $H^k_{P^1/R}(V_{\mathfrak{r}}(\nu))^*$ for $k > 0$ and $H^0_{P^1/R}(V_{\mathfrak{r}}(\nu))^*/V_{\mathfrak{p}^1}(\nu)^*)$ have $\mathfrak{p}^1$-typical highest weights. The reader will easily verify this using Corollary 2.2. Therefore, by Theorem 2, $H^q(G/P^1, \mathcal{H}^p(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))) = 0$ for all $q > 0$ and all p, and furthermore

$$\mathbb{H}^p(G/P^1, \hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))) = H^0(G/P^1, \mathcal{H}^p(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))) \; .$$

Since $\mathbb{H}^{\cdot}(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*)) = H^{\cdot}(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*))$, the coboundary maps in (3.19) provide us with isomorphisms

$$H^0(G/P^1, \mathcal{H}^{j-1}(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_{\mathfrak{r}}(\nu)^*))) \xrightarrow{\sim} H^j(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)^*))$$

for $j \geq 1$, and moreover $H^0(G/P^1, R^0\tau_*\mathcal{O}_{G/R}(V_\mathfrak{r}(\nu)^*)) = 0$. Therefore if $\mu \neq \nu$,

$$\begin{aligned} m_{1,0}^{j-1}(\nu,\mu) &= [H^{j-1}_{P^1/R}(V_\mathfrak{r}(\nu))^* : V_{\mathfrak{p}^1}(\mu)^*] \\ &= [H^0(G/P^1, \mathcal{H}^{j-1}(\hat{R}^{\cdot}\tau_*\mathcal{O}_{G/R}(V_\mathfrak{r}(\nu)^*))) : V(\mu)^*] \\ &= [H^j(G/P^1, \mathcal{O}_{G/P^1}(V(\nu)^*)) : V(\mu)^*] \\ &= m^j(\nu,\mu) \end{aligned}$$

for $j \geq 1$, and

$$m^0(\nu,\mu) = 0 \ .$$

□

Lemma 3.4. *Let $\nu = (\nu_1, \ldots, \nu_n) \in \Lambda^+$, $\nu_1 > \nu_2 + 1$, $\nu_{k-1} > \nu_k + 1$, $\nu_1 + \nu_k = 0$ for some k. Then $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu)) = 0$.*

Proof. Clearly $\Omega^+_{\theta^{\nu+\varepsilon_k}}(\nu) = \{\nu + \varepsilon_k\}$ and thus, assuming that $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu)) \neq 0$ we have $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu)) \simeq V(\nu+\varepsilon_k)$ by Proposition 2.2(c). That implies

$$\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)\hat{\otimes}E)^{\theta^{\nu+\varepsilon_k}} \simeq \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu+\varepsilon_k))$$

and

$$H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu+\varepsilon_k)) \simeq \hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu))) \ .$$

Since $\nu + \varepsilon_k$ is $\mathfrak{p}^1$-typical in $\mathfrak{g}$, $H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu+\varepsilon_k)) = V(\nu+\varepsilon_k)$, which gives $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu))) \simeq V(\nu+\varepsilon_k)$. But Proposition 3.1 implies that

$$[H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu)) : V(\nu - \varepsilon_1 + \varepsilon_k)] = 1 \ .$$

Furthermore, $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu-\varepsilon_1+\varepsilon_k)) = V(\nu+\varepsilon_k)$ by Lemma 3.2 . Since $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}$ is an exact functor, if $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu)) \simeq V(\nu+\varepsilon_k)$ then

$$[\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu)) : V(\nu+\varepsilon_k)] \geq 2,$$

which contradicts

$$\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(H^0_{G/P^1}(V_{\mathfrak{p}^1}(\nu)) = V(\nu+\varepsilon_k).$$

This contradiction proves that $\hat{T}^+_{\theta^\nu,\theta^{\nu+\varepsilon_k}}(V(\nu)) = 0$. □

Corollary 3.1. *If ν is as in Lemma 3.4, then $(\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)\hat{\otimes}E))^{\theta^{\nu+\varepsilon_k}} = 0$.*

□

Lemma 3.5. *Let $\lambda \in \Lambda^+$ satisfy the condition of Theorem 3(c). Then $m^i(\lambda, \mu) \neq 0$ implies $\mu_1 = \lambda_1 - 1$, $\mu_n = \lambda_n + 1$.*

PROOF. It goes by induction on n. Assume that the claim of the Lemma is true for all $n < n_0$ but is wrong for $n = n_0$. Fix corresponding weights λ^0 and μ^0 such that $m^i(\lambda^0, \mu^0) \neq 0$, $\lambda^0_1 = \lambda^0_2 + 1$, $\lambda^0_{n_0} + 1 = \lambda^0_{n_0-1}$, $\mu^0_1 = \lambda^0_1 - k$, $\mu^0_{n_0} = \lambda^0_{n_0} + \ell$, $\ell > 1$, where $k \leq \ell$ and k is minimal possible. (The case $\ell \leq k$, $k > 1$ is completely similar and requires no separate consideration.) Assume also that $\mu^0 \neq 0$, and therefore $\mu^0_1 > 0$. Our assumption implies

$$[\hat{T}^+_{\theta^{\lambda^0}, \theta^{\mu^0+\varepsilon_1}}(H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda^0))) : V(\mu^0 + \varepsilon_1)] \neq 0 \ . \tag{3.20}$$

