ALAG

(Abelian Lagrangian Algebraic Geometry)

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This is a by-product after a series of discussions during our staying at MPI, where the second author has presented his program for linking the Lagrangian and Algebraic geometries via the geometric quantization and the mirror symmetry and the first author tried to explain it in our seminar as far he has understood it himself. In fact, the present paper is just the first chapter of a book that probably will be published in a future. In this chapter we just fix some basic concepts, construct the holomorphic BPU-map, and give the simplest applications to the Legendrian knots in S^3.

Contents

Introduction

0.1. Classical mechanics. A natural mathematical framework for the classical Hamiltonian dynamics is presented by the Symplectic geometry (see [A]). A smooth (compact or not) C^∞-manifold M is called *symplectic*, if it is equipped with a *symplectic structure*, that is, with a *closed* 2-form ω that induces a skew symmetric isomorphism $\omega : TM \xrightarrow{\sim} T^*M$ between the tangent and cotangent bundles (in particular, M should be even dimensional). The symplectic structure provides the commutative algebra of smooth functions $C^\infty(M)$ with the Lie algebra structure given by the Poisson bracket $\{f_1, f_2\} \overset{\text{def}}{=} \omega(H_{f_1}, H_{f_2})$, where $H_f \overset{\text{def}}{=} \omega^{-1}(df)$ is *the Hamiltonian vector field* (or *the symplectic gradient*) of a smooth function $f \in C^\infty(M)$, that is, the unique vector field H_f on M satisfying $\omega(H_f, v) = df(v)$ for all vector fields v. The Hamiltonian flows defined by Hamiltonian vector fields satisfy the well known Hamilton equations and preserve the symplectic form ω.

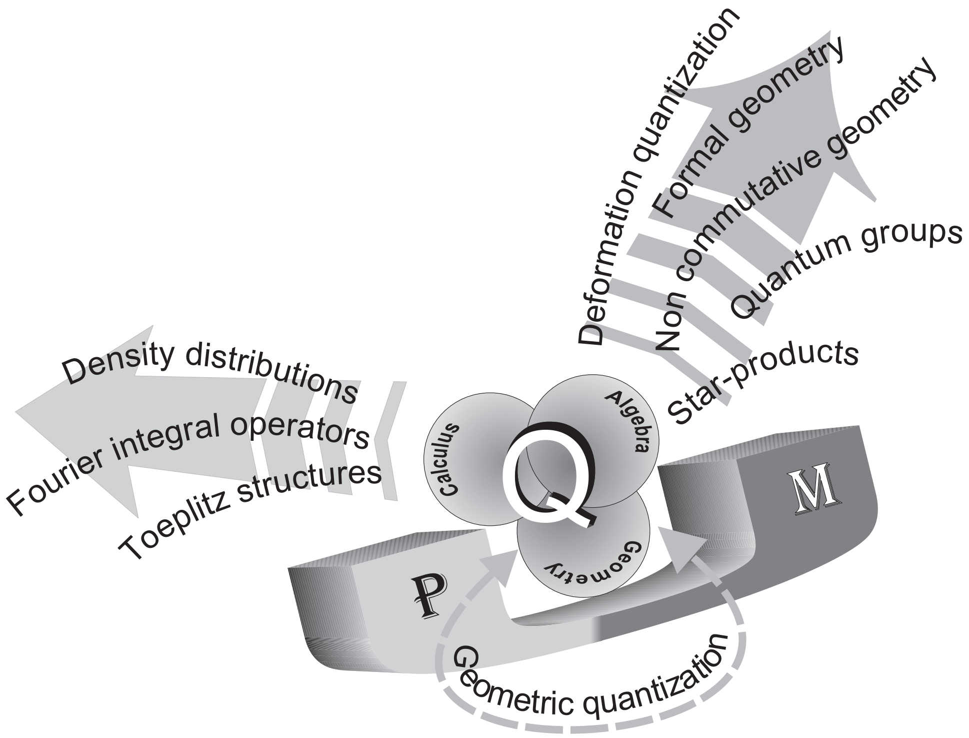

Fig. A. 'Quantum decay'.

