Stratifications of hyperelliptic Jacobians and the Sato Grassmannian

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Abstract

In this paper a one-dimensional family of stratifications on a hyperelliptic Jacobian is introduced. It generalises a well-known stratification, considered in algebraic geometry, in the context of special divisors. The stratification is shown to be related to a natural stratification on the Sato Grassmannian, via an extension of Krichever's map. It is also related to the stratification associated to the Laurent solutions of certain vector fields which can both be seen as living on the Grassmannian or on the Jacobian.

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1. Introduction

In this paper we introduce a one-dimensional family of stratifications on the Jacobian of any hyperelliptic curve and show how it appears naturally in different situations. Some stratifications of Abelian varieties, in particular of Jacobians, have been used and studied in algebraic geometry, in connection with linear systems of (special) divisors on curves. For example, let Γ be a hyperelliptic curve with hyperelliptic involution $P \mapsto P^\sigma$ and let P be a Weierstraß point on it. Then Gunning (see [Gu]) considers for $m = 0, \ldots, g$ the subsets $J_m(\Gamma, P)$ of the Jacobian of Γ, $\mathrm{Jac}(\Gamma)$, defined by

$$J_m(\Gamma, P) = \left\{ \{D\} \mid D = \sum_{i=1}^{m}(P_i - P),\ P_i \in \Gamma \setminus \{P\},\ i \neq j \Rightarrow P_i \neq \imath P_j \right\},$$

where $\{D\}$ denotes the class of all divisors linearly equivalent to D, viewed as a point of $\mathrm{Jac}(\Gamma)$. He shows that they define a stratification of the Jacobian of Γ.

This stratification generalises in a natural way, specific to hyperelliptic Jacobians, to the case where P is any point on the curve Γ. If the point corresponding to P under the hyperelliptic involution is denoted by P^σ, then we define for m and n positive, $m + n \leq g =$ genus (Γ),

$$J_{m,n}(\Gamma, P) = \left\{ \{D\} \mid D = \sum_{i=1}^{g-m-n} P_i + mP + nP^\sigma - gP,\ P_i \in \Gamma \setminus \{P, P^\sigma\} \text{ and } i \neq j \Rightarrow P_i \neq P_j^\sigma \right\}.$$

Remark that in the case $P = P^\sigma$ considered by Gunning, one has $J_m(\Gamma, P) = J_{m-i,i}(\Gamma, P)$ for any $i \leq m$. In the opposite case $P \neq P^\sigma$ however, all $J_{m,n}(\Gamma, P)$ are disjoint and we show that they stratify $\mathrm{Jac}(\Gamma)$, with $i+1$ strata of codimension i, (in total $\frac{(g+1)(g+2)}{2}$ strata) and it is shown how they relate. If the chosen point $P \in \Gamma$ is replaced by P^σ then one obviously obtains the same stratification, up to a translation; therefore the family of stratifications is essentially parametrised by Γ/σ, i.e., by $\mathbb{P}^1$.

It is easily deduced from [SW] that the stratification considered by Gunning arises in the context of an infinite-dimensional Grassmannian, Gr, introduced by Sato (see [SS]). The Grassmannian Gr can be defined as the set of all spaces of formal power series in one variable z (which should be thought of as being large) which have an algebraic base of the form

$$\{w_0(z), w_1(z), w_2(z), \ldots\},$$

where

$$w_i(z) = \sum_{j=-\infty}^{s_i} w_{ij} z^j, \qquad w_{is_i} \neq 0 \text{ and } s_i < s_{i+1},$$

with $i = s_i$ for i sufficiently large. To such a plane W there is associated the (ordered) subset $S_W = \{s_0, s_1, \ldots,\}$ of the integers, which has the property that $s_i = i$ for i sufficiently large. Each such sequence defines in a natural way a (non-empty) subset $\Sigma_S \subset \mathrm{Gr}$, defined as

$$\Sigma_S = \{W \in \mathrm{Gr} \mid S_W = S\}.$$

These (non-intersecting) subsets can be shown to be the strata of a stratification of Gr (see [PS]). To relate this stratification to Gunning's stratification, the Krichever map is used. Roughly speaking this map associates to a point in the Jacobian the family of all sections of its corresponding line bundle, which are holomorphic except at the marked point $P \in \Gamma$. This family is identified with an element of Gr by using a trivialisation of the line bundle. The point is that although this element of Gr depends on the trivialisation, the stratum it belongs to is independent of it, hence we may use the Krichever map to relate both stratifications: we show that (different) strata are mapped into (different) strata so that we may think of the stratifications considered by Gunning as being induced by the natural stratification of Gr via the Krichever map.

The natural question arises whether the stratifications by the subsets $J_{m,n}(\Gamma, P)$ can for every $P \in \Gamma$ be obtained in this way by an appropriate generalisation of the Krichever map. The answer is affirmative and the generalised Krichever map we introduce, associates now to each point in Jac(Γ) two points in Gr, i.e., a point in the product Gr $\times$ Gr, which is equiped with the product stratification. In the special case that $P = P^\sigma$ the map reduces to a diagonal map (i.e., both points are the same) giving the ordinary Krichever map on each component. We also show that the stratification on Gr $\times$ Gr can be weakened to a coarser stratification, which still induces the family of stratifications. This coarser stratification shows up when considering the so-called K-P hierarchy on the Grassmannian (see [SS], [SW] and [DJMK]).

This K-P hierarchy, in particular a distinguished vector field of it, determines a special family of vector fields on Jac(Γ), depending on the marked point P on Γ. As is well-known from the theory of integrable systems, every meromorphic function on Jac(Γ) admits families of Laurent solutions describing the function on the integral curves of the vector field (see [AvM2]). Taking one or several functions a decomposition of Jac(Γ) is given by fixing the way these solutions blow up. This decomposition may be a stratification. We will show that the choice of the very special vector field coming from the K-P hierarchy and a natural choice of functions coming from the symmetric functions on the curve, gives for each choice of the marked point P on the curve, indeed a stratification which coincides again with the stratification by the subset $J_{m,n}(\Gamma, P)$, thereby providing us with a very explicit description of the former stratifications; in particular the leading behaviour of the Laurent solutions to the differential equations which describe the vector field will be computed explicitely by introducing a pair of tau functions which corresponds to the extended Krichever map.

The text is organised as follows. In Section 2 some preliminaries about hyperelliptic curves and their Jacobians are recalled and the stratifications are introduced. We give a detailed description of them since they are fundamental for the whole paper. Section 3 deals with the Sato Grassmannian, which is also recalled, together with its stratification. The Krichever map is explained and extended as needed for our purposes, leading to the main result relating the two stratifications. In the end the coarser stratification is discussed in the context of the K-P hierarchy. In the final Section 4, we look at special vector fields on the Jacobian, associated to a point on the curve; the relation between Laurent solutions to the vector field and stratifications of the Jacobian is explained and related to the stratification in Section 2, relying heavily on some results obtained in Section 3.

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2. The algebraic description of the stratification

In this section we introduce a natural family of stratifications on the Jacobian of a hyperelliptic curve, parametrised by a point on the curve. In the first paragraph we recall some basic results about hyperelliptic curves and their Jacobians (see [GH] or [H]). The stratification is introduced in the second paragraph and its structure is described.

2.1. Preliminaries

Let Γ be a smooth curve of genus g which is *hyperelliptic*, i.e., Γ carries an involution $\sigma: \Gamma \to \Gamma$, $Q \mapsto \sigma(Q) = Q^\sigma$, $\sigma^2 = \mathrm{Id}$, the so-called *hyperelliptic involution.* This involution has $2g+2$ fixed points, the *Weierstraß points* of Γ, which are also the branch points of any 2:1 cover $\pi: \Gamma \to \mathbb{P}^1$. Such a cover gives rise to an equation $y^2 = f(x)$ for (an affine part of) Γ; the degree of f is $2g+1$ or $2g+2$ according to whether or not $\infty \in \mathbb{P}^1$ is the image of a Weierstraß point, i.e., according to whether $p^{-1}(\infty)$ contains one point (with multiplicity two), or two points. These two points correspond under σ, which is given in terms of the coordinates x, y by $(x,y) \mapsto (x,-y)$.

The group of divisors $D = \sum_{\text{finite}} c_i P_i \, (P_i \in \Gamma)$ on Γ is denoted by $\mathrm{Div}(\Gamma)$ and σ extends linearly to $\mathrm{Div}(\Gamma)$ giving an involution $D \mapsto D^\sigma$. There is associated to each meromorphic function $f \in \mathcal{M}(\Gamma)$ its divisor of zeroes minus its divisor of poles, denoted by (f); obviously the map $(\cdot): \mathcal{M}(\Gamma) \to \mathrm{Div}(\Gamma)$ is a homomorphism. In the same way (ω) is defined for any meromorphic differential and one has $(f\omega) = (f) + (\omega)$. For example, let $P \in \Gamma$ and let

$$y^2 = f(x) = \prod_{i=1}^{\deg f} (x - x(B_i))$$

be an equation for Γ such that $x(P) = \infty$. Then

$$(y) = \sum_{i=1}^{\deg f} B_i - \frac{\deg f}{2}(P + P^\sigma) \text{ and } (x) = \sum_{i=1}^{2} \left(0, (-1)^i \sqrt{f(0)}\right) - (P + P^\sigma). \tag{1}$$

Also

$$(dx) = \sum_{i=1}^{2g+1} B_i - 3P \text{ or } (dx) = \sum_{i=1}^{2g+2} B_i - 2(P + P^\sigma), \tag{2}$$

according to whether $P = P^\sigma$ or $P \neq P^\sigma$ (in that order).