Furthermore,

$$\hat{T}^+_{\theta^{\lambda^0}, \theta^{\mu^0+\varepsilon_1}}(H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda^0))) = H^i(G/P^1, (\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda^0)\hat{\otimes}E)^{\theta^{\mu^0+\varepsilon_1}})^*)^* \ .$$

However,

$$\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda^0)\hat{\otimes}E)^{\theta^{\mu^0+\varepsilon_1}} \simeq \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\nu)) \tag{3.21}$$

for a certain $\nu \in \Lambda^+$. Indeed, since $\#\theta^{\mu^0+\varepsilon_1} \geq \#\theta^{\mu^0}$ (because $\mu^0_1 + \mu^0_{n^0} \geq 0$) and $\bar{\theta}^{\mu^0+\varepsilon_1} = \bar{\theta}^{\mu^0}$, Corollary 2.1 implies that $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda^0)\hat{\otimes}E)^{\theta^{\mu^0+\varepsilon_1}}$ is an irreducible $\mathfrak{g}$-linearized $\mathcal{O}_{G/P^1}$-module, which is equivalent to (3.21). Moreover Proposition 2.2(b) gives $\nu = \lambda^0 + \varepsilon_j$, and (3.20) and (3.21) yield $m^i(\nu, \mu^0 + \varepsilon_1) \neq 0$.

We will consider the following possibilities for ν:

(i) $\nu = \lambda^0 + \varepsilon_1$,

(ii) $\nu = \lambda^0 + \varepsilon_2$,

(iii) $\nu = \lambda^0 + \varepsilon_j$, $3 \leq j \leq n^0 - 2$,

(iv) $\nu = \lambda^0 + \varepsilon_{n^0-1}$.

Case (i) is impossible because $\lambda^0 + \varepsilon_1$ satisfies the condition of Theorem 3(b) and thus necessarily $\mu^0_{n^0} = \nu_{n^0} - 1 = \lambda^0_{n^0} - 1$, i.e. $\ell = 1$ which is a contradiction.

If ν is as in (ii), then ν satisfies the condition of Lemma 3.3. Therefore (since $m^i(\nu, \mu^0 + \varepsilon_1) \neq 0$) $i > 0$ and $m^i(\nu, \mu^0+\varepsilon_1) = m^{i-1}_{1,0}(\nu, \mu^0+\varepsilon_1)$. By Theorem 3(b) (applied to $\mathfrak{h}+\mathfrak{p}^1_{ss}$),

$$m^{i-1}_{1,0}(\nu, \mu^0 + \varepsilon_1) = m^{i-1}_{1,1}(\nu + \varepsilon_{n^0} - \varepsilon_2, \mu^0 + \varepsilon_1) \ ,$$

and since $m^{i-1}_{1,1}(\nu + \varepsilon_{n^0} - \varepsilon_2, \mu^0 + \varepsilon_1) \neq 0$, we obtain $\nu_{n^0} + 1 = \mu^0_{n^0}$. This together with the equality $\lambda_{n^0} = \nu_{n^0}$ gives $\ell = 1$ which is a contradiction.

Let us now consider case (iv). Corollary 3.1 and (3.21) imply $\lambda_3^0 = \lambda_2^0 - 1 = \lambda_1^0 - 2$. Furthermore ν satisfies the condition of Theorem 3(b) and thus $m^i(\nu, \mu^0 + \varepsilon_1) = m^i_{1,0}(\nu + \varepsilon_{n^0} - \varepsilon_1, \mu^0 + \varepsilon_1)$. Since $m^i_{1,0}(\nu + \varepsilon_{n^0} - \varepsilon_1, \mu^0 + \varepsilon_1) \neq 0$, we have $\nu_1 - 1 = \mu_1^0 + 1$. However, $\nu + \varepsilon_{n^0} - \varepsilon_1$ satisfies the condition of Theorem 3(c) over the reductive part of $\mathfrak{p}^1$, and thus by the induction assumption $\nu_{n^0} + 1 = \mu^0_{n^0} - 1$. But $\nu_{n^0} = \lambda^0_{n^0}$ and $\nu_1 = \lambda_1^0$ give $\mu_1^0 = \lambda_1^0 - 2$, $\mu^0_{n^0} = \lambda_{n^0} + 2$, and using this the reader will check immediately that $\#\theta^{(\mu^0+\varepsilon_1)} > \#\theta^{\mu^0}$ and $\#\theta^\nu = \#\theta^{\lambda^0}$ which contradicts $\#\theta^{\mu^0} = \#\theta^{\lambda^0}$ and $\#\theta^\nu = \#\theta^{\mu^0+\varepsilon_1}$. This means that (iv) is also impossible.