0.2. Quantization dreams. Ideally, *a quantization* is a procedure Q assigning to each symplectic manifold (M, ω) a Hilbert space[1] $Q(M, \omega)$, and to each function f on M a self-adjoint operator $Q(M, \omega) \xrightarrow{Q(f)} Q(M, \omega)$ such that $Q(f)$ depends on f linearly, the constant functions are represented by the corresponding scalar homotheties, and also some 'irreducibility' and the *Dirac commutator condition* $[Q(f_1), Q(f_2)] = i\, Q(\{f_1, f_2\})$ hold. Unfortunately (or

[1] called the space of *wave functions*

fortunately, maybe), such a procedure can not exist by Van Hove's theorem (see [AM]). Mathematical attempts to budge this problem are concentrated in enlarging a classical area and, simultaneously, relaxing some requirements on a correspondence between the quantum and classical objects.

0.3. The big triple. There are several ways to move amendments to the above quantization rules. The 'quantum mathematics' looks today like a radioactive decay via *algebraic*, *analytic*, and *geometric* emanations shown on fig. A. The first two directions are developed especially wide and extensive. Say, algebraic quantization (deformation quantization, $*$-products, e. t. c.) sometimes completely leaves the original setup and forgets the classical phase space (M, ω). In a contrast, the analytical approach rests very close to the origin and produces a lot of special tricks to keep in mind the classical situation.

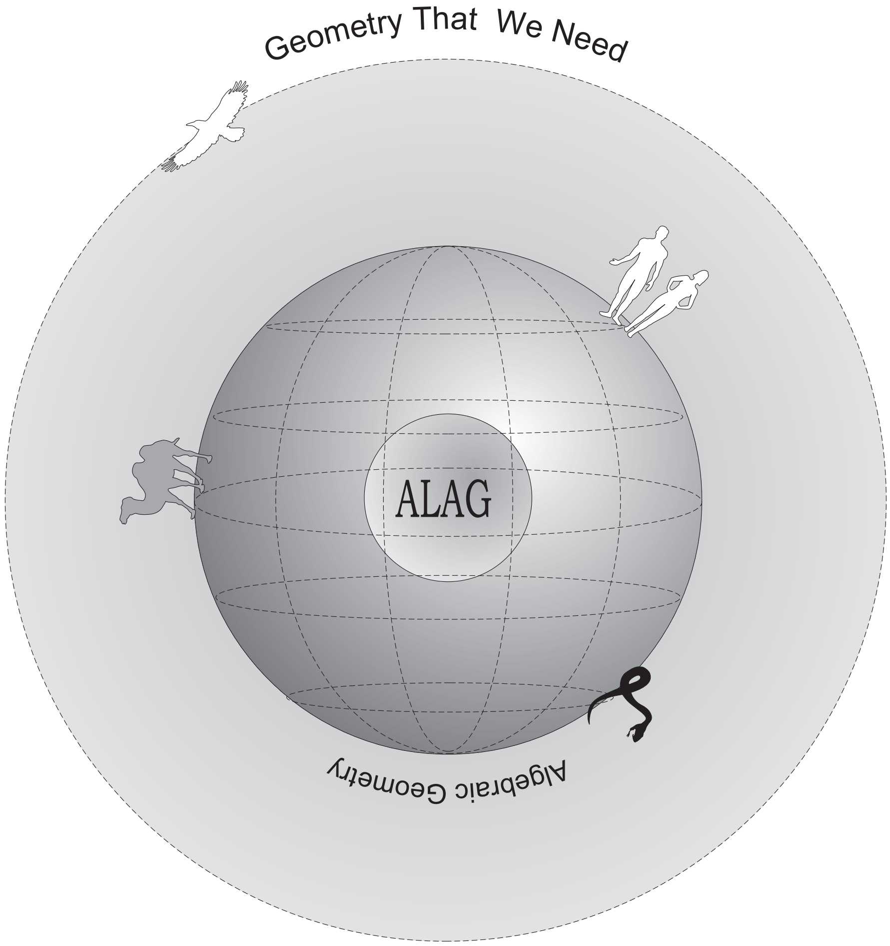

Fig. B. The Geometrical World.