We introduce the spaces $L(D)$ and $\Omega(D)$ for $D \in \mathrm{Div}(\Gamma)$ as

$$L(D) = \{f \mid f \text{ meromorphic function on } \Gamma \text{ and } (f) + D \geq 0\},$$
$$\Omega(D) = \{\omega \mid \omega \text{ meromorphic differential on } \Gamma \text{ and } (\omega) + D \geq 0\}.$$

Their dimensions are related by the Riemann-Roch formula which states (for algebraic curves) that for any $D \in \mathrm{Div}(\Gamma)$,

$$\dim L(D) = \dim \Omega(-D) - g + 1 + \deg(D), \tag{3}$$

the degree of a divisor being defined as $\deg(\sum c_i P_i) = \sum c_i$. In particular, since every holomorphic function on Γ is constant, the space $\Omega = \Omega(0)$ of holomorphic differentials has dimension g and by (1) and (2) has in our case a base

$$\left\{ \frac{dx}{y}, \frac{x\,dx}{y}, \dots, \frac{x^{g-1}dx}{y} \right\}, \tag{4}$$

when $y^2 = f(x)$ is an equation for Γ as above. Remark that it follows from (3) and (4) that if $P_i\,(i = 1, \ldots, n \leq g)$ are such that $i \neq j \Rightarrow P_i \neq P_j^\sigma$ then

$$\dim \Omega\left(\sum_{i=1}^{n} P_i\right) = g - n. \tag{5}$$

For their meromorphic analogues with poles at P and P^σ only we have

$$\dim \Omega(kP + lP^\sigma) = g + k + l - 1 \text{ for } k > 0,\, l \geq 0. \tag{6}$$

To see this in case $P \neq P^\sigma$, first remark that (1) and (2) imply that $x^i dx$ has a pole of order $i + 2$ at P and at P^σ (and no other poles), while $x^{g+i}dx/y$ has at these points poles of order $i + 1$. This gives one differential form with a single pole at P and P^σ and for any $n > 1$ two differential forms with a pole of order n at these points. Since the first set of forms is even with respect to σ and the other set is odd they are all independent (and independent from the holomorphic differentials). They are maximal independent, since having another independent form with poles only at P and P^σ would result in having a meromorphic differential form with a single pole, which contradicts the fact that the sum of the residues of a differential form over all its singular points is always 0. This leads to (6) in case $P \neq P^\sigma$, the proof for the case $P = P^\sigma$ is very similar.

On the group $\mathrm{Div}(\Gamma)$ one introduces the notion of *linear equivalence* by $D \sim_l D'$ iff $D - D' = (f)$ for some meromorphic function f on Γ and the class of D is written as $\{D\}$. The homomorphism deg descends to a homomorphism

$$\deg_l\colon \frac{\mathrm{Div}(\Gamma)}{\sim_l} \to \mathbb{Z}$$

and its kernel, $\ker \deg_l$, is called the *Jacobian* of Γ, $\mathrm{Jac}(\Gamma)$. In the present case of hyperelliptic curves there is a very explicit description of the linear equivalence relation as we state in the following lemma.

Lemma 1. *Let Γ be a hyperelliptic curve of genus g with involution σ and let $P \in \Gamma$ fixed. Then*
1) $D_1 + D_1^\sigma \sim_l D_2 + D_2^\sigma$ for any $D_1, D_2 \in \mathrm{Div}(\Gamma)$ of the same degree,
2) if $\sum_{i=1}^g P_i \sim_l \sum_{i=1}^g Q_i$, then $\sum_{i=1}^g P_i = \sum_{i=1}^g Q_i$ or $P_i = P_j^\sigma$ for some $i \neq j$,
3) if $\deg D = 0$ then $D \sim_l \sum_{i=1}^g (P_i - P)$ for some $P_i \in \Gamma$.

The notion of linear equivalence is natural from the basic relation between divisors and (holomorphic) line bundles on a smooth curve: if a divisor D has local defining functions $(f_\alpha)_{\alpha \in I}$ for some cover $(U_\alpha)_{\alpha \in I}$ of the curve, then the transition functions of a line bundle $[D]$ are given by f_α/f_β on $U_\alpha \cap U_\beta$, and it is a fundamental fact that the line bundle $[D]$ is determined by the (linear) equivalence class $\{D\}$; also every line bundle is the line bundle of a divisor. To a meromorphic section φ of $[D]$ there is associated its divisor (φ) and there exists a section φ for which $(\varphi) = D$; fixing such a section shows that $L(D)$ is isomorphic to the vector space of holomorphic sections of $[D]$, in particular these spaces have the same dimension.

Let the degree of a line bundle be defined as the degree of its corresponding divisor and denote for any $d \in \mathbb{Z}$ the set of all line bundles of degree d by $\mathrm{Pic}^d(\Gamma)$. Then it follows that for any $d \in \mathbb{Z}$, $\mathrm{Pic}^d(\Gamma)$ is isomorphic to $\mathrm{Jac}(\Gamma)$ via $\{D\} \mapsto [D + \mathcal{D}_d]$ where $\mathcal{D}_d$ is any fixed divisor of degree d. Except for $d = 0$ there is no canonical choice for $\mathcal{D}_d$; if however — as in the present paper — the curve has a marked point P then one is led to the natural choice $\mathcal{D}_d = dP$, used exclusively in the sequel.

2.2. The stratification

We now introduce a decomposition of $\mathrm{Jac}(\Gamma)$ with respect to an arbitrary fixed point P on the (hyperelliptic) curve Γ. Let $\mathcal{I}_g$ denote the set

$$\mathcal{I}_g = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid 0 \le m+n \le g\}$$

which we order by $(m,n) \le (m',n')$ iff $m \le m'$ and $n \le n'$. Then for $(m,n) \in \mathcal{I}_g$ we define a subset $\mathrm{Div}_{m,n}(\Gamma,P)$ of $\mathrm{Div}(\Gamma)$ by

$$\mathrm{Div}_{m,n}(\Gamma,P) = \left\{ \sum_{i=1}^{g-m-n} P_i + mP + nP^\sigma - gP \mid P_i \in \Gamma \setminus \{P, P^\sigma\} \text{ and } i \neq j \Rightarrow P_i \neq P_j^\sigma \right\};$$

the term gP is introduced here in order to make every element in $\mathrm{Div}_{m,n}(\Gamma,P)$ of degree 0. We denote

$$\mathrm{Div}_0(\Gamma,P) = \bigcup_{n=0}^{g} \bigcup_{m=0}^{g-n} \mathrm{Div}_{m,n}(\Gamma,P).$$

and show in the following lemma that $\pi\colon \ker\deg \to \ker_l$ restricts to a bijection $\pi\colon \mathrm{Div}_0(\Gamma,P) \to \mathrm{Jac}(\Gamma)$.

Lemma 2.

1) For any $(m,n) \in \mathcal{I}_g$ the restriction of π to $\mathrm{Div}_{m,n}(\Gamma,P)$ is injective.

2) If $P \neq P^\sigma$, then the subsets $\pi(\mathrm{Div}_{m,n}(\Gamma,P))$, $(m,n) \in \mathcal{I}_g$ are all disjoint.

3) If $P = P^\sigma$, then $\mathrm{Div}_{m+1,n}(\Gamma,P) = \mathrm{Div}_{m,n+1}(\Gamma,P)$ if $m+n+1 \le g$. In this case the $g+1$ subsets $\pi(\mathrm{Div}_{m,0}(\Gamma,P))$, $0 \le m \le g$ are all disjoint.

4) $\pi(\mathrm{Div}_0(\Gamma,P)) = \mathrm{Jac}(\Gamma)$.

Proof

Suppose that $P \neq P^\sigma$ and that we are given $D \in \mathrm{Div}_{m,n}(\Gamma,P)$ and $D' \in \mathrm{Div}_{k,l}(\Gamma,P)$, with say $m \ge k$. Then canceling k terms P it follows that we are asked for a meromorphic function f on Γ with at most g poles P_i, no two of which correspond under the hyperelliptic involution. Using (5) and the Riemann-Roch formula (3) the function f must be constant, hence $D = D'$. This proves *1)* and *2)*, and since the first part of *3)* is obvious, also *3)*.

To prove that $\pi(\mathrm{Div}_0(\Gamma,P)) = \mathrm{Jac}(\Gamma)$ we need to show that every divisor D of degree zero is linear equivalent to a divisor inside one of the sets $\mathrm{Div}_{m,n}(\Gamma,P)$. By Lemma 1, $D \sim_l \sum_{i=1}^{g}(P_i - P)$, for some points $P_i \in \Gamma$, but by the same lemma every occurence of $Q + Q^\sigma$ can be replaced by $P + P^\sigma$, so that eventually it must belong to one of the sets $\mathrm{Div}_{m,n}(\Gamma,P)$. ∎

We now prove that the sets $J_{m,n}(\Gamma,P) \stackrel{\text{def}}{=} \pi(\mathrm{Div}_{m,n}(\Gamma,P))$ (or $J_m(\Gamma,P) \stackrel{\text{def}}{=} \pi(\mathrm{Div}_{m,0}(\Gamma,P))$ in case $P = P^\sigma$) define a *stratification* of $\mathrm{Jac}(\Gamma)$, meaning that they are disjoint differentiable manifolds, whose boundary is a finite union of lower-dimensional sets $J_{s,t}(\Gamma,P)$ (resp. $J_s(\Gamma,P)$). To this aim we first need to explain the differential, or even complex, structure of $\mathrm{Jac}(\Gamma)$. It is one of the oldest and most profound results in the theory of algebraic curves that $\mathrm{Jac}(\Gamma)$ has the structure of a complex (algebraic) torus $\mathbb{C}^g/\Lambda$, where Λ is a lattice of maximal rank in $\mathbb{C}^g$. In fact, it was first defined as a complex torus and shown (by Abel) to correspond to the above definition. We sketch the construction of the analytical object. Choose a basis $A_1,\dots,A_g,B_1,\dots,B_g$ for $H_1(\Gamma,\mathbb{Z})$ such that the intersection indices between the cycles obey $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$. Let $\{\omega_1,\cdots,\omega_g\}$ be the normalised basis of holomorphic differentials for which $\int_{A_i}\omega_j = \delta_{ij}$. Then the $2g$ columns of the matrix $(I_g\, Z)$, where $Z_{ij} = \int_{B_i}\omega_j$, define a lattice Λ in $\mathbb{C}^g$, which turns out to be of maximal rank. The quotient $\mathbb{C}^g/\Lambda$ is a complex torus, which is up to isomorphism independent from the choice of basis for $H_1(\Gamma,\mathbb{Z})$. To link this torus with $\mathrm{Jac}(\Gamma)$ defined above, one introduces the Abel map $\mathcal{A}:\mathrm{Jac}(\Gamma)\to\mathbb{C}^g/\Lambda$ by

$$\mathcal{A}\left\{\sum_i (P_i - Q_i)\right\} = \left(\sum_i \int_{Q_i}^{P_i}\omega_1,\dots,\sum_i\int_{Q_i}^{P_i}\omega_g\right) \pmod{\Lambda}$$

and proves that it is a well-defined isomorphism (*Abel's Theorem*).

The subsets $J_{m,n}(\Gamma,P)$ and $J_m(\Gamma,P)$ introduced above can thus be seen as subsets of a complex torus under the Abel isomorphism and we will identify them with their image, writing $J_{m,n}(\Gamma,P)$ for $\mathcal{A}(J_{m,n}(\Gamma,P))$ since no confusion can arise. We show that they are submanifolds of the torus and fit together such that they define a stratification of it. We give separate theorems for the cases $P\neq P^\sigma$ and $P=P^\sigma$.