It remains to consider case (iii). In this case we notice that the weight ν satisfies the condition for λ in Theorem 3(c) and $m^i(\nu, \mu^0 + \varepsilon_1) \neq 0$. Furthermore $\nu_1 - \mu_1^0 - 1 = k - 1$, $\nu_{n^0} - \mu^0_{n^0} = \ell$, which is an obvious contradiction to the minimality of k. This proves Lemma 3.5 for the case $\mu^0 \neq 0$. The case $\mu^0 = 0$ can be done in the same way by using the functor $T^+_{\theta^{\mu^0+\varepsilon_1}}$ instead of $\hat{T}^+_{\theta^\lambda, \theta^{\mu^0+\varepsilon_1}}$. □

We can now complete the proof of Theorem 3(c). The observation that $\Omega^-_{\theta^{\lambda-\varepsilon_1}_{\mathfrak{p}^1}}(\lambda) = \{\lambda - \varepsilon_1\}$ and $\theta^{\lambda-\varepsilon_i} \neq \theta^{\lambda-\varepsilon_1}$ for $i \neq 1$ implies via Proposition 2.2(c) the existence of an isomorphism of $\mathfrak{g}$-linearized $\mathcal{O}_{G/P^1}$-modules

$$\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)\hat{\otimes}E^*)^{\theta^{\lambda-\varepsilon_1}} \simeq \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda - \varepsilon_1)) .$$

Therefore

$$\hat{T}^-_{\theta^\lambda, \theta^{\lambda-\varepsilon_1}}(H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda))) \simeq H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda - \varepsilon_1))$$

for any $i \geq 0$. Lemma 3.5 implies $\Omega^-_{\theta^{\lambda-\varepsilon_1}}(\mu) = \{\mu - \varepsilon_n\}$ for any μ with $m^j(\lambda, \mu) \neq 0$ for some j. Moreover, by Lemma 3.2,

$$\hat{T}^-_{\theta^\lambda, \theta^{\lambda-\varepsilon_1}}(V(\mu)) = V(\mu - \varepsilon_n) .$$

Thus $m^i(\lambda, \mu) = m^i(\lambda - \varepsilon_1, \mu - \varepsilon_n)$ for any $i \geq 0$. However, by Lemma 3.3, $m^i(\lambda - \varepsilon_1, \mu - \varepsilon_n) = m^{i-1}_{1,0}(\lambda - \varepsilon_1, \mu - \varepsilon_n)$ for $i > 0$ and, by Theorem 3(b), $m^{i-1}_{1,0}(\lambda - \varepsilon_1, \mu - \varepsilon_n) = m^{i-1}_{1,1}(\lambda - \varepsilon_1, \mu - \varepsilon_n)$. But obviously $m^{i-1}_{1,1}(\lambda - \varepsilon_1, \mu - \varepsilon_n) = m^{i-1}_{1,1}(\lambda - \alpha, \mu)$. Finally, $m^0(\lambda - \varepsilon_1, \mu - \varepsilon_n) = 0$ by Lemma 3.3 and thus $m^0(\lambda, \mu) = 0$. The proof of Theorem 3(c) is complete.

Proof of Theorem 3(a).

Using Proposition 3.1, Theorem 3(b), (c) and Theorem 4[5] one verifies straightforwardly

[5]The proof of Theorem 4 is presented in **3.5** and is completely independent of Theorem 3.

Lemma 3.6. *Let $\lambda \in \Lambda^+$ satisfy the condition of Theorem 3(a) (with $k = n$).*

(a) $\sum_{i\geq 0} m^i(\lambda, \mu) \leq 1$ *for any μ with $\#\mu = \#\lambda$.*

(b) *If $\sum_{i\geq 0} m^i(\lambda,\mu) > 1$, then $\sum_{i\geq 0} m^i(\lambda,\mu) = 2$ and $\mu = \lambda'$, where $\lambda' := w(\lambda - \lambda_1\alpha) \in \Lambda^+$ for an appropriate $w \in W$.*

□

In order to prove (1.10) and (1.11) (which is all we need to complete the proof of Theorem 3(a)) it suffices to establish

Proposition 3.2. *Let $\psi_\lambda : H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to \hat{T}^+_{\theta^{\lambda-\varepsilon_1},\theta^\lambda}(V(\lambda-\varepsilon_1))$ be the map introduced in the proof of Proposition 3.1.*

(a) *There is the following complex of $\mathfrak{g}^\Pi$-modules:*

$$0 \to V(\lambda - \alpha) \to \mathrm{im}\psi_\lambda \to V(\lambda) \to 0 \tag{3.22}$$

(b) *If $[H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) : V(\lambda')] = 0 \pmod 2$, (3.22) is an exact sequence.*

(c) *If $[H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) : V(\lambda')] = 1 \pmod 2$, the cohomology of the complex (3.22) is isomorphic to $V(\lambda')$.*

PROOF. (a) immediately follows from Proposition 2.2(d). We will prove (b) and (c) together.

