Continuation of hereditary orders in local central simple algebras

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Introduction

The subject of this work can be viewed as a part of the continuation theory of special pseudo valuations in the sense of Benz [Be1].

Let A be a finite-dimensional central simple algebra over a non-archimedean local field F.

A function $\omega : A \to \mathbb{R} \cup \{\infty\}$ *is called a (non-archimedian) pseudo valuation of the algebra* A *if it fulfils the following properties:*

(i) $\omega(x) = \infty$ *iff* $x = 0$*;*

$(i)'$ $0 < \omega(x_0) < \infty$ *for at least one* x_0*;*

(ii) $\omega(ab) \geq \omega(a) + \omega(b)$*;*

$(ii)'$ *for each value* $w := \omega(x)$ *for* $x \in A$*, exists an element* $x_w \in A$ *mit* $\omega(x_w) = w$ *and* $\omega(x_w y) = \omega(y x_w) = \omega(x_w) + \omega(y)$ *for all* $y \in A$*;*

(iii) $\forall x, y \in A : \omega(x - y) \geq min(\omega(x), \omega(y))$.

A pseudo valuation ω *is called an almost valuation if we have in addition:*

(iv) The residual class ring $\mathfrak{A}/\mathfrak{P}$ *of the valuation ring* $\mathfrak{A} := \{x \in A \mid \omega(x) \geq 0\}$ *with the valuation ideal* $\mathfrak{P} := \{x \in A \mid \omega(x) > 0\}$ *is a semisimple (finite) ring.*

The elements x_w fulfilling the condition $(ii)'$ are called the ω-elements of A. (If, for an almost valuation ω, each $x \in A$ is an ω-element, then ω is a valuation in the usual sense).

In our case of a central simple algebra A, the almost valuations ω, which continue the valuation of the ground field, biuniquely correspond to the so-called principal orders $\mathfrak{A}$ of A, and the set of non-zero ω-elements corresponds to the normalizers $\mathcal{K}_{\mathfrak{A}}$ of the $\mathfrak{A}'s$. Benz [Be2] has shown that the valuation of a maximal extension M/F in A has always a unique continuation to an almost valuation of A; hence, there exists a unique principal order normalized by $M^\times$. Fröhlich [F] has given a shorter proof and examined the associated embeddings of maximal fields. More generally to each hereditary order $\mathfrak{A}$ in A, we can associate a pseudo valuation $\omega_{\mathfrak{A}} : A \to \mathbb{Z} \cup \{\infty\}$ by $\omega_{\mathfrak{A}}(x) := \frac{t}{rd} max\{j \in \mathbb{Z} \cup \{\infty\} \mid x \in \mathfrak{P}^{tj}\}$, where $\mathfrak{P}^\infty := \{0\}$, d is the index of A, r the period of $\mathfrak{A}$ as in 0.3 and t is as in 0.5 or 1.2. In fact, t is the smallest positive power of the Jacobson radical $\mathfrak{P}$ of $\mathfrak{A}$ that is a principal ideal and if $t = 1$, then $\mathfrak{A}$ is called a principal order. Again the $A^\times$-normalizer $\mathcal{K}_\mathfrak{A}$ is the set of non-zero $\omega_\mathfrak{A}$-elements.

The most important aim of our work is to show that there always exists a continuation $\omega_\mathfrak{A}$ associated to a hereditary order $\mathfrak{A}$ in A of a pseudo valuation $\omega_\mathfrak{B}$ associated to a hereditary order $\mathfrak{B}$ in a simple subalgebra B (central over a field extension E/F of our main local ground field F) to our main algebra A.

The proof requires the handling of several special cases and uses the results of [Be2], [F] and [Z] about principal orders. So first, in Chapter 1, we consider the relationship between principal and arbitrary hereditary orders. We show in 1.2 that each hereditary order is the intersection of a canonical set of principal orders, such that also its normalizer is the intersection of the normalizers of that set of principal orders. If $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ is that canonical set, then 1.2(ii)(c) means in terms of pseudo valuations $\forall x \in A : \omega_\mathfrak{A}(x) = max(\omega_{\mathfrak{A}_1}(x), \ldots, \omega_{\mathfrak{A}_t}(x))$; hence, in terms of [Be1](8.1), $\omega_\mathfrak{A}$ is the direct sum of the $\omega_{\mathfrak{A}_1}, \ldots, \omega_{\mathfrak{A}_t}$. We introduce the notion of pairs and will explain our main result 1.9 in all details.

In Chapter 2 we treat the special case where the inertial degree $f(E/F)$ and the index of the algebra A are prime to each other, which we call the splittable case, because, for an unramified extension L/F having the same degree as the index of the algebra A, we obtain that $E \otimes_F L/L$ is an extension in the L-algebra $A \otimes_F L$ and by [CR](26.29) $\mathfrak{A} \otimes_{o_F} o_L$ remains hereditary. Therefore, all results can be obtained here by splitting the algebra A by an unramified splitting field and using the results of [BK3]. In 2.2 we obtain that such a continuation exists and is always unique. From that we reduce to the case where E/F is unramified and $[E : F]$ divides the index of the algebra A, which means that E/F can be embedded into the central division algebra D over F, whereof A is a matrix algebra. This is the special case considered in Chapter 3.

In the splittable case and in Lemma 3.2 we have the unramified case of our construction, *where an unramified continuation means that the associated continuation of pseudo valuations is unramified in the usual sense; hence, the pseudo valuation of B and its continuation to A have a common uniformizer in B* (also see Theorem 1.9(i)(b) and notice that the continuation is unramified iff $f(E/F) = l_1$). For the existence of a continuation we can use 2.1. The key to the general case is then the totally ramified case of Lemma 3.3. One of the main ideas of its proof is that each pair $(\mathfrak{A}, \mathfrak{A}_1)$ of a hereditary order $\mathfrak{A}$ and an associated principal order $\mathfrak{A}_1$ as in 1.2 admits a common normal form, as we have realized in 1.5. This suggests the introduction of the notion of pairs and their continuations. For a given hereditary order $\mathfrak{A}$ a partner is given by a principal order $\mathfrak{A}_1$ containing $\mathfrak{A}$ such that the associated pseudo valuations are admit a common uniformizer and, for a given pair $(\mathfrak{B}, \mathfrak{B}_1)$ in B, $(\mathfrak{A}, \mathfrak{A}_1)$, are a continuation iff the pseudo valuation associated to $\mathfrak{B}$ is continued by the pseudo valuation associated to $\mathfrak{A}$, and the almost valuation associated to $\mathfrak{B}_1$ is continued by the almost valuation associated to $\mathfrak{A}_1$. In 3.4 it turns out that this characterizes our construction of a continuation uniquely.

In Chapter 4 we combine our results to obtain the general case of Theorem 1.9, where we state the existence of a unique continuation of a pair $(\mathfrak{B}, \mathfrak{B}_1)$ in B to a pair $(\mathfrak{A}, \mathfrak{A}_1)$ to A.

This work is the compilation of the most important results of a diploma work [G] completed under the supervision of my teacher Ernst-Wilhelm Zink of the Humboldt-University Berlin. I would like to thank him for teaching me the foundations of the theory of hereditary orders and for developing the fundamental ideas of this work in mutual discussions. We generalize here the main results of the work [Z] to the notion of pairs (see the next section). Special thanks are also due to Paul Broussous for his hint to Lemma 3.1 and Bernhard Runge for valuable discussions.

Concerning standard books about the structure theory of hereditary orders, we refer to [R] and [CR]. Additional information can be found in [BF]. Now let us make our notation more concrete.

0 Notation and preparations

Let F be a non-archimedean local field and A a finite-dimensional central simple algebra over F. Let V be a left simple A-module. V is a right $D = \mathrm{End}_A(V)$-vector space and

$$A \simeq \mathrm{End}_D(V) \simeq M_m(D)$$

(where $M_m(D)$ denotes the ring of $m \times m$ matrices in D). Denote by d the index of the division algebra D (that is, $\dim_F(D) = d^2$), fix an isomorphism $A \simeq M_m(D)$, and identify A with $M_m(D)$. Denote by $\mathfrak{o}_F$, $\mathfrak{o}_D$, ... the rings of integers of F, D, ..., by $\mathfrak{p}_F$, $\mathfrak{p}_D$, ... its unique prime ideals and by $\mathfrak{k}_F$, $\mathfrak{k}_D$, ... the associated residual fields.

0.1 *An $\mathfrak{o}_F$-order in A is a subring with the same unit element as A which is a complete $\mathfrak{o}_F$-lattice in A. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$ order in A, i. e., an $\mathfrak{o}_F$-order which is (left- or right-) hereditary as a ring. We denote by $\mathfrak{P}$ its Jacobson radical and by $\mathcal{K}_{\mathfrak{A}}$ its normalizer. It is well known (see* [R](39.12)*) that $\mathfrak{A}$ is hereditary iff $\mathfrak{P}$ is a projective left or right $\mathfrak{A}$-module, which is also the case iff $\mathfrak{P}$ is a fractional invertible ideal of $\mathfrak{A}$. Hence, there exists an $(\mathfrak{A},\mathfrak{A})$-bisubmodule $\mathfrak{P}^{-1}$ of A which is a complete $\mathfrak{o}_F$-lattice in A such that $\mathfrak{P}\mathfrak{P}^{-1} = \mathfrak{P}^{-1}\mathfrak{P} = \mathfrak{A}$.*

An $\mathfrak{o}_D$-lattice chain in V is a family $\{X_j\}_{j\in\mathbb{Z}}$ of complete $\mathfrak{o}_F$-lattices in V that are right $\mathfrak{o}_D$-modules such that

a) $\forall j \in \mathbb{Z}, X_j \supsetneq X_{j+1}$,

b) $\exists r \in \mathbb{N}, \forall j \in \mathbb{Z}, X_j\mathfrak{p}_D = X_{j+r}$.

0.2 *One knows from* [BF] *that* $\mathfrak{A}$ *is also hereditary iff there exists a (unique up to a translation of the index)* $\mathfrak{o}_D$*-lattice chain* $\{X_k\}_{k\in\mathbb{Z}}$ *in* V *such that*

$$\mathfrak{A} = \mathrm{End}_{\mathfrak{o}_D}\{X_k\}_{k\in\mathbb{Z}} = \{x \in A \mid \forall k \in \mathbb{Z}, xX_k \subset X_k\}.$$

0.3 *We have*

$$\mathfrak{A}/\mathfrak{P} \simeq \bigoplus_{i=1}^{r} \mathrm{End}_{\mathfrak{k}_D}(X_{i-1}/X_i) \simeq M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$$

such that the ordered numbers $n_1,\ldots,n_r$ *provide numerical invariants, up to a cyclic permutation, of the conjugacy class of* $\mathfrak{A}$*. These numbers also come from a normal form (see [R] 39.14)*

$$g\mathfrak{A}g^{-1} = \begin{pmatrix} (\mathfrak{o}_D), (\mathfrak{p}_D) & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D), (\mathfrak{o}_D) & \cdots, \ \cdots \\ \cdots \ , \ \cdots & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D), \ \cdots & \cdots, (\mathfrak{o}_D) \end{pmatrix}^{\{n_1,\ldots,n_r\}} ,$$

where $(\mathfrak{o}_D)$ *(and* $(\mathfrak{p}_D)$ *respectively) mean* $n_i \times n_j$*-matrices, where the lower-diagonal and diagonal blocks have entries in* $\mathfrak{o}_D$ *and the upper-diagonal blocks have entries in* $\mathfrak{p}_D$ *(note that* $n_1+\ldots+n_r = m$*) and* g *is a suitable element of* $A^\times$*. Here the number* r *is uniquely determined and called the period of* $\mathfrak{A}$ *and* $\{X_j\}_{j\in\mathbb{Z}}$*, respectively. The numbers* $n_1,\ldots,n_r$ *are unique up to a cyclic permutation.*

The modules $X_0,\ldots,X_{r-1}$ *of the lattice chain of* $\mathfrak{A}$ *in* V *(as above) are a system of representatives of the isomorphy classes of direct indecomposable right* $\mathfrak{A}$*-modules* (1)

(see [BF](1.2.8)(ii) *and* [R](39.23)*).*

We also have

$$g\mathfrak{P}g^{-1} = \begin{pmatrix} (\mathfrak{p}_D), (\mathfrak{p}_D), \cdots, (\mathfrak{p}_D), (\mathfrak{p}_D) \\ (\mathfrak{o}_D), (\mathfrak{p}_D), \cdots, \ \cdots \ , \ \cdots \\ \cdots \ , (\mathfrak{o}_D), \cdots, \ \cdots \ , \ \cdots \\ (\mathfrak{o}_D), \ \cdots \ , \cdots, (\mathfrak{p}_D), \ \cdots \\ (\mathfrak{o}_D), \ \cdots \ , \cdots, (\mathfrak{o}_D), (\mathfrak{p}_D) \end{pmatrix}^{\{n_1,\ldots,n_r\}} .$$

If $\mathfrak{A}$ is given in normal form 0.3, then, for the simple A-module $V := \bigoplus_{i=1}^{m} D$, we can choose the numbering of the chain $\{X_j\}_{j\in\mathbb{Z}}$ such that $\forall j \in \mathbb{Z}, {}^tX_j = \mathfrak{p}_D^l((\bigoplus_{i=1}^{k}(\bigoplus_{\mu=1}^{n_i}\mathfrak{p}_D))\bigoplus(\bigoplus_{i=k+1}^{r}(\bigoplus_{\mu=1}^{n_j}\mathfrak{o}_D)))$, where $k,l \in \mathbb{Z}$ such that $j = lr + k$, $0 \le k < r$, and ${}^t(.)$ means the set of transposed vectors.

An order $\mathfrak{A}$ is called principal if its Jacobson radical $\mathfrak{P}$ is a principal ideal, especially by 0.1 such an $\mathfrak{A}$ is hereditary.