Geometrically, most of modified quantization rules aim at a choice of a suitable function class that can be quantized in a consistent fashion. This choice is nothing but a choice of some auxiliary structure on the manifold. But although several (rather limited) uniqueness results are known in the framework of such the theories, much more interesting and important phenomenon is that we know also many special situations where the result of a quantization procedure does not depend on the choice of the auxiliary data to some extent. So, there is a heuristic fruitful lurking of a quantization scheme, which is more or less unique and *grosso modo* obeys the above rules. The main aim of this paper is to develop the Geometrical part of fig. A in order to tie in all three branches together. Such a fusion requires a new power geometry, which could be putted in the kernel of the geometrical picture of the world (see fig. B). We call it ALAG, that means a synthesis of the Algebraic and Lagrangian geometries. Let us describe briefly its basic ingredients and their interaction with different rays on the fig. A.

0.4. Deformation quantization. Let $A = C^\infty(M)[[q]]$ be an algebra of the formal power series in a variable q over $C^\infty(M)$. An associative $\mathbb{C}[[q]]$-linear product $*$ on A is called *a star product*, if $\forall\, f_1, f_2 \in C^\infty(M)$

$$f_1 * f_2 \equiv f_1 \cdot f_2 \ (\operatorname{mod} q) \tag{0-1}$$

$$\frac{1}{q}\,(f_1 * f_2 - f_2 * f_1) \equiv -i\,\{f_1, f_2\} \ (\operatorname{mod} q) \tag{0-2}$$

The first condition means that $A/qA = C^\infty(M)$ and we can write

$$f_1 * f_2 = f_1 \cdot f_2 + \sum_{i=1}^{\infty} C_i(f_1, f_2) q^i\,,$$

for some $C_i(f_1, f_2) \in C^\infty(M)$ that are $\mathbb{C}$-linear in f_j and satisfy the equation

$$C_1(f_1, f_2) - C_1(f_2, f_1) = -i\{f_1, f_2\}\,, \tag{0-3}$$

because of the second condition (0-2). The algebra A equipped with a star product, that is, with a collection $\{C_i(f_1, f_2)\}$ satisfying (0-3), is called *a deformation quantization* of $C^\infty(M)$.

The concept of deformation quantization has a long story ... About twenty years ago Berezin, Bayen, Flato, Frosdal, Lichnerowicz and Sternheimer consider the quantization procedure as a deformation of the usual commutative product of classical observables into a new non commutative associative product $*_h$ that depend on the Planck constant h and satisfies the requirements listed in n° 0.2 (see [BFFLS]). Subsequently, the theory flows into the pure algebraic framework combining the Hochschild cohomology theory for the associative algebra deformations and the Chevalley - Eilenberg cohomology theory for the Poisson - Lie algebra of classical observables. The problem of existence and classification of $*$-products on a symplectic manifold was solved by Fedosov and De Wilde - Lecomte (see survey [De]). The same problem for any Poisson manifold was solved by M. Kontsevich who presents an universal formula for the deformation quantization of an arbitrary Poisson structure on $\mathbb{R}^n$. His formula expresses a $*$-product as a power series in h in terms of bidifferential operators parametrized by graphs. Actually, it gives a perturbation expansion for a 2-dimensional field theory[1] on a disc with boundary (see [CF]) and the coefficients in h are given by integrals over suitable configuration spaces.

However, all these results (and their proofs) do not exceed the bounds of the formal algebra, that is, do not yield Hilbert spaces and/or allow to reconstruct either manifolds or auxiliary geometric structures. Only in the Kählerian case there is known Karabegov's result on a deformation quantization 'with the separation of variables' where the algebraic machinery respects a complex structure on a manifold.

[1] in the topological sigma-model of B type

So, if we do not want to kill the geometry or give rise a 'non commutative' one, we have to return to the starting point.