Step 1. We consider first the case when λ is "$\mathfrak{p}^1 \cap \mathfrak{p}^{n-1}$"-generic, i.e. when $\lambda_1 - \lambda_2 \gg 0$, $\lambda_{n-1} - \lambda_n \gg 0$. In this case Borel-Weil-Bott's theorem applied to the quotients of the canonical $\mathfrak{g}_0$-sheaf filtration on $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)^*)$ ensures that

$$[H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) : V(\lambda')] = 0.$$

Thus, we have to prove that according to (b) the complex (3.22) is exact. Since λ is "$\mathfrak{p}^1 \cap \mathfrak{p}^{n-1}$-generic", $H^i_{G/P^1}(V_{\mathfrak{p}^1}(\lambda-\alpha)) = 0$ for $i > 0$, and therefore according to (3.11) $\mathrm{im}\psi_\lambda = H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda))$. So we have to show that $m^0(\lambda,\mu) = 0$ for $\mu \neq \lambda, \lambda-\alpha$. Assume that $m^0(\lambda,\mu) \neq 0$ for some $\mu \neq \lambda, \lambda - \alpha$. Then $\theta^\lambda = \theta^\mu$, and Corollary 2.3 implies that $\mu = \lambda - \sum_{\beta_i\in\Delta^+}\beta_i$. Since $\lambda_1 - \lambda_2 \gg 0$, $\lambda_{n-1} - \lambda_n \gg 0$, we have $\mu = \lambda - t\alpha - \sum_j \alpha_j$, with $\alpha_j = 0$ or $\alpha_j = \varepsilon_{j_1} - \varepsilon_{j_2}$, $1 < j_1 < j_2 < n$. Furthermore

$$[H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)\hat{\otimes}E^*)^{\theta^{\mu-\varepsilon_n}} : V(\mu - \varepsilon_n)] \neq 0\ ,$$

and therefore $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)\hat{\otimes}E^*)^{\theta^{\mu-\varepsilon_n}} \neq 0$. This proves that $t \leq 1$, because for $t > 1$ Lemma 2.2 implies $\theta^{\lambda-\varepsilon_i} = \theta^{\mu-\varepsilon_n}$ for at least for one i, $i \neq n$, which contradicts Lemma

2.1. On the other hand, it is obvious that $t \geq 1$ since $H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda))$ is a quotient of the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^1)} V_{\mathfrak{p}^1}(\lambda)$. Thus $t = 1$. Furthermore, clearly the $(\mathfrak{p}^1)^\Pi$-module multiplicity of $V_{\mathfrak{p}^1}(\mu)$ in $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^1)} V_{\mathfrak{p}^1}(\lambda)$ is not equal to zero and therefore $V_{\mathfrak{p}^1}(\mu)$ is a subquotient of the $(\mathfrak{p}^1)^\Pi$-module $[U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^1)} V_{\mathfrak{p}^1}(\lambda)]_{-1}$, where the subscript $_{-1}$ refers to the canonical $\mathbb{Z}$-grading of the generalized Verma module. But note that

$$[U(\mathfrak{g}) \otimes_{V(\mathfrak{p}^1)} V_{\mathfrak{p}^1}(\lambda)]_{-1} \simeq (\mathfrak{g}/\mathfrak{p}^1) \otimes_{\mathbb{C}} V_{\mathfrak{p}^1}(\lambda)$$

as a $(\mathfrak{p}^1)^\Pi$-module. Using now Corollary 2.1 for $\mathfrak{p}^1$ the reader will verify that μ necessarily equals $\lambda - \alpha$.

Since we have showed already $[H^0_{G/P}(V_{\mathfrak{p}^1}(\lambda)) : V(\lambda)] = 1$ and $[H^0_{G/\mathfrak{p}^1}(V_{\mathfrak{p}^1}(\lambda)) : V(\lambda - \alpha)] = 1$, we have the exact sequence

$$0 \to V(\lambda - \alpha) \to H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to V(\lambda) \to 0$$

for a "$\mathfrak{p}^1 \cap \mathfrak{p}^{n-1}$-generic" λ.