0.4 *From* [BF](1.3.2) *we know that a herditary order $\mathfrak{A}$ with invariants $n_1, \ldots, n_r$ is a principal order if and only if $n_1 = \ldots = n_r = s$. In this situation we call s just the invariant of $\mathfrak{A}$.*

0.5 Notation *$\mathfrak{A}$ will always denote a hereditary $\mathfrak{o}_F$-order in A. It is not necessarily given in normal form 0.3, but will have period r and invariants $n_1, \ldots, n_r$. Let t be the smallest period of the numbers $n_1, \ldots, n_r$. Denote by $\{X_j\}_{j \in \mathbb{Z}}$ an associated lattice chain in V, by $\mathfrak{P}$ the Jacobson radical and by $\mathcal{K}_{\mathfrak{A}} = \{x \in A^\times \mid x\mathfrak{A}x^{-1} = \mathfrak{A}\}$ the normalizer. For an extension E/F in A we consider a hereditary $\mathfrak{o}_E$-order $\mathfrak{B}$ in the centralizer $B = C_A(E) := \{x \in A \mid \forall y \in E, yx = xy\}$. We have $B \simeq \mathrm{End}_{\tilde{D}}(W) \simeq M_{m'}(\tilde{D})$ for a right simple B-module and left $\tilde{D}$-vector space W, where $\tilde{D} = \mathrm{End}_B(W)$ is a skew field of index d'. Denote by $\{Y_j\}_{j \in \mathbb{Z}}$ a lattice chain associated to $\mathfrak{B}$ in W, by $\mathfrak{Q}$ the Jacobson radical of $\mathfrak{B}$, by r' the period of $\mathfrak{B}$, by ${n'}_1, \ldots, {n'}_{r'}$ the invariants of $\mathfrak{B}$, by t' the smallest period of the numbers ${n'}_1, \ldots, {n'}_{r'}$, and by $\langle ., \ldots, . \rangle$ the greatest common divisor of integers. From the theory of Brauer groups it is well known that, in the latter situation, we have $d' = \frac{d}{\langle [E:F], d \rangle}$ and $m' = \frac{\langle [E:F], d \rangle m}{[E:F]}$ (see for example [Z](Proposition 1)), where m and d are as in the beginning of this section.*

0.6 Definition *Let E/F be a field extension of F in A such that $E^\times \subset \mathcal{K}_{\mathfrak{A}}$. Let $\mathfrak{B}$ be a hereditary order in B.*

(i) We say that $\mathfrak{A}$ is a continuation of B iff $\mathfrak{A} \cap B = \mathfrak{B}$ and $\mathcal{K}_{\mathfrak{B}} \subset \mathcal{K}_{\mathfrak{A}}$.

(ii) Such a continuation is called unramified iff there exists an $h \in \mathcal{K}_{\mathfrak{B}}$ such that $\mathcal{K}_{\mathfrak{A}} = \langle h \rangle \mathfrak{A}^\times$

Remark: 1) It is immediately ovious that $\mathcal{K}_{\mathfrak{A}} \cap B \subset \mathcal{K}_{\mathfrak{A} \cap B}$ in general; hence, if $\mathfrak{A}$ is a continuation of $\mathfrak{B}$, then $\mathcal{K}_{\mathfrak{A}} \cap B = \mathcal{K}_{\mathfrak{B}}$.

2) The notion of continuation has been chosen due to the notions of the valuation theory of algebras. Continuations of orders are the same as continuations of the associated pseudo valuations, and unramified continuations of orders are equivalent to unramified continuations of the associated pseudo valuations.

Our most important aim of this work stated in terms of orders (not valuations) is then to show that each hereditary order $\mathfrak{B}$ in B has a continuation $\mathfrak{A}$ to A and, secondly, to make this continuation unique by the consideration of pairs. We will also give some information about the associated embedding of E/F in the algebra A in this special case (see 1.9(ii)(iii)).

The general case of embeddings being good with respect to continuation will be investigated in a forthcoming paper, where we will give an explicit version of the criterion of Broussous [Br]. There we will also introduce a more general version of the notion of a pure element as in [Z]. The notion of [Z] could also be generalized by the results of this paper, but with less practicalness to the representation theory of simple algebras.

1 Hereditary orders from principal orders

It is straightforward that:

1.1 Lemma *Let $\mathfrak{B}$, $\mathfrak{Q}$, B, E/F and $\{Y_j\}_{j\in\mathbb{Z}}$ be as in 0.5 and $g \in A^\times$. Then $\mathcal{K}_{g\mathfrak{B}g^{-1}} = g\mathcal{K}_{\mathfrak{B}}g^{-1}$ and $g\mathfrak{B}g^{-1}$ is a hereditary $\mathfrak{o}_{gEg^{-1}}$-order in $gBg^{-1} = C_A(gEg^{-1})$ with the Jacobson radical $g\mathfrak{Q}g^{-1}$ and the same invariants as $\mathfrak{B}$.*

□

1.2 Proposition *Let $\mathfrak{A}$ be as in 0.5 and use the associated notations, except for (at this moment) the notation for t. Let $\nu_{\mathfrak{A}} : A \to \mathbb{Z} \cup \{\infty\}$ be defined by $\nu_{\mathfrak{A}}(x) := max\{j \in \mathbb{Z} \mid x \in \mathfrak{P}^j\}$ if $x \in A\backslash\{0\}$ and $\nu_{\mathfrak{A}}(0) := \infty$. From* [BF](1.2.8)(i) *we know that $\nu_{\mathfrak{A}|\mathcal{K}_{\mathfrak{A}}}$ is a group homomorphism with kernel $\mathfrak{A}^\times$. Consequently, letting $t \in \mathbb{N}$ such that $\nu_{\mathfrak{A}|\mathcal{K}_{\mathfrak{A}}}(\mathcal{K}_{\mathfrak{A}}) = t\mathbb{Z}$, there exists an $h \in \mathcal{K}_{\mathfrak{A}}$ such that $\nu_{\mathfrak{A}}(h) = t$, and, therefore, $\mathcal{K}_{\mathfrak{A}} = \langle h\rangle\mathfrak{A}^\times = \mathfrak{A}^\times\langle h\rangle$ ($\langle h\rangle$ the infinite cyclic group generated by h). Let $\mathfrak{A}_i := \mathrm{End}_{\mathfrak{o}_D}\{X_{(i-1)+tj}\}_{j\in\mathbb{Z}}$ for $i = 1, \ldots, t$. Then*

(i) t is the smallest period of the sequence of natural numbers $n_1, \ldots, n_r$ as in 0.5,

(ii) $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ are principal orders

(a) with invariant $s := n_1 + \ldots + n_t$ and period $\frac{r}{t}$, hence pairwise conjugated,

(b) with Jacobson radicals $\mathfrak{P}_i = h\mathfrak{A}_i$ for $i = 1, \ldots, t$, and

(c) such that for all $j \in \mathbb{Z}$, $\mathfrak{P}^{tj} = \bigcap_{i=1}^{t} \mathfrak{P}_i^j$.

(iii) Let $\mathfrak{O}$ be a hereditary $\mathfrak{o}_F$-order in A such that $\mathfrak{A} \subset \mathfrak{O}$ and $\mathcal{K}_{\mathfrak{A}} \subset \mathcal{K}_{\mathfrak{O}}$. Then there exist $l, j_1, \ldots, j_l \in \{1, \ldots, t\}$ such that $\mathfrak{O} = \mathfrak{A}_{j_1} \cap \ldots \cap \mathfrak{A}_{j_l}$.

(iv) $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ are the unique principal orders such that $\mathfrak{A} \subset \mathfrak{A}_i$ and $\langle h\rangle\mathfrak{A}^\times = \mathcal{K}_{\mathfrak{A}} \subset \mathcal{K}_{\mathfrak{A}_i} = \langle h\rangle\mathfrak{A}_i^\times$ or, equivalently, the unique principal orders such that

(a) $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ are pairwise distinct but conjugate over $A^\times$,

(b) $\mathfrak{A} = \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_t$, and

(c) $\mathcal{K}_{\mathfrak{A}} = \mathcal{K}_{\mathfrak{A}_1} \cap \ldots \cap \mathcal{K}_{\mathfrak{A}_t}$ or,

equivalently, the unique minimal set of principal orders satisfying (a),(b).

<u>Remark</u> *If we look for the maximal set of hereditary orders satisfying (iv)(a),(b), we obtain the set of maximal orders that contains $\mathfrak{A}$, and the order of this set is r (see* [BF]*(1.2.8.(ii)) or* [R]*(39.23)).*

<u>Proof</u>

By 1.1 and replacing $\mathfrak{A}$ by $g\mathfrak{A}g^{-1}$, h by ghg^{-1}, we can restrict ourselves to the case where $\mathfrak{A}$ is given in normal form 0.3.

(ii)(a) Then $\pi_D I_m \in \mathcal{K}_{\mathfrak{A}}$ and $\nu_{\mathfrak{A}}(\pi_D I_m) = r$, hence $t \mid r$ and $\dim_{\mathfrak{k}_D}(X_j/h^{\frac{r}{t}}X_j) = \dim_{\mathfrak{k}_D}(X_j/hX_j) + \ldots + \dim_{\mathfrak{k}_D}(h^{\frac{r}{t}-1}X_j/h^{\frac{r}{t}}X_j) = m$. Because of $X_i/X_j \simeq hX_i/hX_j$ for $i < j$ as $\mathfrak{k}_D$ vector spaces, we obtain $\dim_{\mathfrak{k}_D}(X_{(i-1)+tj}/X_{(i-1)+t(j+1)}) = \dim_{\mathfrak{k}_D}(X_{(i-1)+tj}/hX_{(i-1)+tj}) = \frac{mt}{r} =: s$ for all $j \in \mathbb{Z}$.

(i) Since $\dim_{\mathfrak{k}_D}(X_j/X_{j+t}) = \dim_{\mathfrak{k}_D}(X_j/X_{j+1}) + \ldots + \dim_{\mathfrak{k}_D}(X_{j+t-1}/X_{j+t}) = n_{j+1} + \ldots + n_{j+t}$ for $j = 0, \ldots, r-1$, t is a period of the partition $n_1, \ldots, n_r$ of m. On the other hand, if t' is an arbitrary period of $n_1, \ldots, n_r$, it is easy to see from the explicit description of the chain $\{X_j\}_{j\in\mathbb{Z}}$ for $V = D^m$ (column vectors) given after 0.3 that for, $s' := n_1 + \ldots + n_{t'}$,

$$h_{s'} := \begin{pmatrix} 0 & , \pi_D I_{s'} \\ I_{m-s'} , & 0 \end{pmatrix} \tag{2}$$

belongs to $\mathcal{K}_{\mathfrak{A}}$, hence $t \mid t'$.

(ii)(b) follows from [BF](1.2.9).

(ii)(c) follows from (ii)(b), and $\mathfrak{P}^{tj} = h^j\mathfrak{A} = \bigcap_{i=1}^{t} h^j\mathfrak{A}_i = \bigcap_{i=1}^{t} \mathfrak{P}_i^j$ for all $j \in \mathbb{Z}$.

(iii) Let $\{Z_j\}_{j\in\mathbb{Z}}$ be a lattice chain for $\mathfrak{O}$. The $Z_j{}'s$ span V over F, V is a simple A-module and $\mathfrak{A}$ spans A over F. Moreover, each isomorphism of $\mathfrak{A}$-modules that are complete $\mathfrak{o}_F$-lattices in V extends to an automorphism of V as a simple A-module. Since $\mathfrak{A} \subset \mathfrak{O}$ and because the $Z_j's$ are direct indecomposable $\mathfrak{A}$-modules, we obtain by 0.3(1) and Schur's Lemma that $\{Z_j\}_{j\in\mathbb{Z}} \subset \{X_j\}_{j\in\mathbb{Z}}$. Because of $h \in \mathcal{K}_{\mathfrak{A}} \subset \mathcal{K}_{\mathfrak{O}}$ it follows that, if $Z_k \in \{X_{(i-1)+tj}\}_{j\in\mathbb{Z}}$ for an $i \in \{1, \ldots, t\}$, then $\{X_{(i-1)+tj}\}_{j\in\mathbb{Z}} \subset \{Z_j\}_{j\in\mathbb{Z}}$; hence, our result follows from $\{X_j\}_{j\in\mathbb{Z}} = \bigcup_{i=1}^{t} \{X_{(i-1)+tj}\}_{j\in\mathbb{Z}}$.

(iv) Clearly, the orders $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ fulfil all necessary conditions. The first assertion is clear from (iii) and the choice of h. From the arguments of the proof of (iii) it follows easily that a minimal set of hereditary orders fulfilling (iv)(a),(b) splits the chain $\{X_j\}_{j\in\mathbb{Z}}$ into disjoint subchains, each of them associated to a member of the set. As they all have the same invariants, it is easy to see that the order of this set must be a period of the numbers $n_1, \ldots, n_r$, i. e., t, and the assertion follows from the definition of the $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ in terms of subchains. For the main assertion notice that each principal order contained in a set of principal orders satisfying (iv)(a)-(c) must contain h in its normalizer. From our last considerations and the choice of h it follows that the order of this set can be at most t and is a period of $n_1, \ldots, n_r$. Hence, our result follows from (i).

□

1.3 Definition *(i) In the following we often use the notation $\nu_{\mathfrak{A}}$ for the 'valuation map' associated to a hereditary order $\mathfrak{A}$ as in 1.2.*

(ii) We call t as in 1.2(i) the partition period.

(iii) Because of 1.2(iv), we call $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ (from 1.2) the canonical principal orders associated to $\mathfrak{A}$.

1.4 Corollary

(i) Let $\mathfrak{A}$ be a hereditary order in A with canonical principal orders $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$, and M/F a maximal extension in A (that is, $[M : F] = md$) such that $M^\times \subset \mathcal{K}_\mathfrak{A}$. Then $t = 1$ and $\mathfrak{A} = \mathfrak{A}_1$ is a principal order.

(ii) Let $\mathfrak{B}$, E/F and B be as in 0.5, and assume that $\mathfrak{B}$ is principal. Then it has a unique hereditary continuation to A, which is principal.

Proof

(i) Follows from the uniqueness assertion in [Be2](Chap. 2) or in [F](theorem 1.1) because $\mathcal{K}_\mathfrak{A} = \mathcal{K}_{\mathfrak{A}_1} \cap \ldots \cap \mathcal{K}_{\mathfrak{A}_t}$.

(ii) Follows from (i) and [F]((2.7), Corollary 3).

$\square$

1.5 Proposition

(i) Let $\tilde{\mathfrak{A}}$ be the hereditary order in normal form 0.3, where the order of the numbers in the partition $n_1, \ldots, n_r$ of m is fixed. Let t be the partition period as in 1.2 and $\mathfrak{A}_1$ the principal order in normal form 0.3 with the invariant $s := n_1 + \ldots + n_t$ and the period $\frac{r}{t}$. Then $\tilde{\mathfrak{A}}_1$ is a canonical principal order of $\tilde{\mathfrak{A}}$.