0.5. Geometric Quantization. The original area containing the both classic and quasi-classic systems is the Symplectic geometry. The main heroes here are the *Lagrangian submanifolds* (see the 'symplectic creed' in [We1, p. 5]). Moreover, the symplectic geometry on a given (M,ω) may be treated as a 'category' with the Lagrangian submanifolds as the morphisms (see [We1] again) and this viewpoint is very closed to the motivic ideology in the algebraic geometry. On the other hand, Boutet de Monvel and Guillemin has developed a quantization procedure for the contact manifolds. It leads to a remarkable unification of the spectral theory of the Fourier integral operators with the algebraic-geometric theory of the Hilbert polynomials (comp. with n° 4.4.2 below). So, the advanced 'symplectic' developments *a posteriori* come naturally into some classical algebraic-geometric framework.

0.6. From Symplectic to Algebraic. The straightforward way for sending the Symplectic geometry into the Algebraic one is *a Kähler polarization*, which equips M with an integrable complex structure I compatible with ω, that is, converts M to a Kähler manifold M_I with the Kähler form ω. If we assume, in addition, that the cohomology class $[\omega]$ is integer, then there exists a complex line bundle L on M with $c_1(L)=[\omega]$. Moreover, this bundle admits a Hermitian metric h and a compatible Hermitian connection a with the curvature $2\pi i\,\omega$. Since the Hodge type of the Kähler form ω is (1,1), we get a natural holomorphic structure on L induced by h and a. So, M_I turns to an algebraic variety polarized by L. Now, the Kodaira embedding theorem implies that $L^{\otimes k}$ is *very ample* for $k \gg 0$, that is, a natural map into a projective space:

$$\varphi_k : M_I \xhookrightarrow{x\mapsto Ann\,(x)} \mathbb{P}H^0(M_I, L^{\otimes k})^* \,. \tag{0-4}$$

is a holomorphic embedding. So, we are inside the Projective Algebraic Geometry.

On the other side, $H^0(M_I,L^{\otimes k}) \subset C^\infty(M,L^{\otimes k})$ is a subspace in the space of all smooth sections. $C^\infty(M,L^{\otimes k})$ is normed by the Liouville volume form $\Omega = \omega^{\wedge n}$ on M and the Hermitian form $(\,*\,,\,*\,)_{L^{\otimes k}}$ on $L^{\otimes k}$ via

$$\langle\, s_1\,,\, s_2\,\rangle_k \overset{\text{def}}{=} \int_M (s_1,s_2)_{L^{\otimes k}}\,\Omega \quad , \qquad \|s\|_k \overset{\text{def}}{=} \sqrt{\langle\, s\,,\, s\,\rangle_k}\;,$$

where $s, s_1, s_2 \in C^\infty(M,L^{\otimes k})$. Let us write $L^2(M,L^{\otimes k})$ for the completion of $C^\infty(M,L^{\otimes k})$ w. r. t. this norm. So, we can consider $H^0(M_I,L^{\otimes k}) \subset L^2(M,L^{\otimes k})$ as the Hilbert subspace.

0.6.1. Quantization of phase space. In the Kähler polarization framework, the Hilbert space of the global holomorphic sections

$$Q_k(M_I) \overset{\text{def}}{=} H^0(M_I,L^{\otimes k}) \subset L^2(M,L^{\otimes k})$$

is called *a wave function space* of level k. The orthogonal projector

$$L^2(M,L^{\otimes k}) \xrightarrow{\Pi_k} H^0(M_I,L^{\otimes k})$$

is called *a Szegö projector*. The holomorphic embedding (0-4) composed with the Hermitian isomorphisms $H^0(M_I,L^{\otimes k})^* \xrightarrow{\sim} \overline{H^0(M_I,L^{\otimes k})}$ gives an (antiholomorphic) embedding

$$M_I \xhookrightarrow{\overline{\varphi}_k} \mathbb{P}H^0(M_I,L^{\otimes k}) \,. \tag{0-5}$$

0.6.2. Quantization of functions. Each $f \in C^\infty(M)$ gives *a Toeplitz operator*:

$$H^0(M_I, L^{\otimes k}) \xrightarrow{T_f^{(k)}} H^0(M_I, L^{\otimes k}) \ : \quad s \longmapsto \Pi_k(f \cdot s) \ .$$

So, we get *a Berezin - Toeplitz map* $C^\infty(M) \xrightarrow{T_k} End\big(H^0(M_I, L^{\otimes k})\big)$ sending $f \longmapsto T_f^{(k)}$.