Step 2. Now we will consider the complexes

$$0 \to V(\lambda) \to \mathrm{im}\ \psi_{\lambda+\alpha} \to V(\lambda + \alpha) \to 0, \tag{3.23}$$

$$0 \to V(\lambda - \alpha) \to \mathrm{im}\ \psi_\lambda \to V(\lambda) \to 0 \tag{3.24}$$

and will show that if the cohomology of (3.23) is zero or is isomorphic to $V(\lambda')$, then the cohomology of (3.24) is zero or is isomorphic to $V(\lambda')$. Consider first the exact sequence

$$0 \to \mathrm{im}\ \psi_{\lambda+\alpha} \to \hat{T}^+_{\theta^{\lambda-\varepsilon_n},\theta^\lambda}(V(\lambda - \varepsilon_n)) \to H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to 0 \tag{3.25}$$

The canonical surjection $p_\lambda : H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to V(\lambda)$ and the injection $V(\lambda) \hookrightarrow \mathrm{im}\psi_{\lambda+\alpha}$ induce from (3.25) the complex of $\mathfrak{g}^\Pi$-modules:

$$0 \to V(\lambda) \to \hat{T}^+_{\theta^{\lambda-\varepsilon_n},\theta^\lambda}(V(\lambda - \varepsilon_n)) \to V(\lambda) \to 0.$$

Its cohomology $Q_{\lambda+\alpha}$ is a contragradient $\mathfrak{g}^\Pi$-module and there is an exact sequence

$$0 \to \mathrm{im}\ \psi_{\lambda+\alpha}/V(\lambda) \to Q_{\lambda+\alpha} \to \mathrm{ker}p_\lambda \to 0.$$

If the cohomology of (3.23) is zero, then $(\mathrm{im}\psi_{\lambda+\alpha})/V(\lambda) \simeq V(\lambda + \alpha)$. Since $[Q_{\lambda+\alpha} : V(\lambda+\alpha)] = 1$, $Q_{\lambda+\alpha}/V(\lambda+\alpha) \simeq \mathrm{ker}p_\lambda$ is contragradient. But $\mathrm{im}\psi_\lambda$ has only two $\mathfrak{n}$-singular

$\mathfrak{h}$-submodules: v_λ and $v_{\lambda-\alpha}$. Therefore $\psi_\lambda(\ker p_\lambda)$ is indecomposable and has a unique irreducible submodule isomorphic to $V(\lambda-\alpha)$. Then by the contragradiency of $\ker p_\lambda$, $\ker p_\lambda$ has a submodule P isomorphic to $\psi_\lambda(\ker p_\lambda)^\vee$, such that $\psi_\lambda(P) = V(\lambda-\alpha)$. Therefore, if $[\psi_\lambda(\ker p_\lambda)/V(\lambda-\alpha) : V(\mu)] \neq 0$, then $[\ker p_\lambda : V(\mu)] \geq 2$. This implies that μ could be only equal to λ', and we have either $\psi_\lambda(\ker p_\lambda)/V(\lambda-\alpha) = 0$ or $\psi_\lambda(\ker p_\lambda)/V(\lambda-\alpha) \simeq V(\lambda')$.

If the cohomology of (3.23) is isomorphic to $V(\lambda')$, then the contragradiency of $Q_{\lambda+\alpha}$ and indecomsability of $\mathrm{im}\psi_{\lambda+\alpha}/V(\lambda)$ imply that $\ker p_\lambda$ has a contragradient submodule U_λ such that $\ker p_\lambda/U_\lambda$ is isomorphic to $V(\lambda')$. The multiplicity of each irreducible component in U_λ equals 1 by Lemma 3.6. Therefore U_λ is semi-simple and $\psi_\lambda(U_\lambda)$ is semi-simple. Hence $\psi_\lambda(U_\lambda) = V(\lambda-\alpha)$. But then $\psi_\lambda(\ker p_\lambda)$ is isomorphic to $V(\lambda-\alpha)$ or $\psi_\lambda(\ker p_\lambda)/V(\lambda-\alpha)$ is isomorphic to $V(\lambda')$. In the first case the cohomology of (3.24) is zero, in the second case it is isomorphic to $V(\lambda')$.

Step 3. Steps 1 and 2 imply that the cohomology of the complex

$$0 \to V(\lambda-\alpha) \to \mathrm{im}\psi_\lambda \to V(\lambda) \to 0$$

is either zero or isomorphic to $V(\lambda')$. Indeed, assume this is false for some λ. Then by Step 1 there is a maximal $k \in \mathbb{Z}^+$ such that for $\lambda + k\alpha$ it is not true. But then Step 2 implies that it is false also for $\lambda + (k+1)\alpha$. Contradiction.

Step 4. The following lemma is all we need to complete the proof of Theorem 3(a).

Lemma 3.7. $\sum_{i\geq 0} m^i(\lambda,\lambda') = 0$ *or* 2 *for any* $\lambda \in \Lambda^+$.

PROOF. It follows from Lemma 3.6 that

$$\sum m^i(\lambda,\lambda') \leq 2.$$

Assume that Lemma 3.7 is true for all $n' < n = \dim E$ but is false for n. (Easy computations verify that the Lemma is true for $n = 2, 3$ so $n \geq 4$.) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n)$ be a weight for which $\sum_{i\geq 0} m^i(\lambda,\lambda') = 1$. Since n is minimal, λ satisfies the condition of Theorem 3(a) with $\alpha = \varepsilon_1 - \varepsilon_n$, i.e. $\lambda_1 + \lambda_n = 0$, $\lambda_1 > \lambda_2 + 1$, $\lambda_{n-1} > \lambda_n + 1$. Without loss of generality one can assume that $\lambda_2 + \lambda_{n-1} \geq 0$.