(ii) Each pair $(\mathfrak{A}, \mathfrak{A}_1)$ of a hereditary order $\mathfrak{A}$ in A and an associated principal order $\mathfrak{A}_1$ is conjugate under $A^\times$ to exactly one pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$ in simultanous normal form as in (i), i. e., there exists exactly one such pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$ such that $(g\mathfrak{A}g^{-1}, g\mathfrak{A}_1g^{-1}) = (\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$ for a suitable $g \in A^\times$.

(iii) Let $\mathfrak{A}$, $\{X_j\}_{j\in\mathbb{Z}}$ and $n_1, \ldots, n_r, r$ be as in 0.5 and $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ be the canonical principal orders of $\mathfrak{A}$. Choose the index of the chain and the canonical principal orders in such a way that $n_i = \dim_{\mathfrak{k}_D}(X_{(i-1)+tj}/X_{i+tj})$ for $i = 1, \ldots, t$ ($j \in \mathbb{Z}$ is arbitrarily chosen and $\{X_{(i-1)+tj}\}_{j\in\mathbb{Z}}$ is the chain of $\mathfrak{A}_i$). Then the pair $(\mathfrak{A}, \mathfrak{A}_i)$ is conjugated to a pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$ in simultaneous normal form as in (i), where the invariants of $\tilde{\mathfrak{A}}$ appear in the normal form in the order $n_i, \ldots, n_r, n_1, \ldots, n_{i-1}$.

Proof

(i) Clearly, $\tilde{\mathfrak{A}} \subset \tilde{\mathfrak{A}}_1$. From 0.3 and the explicit descriptions of the chains in $V = D^m$ (column vectors), we obtain that h_s, as in (2), is a common uniformizer for $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}_1$ as required for the first assertion of 1.2(iv).

(ii) First we see that if $(\mathfrak{A}, \mathfrak{A}_1)$ is conjugate to $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$, where the invariants appear in the order of $n_1, \ldots, n_r$ in the normal form of $\tilde{\mathfrak{A}}$, and is also conjugate to another pair in simultanous normal form $(\tilde{\mathfrak{H}}, \tilde{\mathfrak{H}}_1)$, then 0.3 implies that, in the normal form of $\tilde{\mathfrak{H}}$, the invariants has to appear in a certain cyclic permuted order $n_i, \ldots, n_r, n_1, \ldots, n_{i-1}$ for some $i \in \{1, \ldots, t\}$, hence $\tilde{\mathfrak{A}}_1 = \tilde{\mathfrak{H}}_1$ in particular.

Therefore, $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{D}}$ must be conjugated over an element of $A^\times$ that fixes $\tilde{\mathfrak{A}}_1$; hence, over $\mathcal{K}_{\tilde{\mathfrak{A}}_1}$. If $x, y \in A^\times$ conjugate $\tilde{\mathfrak{A}}$ to $\tilde{\mathfrak{D}}$, then $y^{-1}x \in \mathcal{K}_{\tilde{\mathfrak{A}}} \subset \mathcal{K}_{\tilde{\mathfrak{A}}_1}$; hence, if $x \in \mathcal{K}_{\tilde{\mathfrak{A}}_1}$, then $y \in \mathcal{K}_{\tilde{\mathfrak{A}}_1}$, too.

On the other hand, if $i > 1$, it is easy to see from the explicit description of the chain $\{X_j\}_{j\in\mathbb{Z}}$ in the case of $V = D^m$ that $h_{n_1+\ldots+n_{i-1}}$, as in (2), conjugates the chains, hence the orders $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{D}}$, but it cannot be contained in $\mathcal{K}_{\tilde{\mathfrak{A}}_1}$. Thus we must have $i = 1$ and $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1) = (\tilde{\mathfrak{D}}, \tilde{\mathfrak{D}}_1)$. Therefore, $(\mathfrak{A}, \mathfrak{A}_1)$ can be conjugated to at most one such pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$.

For the next step let $\mathfrak{A}$ be as in 0.5, $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$ its canonical principal orders, $s = n_1 + \ldots + n_t$ and $\mathfrak{A}^{(1)}, \ldots, \mathfrak{A}^{(t)}$ the set of all hereditary orders in normal form 0.3 such that the invariants $n_1, \ldots, n_r$ appear one after another in the orders $n_1, \ldots, n_r$; $n_2, \ldots, n_r, n_1$; $\ldots$; $n_t, \ldots, n_r, n_1, \ldots, n_{t-1}$. Clearly, each pair $(\mathfrak{A}^{(i)}, \tilde{\mathfrak{A}}_1)$ for $i = 1, \ldots, t$ is a pair in the sense of (i).

Now, by 0.3, each order $\mathfrak{A}^{(i)}$ is conjugate to $\mathfrak{A}$ by an element $g_i \in A^\times$, and, by 1.1 in the case of $E = F$ and the first assertion of 1.2(iv), $g_i\tilde{\mathfrak{A}}_1 g_i^{-1}$ is a canonical principal order of $g_i\mathfrak{A}^{(i)}g_i^{-1} = \mathfrak{A}$ for $i = 1, \ldots, t$, hence it is equal to one of the $\mathfrak{A}_1, \ldots, \mathfrak{A}_t$.

From the above we know that none of the pairs $(\mathfrak{A}, g_i\tilde{\mathfrak{A}}_1 g_i^{-1})$ can be conjugate; hence, all $g_i\tilde{\mathfrak{A}}_1 g_i^{-1}$ must be different. Thus, there exists a $k \in \{1, \ldots, t\}$ such that $(g_k^{-1}\mathfrak{A}g_k, g_k^{-1}\mathfrak{A}_1 g_k) = (\mathfrak{A}^{(k)}, \tilde{\mathfrak{A}}_1)$, as required.

(iii) By conjugation in $A^\times$, (ii), with 1.1 in the case of $E = F$ and the first assertion of 1.2(iv), we can restrict ourselves to the case where $\mathfrak{A}$ and $\mathfrak{A}_1$ are given in normal form 0.3, and our assertion follows immediatly from the explicit description of the chain $\{X_j\}_{j\in\mathbb{Z}}$ in the simple A-module $V = D^m$.

□

1.6 Definition

(i) The pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}_1)$ *as in 1.5(i) is obviously uniquely determined by the invariant tuple* $((n_1, \ldots, n_t), \frac{r}{t})$ *and our fixed isomorphism* $A \simeq M_m(D)$. *Here we only consider tuples of the form* $((k_1, \ldots, k_u), \kappa)$, *where* $k_1, \ldots, k_u$ *is a sequence of integers with minimal period* u *and* $m = (k_1 + \ldots + k_u)\kappa$; *hence,* κ *is just the number of repetitions for which the sequence* $k_1, \ldots, k_u$ *appears in the invariants of the associated hereditary order, and* $k_1 + \ldots + k_u$ *is the invariant of the associated canonical principal order. We call* $((n_1, \ldots, n_t), \frac{r}{t})$ *the invariants of the pair (of the conjugacy class of the pair, respectively).*

(ii) A pair $(\mathfrak{A}, \mathfrak{A}_1)$ *always means a pair in the sense of 1.5(ii).*

Remark: Note that the order in the invariant tuple $((n_1, \ldots, n_t), \frac{r}{t})$ *is fixed but, in general, the invariants of a hereditary order* $\mathfrak{A}$ *are just unique up to cyclic permutation. Moreover, by the Skolem-Noether Theorem the invariants of a pair are independent of the fixed isomorphism* $A \simeq M_m(D)$.

1.7 Definition *Let* $(\mathfrak{A}, \mathfrak{A}_1)$ *be a pair in* A, E/F *an extension in* A, $B := C_A(E)$ *and* $(\mathfrak{B}, \mathfrak{B}_1)$ *a pair in* B.

(i) We say that $(\mathfrak{A},\mathfrak{A}_1)$ is a continuation of $(\mathfrak{B},\mathfrak{B}_1)$ if $\mathfrak{A}$ continues $\mathfrak{B}$ and $\mathfrak{A}_1$ continues $\mathfrak{B}_1$.

(ii) Let $((n'_1,\ldots,n'_{t'}),\frac{r'}{t'})$ be the invariants of the pair $(\mathfrak{B},\mathfrak{B}_1)$. We say that the pairs $(\mathfrak{A},\mathfrak{A}_1)$ are numerically proportional if there exists $l_1\in\mathbb{N}$ and $l_2\in\mathbb{Q}_{>0}$ such that the pair $(\mathfrak{A},\mathfrak{A}_1)$ has invariants $((n_1,\ldots,n_t),\frac{r}{t})=((n'_1l_1,\ldots,n'_{t'}l_1),\frac{r'}{t'}l_2)$

(iii) We call the pairs $(\mathfrak{A},\mathfrak{A}_1)$, $(\mathfrak{B},\mathfrak{B}_1)$ proportional if $(\mathfrak{A},\mathfrak{A}_1)$ is a continuation of as well as numerically proportional to $(\mathfrak{B},\mathfrak{B}_1)$.

1.8 Corollary *(i) If the pairs $(\mathfrak{A},\mathfrak{A}_1)$ and $(\mathfrak{B},\mathfrak{B}_1)$ are numerically proportional and if we let $s:=n_1+\ldots+n_t$ be the invariant of $\mathfrak{A}_1$, $s':=n'_1+\ldots+n'_{t'}$ the invariant of $\mathfrak{B}_1$, where $((n'_1,\ldots,n'_{t'}),\frac{r'}{t'})$ (resp. $((n_1,\ldots,n_t),\frac{r}{t})=((n'_1l_1,\ldots,n'_{t'}l_1),\frac{r'}{t'}l_2)$) are the invariants of $(\mathfrak{B},\mathfrak{B}_1)$ (resp. $(\mathfrak{A},\mathfrak{A}_1)$) as in 1.7, then $l_1=\frac{s}{s'}$.*

(ii) If $(\mathfrak{A},\mathfrak{A}_1)$ continues $(\mathfrak{B},\mathfrak{B}_1)$, then $\frac{s}{s'}=\langle f(E/F),\frac{r'}{t'}\frac{[E:F]}{\langle[E:F],d\rangle}\rangle$.

(iii) If the pairs $(\mathfrak{A},\mathfrak{A}_1)$, $(\mathfrak{B},\mathfrak{B}_1)$ are proportional, then $l_1=\langle f(E/F),\frac{r'}{t'}\frac{[E:F]}{\langle[E:F],d\rangle}\rangle$ and $l_1l_2=\frac{m}{m'}=\frac{[E:F]}{\langle[E:F],d\rangle}$.

<u>Proof</u>

(i) $l_1=\frac{n'_1l_1+\ldots+n'_{t'}l_1}{n'_1+\ldots+n'_{t'}}=\frac{n_1+\ldots+n_t}{n'_1+\ldots+n'_{t'}}=\frac{s}{s'}$.

(ii) From [Z] (Corollary 3.(ii)) we conclude that $\frac{s}{s'}=\frac{\langle s'f(E/F),m\rangle}{s'}=\frac{\langle s'f(E/F),m'\frac{[E:F]}{\langle[E:F],d\rangle}\rangle}{s'}=\langle f(E/F),\frac{r'}{t'}\frac{[E:F]}{\langle[E:F],d\rangle}\rangle$.

(iii) Follows from (i),(ii) and 0.5.

□

1.9 Theorem (main result) *Let A, d, E/F and B be as in 0.5.*

(i) Let $(\mathfrak{B},\mathfrak{B}_1)$ be a pair in B. Then there exists a unique continuation $(\mathfrak{A},\mathfrak{A}_1)$ to A. Furthermore, for this continuation we have

(a) $(\mathfrak{A},\mathfrak{A}_1)$ and $(\mathfrak{B},\mathfrak{B}_1)$ are numerically proportional and

(b) for the proportional factor l_1 as in 1.8(iii) and all $h\in\mathcal{K}_{\mathfrak{B}}$ we have $\nu_{\mathfrak{A}}(h)=\frac{f(E/F)}{l_1}\nu_{\mathfrak{B}}(h)$.

(ii) Let $(\mathfrak{A},\mathfrak{A}_1)$ be a pair in A. Then the following assertions are equivalent:

(a) $(\mathfrak{A}\cap B,\mathfrak{A}_1\cap B)$ is a pair in B continued by $(\mathfrak{A},\mathfrak{A}_1)$.

(b) There exist field extensions $M/K/E$ in A such that $M^\times\subset\mathcal{K}_{\mathfrak{A}_1}$, $K^\times\subset\mathcal{K}_{\mathfrak{A}}$, $e(K/F)=e(M/F)$, $[M:F]=md$ (M/F is of maximal degree in A) and $d\mid e(M/F)f(E/F)$.

(iii) Let $(\mathfrak{A},\mathfrak{A}_1)$ be a pair in A with invariants $((n_1,\ldots,n_t),\frac{r}{t})$. Then the following assertions are equivalent:

(a) There exists an embedding $E\hookrightarrow A$ over F and field extensions $M/K/E$ of the image of E in A such that $M^\times\subset\mathcal{K}_{\mathfrak{A}_1}$, $K^\times\subset\mathcal{K}_{\mathfrak{A}}$, $e(K/F)=e(M/F)$, $[M:F]=md$ (M/F is of maximal degree in A) and $d\mid e(M/F)f(E/F)$.

(b) $[E:F] \mid d\frac{r}{t}\langle n_1,\ldots,n_t\rangle$, $e(E/F) \mid d\frac{r}{t}$ *and* $f(E/F) \mid \ddot{d}\langle n_1,\ldots,n_t\rangle$, *where* $\ddot{d}$ *is the greatest divisor of* d *that is prime to* $\frac{r}{t}$.

The proof of this theorem will be given in 4.

2 Continuation in the splittable case of $\langle f(E/F), d\rangle = 1$

2.1 Proposition *Let* E/F, B, $\mathfrak{B}$, r', t' *be as in 0.5, further let* $\mathfrak{B}_1,\ldots,\mathfrak{B}_{t'}$ *be the canonical principal orders of* $\mathfrak{B}$ *and* $\mathfrak{A}_1,\ldots,\mathfrak{A}_{t'}$ *the unique continuations as in 1.4(ii). Furthermore assume that the continuations of the* $\mathfrak{B}_i{}'s$ *by the* $\mathfrak{A}_i{}'s$ *are all unramified, then, if* $\mathfrak{A} := \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_{t'}$, $\mathfrak{A}$ *is a hereditary* $\mathfrak{o}_F$*-order in* A *and a continuation of* $\mathfrak{B}$.