0.6.3. Quasi-classic limit. The Berezin - Toeplitz map is just *a linear map* from a vector space underlying the (commutative) algebra of functions to the one underlying the (non commutative) matrix algebra, because it preserves neither Lie nor associative algebra structures. However, the whole infinite family $\{T^{(k)}\}_{k\geq 0}$ 'approximates' the algebra $C^\infty(M)$ in the following sense. Consider the supremum norm $\|f\|_\infty$ on $C^\infty(M)$ and the corresponding operator norm on $End\left(\, H^0(M_I, L^{\otimes k})\,\right)$:

$$\left\| T_f^{(k)} \right\| = \sup_{s \neq 0} \frac{\left\| T_f^{(k)}(s) \right\|}{\|s\|} \ .$$

Then the Bordemann - Meinrenken - Schlichenmaier theorem (see [BMS]) says that the Berezin - Toeplitz maps have nice asymptotics, that is, for every $f \in C^\infty(M)$ there exists a constant $K_f > 0$ such that

$$\|f\|_\infty + \frac{K_f}{k} \leq \left\| T_f^{(k)} \right\| \leq \|f\|_\infty$$

and for every pair $f_1, f_2 \in C^\infty(M)$ we have the following asymptotics as $k \to \infty$

$$\left\| \, k \cdot i \, [\, T_{f_1}^{(k)} \, , \, T_{f_2}^{(k)} \,] - T_{\{f_1, f_2\}}^{(k)} \, \right\| \ \sim \ \left\| \, T_{f_1}^{(k)} \cdot T_{f_2}^{(k)} - T_{f_1 \cdot f_2}^{(k)} \, \right\| \ \sim \ O(1/k) \ .$$

In other words, although the Berezin - Toeplitz maps are not faithful on any finite level k, they tend to be that as $k \to \infty$. This property is usually referred as *a good quasi-classical behavior*.

Returning to the deformation quantization, Schlichenmaier has proved that there exists a *unique* star product $\{C_i(f_1, f_2)\}$ on $C^\infty(M)$ such that for every pair functions f_1, f_2 and every integer $N > 0$ there exists a constant $K = K(f_1, f_2, N)$ satisfying

$$\left\| T_{f_1}^{(k)} \cdot T_{f_2}^{(k)} - \sum_{0 \leq i \leq N} \frac{T_{C_i(f_1, f_2)}^{(k)}}{k^i} \right\| = \frac{K}{k^N} \ . \tag{0-6}$$

So, the Berezin - Toeplitz maps provide a unique strict quantization procedure with the good quasi-classical behavior.

0.7. ALAG. The crucial algebraic-geometric fact observed in many examples of a geometric quantization output is the following: although the wave function spaces $Q_k(M_I)$ do depend *a priori* on a choice of the complex structure I on M, their projectivizations $\mathbb{P}\left(\, Q_k(M_I) \,\right)$ do not, actually. More precisely, there is a holomorphic vector bundle $Q_k \xrightarrow{\pi_k} \mathcal{M}$ over the moduli space $\mathcal{M}$, of polarized complex structures on M, such that its fiber over $I \in \mathcal{M}$ equals $Q_k(M_I)$ and there is a natural *projective flat connection* (see [Hi1]) on this bundle.

This phenomenon has a very simple geometrical reason: certain finite codimensional submanifolds in the space of lagrangian cycles admit some internal[1] additional structures that

[1]that is, induced only by the symplectic structure on M

make them 'polarized algebraic varieties' and put into Algebraic Geometry. These algebraic-geometric structures are *universal* in the sense that an Algebraic Geometry of any Kähler quantization procedure is naturally induced by them. On the other side, these infinite dimensional algebraic moduli spaces, of Lagrangian cycles with some extra data, can be handled by the standard algebraic-geometric tools. We hope to explain in this paper how it works actually.