Let first $\lambda_2 \neq 0$. Consider $\nu = \lambda + \varepsilon_2$. Then by Lemma 3.2,

$$\mathcal{O}_{G/P^1}(V(\lambda)\hat{\otimes}E)^{\theta^\nu} \simeq \mathcal{O}_{G/P^1}(V(\nu)) ,$$

$$\hat{T}^+_{\theta^{\lambda'},\theta^\nu}(V(\lambda')) \simeq V(\nu') .$$

Therefore $m^i(\lambda,\lambda') = m^i(\nu,\nu')$ for any $i \geq 0$. Thus $\sum_{i\geq 0} m^i(\nu,\nu') = 1$, $\nu_1-\nu_2 < \lambda_1-\lambda_2$, $\nu_{n-1}-\nu_n = \lambda_{n-1}-\lambda_n$. But we can repeat this procedure several times by considering $\nu' = \nu+\varepsilon_2$, $\nu'' = \nu'+\varepsilon_2$, etc. We obtain finally $\nu^{(\tau)}$ which does not satisfy the conditions of Theorem 3(a), and this is a contradiction.

Let now $\lambda_2 = 0$. Then $\lambda_2+\lambda_{n-1} \geq 0$ implies $\lambda_{n-1} = 0$ and thus $\lambda = (a,0,\ldots,0,-a)$, $\lambda' = (0,0,\ldots,0)$. For $a = 2$ one verifies immediately that

$$\mathrm{ch}V(\lambda) = \mathrm{ch}(S^2(\mathfrak{g}) \oplus \Pi S^2(\mathfrak{g})) - 2\mathrm{ch}V(0,\ldots,0).$$

Since $\mathrm{ch}V(\lambda)$ is divisible by $2(1+\varepsilon)$, we have $\sum_{i\geq 0} m^i(\lambda,0) = 2$. Moreover, for $\lambda = (2,0,\ldots,0,-2)$ the module $\psi_\lambda(\mathrm{ker}p_\lambda)$ is indecomposable and can be described by the exact sequence

$$0 \to V(\lambda-\alpha) \to \psi_\lambda(\mathrm{ker}p_\lambda) \to V(\lambda') \to 0 \ .$$

But then a simple argument shows that the same is true for $\lambda+\alpha$, i.e. there is a short exact sequence

$$0 \to V(\lambda) \to \psi_{\lambda+\alpha}(\mathrm{ker}p_{\lambda+\alpha}) \to V(\lambda') \to 0 \ .$$

Indeed, if this is not true, then $\psi_{\lambda+\alpha}(\mathrm{ker}p_{\lambda+\alpha}) \simeq V(\lambda)$, and the exact sequence

$$0 \to \mathrm{im}\psi_{\lambda+\alpha} \to T^+_{\theta\lambda}(V(\lambda-\varepsilon_n)) \to H^0_{G/P^1}(V_{\mathfrak{p}^1}(\lambda)) \to 0$$

implies the contragradiency of $\ker p_\lambda$ (as in the proof of step 2), which in turn implies the semi-simplicity of $\ker p_\lambda$ because any irreducible component of $\ker p_\lambda$ has multiplicity 1. This contradicts to the indecomposability of $\psi_\lambda(\mathrm{ker}p_\lambda)$.

This argument can be repeated for any $\lambda+k\alpha$, unless $m^0(\lambda+(k-1)\alpha,\lambda') \neq 2$. Thus the Lemma is true for $\lambda = (a,0,\ldots,0,-a)$ where $a \leq n-1$. But for $a \geq n$ the reader will verify immediately that the argument in Step 1 gives $m^i(\lambda,0) = 0$ for all $i \geq 0$. The proof of Theorem 3(a), and thus also of Theorem 3, is complete. □

3.5. Proof of Theorem 4

It suffices to consider the case when $P = P^1$ since both statements of the theorem are symmetric with respect to the interchange of P^1 and P^{n-1}.

(a) It is a classical fact that

(3.26)

$$H^j(\mathbb{P}(E_0),\Omega^k_{\mathbb{P}_{n-1}(\mathbb{C})}) = \Pi^k S^k(\mathcal{N}^*_{(G/P^1)_{\mathrm{red}}/G/P^1})) = \begin{cases} 0 & \text{for } k \neq j \\ \mathbb{C} & \text{for } j,\ j = 0,1,\ldots,n-1 \ . \end{cases}$$

Consider the spectral sequence with term $E_1^{p,q} = H^{p+q}\left(\mathbb{P}(E_0),\ S^p(\mathcal{N}^*_{(G/P^1)_{\rm red}/G/P^1})\right)$. It abuts to $H^\cdot(\mathbb{P}(E_0),\ \mathcal{O}_{G/P^1})$. But (3.26) implies now that this sequence collapses at the term $E_1^{p,q}$. Indeed all linear maps

$$\delta_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

necessarily equal zero since they are even (parity preserving) linear maps. This gives (1.15) immediately.