Remark: One would like to show here that in this case the $\mathfrak{A}_1,\ldots,\mathfrak{A}_{t'}$ are the canonical principal orders of $\mathfrak{A}$. For that it remains to show that t' is the partition period of $\mathfrak{A}$. This will be done later in the cases 2.2 and 3.2.

Proof

Because of 1.2, we have $\mathfrak{A}\cap B = \bigcap_{i=1}^{t'} \mathfrak{A}_i \cap B = \bigcap_{i=1}^{t'} \mathfrak{B}_i = \mathfrak{B}$. As $\bigcap_{i=1}^{t'} \mathcal{K}_{\mathfrak{A}_i} \subset \mathcal{K}_{\bigcap_{i=1}^{t'}\mathfrak{A}_i}$ in general, $\mathcal{K}_{\mathfrak{A}} \cap B \supset \bigcap_{i=1}^{t'} \mathcal{K}_{\mathfrak{A}_i} \cap B = \bigcap_{i=1}^{t'} \mathcal{K}_{\mathfrak{B}_i} = \mathcal{K}_{\mathfrak{B}}$. Thus, it is only necessary to show that $\mathfrak{A}$ is a hereditary order in A.

As a consequence of 1.2(ii)(b)(iv) and because the continuations of the $\mathfrak{B}_i{}'s$ by the $\mathfrak{A}_i{}'s$ are unramified by assumption,

$$\text{there exists an } h \in \mathcal{K}_{\mathfrak{B}} \text{ such that for } i = 1,\ldots,t',\ \mathcal{K}_{\mathfrak{A}_i} = \langle h\rangle \mathfrak{A}_i^{\times}. \tag{3}$$

$\mathfrak{B}_1,\ldots,\mathfrak{B}_{t'}$ have the same invariants; hence, there exist $g_1,\ldots,g_{t'} \in B^\times$ such that

$$g_1\mathfrak{B}_1 g_1^{-1} = \mathfrak{B}_2, \ldots, g_{t'-1}\mathfrak{B}_{t'-1}g_{t'-1}^{-1} = \mathfrak{B}_{t'} \text{ and } g_{t'}\mathfrak{B}_{t'}g_{t'}^{-1} = \mathfrak{B}_1. \tag{4}$$

Furthermore, 1.1 and [F]((2.7), Corollary 3) imply that $g_i\mathfrak{A}_i g_i^{-1}$ is the unique continuation of $g_i\mathfrak{B}_i g_i^{-1}$ for $i = 1,\ldots,t'$; hence,

$$g_1\mathfrak{A}_1 g_1^{-1} = \mathfrak{A}_2, \ldots, g_{t'-1}\mathfrak{A}_{t'-1}g_{t'-1}^{-1} = \mathfrak{A}_{t'} \text{ and } g_{t'}\mathfrak{A}_{t'}g_{t'}^{-1} = \mathfrak{A}_1. \tag{5}$$

If, for $i = 1,\ldots,t'$, $\{Y_j^i\}_{j\in\mathbb{Z}}$ is the chain of $\mathfrak{B}_i$ in a simple B-module W, then 0.2 (4) implies $\{g_iY_j^i\}_{j\in\mathbb{Z}} = \{Y_j^{i+1}\}_{j\in\mathbb{Z}}$ and $\{g_{t'}Y_j^{t'}\}_{j\in\mathbb{Z}} = \{Y_j^1\}_{j\in\mathbb{Z}}$ for $i = 1,\ldots,t'-1$. Now, let $\{Y_j\}_{j\in\mathbb{Z}}$ be the chain of $\mathfrak{B}$ in W and index the orders $\mathfrak{B}_1,\ldots,\mathfrak{B}_{t'}$ and the associated chains in such a way that $Y_j^i = Y_{(i-1)+jt'}$ for $i = 1,\ldots,t'$ and all $j \in \mathbb{Z}$. So, by changing the $g_1,\ldots,g_{t'}$ via powers of h, one can obtain that $g_iY_j^i = Y_j^{i+1}$ and $g_{t'}Y_j^{t'} = Y_{j+1}^1$ for $i = 1,\ldots,t'-1$ and all $j \in \mathbb{Z}$. In particular, we have

$g_i \in \{y \in B \mid \forall j \in \mathbb{Z}, yY^i_j \subset Y^i_j\} = \mathfrak{B}_i$ for $i = 1, \ldots, t'$. So, by changing g_1 by an element of $\mathfrak{B}_1^\times$, one can obtain $g_{t'} \ldots g_1 = h$. The conditions (5) are stable under all these changes.

Now let $\{X^i_j\}_{j\in\mathbb{Z}}$ be the chain of $\mathfrak{A}_i$ in a simple A-module V for $i = 1, \ldots, t'$. Then, again, (5) means $\{g_i X^i_j\}_{j\in\mathbb{Z}} = \{X^{i+1}_j\}_{j\in\mathbb{Z}}$ and $\{g_{t'} X^{t'}_j\}_{j\in\mathbb{Z}} = \{X^1_j\}_{j\in\mathbb{Z}}$. Hence, because $g_i \in \mathfrak{B}_i \subset \mathfrak{A}_i$ for $i = 1, \ldots, t'$, $\forall j \in \mathbb{Z}$, $X^1_j \supset g_1 X^1_j = X^2_j \supset \ldots \supset g_{t'} X^{t'}_j = g_{t'} \ldots g_1 X^1_j = hX^1_j = X^1_{j+1}$.

Therefore $\bigcup\limits_{i=1}^{t'} \{X^i_j\}_{j\in\mathbb{Z}}$ is an $\mathfrak{o}_D$-lattice chain in V; hence, $\mathfrak{A} = \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_{t'} = \bigcap\limits_{i=1}^{t'} \mathrm{End}_{\mathfrak{o}_D} \{X^i_j\}_{j\in\mathbb{Z}} = \mathrm{End}_{\mathfrak{o}_D} \bigcup\limits_{i=1}^{t'} \{X^i_j\}_{j\in\mathbb{Z}}$ is a hereditary order.

□

2.2 Theorem *Let E/F, A, d, B be as in 0.5 and $\langle f(E/F), d\rangle = 1$. Then*

(i) For $\mathfrak{B}$ being a hereditary $\mathfrak{o}_E$-order in B, there exists a unique hereditary continuation $\mathfrak{A}$ to A. For this continuation we have

(a) $t = t'$, $r = r'\frac{e(E/F)}{\langle e(E/F), d\rangle}$ and $n_i = {n'}_i f(E/F)$ for $i = 1, \ldots, t$, where ${n'}_1, \ldots, {n'}_{r'}, r', t'$ are the invariants of $\mathfrak{B}$ and $n_1, \ldots, n_r, r, t$ are the invariants of $\mathfrak{A}$ as in 0.5,

(b) $\mathfrak{P}^j \cap B = \mathfrak{Q}^j$ for all $j \in \mathbb{Z}$, where $\mathfrak{Q}$ is the Jacobson radical of $\mathfrak{B}$ and $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$.

(ii) Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A. Then the following statements are equivalent:

(a) $\mathfrak{A} \cap B$ is a hereditary $\mathfrak{o}_E$-order in B continued by $\mathfrak{A}$.

(b) $E^\times \subset \mathcal{K}_{\mathfrak{A}}$.

(iii) Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A and $n_1, \ldots, n_r, r, t$ all invariants of $\mathfrak{A}$ as in 0.5. Then the following statements are equivalent:

(a) There exists an embedding j of E over F into A such that $j(E^\times) \subset \mathcal{K}_{\mathfrak{A}}$.

(b) $f(E/F) \mid \langle n_1, \ldots, n_t\rangle$ and $e(E/F) \mid d\frac{r}{t}$.

<u>*Remark*</u>*: 1) As we use 2.1 for the existence part of (i), it follows from the remark to 2.1 and (i)(a) that under the assumptions of (i) all canonical principal orders of $\mathfrak{B}$ are continued by canonical principal orders of $\mathfrak{A}$.*

2) (iii) is a direct generalization of [BK2]*(1.2), and (ii)(iii) is a stronger form of* [F]*(Theorem 3(ii)) in the case of a principal order.*

<u>Proof</u>

(i) Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_{t'}$ be the canonical principal orders of $\mathfrak{B}$ in B and $\mathfrak{A}_1, \ldots, \mathfrak{A}_{t'}$ their unique continuations to A. Choose an unramified extension L/F of degree d in D, then because $\langle f(E/F), d\rangle = 1$, $E \otimes_F L / F \otimes_F L$ is a field extension in the split algebra $A \otimes_F L$ and $B \otimes_F L = C_{A\otimes_F L}(E \otimes_F L)$. Clearly, the $\mathfrak{A}_i \otimes_{\mathfrak{o}_F} \mathfrak{o}'_L s$ are continuations of the $\mathfrak{B}_i \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ and by [BK3](1.2.4) these continuations are unramified. By [BT](4.4) $\mathfrak{B} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ has the Jacobson radical $\mathfrak{Q} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ and, therefore, a common uniformizer with $\mathfrak{B}$ if we consider B to be canonically embedded in $B \otimes_F L$. The same holds true for the relation of the $\mathfrak{B}_i{}'s$ and $\mathfrak{B}_i \otimes_{\mathfrak{o}_F} \mathfrak{o}_L{}'s$ and, finally, for the relation of the $\mathfrak{A}_i{}'s$ and $\mathfrak{A}_i \otimes_{\mathfrak{o}_F} \mathfrak{o}_L{}'s$. We deduce that the continuation of the $\mathfrak{B}_i{}'s$ by the $\mathfrak{A}_i{}'s$ must be unramified and the existence of $\mathfrak{A}$ now follows from 2.1.

To see that it is unique let V, $\{X_j\}_{j\in\mathbb{Z}}$ be as in 0.5. Then, again by [CR](26.29) or [BT](4.4), $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ is a hereditary $\mathfrak{o}_L$-order in $A\otimes_F L \simeq \mathrm{End}_L(V) \simeq M_{md}(L)$ and $\mathfrak{A}\otimes_F L \cap B\otimes_F L = \mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$. Moreover, $\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_L \subset \mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ and the prime element of $E\otimes_F L$ is $\pi_{E\otimes_F L} = \pi_E\otimes 1 \in \mathcal{K}_{\mathfrak{A}}\otimes 1 \subset \mathcal{K}_{\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L}$; hence, $(E\otimes_F L)^\times = < \pi_{E\otimes_F L} > (\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)^\times \subset \mathcal{K}_{\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L}$. Consequently [BK3](1.2.1) implies that $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ and $\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ have the same chain $\{X_j\}_{j\in\mathbb{Z}}$ in the simple $A\otimes_F L$- resp. $B\otimes_F L$-module V. Hence, $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$, and therefore $\mathfrak{A}$, is unique.

(i)(b) Again from [BT](4.4) we know that $\mathrm{Jac}(\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L) = \mathfrak{P}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ and $\mathrm{Jac}(\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L) = \mathfrak{Q}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$; hence, we see from [BK3](1.2.4) that, for all $j\in\mathbb{Z}$, $\mathfrak{P}^j\cap B\otimes_{\mathfrak{o}_F}\mathfrak{o}_L = \mathfrak{P}^j\otimes_{\mathfrak{o}_F}\mathfrak{o}_L\cap B\otimes_F L = \mathrm{Jac}(\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)^j\cap B\otimes_F L = \mathrm{Jac}(\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)^j = \mathfrak{Q}^j\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$. Thus, for all $j\in\mathbb{Z}$, $\mathfrak{P}^j\cap B = \mathfrak{Q}^j$, as required.

(i)(a) Because of 0.2 and 0.3(1), it is obvious that $\mathfrak{A}$ and $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ have the same chain $\{X_j\}_{j\in\mathbb{Z}}$ in the simple A- resp. $A\otimes_F L$-module V. Therefore, $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ has the same partition period as $\mathfrak{A}$, period rd, because the right action of π_D^d has the same period on the chain as the left or right action of π_F and, moreover, the invarints are $n_1,\ldots,n_r,\ldots,n_1,\ldots,n_r$ (d-times), because $\dim_{\mathfrak{k}_L}(X_{i-1}/X_i) = \dim_{\mathfrak{k}_D}(X_{i-1}/X_i)$ for $i=1,\ldots,rd$. Now $\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$, as in the proof of the first assertion of (i), has also the same partition period, period $\frac{rd}{e(E/F)}$, because the left action of $\pi_E^{e(E/F)}$ has the same period on the chain as the left or right action of π_F and, moreover, the invariants are $\frac{n_1}{f(E/F)},\ldots,\frac{n_t}{f(E/F)},\ldots,\frac{n_1}{f(E/F)},\ldots,\frac{n_t}{f(E/F)}$ ($\frac{rd}{e(E/F)t}$-times), because $\dim_{\mathfrak{k}_{E\otimes_F L}}(X_{i-1}/X_i) = \frac{\dim_{\mathfrak{k}_L}(X_{i-1}/X_i)}{f(E/F)}$ for $i=1,\ldots,\frac{rd}{e(E/F)}$.

Next we consider the transformation of the invariants from $\mathfrak{B}$ to $\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$. First note that $\mathfrak{B}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L \simeq \mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_L) \simeq (\mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{L'}))\otimes_{\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{L'}}\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$, where $L/L'/F$ is an intermediate extension of degree $d' = \frac{d}{\langle [E:F],d\rangle} = \frac{d}{\langle e(E/F),d\rangle}$ over F. Hence, because $B\otimes_F L' \simeq M_{m'd'}(L')$ (where m', d' are as in 0.5), one sees from 0.3 that $\mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)$ and $\mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{L'})$ have the same invariants. Again, we see that $\mathfrak{B}$ and $\mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{L'})$ have the same chain $\{Y_j\}_{j\in\mathbb{Z}}$ in a simple B resp. $B\otimes_E(E\otimes_F L')$-module W; hence, again $\mathfrak{B}\otimes_{\mathfrak{o}_E}(\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{L'})$ has the same partition period as $\mathfrak{B}$, the period being $r'd'$, because the right action of $\pi_{\tilde{D}}^{d'}$ has the same period on the chain as the left or right action of π_E and, moreover, the invariants are $n'_1,\ldots,n'_{r'},\ldots,n'_1,\ldots,n'_{r'}$ (d'-times), because $\dim_{\mathfrak{k}_{E\otimes_F L'}}(Y_{i-1}/Y_i) = \dim_{\mathfrak{k}_{\tilde{D}}}(Y_{i-1}/Y_i)$ for $i=1,\ldots,r'd'$.