0.7.1. The text below is organized as follows. In the first two paragraphs we fix some (mostly well known) basic concepts from symplectic and differential geometry: moduli of Lagrangian cycles, Darboux - Weinstein uniformization near a smooth cycle, interaction between symplectic and almost Hermitean structures, Abelian connections and characters of π_1, Jacobians e. t. c. The third paragraph introduces the special submanifolds and extra structures coming from the moduli space of Lagrangian cycles: Bohr - Sommerfeld cycles, Planckian cycles, Berry bundles, half densities e. t. c. We construct here the holomorphic structures on the spaces of half weighted Planckian cycles $\mathcal{P}^{\mathrm{hw}}$ and half weighted Bohr - Sommerfeld cycles of fixed volume $\mathcal{B}_t^{\mathrm{hw}}$. In § 4 we give a detailed explanation of what is the Kähler quantization procedure and construct the holomorphic BPU-maps $\mathcal{B}_{k,t}^{\mathrm{hw}} \longrightarrow \mathbb{P}\left(H^0(L^{\otimes k})^*\right)$. This is the central point of the present paper. The last § 5 illustrates our construction in the simplest possible example, where the initial symplectic system (M,ω) is the Rimannian sphere $S^2 \simeq \mathbb{CP}_1$ equipped with the standard Kähler structure and the Lagrangian cycles are just the immersed circles. The half weighted Planckian cycles space $\mathcal{P}^{\mathrm{hw}}$ turns here to the space of parametrized knots in S^3, which is fibered over S^2 by Hopf. So, although this case is 'almost trivial' with the ALAG-viewpoint, it gives the reasonable geometric envelop for the whole of the Legendrian knots theory, which is so popular now, because of plays especially important role in many CFT-questions.

0.8. Acknowledgments. We would like to express our gratitude to the Max-Planck-Institute für Mathematik in Bonn for the hospitality and very nice atmosphere for a good work.

§ 1. Lagrangian preliminaries.

1.1. Symplectic manifold. Let us fix the notation (M,ω) for a smooth symplectic $2n$-dimensional manifold M with the symplectic structure presented by a smooth closed 2-form ω giving a skew isomorphism

$$\omega : TM \xrightarrow{\sim} TM^* \tag{1-1}$$

between tangent and cotangent bundles on M. Such a manifold M may be considered as a phase space for a classical mechanical system like in [A].

1.2. Lagrangian cycles. Let us fix a smooth compact oriented n-dimensional C^∞-manifold S and also some homotopy class $Map_{\mathrm{LA}}(S,M)$, of smooth maps $S \xrightarrow{\varphi} M$ such that $\varphi^*(\omega) = 0$. Such the maps are called *Lagrangian.* They form an infinite dimensional space with a natural action of the identity component of the diffeomorphism group $Diff_0(S)$ via $h : \varphi \mapsto \varphi \circ h^{-1}$, where $h \in Diff_0(S)$, $\varphi \in Map_{\mathrm{LA}}(S,M)$. The orbit space $\mathfrak{L} \stackrel{\mathrm{def}}{=} Map_{\mathrm{LA}}(S,M)/Diff_0(S)$ is called *a space of Lagrangian cycles* (of a given topological type).

A Lagrangian cycle $S \xrightarrow{f} M$ is called *smooth* (resp. *smoothly immersed*), if φ is a smooth embedding (resp. a smooth immersion). In the first case, $f(S) \subset M$ is called *a smooth Lagrangian submanifold* of M and the condition $\varphi^*\omega = 0$ means that ω restricts to zero on TS. But in the both cases, the symplectic structure isomorphism (1-1) identifies the normal bundle $(N_{M/S}) = \varphi^*(TM)/TS$ with the cotangent bundle $T^*S \simeq \varphi^*(T^*M)/Ann\,(TS)$. To simplify the notations, we will often write S for $\varphi(S)$ as well as soon the mistakes will be impossible (and for embedded or immersed cycles only).