Theorem 3 *If $P\neq P^\sigma$ then $\mathrm{Jac}(\Gamma)$ is stratified by the $(g-m-n)$-dimensional submanifolds $J_{m,n}(\Gamma,P)$, whose closure is given by the (finite) union*

$$\bar{J}_{m,n}(\Gamma,P) = \bigcup_{(k,l)\geq(m,n)} J_{k,l}(\Gamma,P). \tag{7}$$

Each stratum $J_{m,n}(\Gamma,P)$ has two boundary components which are translates of each other by

$$\vec{e} = \mathcal{A}\{P^\sigma - P\} = \left(\int_P^{P^\sigma}\omega_1,\dots,\int_P^{P^\sigma}\omega_g\right) \pmod{\Lambda}.$$

More generally, all $i+1$ strata of dimension $g-i$ are translates of each other by $n\vec{e}$ for some $n\in\{1,\dots,i\}$. The closures of the $(g-1)$-dimensional strata $J_{1,0}(\Gamma,P)$ and $J_{0,1}(\Gamma,P)$ are translates of the theta divisor and are tangent along their intersection $\bar{J}_{1,1}(\Gamma,P)$.

Proof

We first show that each $J_{m,n}(\Gamma,P)$ is a submanifold of $\mathrm{Jac}(\Gamma)$ of dimension $g-m-n$. Let $d=g-m-n>0$ (otherwise there is nothing to prove) and consider the d-fold symmetric product of Γ with itself, denoted $\otimes_s^d\Gamma$. This space is known to have a (complex) differential structure, with coordinates which derive from coordinates on Γ. Namely, on a neighborhood of a generic point $\langle P_1,\dots,P_d\rangle\in\otimes_s^d\Gamma$ for which all P_i are distinct, the coordinates z_i centered at P_i serve as coordinates; when two or more of the P_i coincide however, their corresponding coordinates need to be replaced by the symmetric functions of these coordinates, for example, if $P_1=P_2$ then take z_1+z_2 and z_1z_2 instead of z_1 and z_2. It is clear that as a subset of the torus, $J_{m,n}(\Gamma,P)$ is given by the image of the (Abel map-like) map $\mathcal{A}_s$ defined by

$$\mathcal{A}_s\langle P_1,\dots,P_d\rangle = n\vec{e} + \left(\sum_{i=1}^d\int_P^{P_i}\omega_1,\dots,\sum_{i=1}^d\int_P^{P_i}\right) \pmod{\Lambda},$$

on the open set $\mathcal{U}_d\subset\otimes_s^d\Gamma$ for which all $P_i\notin\{P,P^\sigma\}$ and $i\neq j\Rightarrow P_i\neq P_j^\sigma$. Therefore it suffices to show that the Jacobian of this map is nowhere singular on $\mathcal{U}_d$. If the holomorphic differentials ω_i are written as $f(z_j)dz_j$ around P_j, then the Jacobian matrix of $\mathcal{A}_s$ has at the generic point $\langle P_1,\dots,P_g\rangle$ entries $f_i(P_j)$ and its rank is maximal since otherwise there would be at least a $(g-r+1)$-dimensional family of holomorphic differentials vanishing at the r points P_i in contradiction with (5) and the domain of $\mathcal{A}_s$. If some of the points P_i coincide we arrive at the same conclusion (including multiplicities): if, say, P_1 occurs n times then the the i-th column of the matrix is to be replaced by the $(i-1)$-th derivative of f_i, evaluated at P_j; then the rank being not maximal would mean that there is a $(g-r+1)$-dimensional family of holomorphic differentials vanishing n times at P_1 and vanishing simply at the other points, again in contradiction with (5).

We now compute the boundary $\bar{J}_{m,n}(\Gamma,P)$ of the strata $J_{m,n}(\Gamma,P)$. Since $\mathrm{Jac}(\Gamma)$ is given under the Abel isomorphism $\mathcal{A}$ the quotient topology coming from $\otimes_s^g\Gamma$, it is sufficient to compute the closure of each subset $J_{m,n}(\Gamma,P)$ for this topology (recall that we identified $J_{m,n}(\Gamma,P)$ with its image $\mathcal{A}(J_{m,n}(\Gamma,P))$). Let us define the set

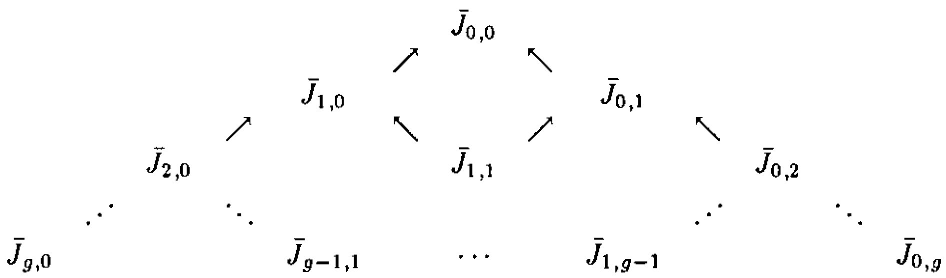
$$K_{m,n}(\Gamma,P)=\left\{\sum_{i=1}^{g-m-n}P_i+mP+nP^\sigma-gP \mid P_i\in\Gamma\right\},$$

which is compact since it is just $\otimes_s^{g-m-n}\Gamma$. By continuity of π, its image $\pi(K_{m,n}(\Gamma,P))$ is also compact, hence closed; obviously it is contained in $\bar{J}_{m,n}(\Gamma,P)$ hence $\bar{J}_{m,n}(\Gamma,P)=\pi(K_{m,n}(\Gamma,P))$; moreover

$$\pi(K_{m,n}(\Gamma,P))=\bigcup_{(k,l)\geq(m,n)}J_{k,l}(\Gamma,P).$$

which proves (7).

Thus the different strata fit together as dictated by the partial order $\leq$ on $\mathcal{I}_g$: if we represent the different spaces $\bar{J}_{m,n}(\Gamma,P)$ by $\bar{J}_{m,n}$, put those of equal dimension on the same horizontal line and depict inclusions by arrows, then we find the following.



Remark that the intersection of two spaces $\bar{J}_{m,n}(\Gamma,P)$ and $\bar{J}_{k,l}(\Gamma,P)$ is given by the set $\bar{J}_{s,t}(\Gamma,P)$ where (s,t) is the supremum of $\{(k,l),\geq(m,n)\}$ (if it exists, otherwise the intersection is empty). Therefore it is read off immediately from the diagram as follows: if say $m\leq k$, then draw on the diagram a diagonal line (of slope 1) through $\bar{J}_{m,n}$ and another one (of slope -1) through $\bar{J}_{k,l}$; then their intersection point (if any) corresponds to the intersection of these lines.

There is exactly one big stratum (i.e., a stratum of maximal dimension g) namely $J_{0,0}(\Gamma,P)$, and its boundary consists of two strata of codimension one, namely $\bar{J}_{1,0}(\Gamma,P)$ and $\bar{J}_{0,1}(\Gamma,P)$, and so on. Since

$$\mathrm{Div}_{m,n}(\Gamma,P)=\mathrm{Div}_{m-1,n+1}(\Gamma,P)+P-P^\sigma$$

for $m\geq 1, n\geq 0$ the sets $\bar{J}_{1,0}(\Gamma,P)$ and $\bar{J}_{0,1}(\Gamma,P)$ are translates of each other by $\vec{e}=\mathcal{A}\{P^\sigma-P\}$, namely $\bar{J}_{0,1}(\Gamma,P)=\bar{J}_{1,0}(\Gamma,P)+\vec{e}$, and it can be shown that they are translates of the theta divisor (see below). In general each stratum $J_{m,n}(\Gamma,P)$ (except the zero-dimensional ones) has two boundary components, $\bar{J}_{m+1,n}(\Gamma,P)$ and $\bar{J}_{m,n+1}(\Gamma,P)$, which are obviously also translates of each other by $\vec{e}$. Therefore the sets $\bar{J}_{m,n}(\Gamma,P)$ of the same dimension are all translates of each other by some integer multiple of $\vec{e}$, for example for the points $\bar{J}_{g,0}(\Gamma, P)$ and $\bar{J}_{0,g}(\Gamma, P)$ it follows immediately that $\bar{J}_{0,g}(\Gamma, P) = \bar{J}_{g,0}(\Gamma, P) + g\vec{e}$.

In [Gu] (Chapter 4, p. 143) explicit formulas are found for calculating the intersection of two translates of the theta divisor. These show that in general the intersection of two translates of the Riemann theta divisor is reducible and has two components. Since in our case $\bar{J}_{1,0}(\Gamma, P) \cap \bar{J}_{0,1}(\Gamma, P) = \bar{J}_{1,1}(\Gamma, P)$ is irreducible, these components coincide, hence $\bar{J}_{1,0}(\Gamma, P)$ and $\bar{J}_{0,1}(\Gamma, P)$ are tangent along $\bar{J}_{1,1}(\Gamma, P)$. ∎

The corresponding theorem for $P = P^\sigma$ is stated as follows and proven in the same way.

Theorem 4 *If $P = P^\sigma$ then* Jac(Γ) *is stratified by the $(g-m)$-dimensional subsets $J_m(\Gamma, P)$, whose closure is given by the (finite) union*

$$\bar{J}_m(\Gamma, P) = \bigcup_{k \geq m} \bar{J}_k(\Gamma, P).$$

and each stratum $\bar{J}_m(\Gamma, P)$ has just one boundary component. Here the stratification is simply depicted as

$$\bar{J}_g \to \bar{J}_{g-1} \to \bar{J}_{g-2} \to \cdots \to \bar{J}_1 \to \bar{J}_0$$

$\bar{J}_0 = \text{Jac}(\Gamma)$, $\bar{J}_1$ *is a translate of the theta divisor and $\bar{J}_g$ is the origin in* Jac(Γ).

∎

In Theorems 3 and 4 we claimed that $\bar{J}_{1,0}(\Gamma, P)$ and $\bar{J}_1$ were translates of the theta divisor; this is the divisor of the classical Riemann theta function for Jac(Γ), which is the entire function on $\mathbb{C}^g$ defined as

$$\theta(z) = \sum_{l \in \mathbb{Z}^g} e^{\pi i \langle l, Al \rangle} e^{2\pi i \langle l, z \rangle} \tag{8}$$

when the lattice Λ of $\text{Jac}(\Gamma) \cong \mathbb{C}^g/\Lambda$ is written as $(I_g A)$. Remark that although θ is only defined on $\mathbb{C}^g$, the theta divisor is well-defined as its zero locus on Jac(Γ). Riemann showed (see [M]) that there is a constant $\vec{\Delta} \in \mathbb{C}^g$ such that

$$\theta(Z) = 0 \iff \exists P_i \in \Gamma : Z = \mathcal{A}\left\{\sum_{i=1}^{g-1}(P_i - P)\right\} - \vec{\Delta} \pmod{\Lambda}. \tag{9}$$

The important condition in the right hand side is that the sum runs over $g-1$ points only. Formula (9) leads at once to the cited claims.