(b) The injection of $\mathfrak{p}^1$-modules

$$V_{\mathfrak{p}^1}(1,0,\ldots,0) \otimes_{\mathbb{C}} V_{\mathfrak{p}^1}(-1,0,\ldots,0) \hookrightarrow V(1,0,\ldots,0) \otimes_{\mathbb{C}} V_{\mathfrak{p}^1}(-1,0,\ldots,0)$$

gives rise to the following exact sequence of G-linearized $\mathcal{O}_{G/P^1}$-modules

$$(3.27)\quad \begin{aligned} &0 \to \mathcal{E} := \mathcal{E}nd(\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))) \to \\ &\to V(1,0,\ldots,0) \otimes_{\mathbb{C}} \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)) \to \mathcal{O}_{G/P^1} V_{\mathfrak{p}^1}(-1,1,0,\ldots,0)) \to 0\ . \end{aligned}$$

We claim that

$$H^i(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))) = \begin{cases} V(0,\ldots,0,-1) & \text{for } i = 0 \\ 0 & \text{for } i > 0\ . \end{cases}$$

The vanishing of all higher cohomology groups of $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))$ follows directly from the observation that $(\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)))_{\rm red} \simeq \mathcal{O}_{\mathbb{P}(E_0)}(1) \oplus \Pi\mathcal{O}_{\mathbb{P}(E_0)}(1)$, where $(\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)))_{\rm red}$ is the restriction of $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))$ to $\mathbb{P}(E_0) = (G/P^1)_{\rm red}$ and $\mathcal{O}_{\mathbb{P}(E_0)}(1)$ is the line bundle dual to the tautological bundle on $\mathbb{P}(E_0)$. Indeed, this implies that $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))$ has a $G_{\rm red}$-equivariant sheaf filtration with adjoint factors

$$(3.28)\qquad \Pi\Omega^i_{\mathbb{P}(E_0)} \otimes_{\mathcal{O}_{\mathbb{P}(E_0)}} \mathcal{O}_{\mathbb{P}(E_0)}(1)$$

for $i = 0, 1, \ldots, n-1$, and it is well known that the higher cohomology of all the sheaves (3.28) vanishes. The isomorphism

$$V(0,\ldots,0,-1) = H^0(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)))$$

is nothing but the canonical injection

$$V(0,\ldots,0,-1) \hookrightarrow H^0(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)))\ ,$$

see **1.2**. The latter is necessarily an isomorphism because a straightforward calculation verifies that

$$\text{(3.29)} \qquad \mathrm{Ech}\, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)) = \mathrm{ch} V(0,\ldots,0,-1)\ ,$$

and the left-hand side of (3.29) is nothing but ch $H^0(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0)))$ since all higher cohomology groups of $\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,0,\ldots,0))$ equal zero.

Therefore the long exact sequence of cohomologies of (3.27) gives

$$\text{(3.30)} \quad \begin{aligned} 0 \to H^0(G/P^1,\mathcal{E}) \to V(1,0,\ldots,0) \otimes_{\mathbb{C}} V(0,\ldots,0,-1) = \mathfrak{g} \oplus \Pi\mathfrak{g} \to \\ \to H^0(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,1,0,\ldots,0))) \to H^1(G/P^1,\mathcal{E}) \to 0 \end{aligned}$$

and

$$\text{(3.31)} \qquad H^j(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,1,0,\ldots,0))) \simeq H^{j+1}(G/P^1,\mathcal{E})$$

for $j \geq 1$.

Consider now the exact sequence of G-linearized $\mathcal{O}_{G/P^1}$-modules

$$\text{(3.32)} \qquad 0 \to \mathcal{O}_{G/P^1} \oplus \Pi\mathcal{O}_{G/P^1} \xrightarrow{d} \mathcal{E} \to \mathcal{O}_{G/P^1} \oplus \Pi\mathcal{O}_{G/P^1} \to 0\ ,$$

d being the inclusion of the sheaf of diagonal endomorphisms into the sheaf of endomorphisms. It is obvious from the claim of (a) that the following two statements are equivalent:

(i) all coboundary morphisms in the long exact sequence of cohomologies of (3.32) are isomorphisms;

(ii) $H^0(G/P^1,\mathcal{E}) \simeq H^{n-1}(G/P^1,\mathcal{E}) \simeq V(0,\ldots,0)$ $(= \mathbb{C}^{1+\varepsilon} = \mathbb{C} \oplus \Pi\mathbb{C})$,
$H^i(G/P^1,\mathcal{E}) = 0$ for $i = 1,\ldots,n-2$.