1.2.1. Methodological remark. We will not investigate any details connected with the topology of $\mathfrak{L}$, its singularities, stability conditions, and so on. We use the quotient notation $Map_{\text{LA}}(S,M)/Diff_0(S)$ rather as a very convenient tool to activate the geometrical intuition. All our 'global' constructions always can be formulated in terms of some $Diff_0(S)$ - invariant objects on $Map_{\text{LA}}(S,M)$. So, we treat $\mathfrak{L}$ rather in a framework of the invariant theory and, after S. Donaldson, interpret special structures on the space of Lagrangian cycles in terms of moment maps and symplectic and/or Kähler reduction.

On the other hand, each smooth Lagrangian cycle gives a smooth point on $\mathfrak{L}$ together with a standard open neighborhood that can be modelled by the following very tame example.

1.3. Regular local example: the cotangent bundle. Let M be the total space of the cotangent bundle $T^*S \xrightarrow{p} S$ on a smooth manifold S. Then there is a tautological 1-form η on M. It is called *an action form* and sends a tangent vector $w \in T_{(\xi,x)}T^*S$, where $x \in S$, $\xi \in T^*_xS$, to

$$\eta(w) \stackrel{\text{def}}{=} p^*\xi(w) \;. \tag{1-2}$$

The differential of the action form $\omega \stackrel{\text{def}}{=} d\eta$ gives a canonical symplectic structure on M. If we take some local coordinates $(q_1, q_2, \dots, q_n)$ on a small ball $U \subset S$ and coordinate $T^*S|_U$ by $(p_1, p_2, \dots, p_n\,,\; q_1, q_2, \dots, q_n)$, where p_i are linear coordinates along $\partial/\partial q_i$ in the fibers of the cotangent bundle, then

$$\eta(p,q) = \sum_{j=1}^{n} p_j dq_j \quad , \qquad \omega(p,q) = \sum_{j=1}^{n} dp_j \wedge dq_j \;. \tag{1-3}$$

So, such coordinates are the symplectic ones. They are called *canonical*. One checks immediately that any smooth q-coordinates change on S induces a symplectic (p,q)-coordinates change on M.

Clearly, all fibres of the canonical projection $M \twoheadrightarrow S$ and also the zero section $S \xhookrightarrow{s_0} M$ are Lagrangian submanifolds. It is not difficult to describe all Lagrangian sections $S \xhookrightarrow{s} M$. Namely, any section is presented by a differential form $\alpha_s = s^*(\eta)$, such that

$$s(x) = (x, \alpha_s(x)) \in M = T^*S \;.$$

1.3.1. PROPOSITION. *A section $S \xhookrightarrow{s} T^*S$ is Lagrangian iff the corresponding form $\alpha_s \in C^\infty(X, T^*S)$ is closed.*

PROOF. $X \xrightarrow{s} M$ is Lagrangian iff $0 = s^*(\omega) = s^*(d\eta) = ds^*(\eta) = d\alpha_s$. □

In fact, the zero section $S \xhookrightarrow{s_0} M$ gives the universal local model for a smooth Lagrangian cycle because of the famous Darboux - Weinstein theorem discussed in the next section.

1.4. Local structure of $\mathfrak{L}$ near a smooth cycle. Let $S \xrightarrow{\varphi} M$ be a smooth Lagrangian immersion. The tangent space $T_\varphi Map(S,M)$ to the space of all smooth maps at a given map $S \xrightarrow{\varphi} M$ coincides with the space $C^\infty(S, \varphi^*TM)$ of smooth sections of the pulled back tangent bundle on M: a small deformation of φ given by a local vector field v near $\varphi(S)$ is the shift by its integral flow. Such a shift preserves the condition $\varphi^*(\omega) = 0$ iff $\varphi^*(\mathcal{L}_v\omega) = 0$ and is tangent to $Diff_0(S)$-orbit of φ iff $v|_{\varphi(S)}$ is tangent to $\varphi(S)$. So,

$$T_\varphi\big(Map(S,M)/Diff_0(S)\big) = \{\, v \in \mathcal{C}^\infty\,(\,S,\ N_{M/S}\,) \mid \varphi^*(\mathcal{L}ie_v\omega) = 0\,\}\ ,$$

where v is considered as a class of local vector fields near $\varphi(S)$ modulo such the fields tangent to $\varphi(S)$ (this interpretation for the sections of the normal bundle is correct, because any tangent field on $\varphi(S)$ can be extended onto a small tube around $\varphi(S)$).