3. The Sato Grassmannian

We show in this section how the stratification from the preceeding section is induced by a natural stratification of the Sato Grassmannian via an extension of the Krichever map. In the first paragraph we recall from [SS], [SW] and [PS] the Sato Grassmannian, its stratification and the Krichever map, which relates the Grassmannian to algebraic curves. In the second paragraph, we introduce an extension of this map in the case of hyperelliptic curves and relate both stratifications. A coarser stratification of the Grassmannian is introduced in the last paragraph; it appears in a natural way when the K-P hierarchy is introduced on the Grassmannian.

3.1. The Grassmannian and its stratification

In this paragraph Γ denotes any smooth curve of genus g (i.e., Γ needs not to be hyperelliptic), with a marked point P on it. We also fix a small coordinate neighborhood $(s,\mathcal{U})$ centered at P, for which $s(\mathcal{U})$ is the unit disk in $\mathbb{C}$. Then the boundary $\partial\mathcal{U}$ is diffeomorphic to a circle and $L^2(\partial\mathcal{U},\mathbb{C})$ is a Hilbert space, with a base

$$\{\ldots,z^{-2},z^{-1},1,z,z^2,\ldots\},$$

where $z = s^{-1}$. The Hilbert space decomposes as $L^2(\partial\mathcal{U},\mathbb{C}) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ = \overline{\{1,z,z^2,\ldots\}} \text{ and } \mathcal{H}_- = \overline{\{z^{-1},z^{-2},\ldots\}},$$

(the closure is here the L^2-closure). Let Gr denote the set of all closed subspaces $W \subset L^2(\partial\mathcal{U},\mathbb{C})$ which have an algebraic base of the form $\{f_i\}_{i\in\mathbb{N}}$, with

$$f_i = \sum_{j=-\infty}^{s_i} c_k z^k \qquad 0 \neq c_{s_i} \in \mathbb{C},\ s_i < s_{i+1}, s_i = i \text{ for } i \text{ sufficiently large.} \tag{10}$$

We call Gr the *(Sato) Grassmannian* of $L^2(\partial\mathcal{U},\mathbb{C})$; it is a connected† Banach manifold, modelled on the Hilbert space of all Hilbert-Schmidt operators $\mathcal{H}_+ \to \mathcal{H}_-$. For f_i as in (10) we define its order to be s_i and we associate to W the (ordered) subset $S_W = \{s_0,s_1,s_2,\ldots\}$. We call such a subset of $\mathbb{Z}$ with $s_i < s_{i+1}$ and $s_i = i$ for i sufficiently large, a *sequence*. The set of all points in Gr which have as sequence S will be denoted by Σ_S,

$$\Sigma_S = \{W \in \text{Gr} \mid S_W = S\}.$$

We define a partial order on sequences by $S \leq S'$ if the entries s_i and s'_i of S and S' satisfy $s_i \geq s'_i$ for all $i \in \mathbb{N}$, and define the length $l(S)$ of a sequence S as the finite sum $l(S) = \sum_{i\geq 0}(i - s_i)$. Then $S \leq S'$ obviously implies $l(S) \leq l(S')$. Denoting by U_S the set

$$U_S = \left\{W \in \text{Gr} \mid \text{proj}\left(W \to \overline{\{z^i \mid i \in S\}}\right) \text{ is an isomorphism}\right\},$$

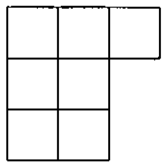
the stratification of Gr is described as follows (see [PS]).

Theorem 5 *For any sequence S, the set Σ_S is a closed subspace of U_S and the collection of all U_S forms an open cover of Gr. The big stratum is given by $\Sigma_{\mathbb{N}}$ and all Σ_S are smooth manifolds of codimension $l(S)$. The closure in Gr of each Σ_S is the union of the strata $\Sigma_{S'}$ for which $S' \geq S$.*

† by the last condition in (10) we singeled out the connected component containing $\mathcal{H}_+$ of what [PS] and [SS] call the Grassmannian

Sequences are in bijection with partitions. By a *partition* ν we mean a finite, non-increasing sequence of positive integers $\nu_0 \geq \nu_1 \geq \cdots \geq \nu_r \geq 0$. The bijection is simply given by $\nu_i = i - s_i$ and we see that $l(S) = \sum_{i=0}^{r} \nu_i$. The sequence corresponding to a partition ν will be denoted by S_ν. Also we define $l(\mu) = l(S_\mu)$ and $\mu \leq \nu$ iff $S_\mu \leq S_\nu$.

Partitions in turn are in bijection with Young diagrams, by which they are best visualised; a *Young diagram* is a finite (left aligned) arrangement of squares such that each row has at most as many squares as the preceeding row and the Young diagram corresponding to $\nu_0 \geq \nu_1 \geq \cdots \geq \nu_r \geq 0$ is given by drawing ν_i squares in the i-th row. Then the number of squares in a Young diagram (called its *weight*) equals the length of its partition. For example, if ν is the partition $3 \geq 2 \geq 2 \geq 0$ then $S_\nu = \{-3, -1, 0, 3, 4, \ldots\}$ and its Young diagram is drawn as follows.



We finally recall the *Krichever map*. The curve Γ, the point P and a local parameter s around P being fixed, there is associated to a line bundle $\mathcal{L} \in \mathrm{Pic}^g(\Gamma)$ and a trivialisation ϕ of $\mathcal{L}$ (say over a neighborhood $\mathcal{V}$ of the closure of the coordinate neighborhood $\mathcal{U}$ of s), a point $W(\mathcal{L}, \phi)$ in Gr as follows. Using ϕ we may think of sections of $\mathcal{L}$ over $\mathcal{V}$ as functions on $\mathcal{V}$, in particular such a section determines an element of $L^2(\partial\mathcal{U}, \mathbf{C})$. Then $W(\mathcal{L}, \phi)$ is defined as the closure of the set of all elements of $L^2(\partial\mathcal{U}, \mathbf{C})$ obtained in this way from meromorphic sections of $\mathcal{L}$ which are holomorphic away from P. Then the pole which the section has at P coincides with the order of the section at P and in particular is independent of the trivialisation ϕ. It follows that although $W(\mathcal{L}, \phi)$ depends on ϕ, the stratum of Gr it belongs to, is independent of ϕ. Therefore the Krichever map induces a decomposition (possibly a stratification) on $\mathrm{Pic}^g(\Gamma)$, hence also of $\mathrm{Jac}(\Gamma)$. We will generalise the Krichever map in the case that Γ is hyperelliptic to obtain a map which induces the stratifications on $\mathrm{Jac}(\Gamma)$ which we considered in the previous section.

3.2. Relating the stratifications

We now return to the case for which Γ is hyperelliptic, s a local parameter on a small neighborhood $\mathcal{U}$ of a fixed point P; the Grassmannian built using these data is just denoted by Gr. For a point $\{D\} \in \mathrm{Jac}(\Gamma)$, let $\mathcal{L}_+$ be the corresponding element in $\mathrm{Pic}^g(\Gamma)$ under our identification $\mathrm{Jac}(\Gamma) \overset{\otimes[gP]}{\longrightarrow} \mathrm{Pic}^g(\Gamma)$, i.e., $\mathcal{L}_+ = [D + gP]$ and let $\mathcal{L}_- = \mathcal{L}_+ \otimes [P - P^\sigma]$; also choose a trivialisation ϕ_+ of $\mathcal{L}_+$ over $\mathcal{U}$ and choose a trivialisation of $\mathcal{L}_-$ as $\phi_- = \phi_+ s$ if $P \neq P^\sigma$ and $\phi_- = \phi_+$ otherwise. Then we obtain two points $W_+(D) \overset{\text{not}}{=} W(\mathcal{L}_+, \phi_+)$ and $W_-(D) \overset{\text{not}}{=} W(\mathcal{L}_-, \phi_-)$, each belonging to a stratum which is independent of $\phi_\pm$. Thus, Γ, P and $(s, \mathcal{U})$ being fixed, there is associated to a point in $\mathrm{Jac}(\Gamma)$ and a trivialisation of its line bundle a point in Gr $\times$ Gr; if P is a Weierstraß point, then the image of this map is contained in the diagonal of Gr $\times$ Gr and we get the Krichever map; therefore we call our map an *extension* of the Krichever map. The two sequences of these strata will be denoted by $S_+(D)$ and $S_-(D)$, since they depend on D only. We will show that the stratification of $\mathrm{Jac}(\Gamma)$ with respect to P, as defined in Section 2 is induced from the product stratification on Gr $\times$ Gr via this map.

Proposition 6 *If* $\deg D = 0$ *then the sequences* $S_+(D)$ *and* $S_-(D)$ *are computed as follows:*

$$S_+(D) = \{n \in \mathbb{Z} \mid \dim L(D + (g+n)P) > \dim L(D + (g+n-1)P)\},$$
$$S_-(D) = \{n \in \mathbb{Z} \mid \dim L(D + (g+n+1)P - P^\sigma) > \dim L(D + (g+n)P - P^\sigma)\}.$$

Proof

Since $\deg D = 0$, $\{D\} \in J_{k,l}(\Gamma, P)$ for some $k, l \geq 0$, $k + l \leq g$. By Lemma 1, $\{D\}$ is written as $\{D_g - gP\}$ for a unique $D_g = \sum_{i=1}^{g-m-n} P_i + mP + nP^\sigma$ of degree g, with $P_i \in \Gamma \setminus \{P, P^\sigma\}$, no two P_i corresponding under σ. Let φ be a holomorphic section of $[D_g]$ for which $(\varphi) = D_g$. Then the map $f \to \varphi f$ determines an isomorphism between the meromorphic functions on Γ with (simple) poles on the points of D_g and an arbitrary pole at P on the one hand, and meromorphic sections of $[D_g]$, holomorphic away from P at the other hand. Consequently we will find a function in $W_+(D) = W([D + gP], \phi)$ of order n exactly when there exists a meromorphic function with poles on D_g and a pole of order n at P, i.e.,

$$n \in S_+(D) \text{ iff } \dim L(D_g + nP) > \dim L(D_g + (n-1)P), \tag{11}$$

which shows that $S_+(D)$ can be read off from the dimensions $\dim L(D_g + nP)$. The formula for $S_-(D)$ follows immediately from $S_-(D) = S_+(D + P - P^\sigma)$. ∎

The following lemma will give us neat formulas to compute the sequences $S_+(D)$ and $S_-(D)$.