Summarizing we obtain the required relations for $\mathfrak{A}$ and $\mathfrak{B}$.

(ii) (a)$\Rightarrow$(b) $E^\times \subset \mathcal{K}_{\mathfrak{A}\cap B} \subset \mathcal{K}_{\mathfrak{A}}$.

(b)⇒(a) As in the proof of (i) we see that $(E \otimes_F L)^\times \subset \mathcal{K}_{\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L}$ and that $\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L \cap B \otimes_F L \simeq \mathfrak{A} \cap B \otimes_{\mathfrak{o}_E} (\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)$ is hereditary; therefore, by [CR](26.29), $\mathfrak{A} \cap B$ is hereditary. $\mathfrak{A} \cap B$ has a unique continuation $\mathfrak{A}'$ to A and, clearly, again $E^\times \subset \mathcal{K}_{\mathfrak{A}'}$. However, in the same situation the latter condition was sufficient to prove in (i) that $\mathfrak{A}$ is unique; hence, $\mathfrak{A} = \mathfrak{A}'$, as required.

(iii) (a)⇒(b) Follows from (ii) and the relation of the invariants in (i)(a). Here we must note that the partition period is always a divisor of the period of an order.

(b)⇒(a) We embed E first arbitrarily over F into A. If the relations of the invariants hold, there exists a hereditary order $\mathfrak{C}$ in $C := C_A(E)$ with the same invariants as $\mathfrak{B}$ in (i)(a). $\mathfrak{C}$ has a continuation $\mathfrak{A}'$ to A with the same invariants as $\mathfrak{A}$; hence, by 0.3, there exist $g \in A^\times$ such that $g\mathfrak{A}'g^{-1} = \mathfrak{A}$. By 1.1 we have $(gEg^{-1})^\times \subset \mathcal{K}_{\mathfrak{A}}$; hence, if we change our first embedding according to the adjoint action of g, we obtain an embedding j, as required.

□

3 Continuation of pairs in the case of an unramified extension in the division algebra D

3.1 Lemma *Let $E \subset L \subset D$, $L/E/F$ unramified extensions and L/F maximal in D (i.e., $[L : F] = d$). Let $A = M_m(D)$, $B = M_m(\tilde{D}) = C_A(E)$; hence, $\tilde{D} = C_D(E)$. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A and $\{X_j\}_{j\in\mathbb{Z}}$ its chain, as in 0.5, in $V = D^m$ (column vectors) and $\mathfrak{B} := \mathfrak{A} \cap B$.*

(i) $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in B.

(ii) Let f:=[E:F], $\{Y_j\}_{j\in\mathbb{Z}}$ be the chain of $\mathfrak{B}$ in $W = \tilde{D}^m$ (column vectors) and π be a prime element of D which normalizes L under the adjoint action. Then we have a decomposition $V = \bigoplus_{i=1}^{f} W\pi^{i-1}$ of V as a ($\tilde{D}$,B)-bimodule and, under this decomposition, X_j decomposes for all $j \in \mathbb{Z}$ into a direct sum of indecomposable ($\mathfrak{B}$,$\mathfrak{o}_{\tilde{D}}$)-bimodules to $X_j = Y_j^{(1)} \oplus Y_j^{(2)}\pi \oplus \ldots \oplus Y_j^{(f)}\pi^{f-1}$, where $\{Y_j^{(i)} \mid j \in \mathbb{Z}\} = \{Y_j \mid j \in \mathbb{Z}\}$ for $i = 1, \ldots, f$.

<u>Proof</u>

We give here a quick proof of these assertions, which are just special versions of the more general results of [Br].

$E \otimes_F L$ is a commutative semisimple algebra over L isomorphic to the direct sum of f copies of L. This gives us a decomposition $1_A = e_1 + \ldots + e_f$ of the unit element of the algebra A (resp. $A \otimes_F L$) into a direct sum of central primitive idempotents due to the decomposition $E \otimes_F L \simeq \bigoplus_{i=1}^{f} E_i$ ($E_i \simeq L$ for $i = 1, \ldots, f$) into simple components. On each $E \otimes_F L$-module the e_i act as projectors to the isotypic components, which are the sum of all simple submodules of the same isomorphy class of simple $E \otimes_F L$-modules. It is easy to see that L is an $E \otimes_F L$-module in f non-isomorphic ways via $(e \otimes l)l' := el'\sigma^i(l)$ $(i \in \{1, \ldots, f\})$, where σ is the generating element of $\mathrm{Gal}(L/F)$ induced by the adjoint action of π on L. Therefore, $L, L\pi, \ldots, L\pi^{f-1}$ comprise a full set of simple $E \otimes_F L$-modules and the isotypic components of D (as an $E \otimes_F L$-module via $(e \otimes l)v := evl$) are $\tilde{D}, \tilde{D}\pi, \ldots, \tilde{D}\pi^{f-1}$; hence,

$$V = \bigoplus_{i=1}^{f} e_i V = \bigoplus_{i=1}^{f} W\pi^{i-1}, \tag{6}$$

as required.

Clearly, $X_j = \bigoplus_{i=1}^{f} e_i X_j = \bigoplus_{i=1}^{f} Y_j^{(i)}\pi^{i-1}$ for all $j \in \mathbb{Z}$, where the $Y_j^{(i)}$ are indecomposable $\mathfrak{B}$-modules. Clearly $\mathfrak{B}$ is an o_E-order in B. Since π^f is a prime element of $\tilde{D}$, its right action maps the inclusion ordered sets $\{Y_j^{(i)} \mid j \in \mathbb{Z}\}$ bijectively to themselves (with inverse π^{-f}), and the action of π permutes the components of (6). We conclude that the inclusion ordered sets $\{Y_j^{(i)} \mid j \in \mathbb{Z}\}$ are all pairwise equal and form $\mathfrak{o}_{\tilde{D}}$-lattice chains in W.

Finally, by 0.2, $\mathfrak{B}$ is hereditary.

□

3.2 Lemma Let E/F, B, $\mathfrak{B}$, r', t' be as in 0.5. Assume further that E/F is unramified and $[E:F] \mid \langle \frac{r'}{t'}, d\rangle$. Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_{t'}$ be the canonical principal orders of $\mathfrak{B}$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_{t'}$ their unique continuations, then $\mathfrak{A} := \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_{t'}$ is a hereditary order with canonical principal orders $\mathfrak{A}_1, \ldots, \mathfrak{A}_{t'}$. Moreover, each pair $(\mathfrak{A}, \mathfrak{A}_i)$ is numerically proportional to the pair $(\mathfrak{B}, \mathfrak{B}_i)$ for $i = 1, \ldots, t'$.

<u>Proof</u>

Note that we fixed an isomorphism $A \simeq M_m(D)$ and consider D to be canonically embedded into $M_m(D)$. By 1.1 and conjugation in $A^\times$, we can restrict ourselves to the case of $E \subset D$. Let $f := [E:F]$, $V := D^m$, $\tilde{D} := C_D(E)$, $W := \tilde{D}^m$ and $\{Y_j\}_{j\in\mathbb{Z}}$ be a lattice chain of $\mathfrak{B}$ in W. Set for all $j \in \mathbb{Z}$ and $i = 1, \ldots, t'$: $Y_j^i := Y_{(i-1)+jt'}$. Let ${n'}_1, \ldots, {n'}_{r'}$ be the invariants of $\mathfrak{B}$ and $s' := {n'}_1 + \ldots + {n'}_{t'}$. Assume that the index of the chain $\{Y_j\}_{j\in\mathbb{Z}}$ was chosen in such a way that $\dim_{\mathfrak{k}_{\tilde{D}}}(Y_0/Y_1) = n'_1$.

We construct the chains of the $\mathfrak{A}'_i s$ explicitly and will see that $\mathfrak{A}$ has the same partition period as $\mathfrak{B}$. The result will then follow from 1.2(iv) (see also the remark to 2.1). Recall the canonical decomposition (6) in 3.1 and let

$$X_j^i := \bigoplus_{k=0}^{f-1} Y^i_{j-\frac{r'}{t'f}k}\pi^k. \tag{7}$$

We want to show that the sequences $\{X_j^i\}_{j\in\mathbb{Z}}$ are the lattice chains of the $\mathfrak{A}'_i s$ for $i = 1, \ldots, t'$.

As π^f is a prime element of $\mathfrak{o}_{\tilde{D}}$, one sees that the $\{X_j^i\}_{j\in\mathbb{Z}}$ are lattice chains with period $\frac{r'}{t'f}$ in V, and if we set ${\mathfrak{A}'}_i := \mathrm{End}_{\mathfrak{o}_D}\{X_j^i\}_{j\in\mathbb{Z}}$, clearly ${\mathfrak{A}'}_i \cap B = \mathfrak{B}_i$. From [BF](1.2.8 (i)) and because $\mathcal{K}_{\mathfrak{B}}$ acts componentwise with respect to the decomposition (7), one obtains that the $\mathfrak{A}'^{\prime}_i s$ are continuations of the $\mathfrak{B}'_i s$; hence, by 1.4(ii) $\mathfrak{A}'_i = \mathfrak{A}_i$ for $i = 1, \dots, t'$, as required.

Moreover, again by [BF](1.2.8 (i)), a uniformizer of $\mathfrak{B}_i$ acts by definition of the chains as a uniformizer of $\mathfrak{A}_i$, too; hence, the $\mathfrak{A}_i{}'s$ are unramified continuations of the $\mathfrak{B}_i{}'s$.

We know from the proof of 2.1 that the inclusion ordered set $\bigcup_{i=1}^{f} \{X_j^i \mid j \in \mathbb{Z}\}$ can be indexed as a lattice chain for $\mathfrak{A}$ in V, and it is easy to see that this is possible in such a way that $X_{i-1} := X_0^i$ for $i = 1, \dots, t'$ and $X_{t'} := X_1^1$; hence, we obtain for the first numbers of the invariants of $\mathfrak{A}$ that $n_i = \dim_{\mathfrak{k}_D}(X_{i-1}/X_i) = \dim_{\mathfrak{k}_D}(X_0^i/X_0^{i+1}) =$ (resp. $\dim_{\mathfrak{k}_D}(X_0^{t'}/X_1^1)$ $=$) $\sum_{k=0}^{f-1} \dim_{\mathfrak{k}_{\tilde{D}}}(Y^i_{0-\frac{r'}{t'f}k}/Y^{i+1}_{0-\frac{r'}{t'f}k}) =$ (resp. $\sum_{k=0}^{f-1} \dim_{\mathfrak{k}_{\tilde{D}}}(Y^{t'}_{0-\frac{r'}{t'f}k}/Y^1_{1-\frac{r'}{t'f}k})$ $=$) $n'_i f$ for $i = 1, \dots t'-1$ (resp. for $i = t'$). Therefore, by 1.2(i) the partition period of $\mathfrak{A}$ must be at least t'. In the situation of 2.1 with the condition of the last assertion of 1.2(iv), we obtain that this can be at most t', which finishes the proof by the remark following 2.1.

□

3.3 Lemma *Let A, d, E/F, B be as in 0.5 and $\mathfrak{B}$ be a hereditary $\mathfrak{o}_E$-order in B with canonical principal orders $\mathfrak{B}_1, \dots, \mathfrak{B}_{t'}$ and period r'. Assume that E/F is unramified, $[E : F] \mid d$ and $\langle [E : F], \frac{r'}{t'} \rangle = 1$. Then, to a fixed canonical principal order $\mathfrak{B}_1$, there exists a pair $(\mathfrak{A}, \mathfrak{A}_1)$ in A which is proportional to $(\mathfrak{B}, \mathfrak{B}_1)$.*

<u>Proof</u>

Because of 1.1, we can restrict ourselves to the case where $E \subset D$ and $(\mathfrak{B}, \mathfrak{B}_1)$ is given in simultaneous normal form 1.5. Here we identify A with $M_m(D)$ and B with $M_m(\tilde{D})$, where $\tilde{D} = C_D(E)$. Let $((n'_1, \dots, n'_{t'}), \frac{r'}{t'})$ be the invariants of $(\mathfrak{B}, \mathfrak{B}_1)$ and $\mathfrak{A}_1$ be the unique continuation of $\mathfrak{B}_1$ as in 1.4(ii). Let $s' := n'_1 + \dots + n'_{t'}$ and s be the invariant of $\mathfrak{A}_1$. Then, by the assumptions on our field extension E/F and the calculations in 1.8(ii), we see that $\frac{s}{s'} = 1$; hence,

$$\mathfrak{A}_1 \text{ and } \mathfrak{B}_1 \text{ have the same invariant.} \tag{8}$$

Let $f := [E : F]$, $\{X_j^1\}_{j \in \mathbb{Z}}$ be the chain of $\mathfrak{A}_1$ in $V = D^m$ and $\{Y_j^1\}_{j \in \mathbb{Z}}$ be the chain of $\mathfrak{B}_1$ in $W = \tilde{D}^m$. Then, by 3.1(ii) and with its notations, there exist $Y_j^{(1)}, \dots, Y_j^{(f)}$ such that $X_j^1 = \bigoplus_{i=1}^{f} Y_j^{(i)} \pi^{i-1}$ for all $j \in \mathbb{Z}$, where this is a decomposition into a direct sum of indecomposable $(\mathfrak{B}_1, \mathfrak{o}_{\tilde{D}})$-bimodules due to (6).