Via (3.30) and (3.31), (ii) implies that

$$\text{(3.33)} \quad H^i(G/P^1, \mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,1,0,\ldots,0))) = \begin{cases} 0 & \text{for } i = 1,\ldots,n-2 \\ V(0,\ldots,0) & \text{for } i = n-1 \end{cases}$$

and that there is an exact sequence

$$\text{(3.34)} \qquad 0 \to (\mathfrak{g} \oplus \Pi\mathfrak{g})/\mathbb{C}^{1+\varepsilon} \to H^0(\mathcal{O}_{G/P^1}(V_{\mathfrak{p}^1}(-1,1,0,\ldots,0))) \to \mathbb{C}^{1+\varepsilon} \to 0\ .$$

But (3.33) and (3.34) are precisely equivalent to the claim of Theorem 4(b). Therefore (since (i) and (iii) are equivalent) it suffices to establish (i), and this is what we will do in the rest of the proof.

In order to prove (i) it is crucial to note that the exact sequence (3.34) when restricted to $G'/(P^1)'$, G' being the Lie subsupergroup $Q(E')$ for any Π-invariant linear subspace E' of E (and $(P^1)'$ being the stabilizer of a Π-invariant subspace in E' of dimension $1+\varepsilon$), goes into the same exact sequence but defined for $G'/(P^1)'$. In other words, we have the commutative diagram

(3.35)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{G/P^1} \oplus \Pi\mathcal{O}_{G/P^1} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_{G/P^1} \oplus \Pi\mathcal{O}_{G/P^1} & \longrightarrow & 0 \\ & & {\scriptstyle r_{\mathcal{O}}}\downarrow & & {\scriptstyle r_{\mathcal{E}}}\downarrow & & {\scriptstyle r_{\mathcal{O}}}\downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{G'(P^1)'} \oplus \Pi\mathcal{O}_{G'/(P^1)'} & \longrightarrow & \mathcal{E}_{G'/(P^1)'} & \longrightarrow & \mathcal{O}_{G'/(P^1)'} \oplus \Pi\mathcal{O}_{G'/(P^1)'} & \longrightarrow & 0 , \end{array}$$

where $\mathcal{E}_{G'/(P^1)'} := \mathcal{E}nd(\mathcal{O}_{G'(P^1)'}(V_{(\mathfrak{p}^1)'}(1,0,\ldots,0)))$, $(\mathfrak{p}^1)' = \mathrm{Lie}(P^1)'$.

Our next observation is that $r_{\mathcal{O}}$ induces isomorphisms on all cohomology groups except the top ones. Indeed, by considering the canonical filtrations on $\mathcal{O}_{G/P^1}$ and $\mathcal{O}_{G'(P^1)'}$ one reduces this statement to the claim that the composition

$$\Omega^j_{\mathbb{P}(E_0)} \to \Omega^j_{\mathbb{P}(E_0)|\mathbb{P}(E_0')} \to \Omega^j_{\mathbb{P}(E_0')}$$

induces an isomorphism

$$\mathbb{C} = H^j(\mathbb{P}(E_0), \Omega^j_{\mathbb{P}(E_0)}) \xrightarrow{\sim} H^j(\mathbb{P}(E_0')), \Omega^j_{\mathbb{P}(E_0')}) = \mathbb{C}$$

for any $j < n-1$. But this latter claim is well known and easily verifiable.

Clearly, we can now finish the proof of (b) by induction on n. The third and final observation needed is that (i) is indeed true for $n = 3$. Note first that for any n ($n \geq 2$)

$$H^0(G/P^1, \mathcal{E}) = V(0,\ldots,0) . \tag{3.36}$$

This is because $V(0,\ldots,0)$ is obviously a $\mathfrak{g}^{\Pi}$-submodule of $H^0(G/P^1,\mathcal{E})$ and the sequence (3.30), together with the well known fact that $V(0,\ldots,0)$ is the maximal trivial submodule of $\mathfrak{g} \oplus \Pi\mathfrak{g}$, implies (3.36) immediately. For $n = 3$ we obtain now by Serre duality (see [P1] or [M3]) that $H^2(G/P^1, \mathcal{E}) = V(0,0,0)$ because the dualizing sheaf on G/P^1 is isomorphic to $\mathcal{O}_{G/P^1}$ and $\mathcal{E}$ is self-dual. But

$$\mathrm{Ech}\mathcal{E} = 2\mathrm{ch}V(0,0,0) ,$$

and, since by the above, $\mathrm{ch}\, H^0(G/P^1, \mathcal{E}) + \mathrm{ch} H^2(G/P^1, \mathcal{E}) = 2\mathrm{ch} V(0,0,0)$, we have necessarily

$$H^1(G/P^1, \mathcal{E}) = 0 \ .$$

This establishes (ii) for $n = 3$ and therefore also (i) for $n = 3$. In order to finish the proof of (i) for $n > 3$ it remains to consider the diagram whose rows are the long exact sequences corresponding to the rows of (3.35) and to do some straightforward diagram chasing. This can be left to the reader. In this way we have completed the proof of Theorem 4(b) for $n \geq 3$. (For $n = 2$, Theorem 4(b) is obvious.)

(c) The proof is a trivial calculation based on Borel-Weil-Bott's theorem for the projective line. □

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