Lemma 7. *Suppose there are given* $n \leq g$ *points* $P_1, \ldots, P_n \in \Gamma \setminus \{P, P^\sigma\}$ *such that* $i \neq j \Rightarrow P_i \neq P_j^\sigma$. *If* $P \neq P^\sigma$, *let* D *be a divisor of the form* $D = \sum_{i=1}^n P_i + kP + lP^\sigma$ $(k, l \in \mathbb{Z})$. *Then* $\dim L(D)$ *is given by*

$$\dim L(D) = \begin{cases} \max\{g - n - k - l - 1, 0\} + n + k + l + 1 - g & \text{for } k < 0 \text{ or } l < 0, \\ \max\{g - n - \max\{k, l\}, 0\} + n + k + l + 1 - g & \text{for } k, l \geq 0. \end{cases}$$

If alternatively $P = P^\sigma$, *then* $\dim L(D)$ *is given for any divisor of the form* $D = \sum_{i=1}^n P_i + kP$ $(k \in \mathbb{Z})$ *by*

$$\dim L(D) = \begin{cases} \max\{g - n - k - 1, 0\} + n + k + 1 - g & \text{for } k < 0, \\ \max\{g - n - \lceil k/2 \rceil, 0\} + n + k + 1 - g & \text{for } k \geq 0. \end{cases}$$

Proof

We first consider the case $P \neq P^\sigma$. Let $D = \sum_{i=1}^n P_i + kP + lP^\sigma$ as above and suppose that $k < 0$. Then by (6), $\dim \Omega(-kP) = g - k - 1$. If l is non-negative, then the divisor $\sum P_i + lP^\sigma$ is of the form $\sum_{i=1}^{n+l} Q_i$ where $i \neq j \Rightarrow Q_i \neq Q_j^\sigma$, which amounts to $n+l$ linearly independent conditions. If l is negative then by (6), $\dim \Omega(-kP - lP^\sigma) = g - k - l - 1$ and there are n linearly independent conditions coming from the points P_i $(i = 1, \ldots, n)$. It follows as in (5) that in both cases there are $g - n - k - l - 1$ independent differentials in $\Omega(-D)$ as long as this number is positive, otherwise there are no such differentials. By Riemann-Roch,

$$\begin{aligned} \dim L(D) &= \dim \Omega(-D) + n + k + l + 1 - g, \\ &= \max\{g - n - k - l - 1, 0\} + n + k + l + 1 - g, \end{aligned}$$

for $k < 0$. The case $l < 0$ is deduced from the above case by replacing D by D^σ.

It remains to prove the case $k, l \geq 0$. Then we look for holomorphic differentials with zeroes at n general points, with k zeroes at P and l zeroes at P^σ. These are $n + k + l$ conditions, but since $\min\{k, l\}$ of them are the same, we arrive at $n + k + l - \min\{k, l\} = n + \max\{k, l\}$ independent conditions. It follows from (5) that we end up with $g - n - \max\{k, l\}$ differentials, as long this number is positive, otherwise there are no such differentials. Using Riemann-Roch again, we conclude

$$\dim L(D) = \max\{g - n - \max\{k, l\}, 0\} + n + k + l + 1 - g$$

for $k, l \geq 0$. This completes the proof in case $P \neq P^\sigma$.

Suppose now $P = P^\sigma$ and let $D = \sum_{i=1}^n P_i + kP$. If $k < 0$ then it follows from (6) that $\dim \Omega(-kP) = g - k - 1$. The n points P_i impose n independent conditions on these differentials, giving $\dim \Omega(-\sum_{i=1}^n P_i - kP) = \max\{g - n - k - 1, 0\}$. Using Riemann-Roch we find

$$\dim L(D) = \max\{g - n - k - 1, 0\} + n + k + 1 - g,$$

for $k < 0$. If $k \geq 0$ then there are $g - \lceil k/2 \rceil$ holomorphic differentials in $\Omega(-kP)$ (as long as this number is positive), since in this case all the holomorphic differentials vanish to even order at P, as is seen from (1), (2) and (4). Therefore the dimension of $\Omega(-D)$ is given by $\max\{g - \lceil k/2 \rceil - n, 0\}$ and $L(D)$ is computed from the Riemann-Roch theorem as

$$\dim L(D) = \max\{g - \lceil k/2 \rceil - n, 0\} + n + k + 1 - g$$

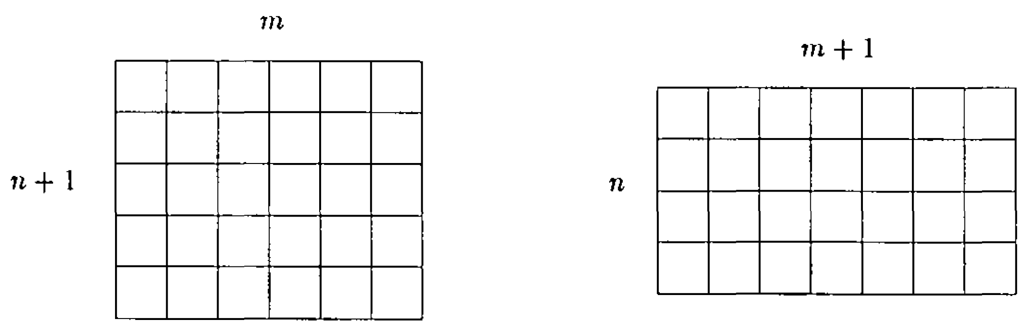
for $k \geq 0$. ∎

We combine Proposition 6 with the previous lemma to compute the sequences $S_+(D)$ and $S_-(D)$ and their Young diagrams. The basic relation between the stratifications of $\mathrm{Jac}(\Gamma)$ and $\mathrm{Gr} \times \mathrm{Gr}$ will follow immediately from it.

Theorem 8 *Suppose $P \neq P^\sigma$ and $\{D\} \in J_{m,n}(\Gamma, P)$. Then $S_+(D)$ and $S_-(D)$ are sequences which depend only on the stratum (i.e., on m and n) and are given by*

$$S_+(D) = \{-m, 1 - m, 2 - m, \ldots, n - m, n + 1, n + 2, n + 3, \ldots\},$$
$$S_-(D) = \{-m - 1, -m, 1 - m, \ldots, n - m - 2, n, n + 1, \ldots\}.$$

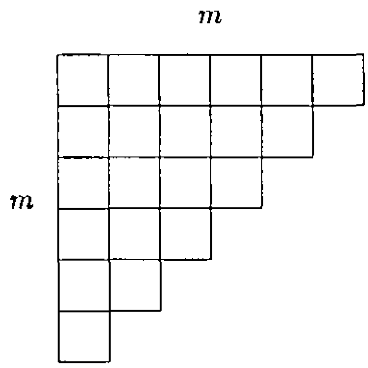
The corresponding Young diagrams are rectangles with m columns and $n + 1$ rows for $S_+(D)$ and $m + 1$ columns and n rows for $S_-(D)$, and their weights are simply given by $l(S_+(D)) = m(n + 1)$ and $l(S_-(D)) = n(m + 1)$. They look as follows.



Secondly, suppose that $P = P^\sigma$ and $\{D\} \in J_m(\Gamma, P)$. Then $S_+(D) = S_-(D)$ is a sequence which depends only on the stratum (i.e., on m) and is given by

$$S_+(D) = \{-m, 2-m, 4-m, \ldots, m-2, m, m+1, m+2, \ldots\}.$$

The corresponding Young diagram is a rotated stairs of height m, i.e., the first row has m squares and every other row has one square less then the preceeding row, hence it has weight $l(S_+(D)) = \frac{m(m+1)}{2}$ and is depicted as follows.



Proof

Suppose at first that $P \neq P^\sigma$. For $D \in \mathrm{Div}_{m,n}(\Gamma, P)$ let $D_g = D + gP$, then by Lemma 7,

$$\dim L(D_g + kP) = \max\{\min\{k+m, n\}, 0\} + 1 + k,$$

if $k + m \geq 0$, otherwise this dimension is zero. Since $S_+(D) = \{k \mid \dim L(D_g + kP) > \dim L(D_g + (k-1)P)\}$ we see that

$$S_+(D) = \{-m, 1-m, 2-m, \ldots, n-m, n+1, n+2, n+3, \ldots\}.$$

Also, since $S_-(D) = S_+(D + P - P^\sigma)$ and since $D + P - P^\sigma \in \mathrm{Div}_{m+1,n-1}(\Gamma, P)$ if $n \geq 1$, the formula for $S_-(D)$ is found in this case by substituting $m+1$ for m and $n-1$ for n in the formula for $S_+(D)$. The proposed formula above for $S_-(D)$ gives for $n = 0$, when properly interpreted, $S_-(D) = \mathbb{N}$. To see its validity, remark that in this case

$$D_g + P - P^\sigma = \sum_{i=1}^{g-m} P_i + mP + P - P^\sigma \sim_l \sum_{i=1}^{g} Q_i$$

for unique Q_i, all different from P, P^σ and no two of which correspond under the hyperelliptic involution (using Lemma 1 again), hence $S_-(D) = \mathbb{N}$. The proof for $P = P^\sigma$ goes exactly along the same lines. ∎

This theorem leads immediately to the main result of this section.

Theorem 9 *The natural stratification of* Jac(Γ) *given by the subsets $J_{m,n}(\Gamma, P)$, $(m,n) \in \mathcal{I}_g$, is induced by the (product) stratification on $Gr \times Gr$ given by the sets $\Sigma_S \times \Sigma_T$ (S, T sequences) via the "map"*

$$F: \mathrm{Jac}(\Gamma) \to Gr \times Gr$$
$$\{D\} \mapsto (W_+(D), W_-(D)).$$

Proof

From the previous theorem it follows that the strata $J_{m,n}(\Gamma, P)$ are mapped into strata of the stratified space Gr × Gr. Also it follows from this theorem that no two different strata $J_{m,n}(\Gamma, P)$ and $J_{m',n'}(\Gamma, P)$ are mapped in the same stratum. To prove this it suffices to show that the numbers $(m, n) \in \mathcal{I}_g$ can be reconstructed from $S_+(D)$ and $S_-(D)$ (or equivalently from their Young diagrams). If both Young diagrams are empty then $(m, n) = (0, 0)$. Otherwise m and n are found by counting rows and columns in one of the non-empty diagrams (remark that for $m = 0$ or $n = 0$ it is essential to have both diagrams). In the case $P = P^\sigma$ both Young diagrams are obviously the same (since $W_+(D) = W_-(D)$) and the theorem can be simplified using only the subsets $J_m(\Gamma, P)$ and the planes $W_+(D) \in$ Gr. ∎

3.3. The K-P hierarchy on Gr and another stratification

There is another stratification on Gr, (and on Gr × Gr) coarser than the previous one, which shows up when a certain natural vector field on Gr is considered. Its strata consist of those points in Gr for which the associated Young diagrams have a given weight. To see that it is also a stratification, remark that each stratum is a finite union of the strata of the original stratification, and the boundary of a stratum now consists of those strata whose Young diagram has more weight than the Young diagram of the given stratum; we call it the *coarser* stratification (on Gr as well as on Gr × Gr where again the product stratification is considered). The following proposition follows at once from Theorem 8.