Let

$$h := \begin{pmatrix} 0 & , \pi^f I_{s'} \\ I_{m-s'} , & 0 \end{pmatrix}$$

be the usual uniformizer for $\mathfrak{B}_1$, i. e., $h \in \mathcal{K}_{\mathfrak{B}_1}$ and $\nu_{\mathfrak{B}_1}(h) = 1$. Then there exist functions $[.]_i : \mathbb{Z} \to \mathbb{Z}$ for $i = 1, \ldots, f$ such that $X_j^1 = \bigoplus_{i=1}^{f} h^{[j]_i} Y_0^{(i)} \pi^{i-1}$. Because of (8), we obtain $s' = \dim_{\mathfrak{k}_D}(X_j/X_{j+1}) = \sum_{i=1}^{f} \dim_{\mathfrak{k}_{\tilde{D}}}(h^{[j]_i} Y_0^{(i)} / h^{[j+1]_i} Y_0^{(i)})$ for all $j \in \mathbb{Z}$; hence, because of the uniqueness of the decomposition (6) and since the chain $\{X_j^1\}_{j\in\mathbb{Z}}$ is ordered by inclusion, we conclude that

$$\text{for } j \in \mathbb{Z}, \text{there exists an } i_j \in \{1, \ldots, f\} \text{ such that } [j+1]_{i_j} = [j]_{i_j} + 1 \text{ and } [j+1]_i = [j]_i \text{ for } i \neq i_j. \tag{9}$$

Since both $\mathfrak{B}_1$ and $\mathfrak{A}_1$ have period $\frac{r'}{t'}$ and π^f is a prime element of $\mathfrak{o}_{\tilde{D}}$, $X_j^1 \pi = X^1_{j+\frac{r'}{t'}} = \bigoplus_{i=1}^{f} h^{[j+\frac{r'}{t'}]_i} Y_0^{(i)} \pi^{i-1}$, $Y_j^1 \pi^f = Y^1_{j+\frac{r'}{t'}}$ and, thus, $X_j^1 \pi = \bigoplus_{i=1}^{f} h^{[j]_i} Y_0^{(i)} \pi^i = Y_0^{(f)} \pi^f \bigoplus_{i=1}^{f-1} h^{[j]_i} Y_0^{(i)} \pi^i$. Comparing the coefficients one sees from the action of π on X_j^1 that

$$[j + \frac{r'}{t'}]_1 = [j]_f + \frac{r'}{t'}, [j + \frac{r'}{t'}]_2 = [j]_1, \ldots, [j + \frac{r'}{t'}]_f = [j]_{f-1}. \tag{10}$$

Let $(\mathfrak{A}', \mathfrak{A}'_1)$ be the pair in A which is given in normal form 1.5, and let $\{X_j'^1\}_{j\in\mathbb{Z}}$ be the chain of $\mathfrak{A}'_1$ in $V = D^m$. We can index the chain such that ${}^tX_j'^1 = \mathfrak{p}_D^k((\bigoplus_{\nu=1}^{\mu s'} \mathfrak{p}_D) \oplus (\bigoplus_{\nu=1}^{m-\mu s'} \mathfrak{o}_D))$, where $j \in \mathbb{Z}$ and where $k \in \mathbb{Z}, 0 \leq \mu < \frac{r'}{t'}$ are such that $j = k\frac{r'}{t'} + \mu$.

Next, we show the existence of $g \in A^\times$ such that $g\mathfrak{A}_1 g^{-1} = \mathfrak{A}'_1$, where $g = PM$ is an $\frac{r'}{t'} \times \frac{r'}{t'}$-$(s' \times s')$-block monomial matrix, which means that $M = \operatorname{diag}(\pi^{k_1} I_{s'}, \ldots, \pi^{k_{\frac{r'}{t'}}} I_{s'})$, where $k_1, \ldots, k_{\frac{r'}{t'}} \in \mathbb{Z}$ and $P = (p_{i,j} I_{s'})_{i,j=1,\ldots,\frac{r'}{t'}}$, where $(p_{i,j})_{i,j=1,\ldots,\frac{r'}{t'}}$ is an $\frac{r'}{t'} \times \frac{r'}{t'}$-permutation matrix. By (10) one sees that $h^{\frac{r'}{t'}} Y_0^{(f)} = h^{[\frac{r'}{t'}]_1} Y_0^{(1)}, Y_0^{(1)} = h^{[\frac{r'}{t'}]_2} Y_0^{(2)}, \ldots, Y_0^{(f-1)} = h^{[\frac{r'}{t'}]_f} Y_0^{(f)}$; hence, for all $k \in \mathbb{Z}$, $Y_0^{(2)} = h^k h^{-[\frac{r'}{t'}]_2} Y_0^{(1)}, \ldots, Y_0^{(f)} = h^k h^{-\sum_{i=2}^{f} [\frac{r'}{t'}]_i} Y_0^{(1)}$. For $k \in \mathbb{Z}$ such that $h^k Y_0^{(1)} = \bigoplus_{i=1}^{m} \mathfrak{p}_{\tilde{D}}$ and noting that $\mathfrak{o}_D = \mathfrak{o}_{\tilde{D}} + \mathfrak{o}_{\tilde{D}}\pi + \ldots + \mathfrak{o}_{\tilde{D}}\pi^{f-1}, \mathfrak{p}_D = \mathfrak{p}_{\tilde{D}} + \mathfrak{o}_{\tilde{D}}\pi + \ldots + \mathfrak{o}_{\tilde{D}}\pi^{f-1}, \ldots, \mathfrak{p}_D^f = \mathfrak{p}_{\tilde{D}} + \mathfrak{p}_{\tilde{D}}\pi + \ldots + \mathfrak{p}_{\tilde{D}}\pi^{f-1}$, we obtain ${}^t(h^k X_0^1) = \bigoplus_{i=1}^{s'[\frac{r'}{t'}]_1} \mathfrak{p}_D^f \bigoplus_{i=1}^{s'[\frac{r'}{t'}]_f} \mathfrak{p}_D^{f-1} \ldots \bigoplus_{i=1}^{s'[\frac{r'}{t'}]_2} \mathfrak{p}_D^1$. As the index in the chain $\{X_j^1\}_{j\in\mathbb{Z}}$ can be shifted and X_0^1 was arbitrarily chosen as a finitely generated direct indecomposable $\mathfrak{A}_1$-submodule of V, we obtain for all $j \in \mathbb{Z}$ the existence of $k_1^j, \ldots, k^j_{\frac{r'}{t'}} \in \mathbb{Z}$ such that ${}^tX_j^1 = (\bigoplus_{i=1}^{s'} \mathfrak{p}_D^{k_1^j}) \oplus \ldots \oplus (\bigoplus_{i=1}^{s'} \mathfrak{p}_D^{k^j_{\frac{r'}{t'}}})$.

Let $M := \operatorname{diag}(\pi^{-k_1^0} I_{s'}, \ldots, \pi^{-k^0_{\frac{r'}{t'}}} I_{s'})$. Then $MX_0^1 = \bigoplus_{i=1}^{m} \mathfrak{o}_D = X_0'^1$ and $MX_0^1\pi = \bigoplus_{i=1}^{m} \mathfrak{p}_D = X'^1_{\frac{r'}{t'}}$. Since $\dim_{\mathfrak{k}_D}(X_j^1/X_{j+1}^1) = s'$ for all $j \in \mathbb{Z}$ (see (8)), there exists an $\frac{r'}{t'} \times \frac{r'}{t'}$-$(s' \times s')$-block permutation matrix P such that $PMX_j^1 = X_j'^1$ for $j = 0, \ldots, \frac{r'}{t'}$. Therefore, as $X_j^1\pi = X^1_{j+\frac{r'}{t'}}$ for all $j \in \mathbb{Z}$, $\{PMX_j^1\}_{j\in\mathbb{Z}} = \{X_j'^1\}_{j\in\mathbb{Z}}$. Consequently, for $g := PM$ we obtain $g\mathfrak{A}_1 g^{-1} = \mathfrak{A}'_1$, as required.

Clearly, $(g^{-1}\mathfrak{A}'g, g^{-1}\mathfrak{A}'_1 g) = (g^{-1}\mathfrak{A}'g, \mathfrak{A}_1)$ is a pair, and it remains to show that $\mathfrak{A} := g^{-1}\mathfrak{A}'g$ is a continuation of $\mathfrak{B}$. To see this we consider the order $\mathfrak{A}_{|s'}$, which denotes the hereditary order given in normal form 0.3 in $A_{s'} = M_{s'}(D)$ with invariants $n'_1, \ldots, n'_{t'}$ (in this order), as well as

$\mathfrak{B}_{|s'}$, which denotes the hereditary order given in normal form 0.3 in $B_{s'} = M_{s'}(\tilde{D}) = C_{A_{s'}}(E)$ with invariants $n'_1, \ldots, n'_{t'}$ (in this order). Because of $\mathfrak{o}_D \cap \tilde{D} = \mathfrak{o}_{\tilde{D}}$, $\mathfrak{p}_D \cap \tilde{D} = \mathfrak{p}_{\tilde{D}}$, $\mathcal{K}_{\mathfrak{A}_{|s'}} =< \pi > \mathfrak{A}^{\times}_{|s'}$ and $\mathcal{K}_{\mathfrak{B}_{|s'}} =< \pi^f > \mathfrak{B}^{\times}_{|s'}$, it is clear that $\mathfrak{A}_{|s'}$ is a continuation of $\mathfrak{B}_{|s'}$. As conjugation by $\frac{r'}{t'} \times \frac{r'}{t'}$-$(s' \times s')$-block permutation matrices just permutes the diagonal $s' \times s'$-blocks and conjugates them by certain powers of π, $\mathfrak{A}$ has the same diagonal blocks as $\mathfrak{A}'$ and $\mathfrak{A}_1$ has the same diagonal blocks as ${\mathfrak{A}'}_1$. Therefore, $\mathfrak{A}$ can be obtained from $\mathfrak{A}_1$ just by replacing the diagonal $(s' \times s')$-blocks by $\mathfrak{A}_{|s'}$. As also $\mathfrak{B}$ can be obtained from $\mathfrak{B}_1$ just by replacing the diagonal $s' \times s'$-blocks by $\mathfrak{B}_{|s'}$, we see immediately that $\mathfrak{A} \cap B = \mathfrak{B}$. We also know that conjugation of $\mathfrak{A}$ by an $\frac{r'}{t'} \times \frac{r'}{t'}$-$(s' \times s')$-block monomial matrix leads to a permutation action on the diagonal blocks combined with the adjoint action of certain powers of π on each diagonal block. Since h is an $\frac{r'}{t'} \times \frac{r'}{t'}$-$(s' \times s')$-block monomial matrix that normalizes $\mathfrak{A}_1$, we obtain that h normalizes $\mathfrak{A}$, too. It follows that $\mathcal{K}_{\mathfrak{B}} \subset \mathcal{K}_{\mathfrak{A}}$; hence, $\mathfrak{A}$ is a continuation of $\mathfrak{B}$, as required.

□

3.4 Lemma *Let A, d, E/F, B be as in 0.5, and assume that E/F is unramified with $[E:F] \mid d$. Let $\mathfrak{B}_1$ be a principal order with the Jacobson radical $\mathfrak{Q}_1$ and $\mathfrak{B}$ a hereditary order in B such that $\mathfrak{B}_1$ contains $\mathfrak{B}$. Then there exists at most one pair $(\mathfrak{A}, \mathfrak{A}_1)$ in A such that*

(a) $E^{\times} \subset \mathcal{K}_{\mathfrak{A}}$,

(b) $\mathfrak{A}_1$ is a continuation of $\mathfrak{B}_1$,

(c) $\mathfrak{B} = \mathfrak{A} \cap B$,

(d) there exists an $h \in \mathcal{K}_{\mathfrak{A}} \cap \mathcal{K}_{\mathfrak{B}_1}$ such that $h\mathfrak{B}_1 = \mathfrak{Q}_1$,

and if that pair exists, then $(\mathfrak{B}, \mathfrak{B}_1)$ is a pair, too, and is continued by $(\mathfrak{A}, \mathfrak{A}_1)$.

Proof

Again taking in account our fixed isomorphism $A \simeq M_m(D)$, by 1.1 we can restrict ourselves to the case of $E \subset D$; hence, in particular, we can identify B with $M_m(\tilde{D})$, where $\tilde{D} = C_D(E)$. Assume that the pair $(\mathfrak{A}, \mathfrak{A}_1)$ exists, and denote by $((n_1, \ldots, n_t), \frac{r}{t})$ its invariants. Let $n'_1, \ldots, n'_{r'}, r'$ be the invariants of $\mathfrak{B}$, t' its partition period, $\tilde{r}$ the period of $\mathfrak{B}_1$, and $E/K/F$ an intermediate extension such that $[E:K] = \langle [E:F], \tilde{r} \rangle$. Let L/E be a maximal, unramified extension in D, π be a prime element of $\mathfrak{o}_D$ that normalizes L, $f := [K:F]$, $f' := [E:K]$ and $C := C_A(K) = M_m(\bar{D})$, where $\bar{D} = C_D(K)$. Then π^f is a prime element of $\mathfrak{o}_{\bar{D}}$ and $\pi^{ff'}$ a prime element of $\mathfrak{o}_{\tilde{D}}$. Let $V := D^m$, $T := \bar{D}^m$, $W := \tilde{D}^m$, $\{X_k\}_{k \in \mathbb{Z}}$ be a lattice chain of $\mathfrak{A}$ in V, $\{X^1_k\}_{k \in \mathbb{Z}}$ be a lattice chain of $\mathfrak{A}_1$ in V, $\{Y^1_k\}_{k \in \mathbb{Z}}$ be a lattice chain of $\mathfrak{B}_1$ in W and $\{Y_k\}_{k \in \mathbb{Z}}$ be a lattice chain of $\mathfrak{B}$ in W.

By [Z](Theorem 2), $\mathfrak{C}_1 := \mathfrak{A}_1 \cap C$ is a principal order in C which continues $\mathfrak{B}_1$ and is continued by $\mathfrak{A}_1$. Clearly, $h \in \mathcal{K}_{\mathfrak{B}_1} \subset \mathcal{K}_{\mathfrak{C}_1} \subset \mathcal{K}_{\mathfrak{A}_1}$, and from [Z](Theorem 2(iii) and Corollary 2(ii)(and its proof)) we obtain $\nu_{\mathfrak{C}_1}(h) = 1$ and $\nu_{\mathfrak{A}_1}(h) = f$; hence, because of $\nu_{\mathfrak{A}} = t\nu_{\mathfrak{A}_1}$,

$$\nu_{\mathfrak{A}}(h) \mid fr. \tag{11}$$

We have decompositions

$$V = T \oplus T\pi \oplus \ldots \oplus T\pi^{f-1}, \tag{12}$$

$$T = W \oplus W\pi^f \oplus \ldots \oplus W\pi^{f(f'-1)} \tag{13}$$

and associated projections $\rho_i : V \to T\pi^{i-1} \to T$ for $i = 1, \ldots, f$, and $\sigma_i : T \to W\pi^{f(i-1)} \to W$ for $i = 1, \ldots, f'$. Let $\{Z_k^1\}_{k\in\mathbb{Z}}$ be a lattice chain of $\mathfrak{C}_1$ in T. Because of 3.1(i), $\mathfrak{C} := \mathfrak{A} \cap C$ is a hereditary order, and we denote by $\{Z_k\}_{k\in\mathbb{Z}}$ its lattice chain in T. From 3.1(ii), we obtain $\rho_i(X_j^1) \in \{Z_k^1\}_{k\in\mathbb{Z}}$ and $\rho_i(X_j) \in \{Z_k\}_{k\in\mathbb{Z}}$ for all $j \in \mathbb{Z}$ and $i = 1, \ldots, f$. As $\langle f, \frac{\tilde{r}}{f'} \rangle = 1$ and $\mathfrak{C}_1$ has period $\frac{\tilde{r}}{f'}$ (by [F], [Z], or the proof of 3.2), we deduce, as in the proof of 3.3, the existence of $(\mathfrak{C}_1, \mathfrak{o}_{\tilde{D}})$-bimodules $Z^{(1)}, \ldots, Z^{(f)} \in \{Z_k^1\}_{k\in\mathbb{Z}}$ and functions $[.]_1, \ldots, [.]_f : \mathbb{Z} \to \mathbb{Z}$ such that for all $j \in \mathbb{Z}$,

$$X_j^1 = h^{[j]_1} Z^{(1)} \oplus h^{[j]_2} Z^{(2)}\pi \oplus \ldots \oplus h^{[j]_f} Z^{(f)}\pi^{f-1}, \tag{14}$$

where, as in (9), there exists exactly one $i_j \in \{1, \ldots, f\}$ for each $j \in \mathbb{Z}$ such that $[j+1]_{i_j} = [j]_{i_j} + 1$ and $[j+1]_i = [j]_i$ for all the other $i \in \{1, \ldots, f\} \backslash \{i_j\}$.