Proposition 10 *The natural stratification of* Jac(Γ) *given by the subsets* $J_{m,n}(\Gamma, P)$ *is also induced by the coarser stratification on Gr × Gr via our extension of Krichever's map.*

Proof

Clearly we only need to prove that no two strata are mapped in the same stratum. If $P = P^\sigma$, then the stratum which corresponds to $J_m(\Gamma, P)$ has weight $\frac{m(m+1)}{2}$, which is different for all $m \in \mathbb{N}$. If $P \neq P^\sigma$, then we need to reconstruct m and n from $w_1 = m(n+1)$ and $w_2 = n(m+1)$. However, given w_1 and w_2 there are only two solutions to this, namely (m, n) and $(-n-1, -m-1)$, only one of which is positive. ∎

The group $\mathbb{C}^\infty$ acts on Gr in an obvious way by $W \mapsto e^{-t_n z^n} W$, $(t_n \in \mathbb{C})$, and its infinitimal action determines an infinite number of commuting vector fields $\partial/\partial t_n$ on Gr, called the *K-P hierarchy* (this hierarchy can be written down in many equivalent forms, see [DMKS], [SS] and [SW]). The point $e^{-\sum_{j=1}^\infty t_j z^j} W$ is denoted by W^t, in particular $W = W^0$. It leads to the so-called *tau function*, also introduced by Sato (see [SS] and [SW]), which is defined for a generic point $W \in$ Gr by

$$\tau_W(t) = \frac{\sigma(W^t)}{e^{-\sum_{j=1}^\infty t_j z^j} \sigma(W)} = \frac{\sigma(e^{-\sum_{j=1}^\infty t_j z^j} W)}{e^{-\sum_{j=1}^\infty t_j z^j} \sigma(W)}.$$

Here $\sigma(W)$ is a canonical global section of the dual Det* of the determinant bundle Det over Gr, which can be defined — with some care — as one defines the determinant bundle over a finite dimensional manifold. For a point for which $\sigma(W) = 0$, this section is replaced by another (non-vanishing) section of Det*. It is a fundamental fact that in the case $W = W(\mathcal{L}, \phi)$ as in the previous paragraph, one has $W^t(\mathcal{L}, \phi) = W(\mathcal{L} \otimes \zeta_t, \phi_t)$ where ζ_t is the line bundle defined by the transition function $e^{\sum_{j=1}^\infty t_j s^{-j}}$ on the overlap of $\mathcal{W} = \Gamma \setminus \{P\}$ and $\mathcal{U}$; moreover, $t \mapsto \zeta_t$ defines a surjective homomorphism (see [Sh]). It follows that $\mathbb{C}^\infty$ acts on the set $\mathrm{Pic}^g(\Gamma)$ by tensoring with ζ_t, hence the vector fields $\partial/\partial t_n$ give linear vector fields on any Jacobian Jac(Γ) under our identification with Pic$^g(\Gamma)$ by $\{D\} \leftrightarrow [D+gP]$.

We apply this to our case in which Γ is hyperelliptic, and we concentrate on the vector field $\partial/\partial t_1$. As before, s is a local parameter around $P \in \Gamma$. Consider the inclusion

$$\imath_P : \Gamma \to \mathrm{Jac}(\Gamma) : Q \mapsto \{Q-P\}.$$

Then $\partial/\partial t_1$, as a vector field on Jac(Γ) has the following property.

Proposition 11 *The first K-P vector field $\partial/\partial t_1$, considered as a vector field on* Jac(Γ), *is tangent to the curve $\imath_P(\Gamma)$ at the origin of* Jac(Γ).

Proof

Let $t = (t_1, 0, 0, \ldots)$ with t_1 small. The line bundle in Pic$^g(\Gamma)$ corresponding to the origin of Jac(Γ) is $\mathcal{L} = [gP]$, with transition functions $g_{\mathcal{U}\mathcal{W}} = s^g$ ($\mathcal{W} = \Gamma \setminus \{P\}$), hence $\mathcal{L}_t = [gP] \otimes \zeta_t$ has transition functions

$$g^t_{\mathcal{U}\mathcal{W}} = s^g \exp(-t_1/s) = s^{g-1}(s-t_1) + \mathcal{O}(t_1^2),$$

and since t_1 is small, the divisor corresponding to it (up to $\mathcal{O}(t_1^2)$) is $(g-1)P + P_{t_1}$, where P_{t_1} is the point in $\mathcal{U}$ for which $s = t_1$. As a point in the Jacobian this is the point $\{P_{t_1} - P\}$ on the embedded curve $\imath_P(\Gamma)$. Therefore, around P, $\imath_P(\Gamma)$ coincides with the integral curve (which is just a straight line in the torus) of $\partial/\partial t_1$ at least to first order, hence they are tangent. The components of this vector in the direction of the holomorphic differentials $x^k dx/y$, ($k = 0, \ldots, g-1$) are easily computed; take for example $P = P^\sigma$ then $x = s^{-2}$, $y = s^{-2g-1} + \mathcal{O}(s^{-2g})$ hence,

$$\lim_{t_1 \to 0} \frac{1}{t_1} \int_P^{P_{t_1}} \frac{x^k dx}{y} = -2 \lim_{s \to 0} \frac{1}{s} \int_0^s s^{2(g-k-1)}(1 + \mathcal{O}(s))ds = -2\delta_{k,g-1}. \tag{12}$$

∎

Of interest to us is also how the tau function, associated to $W \in$ Gr, vanishes in the t_1-direction. This is given by the following proposition, due to [SW].

Proposition 12 *For any $W \in Gr$,*

$$\tau_W(t_1, 0, 0, \ldots) = c t_1^l + \mathcal{O}(t_1^{l+1}),$$

where $c \neq 0$ and l is the codimension of the stratum of Gr containing W, i.e., it is the weight $l(S_W)$ of the Young diagram of W.

∎

Having associated two points $W_+(D)$ and $W_-(D)$ to a point $\{D\}$, we have also two corresponding tau functions $\tau_{W_+(D)}$ and $\tau_{W_-(D)}$. They relate to the theta function as follows.

Theorem 13 *Let A be the $g \times \infty$-matrix with entries A_{ij} defined by expanding the holomorphic differential forms ω_i in terms of s (around P), $\omega_i = \sum_{j=1}^{\infty} A_{ij} s^{j-1} ds$. Then for any divisor D of degree 0,*

$$\tau_{W_+(D)}(t) = \exp(Q(t))\theta\left(\vec{\Delta} - At - \mathcal{A}(D)\right),$$

$$\tau_{W_-(D)}(t) = \exp(Q(t))\theta\left(\vec{\Delta} + \vec{e} - At - \mathcal{A}(D)\right),$$

where $Q(t)$ is a quadratic form in t which is independent of t_1.

Proof

The proof is essentially due to Krichever (see [K]), who shows that if $\mathcal{L}$ is a line bundle of degree g, then

$$\tau_{W(\mathcal{L},\phi)}(t) = \exp(Q(t))\theta(At + Z(\mathcal{L})),$$

for some vector Z which depends "linear" on $\mathcal{L}$ in the sense that

$$Z(\mathcal{L} \otimes [D]) = Z(\mathcal{L}) + \mathcal{A}(D), \tag{13}$$

for any divisor D of degree 0 (see also [Sh]). We determine Z. By the preceeding proposition and Theorem 8, $\tau_{W_+(D)}(0) = 0$ iff $l(S_+(D)) \neq 0$ iff $\{D\} \notin J_{0,0}(\Gamma, P)$. At the other hand, by (9) (Riemann's theorem), $\theta(Z)$ vanishes for the points $\mathcal{A}(D) - \vec{\Delta}$ for which $\mathcal{A}(D) = \{D\} \notin J_{0,0}(\Gamma, P)$. Using (13), $Z(\mathcal{L}) = \mathcal{A}(D) - \vec{\Delta}$ for all D of degree 0, leading to the first formula. The second formula follows at once form the first one. ∎

4. The master systems

4.1. The master systems

Consider for a fixed hyperelliptic curve Γ (of genus g), $P \in \Gamma$ and s a local parameter around P the map

$$\phi_P: \Gamma \to \mathrm{Jac}(\Gamma): Q \mapsto \{Q - P\}.$$

Then $d\phi_P\left(\frac{\partial}{\partial s}\right)_{s=0}$ is a tangent vector at the origin of $\mathrm{Jac}(\Gamma)$, tangent to the embedded curve $\phi_P(\Gamma)$, and we have seen that it determines the unique holomorphic vector field on this torus, which coincides with the first K-P vector field, under the identification of $\mathrm{Jac}(\Gamma)$ with $\mathrm{Pic}^g(\Gamma)$, given by $\{D\} \leftrightarrow [D + gP]$. Natural coordinates can be picked for (an affine part of) $\mathrm{Jac}(\Gamma)$ in which the differential equations describing the vector field take a nice form. This was done by Mumford in case P is a Weierstaß point on Γ (see [M]), and by us in the opposite case (see [V]). The result can be written in a compact form as a so-called Lax pair

$$\frac{dA}{dt} = [A, B], \qquad A = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \tag{14}$$

where

$$u(x) = x^g + \sum_{i=1}^{g} u_i x^{g-i}, \quad v(x) = \sum_{i=1}^{g} v_i x^{g-i}, \quad w(x) = \sum_{i}^{g} w_i x^{g-i}.$$

The sum in $w(x)$ starts from -1 if P is a Weierstraß point and from -2 in the other case; in any case $w(x)$ is taken monic. As for the entry b in B, it is given by

$$b = x - 2u_1, \qquad \text{or} \qquad b = x^2 - 2u_1 x + 2u_1^2 - u_2 + w_0,$$

again according to whether P is, or is not, a Weierstraß point of Γ. In [V] we called the vector field (14) the *odd master system* in case $P = P^\sigma$ and the *even master system* otherwise.

The coefficients of $u(x)$, $v(x)$ and $w(x)$ are meromorphic functions on $\mathrm{Jac}(\Gamma)$, which serve as (a complete set of) coordinates for an affine part of $\mathrm{Jac}(\Gamma)$; for example the polynomial $u(x)$ associated to a generic[†] point $\{D\} = \{\sum_{i=1}^{g} P_i - gP\} \in \mathrm{Jac}(\Gamma)$, is just $u(x) = \prod(x - x(P_i))$, hence its coefficients are symmetric functions on the curve; also $v(x)$ is the unique polynomial of degree $g-1$ which records the y-values of the points P_i, i.e., $v(x(P_i)) = y(P_i)$ for $i = 1, \ldots, g$. It follows that $f(x) - v^2(x)$ is divisible by $u(x)$ and $w(x)$ is by definition the quotient. Remark that in particular an equation for the curve Γ is given by

$$y^2 = f(x) = u(x)w(x) + v^2(x) \tag{15}$$

and the coefficients of $u(x)w(x) + v^2(x)$ are constants. Also the points P and P^σ are points at infinity with respect to this equation. It is easy to deduce from this that the vector field (14) coincides with the vector field given by $d\phi_P\left(\frac{\partial}{\partial s}\right)_{|s=0}$, hence with the first K-P vector field, as we show now.

Proposition 14 *The vector field (14) which describes the master systems coincides with the first K-P vector field $\partial/\partial t_1$.*

[†] generic means here that the point lies in $J_{0,0}(\Gamma, P)$

Proof

Take a generic divisor $P_1+\cdots+P_g$, $P_i=(x_i,y_i)$ and let $u(x)$, $v(x)$ and $w(x)$ be its associated polynomials. Using (14),

$$y_i=v(x_i)=\frac{1}{2}\frac{du}{dt}(x_i)=-\frac{1}{2}\prod_{j\neq i}(x_i-x_j)\frac{dx_i}{dt},$$

hence

$$\sum_{i=1}^{g}\frac{x_i^k dx_i}{y_i}=-2\sum_{i=1}^{g}\frac{x_i^k dt}{\prod_{j\neq i}(x_i-x_j)}=-2\delta_{k,g-1}dt.$$

It follows that the vector field vanishes in the direction of $dx/y,\ldots,x^{g-2}dx/y$ and takes the value -2 for $x^{g-1}dx/y$ exactly as in (12). ∎

4.2. The Laurent solutions for the master systems

The differential equations describing a vector field such as (14) are known to posses families of Laurent solutions (see [AvM3]). We explain this by recalling the argument. Let Z be any point on $\mathrm{Jac}(\Gamma)$ and let us denote for simplicity the functions u_i, v_i and w_i by $z_1,\ldots,z_m$, ($m=3g+1$ or $m=3g+2$). If all functions z_i are holomorphic in this point then the solution $z_i(t)$ is obviously given by power series; therefore suppose that one or more functions z_i blow up at Z, say the blow-up locus of z_1 contains Z. We write the divisor of z_1 as

$$(z_1)=\sum_{i=1}^{k}n_iD_i-\sum_{i=1}^{l}m_iD_i',\qquad (m_i,n_i\in\mathbb{N}\setminus\{0\}),$$

where all D_i and D_i' are different and irreducible. Then Z belongs to one or more D_i', but may belong as well to some of the D_i. In any case, if we pick for each divisor a local defining function around Z, say f_i for D_i and g_i for D_i' (if Z does not belong to some divisor then the local defining function may be taken as the constant function 1), then z_1 is written around Z as

$$z_1=f\frac{f_1^{n_1}f_2^{n_2}\cdots f_k^{n_k}}{g_1^{m_1}g_2^{m_2}\cdots g_l^{m_l}}.$$

We may take linear coordinates $x_1=t,x_2,\ldots,x_n$ for the torus, and think of the local defining functions as being expressed in terms of these. If the t-axis is not contained in any of the divisors D_i or D_i' then all these functions can (again up to a non-vanishing holomorphic function) be written as a (Weierstrass) polynomial in t (by the Weierstrass Preparation Theorem) and we see that the zero or pole z_1 has in Z depends on the components of the divisor of z_1 to which Z belongs but also on the singularity these divisors have in Z (since then the first few terms in the series vanish) and on the contact the vector field d/dt has with these divisors (for the same reason). Proceeding in this way for all functions z_i we find a Laurent solution to the differential equations, which starts from Z. The case in which the t-axis is contained in the divisor of one of the functions corresponds to the exceptional case that both the torus is reducible and one of the functions blows up on a subtorus, a case which will not be encountered here.

The Laurent series organise themselves naturally in families as follows: for every z_i, fix an intersection of some divisors (contained in the divisor of poles of (z_i)), fix an order of singularity and an order of tangency of the vector field. On this set all z_i are written as Laurent series depending on a number of free parameters, equal to the dimension of this set (corresponding to the starting point of the series which can be chosen in it) and in a dense subset the order of pole each expansion experiences is fixed. The pole may however become less severe in an analytic subset, obtained from the intersection with one of the divisors on which z_i has a zero; in such a case the leading coefficient of the Laurent series must be (dependend on) a free parameter, so that it can in particular take the value 0. The different sets obtained in this way do *not* give a stratification of the torus in general; indeed, if, for example, z_1 and z_2 both have a pole on some smooth divisor and the intersection of these divisors is singular, then this singularity will not be seen by the Laurent series.

Finding all Laurent solutions in a direct way is in general a hard problem. At first it is not clear when looking at the differential equations where to start with the solution. For a given choice one needs to solve a non-linear system of algebraic equations for the leading term (which may be very difficult, especially in the present case where the number of variables is indefinite; here this number is $3g+1$ or $3g+2$); the presence of free parameters (giving information about the dimension of the corresponding subset) can in favorable cases be detected by computing the eigenvalues of a matrix, depending on these leading terms, but this is again very difficult when the number of variables, hence the size of the matrix, is indefinite. One also has to show convergence of all Laurent solutions and see how the different sets they correspond to are related (see [AvM3]).

Our method to find to Laurent solutions for the master systems does not use this scheme. Instead we combine Theorem 12 with the following theorem which expresses the symmetric functions u_i in terms of the Riemann theta function. The result is most easily expressed in terms of alternative symmetric functions U_i (on the curve, given by (15)), defined for a generic point $\{D\} = \{\sum_{i=1}^{g} P_i - gP\} \in \text{Jac}(\Gamma)$, as

$$U_i = U_i^D = \sum_{j=1}^{g} x^i(P_j) \qquad (i = 1, \ldots, g).$$

Remark that u_i is a weight homogeneous polynomial in $U_1, \ldots, U_i$ when U_k is given weight k. We also introduce the *Schur polynomials* $p_i(x)$, $x = (x_1, x_2, \ldots)$ defined by

$$\exp\left(\sum_{i=1}^{\infty} x_i \xi^i\right) = \sum_{i=0}^{\infty} p_i(x)\xi^i.$$

In order to simplify the notation we will abbreviate

$$\tilde{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right).$$

Theorem 15 *If $P = P^{\sigma}$ then the symmetric functions U_i are expressed in terms of the Riemann theta function by*

$$U_i^D = c_i - \sum_{j=0}^{2i-1} \frac{\partial}{\partial t_{2i-j}} p_j(\tilde{\partial})(\log\theta)(\vec{\Delta} - \mathcal{A}(D)), \qquad (c_i \in \mathbb{C}). \tag{16}$$

In particular, since the Schur polynomial $p_j(x)$ has degree j in x_1, the Laurent expansion in t_1 for U_i (and hence also for u_i) will have a leading behaviour which is not worse than t_1^{-2i}.

Alternatively, if $P \neq P^\sigma$ then the symmetric functions U_i are expressed in terms of the Riemann theta function by

$$U_i^D = c_i - \sum_{j=0}^{i-1} \frac{\partial}{\partial t_{i-j}} p_j(\tilde{\partial})(\log\theta)(\vec{\Delta} - \mathcal{A}(D)) - \frac{\partial}{\partial t_{i-j}} p_j(-\tilde{\partial})(\log\theta)(\vec{\Delta} - \mathcal{A}(D) + \vec{e}). \tag{17}$$

so that in this case any Laurent expansion in t_1 for U_i (and hence also for u_n) will have a leading behaviour which is not worse than t_1^{-i}.

Proof

The formulas (16) and (17) generalise analogous formulas that have been obtained by several methods for small n (see [D], [MvM]); our proof is a residue calculation as in [D].

The fundamental formula used here is that, if $Z = \mathcal{A}(P_1 + \cdots + P_g - gP)$ with $P_1 + \cdots + P_g$ a generic divisor on Γ, then

$$\theta(\mathcal{A}(Q-P) - Z + \vec{\Delta}) = 0 \text{ iff } Q \in \{P_1, \ldots, P_g\},$$

an easy consequence of (9) (Riemann's Theorem). We start with the case $P = P^\sigma$. Then it follows from this formula that U_i^D is given by

$$\begin{aligned} U_i^D &= c_i - \mathrm{Res}_{Q=P} x^i(Q) d\log\theta(\mathcal{A}(Q-P) - \mathcal{A}(D) + \vec{\Delta}), \\ &= c_i - \mathrm{Res}_{Q=P} x^i(Q) \sum_{l=1}^{g} \omega_l(Q) \left(\frac{\partial}{\partial z_l}\log\theta\right)\left(\mathcal{A}(Q-P) - \mathcal{A}(D) + \vec{\Delta}\right), \end{aligned} \tag{18}$$

for some $c_i \in \mathbb{C}$. As before, we expand ω_i and the components $\mathcal{A}_i$ of the Abel map for Q close to P, say $x(Q) = s^{-2}$ in terms of s,

$$\omega_i(Q) = \sum_{j=1}^{\infty} A_{ij} s^{j-1} ds \qquad \mathcal{A}_i(Q) = \sum_{j=1}^{\infty} \frac{1}{j} A_{ij} s^j ds.$$

We use Taylor's Theorem,

$$F(\vec{z} + \vec{h}) = \exp\left(\sum_{i=1}^{g} h_i \frac{\partial}{\partial z_i}\right) F(\vec{z}), \qquad (h \text{ small}),$$

for

$$F = \frac{\partial}{\partial z_l}(\log\theta),\ \vec{z} = \vec{\Delta} - \mathcal{A}(D),\ \vec{h} = \mathcal{A}(Q-P),\ Q \text{ near } P.$$

This gives

$$\begin{aligned} \left(\frac{\partial}{\partial z_l}\log\theta\right)\left(\mathcal{A}(Q-P) - \mathcal{A}(D) + \vec{\Delta}\right) &= \exp\left[\sum_{j=1}^{\infty}\left(\sum_{i=1}^{g} \frac{1}{j} A_{ij} \frac{\partial}{\partial z_i}\right) s^j\right] \left(\frac{\partial}{\partial z_l}\log\theta\right)\left(\vec{\Delta} - \mathcal{A}(D)\right), \\ &= \exp\left[\sum_{j=1}^{\infty} \frac{1}{j}\frac{\partial}{\partial t_j} s^j\right] \left(\frac{\partial}{\partial z_l}\log\theta\right)\left(\vec{\Delta} - \mathcal{A}(D)\right), \\ &= \sum_{j=0}^{\infty} s^j p_j(\tilde{\partial}) \left(\frac{\partial}{\partial z_l}\log\theta\right)\left(\vec{\Delta} - \mathcal{A}(D)\right). \end{aligned}$$