Let $M_j^1 := h^{[j]_{i_j}} Z^{(i_j)}$. For all $j \in \mathbb{Z}$ there exists exactly one l_j such that $X_j^1 \supsetneq X_{l_j} \supsetneq X_{l_j+1} \supsetneq \ldots \supsetneq X_{l_j+t-2} \supsetneq X_{j+1}^1$. These $(\mathfrak{A}, \mathfrak{o}_D)$-bimodules all have the same summands $Z^{(i)}\pi^{i-1}$ in the component $T\pi^{i-1}$ of (12) for $i \neq i_j$, but if we set $M_j^u := \rho_{i_j}(X_{l_j+u-2})$ $(u = 2, \ldots, t)$, then

$$M_j^1 \supsetneq M_j^2 \supsetneq \ldots \supsetneq M_j^t \supsetneq hM_j^1. \tag{15}$$

Let $X_j^u := X_{l_j+u-2}$ for $u = 2, \ldots, t$. We have $h \in \mathcal{K}_{\mathfrak{A}} \cap \mathcal{K}_{\mathfrak{B}_1} \subset \mathcal{K}_{\mathfrak{A}} \cap C \subset \mathcal{K}_{\mathfrak{C}}$. Since $\{Z_k^1\}_{k\in\mathbb{Z}} \subset \{Z_k\}_{k\in\mathbb{Z}}$ and $hZ_j^1 = Z_{j+1}^1$ for all $j \in \mathbb{Z}$, we obtain $\nu_{\mathfrak{C}}(h) > 0$. Hence, there exist $\bar{\nu} \in \mathbb{N}$ such that $\nu_{\mathfrak{C}}(h) = \bar{\nu}\bar{t}$, where $\bar{t}$ is the partition period of $\mathfrak{C}$. For the residual fields we have canonical isomorphisms

$$\mathfrak{k}_D \simeq \mathfrak{k}_{\bar{D}} \simeq \mathfrak{k}_{\tilde{D}}; \tag{16}$$

hence, for $i = 1, \ldots, t-1$,

$$n_i = \dim_{\mathfrak{k}_D}(X_j^i / X_j^{i+1}) = \dim_{\mathfrak{k}_{\bar{D}}}(M_j^i / M_j^{i+1}) \text{ and } n_t = \dim_{\mathfrak{k}_D}(X_j^t / X_{j+1}^1) = \dim_{\mathfrak{k}_{\bar{D}}}(M_j^t / hM_j^1) \tag{17}$$

for all $j \in \mathbb{Z}$. Since all $M_j^1, hM_j^1 \in \{Z_k^1\}_{k\in\mathbb{Z}}$, which is a subchain of $\{Z_k\}_{k\in\mathbb{Z}}$ of period $\bar{\nu}\bar{t}$, and for all $\alpha < \beta$ we have an isomorphism of $\mathfrak{k}_{\bar{D}}$ vector spaces $Z_\alpha / Z_\beta \simeq hZ_\alpha / hZ_\beta$, the dimension (17) determines the subchains of period $\bar{\nu}\bar{t}$ where the modules M_j^i lie in. Therefore, there exists a number $\eta(i) \in \{1, \ldots, \bar{\nu}\bar{t}\}$ depending only on i for $i = 1, \ldots, t$ such that $M_j^i \in \{Z_{(\eta(i)-1)+\bar{\nu}\bar{t}k}\}_{k\in\mathbb{Z}}$ for all $j \in \mathbb{Z}$.

As the right action of π^f on the subchain $\{X_k^1\}_{k\in\mathbb{Z}}$ of $\{X_k\}_{k\in\mathbb{Z}}$ is componentwise with respect to the decomposition (14) (resp. componentwise on the decomposition of the X_j with respect to (12)) and has period fr on $\{X_k\}_{k\in\mathbb{Z}}$ from (11) and the componentwise left action of h we obtain

$$\nu_{\mathfrak{C}}(h) \mid \bar{r}, \tag{18}$$

where $\bar{r}$ is the period of $\mathfrak{C}$.

On the other hand, we have

$$\mathfrak{A} \cap C = \{x \in C \mid \forall j \in \mathbb{Z},\, i = 1, \ldots, f,\, x\rho_i(X_j) \subset \rho_i(X_j)\} = \mathfrak{C}. \tag{19}$$

However, the $\rho_i(X_j)$ are either contained in the chain $\{Z_k^1\}_{k\in\mathbb{Z}}$ or equal to some M_j^i. Because of 0.3(1), all isomorphy classes of direct indecomposable $\mathfrak{C}$-modules must appear under the $\rho_i(X_j)$ to enable equation (19). It follows that all modules of the chain $\{Z_k\}_{k\in\mathbb{Z}}$ between M_j^1 and hM_j^1 must occur in the list (15). With the equations (17), it follows that for an appropriate choice of the invariants $\bar{n}_1, \ldots \bar{n}_{\bar{r}}$ of $\mathfrak{C}$, which are only unique up to cyclic permutation, we have $\bar{n}_1 = n_1, \ldots, \bar{n}_t = n_t$. By componentwise action of $h \in \mathcal{K}_{\mathfrak{A}}$ with respect to (12), all modules of the chain $\{Z_k^1\}_{k\in\mathbb{Z}}$ occur as summands of the modules of $\{X_k\}_{k\in\mathbb{Z}}$, and $\{Z_k^1\}_{k\in\mathbb{Z}}$ is a subchain of period $\bar{\nu}\bar{t}$ in $\{Z_k\}_{k\in\mathbb{Z}}$. We deduce that $\bar{\nu} = 1$, $\bar{t} = t$ and $\bar{r} = r$. In particular, h is a uniformizer for $\mathfrak{C}$, and consequently $\mathcal{K}_{\mathfrak{C}} = \langle h\rangle\mathfrak{C}^\times \subset \langle h\rangle\mathfrak{C}_1^\times = \mathcal{K}_{\mathfrak{C}_1}$ and $\mathcal{K}_{\mathfrak{C}} = \langle h\rangle\mathfrak{C}^\times \subset \langle h\rangle\mathfrak{A}^\times \subset \mathcal{K}_{\mathfrak{A}}$; hence

$$(\mathfrak{C}, \mathfrak{C}_1) \text{ is a pair continued by } (\mathfrak{A}, \mathfrak{A}_1). \tag{20}$$

Since $\mathfrak{C}_1 \cap B = \mathfrak{B}_1$ and $\mathfrak{C} \cap B = \mathfrak{B}$, 3.1 applies, and thus $\sigma_i(Z_j^1) \in \{Y_k^1\}_{k\in\mathbb{Z}}$ and $\sigma_i(Z_j) \in \{Y_k\}_{k\in\mathbb{Z}}$ for all $j \in \mathbb{Z}$ and $i = 1, \ldots, f'$. For all $j \in \mathbb{Z}$ there exists exactly one $l_j \in \mathbb{Z}$ such that $Z_j^1 \supsetneq Z_{l_j} \supsetneq Z_{l_j+1} \supsetneq \ldots \supsetneq Z_{l_j+t-2} \supsetneq Z_{j+1}^1$. Let $Y_j^{(i,1)} := \sigma_i(Z_j^1)$ and $Y_j^{(i,u)} := \sigma_i(Z_j^u)$ for $i = 1, \ldots, f'$ and all $j \in \mathbb{Z}$, where $Z_j^u := Z_{l_j+u-2}$ for $u = 2, \ldots, t$. Then $Y_j^{(i,u)} \in \{Y_k\}_{k\in\mathbb{Z}}$ for $i = 1, \ldots, f'$, $u = 1, \ldots, t$ and $j \in \mathbb{Z}$. For $u = 1, \ldots, t$ and all $j \in \mathbb{Z}$ we obtain a decomposition

$$Z_j^u = Y_j^{(1,u)} \oplus Y_j^{(2,u)}\pi^f \oplus \ldots \oplus Y_j^{(f',u)}\pi^{f(f'-1)} \tag{21}$$

into a direct sum of indecomposable $(\mathfrak{B}, \mathfrak{o}_{\tilde{D}})$-bimodules due to (13).

By assumption (d) we have $h \in \mathcal{K}_{\mathfrak{A}} \cap \mathcal{K}_{\mathfrak{B}_1} \subset \mathcal{K}_{\mathfrak{B}}$, $\nu_{\mathfrak{B}_1}(h) = 1 > 0$, and by (b)(c), $\mathfrak{B} \subset \mathfrak{B}_1$; hence, there exist $\nu' \in \mathbb{N}$ such that $\nu_{\mathfrak{B}}(h) = \nu' t' > 0$, and using arguments as in the proof of 1.2(iii) we see that $\{Y_k^1\}_{k\in\mathbb{Z}}$ is a subchain of period $\nu' t'$ of $\{Y_k\}_{k\in\mathbb{Z}}$. As $\pi^{ff'}$ (right action) and h (left action) act componentwise on the Y_j^u with respect to the decomposition (21), we obtain from (18) that

$$\nu_{\mathfrak{B}}(h) \mid r'. \tag{22}$$

The decomposition (21) shows that $Z_j^u\pi^f = Y_j^{(f',u)}\pi^{ff'} \oplus Y_j^{(1,u)}\pi^f \oplus \ldots \oplus Y_j^{(f'-1,u)}\pi^{f(f'-1)}$; hence, by (18),(22) and the fact that h acts componentwise with respect to (21), we obtain from the iterated action of π^f that, for fixed $j \in \mathbb{Z}$ and $u \in \{1, \dots, t\}$, all the $Y_j^{(i,u)}$ lie in the same subchain $\{Y_{v+\nu' t' k}\}_{k \in \mathbb{Z}}$ ($v \in \{0, \dots, \nu' t' - 1\}$) of $\{Y_k\}_{k\in\mathbb{Z}}$. Together with (16) for all $j \in \mathbb{Z}$ and an arbitrary $i \in \{1, \dots, f'\}$ this leads to

$$\begin{aligned} &f' \dim_{\mathfrak{k}_{\tilde{D}}}(Y_j^{(i,u)}/Y_j^{(i,u+1)}) = \dim_{\mathfrak{k}_{\tilde{D}}}(Z_j^u/Z_j^{u+1}) = n_u \text{ for } u = 1, \dots, t-1 \text{ and} \\ &f' \dim_{\mathfrak{k}_{\tilde{D}}}(Y_j^{(i,t)}/Y_{j+1}^{(i,t)}) = \dim_{\mathfrak{k}_{\tilde{D}}}(Z_j^t/Z_{j+1}^1) = n_t. \end{aligned} \tag{23}$$

The last equation implies that, for $u = 1, \dots, t$, there exists a number $\mu(u) \in \{1, \dots, \nu' t'\}$ only depending on u such that for all $j \in \mathbb{Z}$ and $i = 1, \dots, f'$, $Y_j^{(i,u)} \in \{Z_{(\mu(u)-1)+\nu' t k}\}_{k\in\mathbb{Z}}$. For all $j \in \mathbb{Z}$ and $i = 1, \dots, f'$ we have, moreover,

$$Y_j^{(i,1)} \supsetneq Y_j^{(i,2)} \supsetneq \dots \supsetneq Y_j^{(i,t)} \supsetneq Y_{j+1}^{(i,1)}. \tag{24}$$

On the other hand, we have

$$\mathfrak{C} \cap B = \{x \in B \mid \forall j \in \mathbb{Z}, \forall i = 1, \dots, f', x\sigma_i(Z_j) \subset \sigma_i(Z_j)\} = \mathfrak{B}. \tag{25}$$

Due to 0.3, (1), (25) implies that all isomorphy classes of direct indecomposable $\mathfrak{B}$-modules have to appear under the $\sigma_i(Z_j)$; hence, by (22) all modules of the chain $\{Y_k\}_{k\in\mathbb{Z}}$ that lie between $Y_j^{(i,1)}$ and $Y_{j+1}^{(i,1)}$ must occur in the list (24). The equations (23) now provide that $\nu' = 1$. We obtain $f' n_1' = n_1, \dots, f' n_t' = n_t$, for an appropriate numbering of the invariants $n_1', \dots, n_{r'}'$ of $\mathfrak{B}$, which are unique up to cyclic permutation only, and using (22) also $t' = t$. Finally, by $n_1 + \dots + n_r = m$, we obtain $\frac{r'}{f'} = r$. In particular, h is a uniformizer for $\mathfrak{B}$; hence, $\mathcal{K}_{\mathfrak{B}} = \langle h \rangle \mathfrak{B}^\times \subset \langle h \rangle \mathfrak{B}_1^\times = \mathcal{K}_{\mathfrak{B}_1}$ and $\mathcal{K}_{\mathfrak{B}} = \langle h \rangle \mathfrak{B}^\times \subset \langle h \rangle \mathfrak{C}^\times = \mathcal{K}_{\mathfrak{C}}$. Therefore, $(\mathfrak{B}, \mathfrak{B}_1)$ is a pair continued by $(\mathfrak{C}, \mathfrak{C}_1)$; hence, by (20) and the transitivity of continuation, the pair $(\mathfrak{B}, \mathfrak{B}_1)$ is continued by $(\mathfrak{A}, \mathfrak{A}_1)$. This proves the last assertion of our proposition.