We have used that $\sum_{i=1}^{g} A_{ij}\frac{\partial}{\partial z_i} = \frac{\partial}{\partial t_j}$, which follows from $z = At + \zeta$ in Theorem 13. We have now expressed everything in terms of s and can compute the residue:

$$\begin{aligned}
U_i^D &= c_i - \operatorname{Res} s^{-2i} \sum_{j=0}^{\infty} p_j(\tilde{\partial}) s^j \left(\frac{\partial}{\partial z_l} \log\theta\right) \left(\vec{\Delta} - \mathcal{A}(D)\right) \sum_{k=1}^{\infty} A_{lk} s^{k-1} ds, \\
&= c_i - \operatorname{Res} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} s^{j+k-2i} p_j(\tilde{\partial}) \frac{\partial}{\partial t_k} (\log\theta) \left(\vec{\Delta} - \mathcal{A}(D)\right) \frac{ds}{s}, \\
&= c_i - \sum_{j=0}^{2i-1} \frac{\partial}{\partial t_{2i-j}} p_j(\tilde{\partial}) (\log\theta) \left(\vec{\Delta} - \mathcal{A}(P)\right).
\end{aligned}$$

The modifications for the case $P \neq P^\sigma$ are the following. In (18) there is an extra term corresponding to the residue in P^σ,

$$\operatorname{Res}_{Q'=P} x^i(Q') \sum_{l=1}^{g} \omega_l(Q') \left(\frac{\partial}{\partial z_l} (\log\theta)\right) \left(\mathcal{A}(Q' - P) - \mathcal{A}(D) + \vec{\Delta}\right).$$

Letting $Q^\sigma = Q'$ it is rewritten as a residue in P upon using $x(Q^\sigma) = x(Q)$ and $\omega(Q^\sigma) = -\omega(Q)$ for all holomorphic differentials ω (hence also $\mathcal{A}(Q^\sigma - P^\sigma) = -\mathcal{A}(Q - P)$), giving:

$$-\operatorname{Res}_{Q=P} x^i(Q) \sum_{l=1}^{g} \omega_l(Q) \left(\frac{\partial}{\partial z_l} \log\theta\right) \left(-\mathcal{A}(Q - P) - \mathcal{A}(D) + \vec{\Delta} + \vec{e}\right).$$

A second mayor difference with the case $P = P^\sigma$ is that now $x(Q) = s^{-1}$ in terms of the local parameter s. Taylor's Theorem gives the same result as above for the residue in P, while for the extra residue term we find

$$\left(\frac{\partial}{\partial z_l} \log\theta\right) \left(-\mathcal{A}(Q - P) - \mathcal{A}(D) + \vec{\Delta} + \vec{e}\right) = \sum_{j=0}^{\infty} s^j p_j(-\tilde{\partial}) \left(\frac{\partial}{\partial z_l} \log\theta\right) \left(\vec{\Delta} + \vec{e} - \mathcal{A}(D)\right),$$

so that finally both residue term are given by

$$\begin{aligned}
&- \operatorname{Res} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} s^{j+k-i} \left(\frac{\partial}{\partial t_k} p_j(\tilde{\partial}) (\log\theta) \left(\vec{\Delta} - \mathcal{A}(D)\right) - \frac{\partial}{\partial t_k} p_j(-\tilde{\partial}) (\log\theta) \left(\vec{\Delta} + \vec{e} - \mathcal{A}(D)\right)\right) \frac{ds}{s}, \\
&= c_i - \sum_{j=0}^{i-1} \left(\frac{\partial}{\partial t_{i-j}} p_j(\tilde{\partial}) (\log\theta) \left(\vec{\Delta} - \mathcal{A}(D)\right) - \frac{\partial}{\partial t_{i-j}} p_j(-\tilde{\partial}) (\log\theta) \left(\vec{\Delta} + \vec{e} - \mathcal{A}(D)\right)\right).
\end{aligned}$$

∎

The above theorem is very helpful to determine the Laurent solutions for the master systems. Since $t = t_1$, we may now make the ansatz

$$u_i = \frac{1}{t^{\rho(i)}} \sum_{j=1}^{\infty} u_{ij} t^j \tag{19}$$

where $\rho(i)$ is given by the theorem, namely $\rho(i) = 2i$ if $P = P^\sigma$ and $\rho(i) = i$ otherwise, and we will find all the Laurent solutions. We show that they lead indeed to the stratification of $\text{Jac}(\Gamma)$ which coincides with the one by the subsets $J_{m,n}(\Gamma, P)$. We give separate propositions for the cases $P = P^\sigma$ and $P \neq P^\sigma$.

Proposition 16 *For the odd master system there are $g+1$ families of Laurent solutions. The m-th family corresponds to the stratum $J_m(\Gamma, P)$ and the functions $u_1, \ldots, u_g$ blow up as*

$$\begin{aligned} u_i &= (-1)^i \frac{(2i-1)!!}{2^i i!} \frac{(m+i)!}{(m-i)!} \frac{1}{t^{2i}} + \mathcal{O}(t^{-2i+1}), \qquad &(i = 1, \ldots, m),\\ u_i &= \mathcal{O}(t^{-2i+1}), &(i = m+1, \ldots, g), \end{aligned} \tag{20}$$

In particular, the odd master system induces a stratification on $\text{Jac}(\Gamma)$ *which coincides with the stratification by the subsets $J_m(\Gamma, P)$.*

Proof

The equations (14) are written out in the case of the odd master system (corresponding to $P = P^\sigma$) as

$$\begin{aligned} \dot{u}(x) &= 2v(x),\\ \dot{v}(x) &= -w(x) + (x - 2u_1)u(x),\\ \dot{w}(x) &= -2(x - 2u_1)v(x), \end{aligned}$$

or just as a third order equation,

$$\dddot{u}_i(x) = 4\left(\dot{u}_{i+1} - 2u_1\dot{u}_i - \dot{u}_1 u_i\right), \qquad (i = 1, \ldots, g;\ u_{g+1} = 0). \tag{21}$$

Then the ansatz (19) leads to the recursion relation

$$a_{i+1} = \frac{2i+1}{i+1}\left[\frac{i(i+1)}{2} + a_1\right] a_i. \tag{22}$$

To solve this recursion relation, remark that if $a_i = 0$ then $a_{i+1} = 0$; since $a_i = 0$ for at least one $i \leq g+1$, we find that

$$a_1 = -\frac{1}{2}m(m+1) \tag{23}$$

for some $m \in \{0, \ldots, g\}$ which leads by induction immediately to the formula

$$a_i = (-1)^i \frac{(2i-1)!!}{2^i i!} \frac{(m+i)!}{(m-i)!}, \qquad (i = 1, \ldots, m),$$

and $a_{m+1} = \cdots = a_g = 0$, hence also to (20). The series for v_i and w_i follow immediately from it by differentiation, in particular they do not give rise to separate families of Laurent solutions.

We now show that the m-th solution corresponds to $J_m(\Gamma, P)$. Take $\{D\} \in J_m(\Gamma, P)$ and let $\{D^t\}$ be the integral curve of $d/dt = \partial/\partial t_1$ with $D^0 = D$. Denote by $u^{D^t}(x)$ and $U^{D^t}(x)$ the associated polynomials, as above. Since it follows from the definition of A that $At + \mathcal{A}(D) = \mathcal{A}(D^t)$, we may compute, using Theorems 15, 13 and Proposition 12 (in that order),

$$\begin{aligned}
u_1^{D^t} &= (\log\theta)^{\cdot\cdot}\left(\vec{\Delta} - \mathcal{A}(D^t)\right) - c_1, \\
&= (\log\theta)^{\cdot\cdot}\left(\vec{\Delta} - \mathcal{A}(D) - At\right) - c_1, \\
&= \left(\log\tau_{W_+(D)}\right)^{\cdot\cdot}(t) - c_1, \\
&= \frac{d^2}{dt^2}\log\left(ct^{l(S_+(D))} + \mathcal{O}(t^{l(S_+(D))+1})\right) - c_1, \qquad (c \neq 0), \\
&= -\frac{l(S_+(D))}{t^2} + \mathcal{O}(1).
\end{aligned}$$

If $\{D\} \in J_m(\Gamma, P)$, then we know from Theorem 8 that $l(S_+(D)) = \frac{m(m+1)}{2}$, so we find by (23) that the m-th stratum corresponds to J_m. ∎

We will now prove the equivalent result to Proposition 16 for the case of the even master system, i.e., for the case $P \neq P^\sigma$.

Proposition 17 *For the even master system there are $\frac{(g+1)(g+2)}{2}$ families of Laurent solutions one for each element of the set $\mathcal{I}_g$. The (m,n)-th family corresponds to the stratum $J_{m,n}(\Gamma, P)$ and the functions $u_1, \ldots, u_g$ blow up as*

$$\begin{aligned}
u_1 &= \frac{m-n}{t} + \mathcal{O}(1), \\
u_i &= \mathcal{O}(t^{-i}), \qquad (i = m+1, \ldots, g),
\end{aligned} \tag{24}$$

In particular, the even master system induces a stratification on $\mathrm{Jac}(\Gamma)$ *which coincides with the stratification by the subsets $J_{m,n}(\Gamma, P)$.*

Proof

The proof goes along the same lines as the proof of Proposition 16. However one finds using the ansatz in this case a recursion relation

$$a_{k+2} = \frac{2k+3}{k+2}a_1 a_{k+1} + \frac{k+1}{k+2}[(k+2)k - (3a_1^2 - 2a_2)]a_k,$$

which is solved at once for $g = 1, 2, 3, \ldots$, but seems to be very hard to be solved for general g. Therefore we compute as in the previous proposition for $\{D\} \in J_{m,n}(\Gamma, P)(\Gamma, P)$ with $(m,n) \in \mathcal{I}_g$:

$$\begin{aligned}
u_1^{D^t} &= (\log\theta)^{\cdot}\left(\vec{\Delta} - \mathcal{A}(D^t)\right) - (\log\theta)^{\cdot}\left(\vec{\Delta} - \mathcal{A}(D^t) + \vec{e}\right) - c_1, \\
&= \left(\log\tau_{W_+(D)}\right)^{\cdot}(t) - \left(\log\tau_{W_-(D)}\right)^{\cdot}(t) - c_1, \\
&= \frac{l(S_+(D)) - l(S_-(D))}{t} + \mathcal{O}(1), \\
&= \frac{m-n}{t} + \mathcal{O}(1).
\end{aligned}$$

The formula for the other u_i follows from Theorem 15. ∎

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