To prove the uniqueness of $(\mathfrak{A}, \mathfrak{A}_1)$, we consider the way in which the chain of $\mathfrak{A}$ is built up from the chain of $\mathfrak{A}_1$ and $\mathfrak{B}$. We have $\mathfrak{A} \cap B = \{x \in B \mid \forall j \in \mathbb{Z}, \forall i = 1, \dots, f, \forall u = 1, \dots, f', x\sigma_u \circ \rho_i(X_j) \subset \sigma_u \circ \rho_i(X_j)\} = \mathfrak{B}$. By (14), (15) and (21) the $\sigma_u \circ \rho_i(X_j)$ are either contained in the chain $\{Y_k^1\}_{k\in\mathbb{Z}}$ of $\mathfrak{B}_1$ or direct summands of an M_j^u (for a certain other choice of $u \in \{2, \dots, t\}$ and $j \in \mathbb{Z}$). To consider the latter case, choose for each $j \in \mathbb{Z}$ and $i \in \{1, \dots, f'\}$ an l_j such that $\sigma_i(M_j^u) = Y_{l_j}^{(i,u)}$. One can see from the structure of the chain $\{Y_k\}_{k\in\mathbb{Z}}$ that the $Y_{l_j}^{(i,u)}$ are uniquely determined by the relations (24); hence, the modules M_j^u for $u = 2, \dots, t$ are uniquely determined by knowing the modules M_j^1 and hM_j^1. Finally, the modules X_j^u for $u = 2, \dots, t$ are uniquely determined by knowing the modules X_j^1 and X_{j+1}^1, because they conside in all components except for one and are determined there by the M_j^u for $u = 2, \dots, t$. Thus, because $\{X_k\}_{k\in\mathbb{Z}} = \bigcup_{u=1}^{t} \{X_k^u\}_{k\in\mathbb{Z}}$, the chain $\{X_k\}_{k\in\mathbb{Z}}$ is (up to a shift of the index) uniquely determined by the chains $\{X_k^1\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$. Therefore, $\mathfrak{A}$ is uniquely determined by $\mathfrak{A}_1$ and $\mathfrak{B}$. Because of [F](Corollary 3), $\mathfrak{A}_1$ is uniquely determined by $\mathfrak{B}_1$. Consequently, $(\mathfrak{A}, \mathfrak{A}_1)$ is uniquely determined by $(\mathfrak{B}, \mathfrak{B}_1)$, as required.

□

4 Continuation of pairs in the general case

<u>Proof of Theorem 1.9</u>

(i),(i)(a) <u>Existence:</u> Let $E/L/F$ be an unramified intermediate extension over F of degree $[L:F] = \langle f(E/F), d\rangle$. The algebra $C := C_A(L)$ has index $d' := \frac{d}{[L:F]}$ and

$$\langle f(E/L), d'\rangle = 1; \tag{26}$$

hence, 2.2 applies and there exists a numerically proportional continuation $(\mathfrak{C}, \mathfrak{C}_1)$ of $(\mathfrak{B}, \mathfrak{B}_1)$ to B. If $\mathfrak{C}_1$ has period $\tilde{r}$, then let $L/K/F$ be the unique intermediate extension of degree $[L:K] = \langle [L:F], \tilde{r}\rangle$. By 0.5, $H := C_A(K)$ is a central simple algebra of index $\frac{d}{[K:F]}$; hence, 3.2 applies and there exists a numercially proportional continuation $(\mathfrak{O}, \mathfrak{O}_1)$ of $(\mathfrak{C}, \mathfrak{C}_1)$ to H, where $\mathfrak{O}_1$ has period $\frac{\tilde{r}}{[L:K]}$. Clearly, $\langle [K:F], \frac{\tilde{r}}{[L:K]}\rangle = 1$; hence, 3.3 applies and there exists a numerically proportional continuation $(\mathfrak{A}, \mathfrak{A}_1)$ of $(\mathfrak{O}, \mathfrak{O}_1)$ to A. By transitivity $(\mathfrak{A}, \mathfrak{A}_1)$ is a numerically proportional continuation of $(\mathfrak{B}, \mathfrak{B}_1)$ to A.

<u>Uniqueness:</u> Again let $E/L/F$ and C as in the existence part of our proof and, moreover, $h \in \mathcal{K}_{\mathfrak{B}}$ such that $\nu_{\mathfrak{B}}(h) = t'$; hence, $\nu_{\mathfrak{B}_1}(h) = 1$. By [Z](Theorem 2), $\mathfrak{C}_1 := \mathfrak{A}_1 \cap C$ is a principal order continued by $\mathfrak{A}_1$ and continuing $\mathfrak{B}_1$. By [Br](Theorem 1) (or using 1.1 and 3.1), $\mathfrak{C} := \mathfrak{A} \cap C$ is a hereditary order, and, by (26) and 2.2(i)(b), $\nu_{\mathfrak{C}_1}(h) = 1$, where $h \in \mathcal{K}_{\mathfrak{A}} \cap \mathcal{K}_{\mathfrak{C}_1}$; hence, 3.4 applies and $(\mathfrak{C}, \mathfrak{C}_1)$ is a pair continued uniquely by $(\mathfrak{A}, \mathfrak{A}_1)$. Because of (26), 2.2 applies, and $(\mathfrak{C}, \mathfrak{C}_1)$ is the unique continuation of $(\mathfrak{B}, \mathfrak{B}_1)$. This proves uniqueness.

(i)(b) From (a) we know that $\mathfrak{A}$ and $\mathfrak{B}$ have the same partition period t. Furthermore, $\nu_{\mathfrak{A}} = t\nu_{\mathfrak{A}_1}$ and $\nu_{\mathfrak{B}} = t\nu_{\mathfrak{B}_1}$; hence, our assertion follows from [Z](Theorem 2(iii), proof of Corollary 3(ii)).

(ii) (a)⇒(b) Let K/E be a totally ramified extension of E of degree $\frac{r'}{t'}d'$, where $d' = \frac{d}{\langle [E:F], d\rangle}$ is the index of B. Because of 2.2(iii), we can embed $K^\times$ over $E^\times$ in $\mathcal{K}_{\mathfrak{B}}$, and, because of 2.2(ii), $\mathfrak{C}_1 := \mathfrak{B}_1 \cap C$ is a principal order in $C := C_B(K)$ continued by $\mathfrak{B}_1$. By (i)(a), $\mathfrak{C}_1$ has period $\bar{r} = \frac{r'}{t'}\frac{\langle e(K/E), d'\rangle}{e(K/E)} = 1$. Because of [F](Corollary 1), there exists an unramified maximal extension M/K in C such that $E^\times \subset \mathcal{K}_{\mathfrak{C}_1}$.

Now we have $E^\times \subset K^\times \subset M^\times \subset \mathcal{K}_{\mathfrak{A}_1}$, $K^\times \subset E^\times \subset \mathcal{K}_{\mathfrak{A}}$ and $e(K/F) = e(M/F)$. As $e(M/F)f(E/F) = d'\frac{r'}{t'}e(E/F)f(E/F) = d\frac{r'}{t'}\frac{[E:F]}{\langle [E:F], d\rangle}$, we obtain $d \mid e(M/F)f(E/F)$. Clearly, M/F is also of maximal degree in A.

(b)⇒(a) Let $E/L/F$ be an unramified intermediate extension over F of degree $[L:F] = \langle f(K/F), d\rangle$ and $C = C_A(L)$, then, because of [Z](Theorem 2), $\mathfrak{C}_1 := \mathfrak{A}_1 \cap C$ is a principal order in C continued by $\mathfrak{A}_1$, and, because of Lemma 1.1 and Lemma 3.1, $\mathfrak{C} := \mathfrak{A} \cap C$ is a hereditary order. Let π be a prime element of o_K, whence $(e(M/F) = e(K/F))$ of o_M. Let $\bar{r}$ be the period of $\mathfrak{C}_1$. Clearly,

$\pi \in \mathcal{K}_{\mathfrak{A}} \cap \mathcal{K}_{\mathfrak{C}_1}$. By [F](Theorem 1(iii), Remark 1) we obtain for $d' := \frac{d}{[L:F]}$ (the index of C) that $\nu_{\mathfrak{C}_1}(\pi) = \frac{\bar{r}d'}{e(M/F)} = \frac{\frac{e(M/F)}{\langle e(M/F),d'\rangle}d'}{e(M/F)} = \frac{d'}{\langle e(M/F),d'\rangle} = \frac{d}{\langle e(M/F)f(K/F),d\rangle} = 1$, because of $d \mid e(M/F)f(E/F)$.

Therefore, 3.4 applies, and $(\mathfrak{C}, \mathfrak{C}_1)$ is a pair in C continued by $(\mathfrak{A}, \mathfrak{A}_1)$. As $\langle f(K/L), d'\rangle = 1$, 2.2 also applies, and $(\mathfrak{A} \cap B, \mathfrak{A}_1 \cap B)$ is a pair in B continued by $(\mathfrak{C}, \mathfrak{C}_1)$; hence, our result follows by the transitivity of continuation.

(iii) Let $((n_1, \ldots, n_t), \frac{r}{t})$ be the invariants of $(\mathfrak{A}, \mathfrak{A}_1)$.

(a)$\Rightarrow$(b) By assumption and (ii) $(\mathfrak{A} \cap B, \mathfrak{A}_1 \cap B)$ is a pair in B continued by and numerically proportional to $(\mathfrak{A}, \mathfrak{A}_1)$. We denote its invariants by $((n'_1, \ldots, n'_{t'}), \frac{r'}{t'})$. We know from [F](Corollary 1) that there exists a divisor $\tilde{d}$ of $\ddot{d}$ such that $f(M/F) = (n_1 + \ldots + n_t)\tilde{d}$ and, clearly,

$$e(E/F) \mid e(M/F) = \frac{md}{f(M/F)} = \frac{r}{t}\frac{d}{\tilde{d}}. \tag{27}$$

As in 1.8 we calculate $l_1 = \frac{n_1+\ldots+n_t}{n'_1+\ldots+n'_t} = \frac{n_1+\ldots+n_t}{\langle f(M/E), \frac{m\langle [E:F],d\rangle}{[E:F]}\rangle} = \frac{(n_1+\ldots+n_t)f(E/F)}{\langle f(M/F), \frac{\langle [E:F],d\rangle m}{e(E/F)}\rangle} = \frac{f(E/F)}{\langle \tilde{d}, \frac{\frac{r}{t}\langle [E:F],d\rangle}{e(E/F)}\rangle} = \frac{f(E/F)}{\langle \tilde{d}, \frac{\langle [E:F],d\rangle}{\langle e(E/F),d\rangle}\rangle} = \frac{f(E/F)}{\langle \tilde{d}, \langle f(E/F), \frac{d}{\langle e(E/F),d\rangle}\rangle\rangle}$; hence, $l_1 \mid f(E/F) \mid l_1\tilde{d}$ and, according to (27), $[E:F] \mid \frac{r}{t}dl_1$. Now everything follows from $l_1 \mid \langle n_1, \ldots, n_t\rangle$ (see 1.8).

(b)$\Rightarrow$(a) From our assumptions, the existence of a divisor l_1 of $\langle n_1, \ldots, n_t\rangle$ such that $l_1 \mid f(E/F) \mid l_1\ddot{d}$ immediately follows and if we choose l_1 to be maximal with this property, we have $\langle \frac{f(E/F)}{l_1}, \frac{\langle n_1,\ldots,n_r\rangle}{l_1}\rangle = 1$. From $[E:F] \mid d\frac{r}{t}\langle n_1, \ldots, n_r\rangle$ and $e(E/F) \mid d\frac{r}{t}$ it follows that $\frac{f(E/F)}{l_1} \mid \frac{d\frac{r}{t}}{e(E/F)}$; hence, $[E:F] \mid d\frac{r}{t}l_1$. Let $E \hookrightarrow A$ be an arbitrary embedding of E over F into A (which exists because of $[E:F]|md$) and $B = C_A(E)$, where we identify E with its image in A. There exists a pair $(\mathfrak{B}, \mathfrak{B}_1)$ in B with invariants $((\frac{n_1}{l_1}, \ldots, \frac{n_t}{l_1}), \frac{rl_1\langle [E:F],d\rangle}{t[E:F]})$.

Choose a totally ramified extension K/E such that $e(K/E) = \frac{rl_1\langle [E:F],d\rangle}{t[E:F]}d'$, where $d' = \frac{d}{\langle [E:F],d\rangle}$. Then, by 2.2(ii), there exists an embedding $K^\times \hookrightarrow \mathcal{K}_{\mathfrak{B}}$ over $E^\times$. Because of 2.2(ii), for $C = C_B(K)$, $(\mathfrak{C}, \mathfrak{C}_1) = (\mathfrak{B} \cap C, \mathfrak{B}_1 \cap C)$ is a pair in C with invariants $((\frac{n_1}{l_1}, \ldots, \frac{n_t}{l_1}), 1)$. By [F](Corollary 1), we can find an unramified maximal extension M/K in C such that $M^\times \subset \mathcal{K}_{\mathfrak{C}_1}$. Part (i) shows that there exists a proportional continuation $(\mathfrak{A}', \mathfrak{A}'_1)$ of $(\mathfrak{B}, \mathfrak{B}_1)$.

Because of 1.8(iii), the choice of the invariants of $(\mathfrak{B}, \mathfrak{B}_1)$ and the definition of l_1, we deduce that $(\mathfrak{A}', \mathfrak{A}'_1)$ has the same invariants as $(\mathfrak{A}, \mathfrak{A}_1)$; hence, both pairs are conjugated over $A^\times$. So, by conjugating the first embedding and the fields M, K, we can assume without loss of generality that $(\mathfrak{A}', \mathfrak{A}'_1) = (\mathfrak{A}, \mathfrak{A}_1)$, $M^\times \subset \mathcal{K}_{\mathfrak{A}_1}$, and $K^\times \subset \mathcal{K}_{\mathfrak{A}}$. Finally, we obtain $d|\frac{r}{t}l_1 d = \frac{rl_1\langle [E:F],d\rangle}{t[E:F]}d'e(E/F)f(E/F) = e(M/F)f(E/F)$ from our assumptions.

□

5 References

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