1.4.1. PROPOSITION. *If $S \xrightarrow{\varphi} M$ is a smooth Lagrangian immersion, then $T_\varphi\mathfrak{L}$ is naturally identified with the space Z^1_{DR} of all closed 1-forms on S.*

PROOF. Sending a normal vector field v to the 1-form $\xi = \varphi^*(i_v\omega) \in C^\infty(S, T^*S)$, we identify the normal bundle $N_{M/S}$ with the cotangent bundle $T^*\varphi(S)$ (like in n° 1.2). Under this identification, the tangent fields $v \in T_\varphi\mathfrak{L}$ go to the *closed* 1-forms, because of

$$d\xi = d\varphi^*(i_v\omega) = \varphi^*(di_v\omega) = \varphi^*(\mathcal{L}_v\omega + i_v d\omega) = \varphi^*(\mathcal{L}_v\omega) = 0\,.$$

□

1.4.2. The Darboux - Weinstein theorem (see [GS, Ch.4, § 1], [We3, § 6], and [We4, § 4]) implies that for any smoothly immersed Lagrangian cycle $S \overset{\varphi}{\hookrightarrow} M$ there exist a tube neighborhood U of the zero section in the cotangent bundle T^*S and a smooth immersion $U \overset{\widehat{\varphi}}{\hookrightarrow} M$ such that $\widehat{\varphi}^*\omega = d\eta$ is the canonical symplectic form on T^*S and φ is the restriction of $\widehat{\varphi}$ on the zero section. Moreover, If φ is the smooth embedding, then we can take $U = T^*S$ and $\widehat{\varphi}$ being a symplectomorphism between the whole of T^*S and some smooth open tubular neighborhood of $f(S)$ in M.

Further, the immersion $T^*S \supset U \xrightarrow{\widehat{\varphi}} M$ induces (by sending a closed 1-form to its graph) a smooth map

$$T_S\mathfrak{L} \supset \mathfrak{U} \xrightarrow{\varepsilon} \mathfrak{L}\,, \tag{1-4}$$

where $\mathfrak{U} \subset Z^1_{\text{DR}}(S)$ consist of closed 1-forms ξ whose graphs lay inside U. It follows from the Weinstein's results (see [We3, § 6], [We4, § 4] already cited) that the image of (1-4) contains an open neighborhood of S in $\mathfrak{L}$ considered in suitable topology. So, we can release any small enough deformation of a smoothly immersed Lagrangian cycle as a graph of a closed 1-form laying in a small neighborhood of zero in $Z^1_{\text{DR}}(S)$. We will refer the map (1-4) as *the exponential map.*

1.4.3. Isodrastic deformations. Let $b_1 = \dim H^1(S,\mathbb{R})$. The subspaces of exact 1-forms

$$B^1_{\text{DR}}(S) \subset Z^1_{\text{DR}}(S) = T_S\mathfrak{L}$$

form a plane distribution of codimension b_1 in the tangent bundle to the space of Lagrangian cycles. The exponential map (1-4) shows that (locally, nearby a smooth cycle) this distribution is integrable to a foliation whose leafs have codimension b_1: if we identify a small neighborhood of a given cycle with a small neighborhood of zero in Z^1_{DR}, then small deformations tangent to Z^1_{DR} will be just the shifts by exact forms. Experts call such the deformations *isodrastic* (exhausting linguistic comments may be found in [We2]). Clearly, the graphs of two closed 1-forms may be obtained from each other by an isodrastic deformation iff these forms are cohomologous.

1.5. Global geometric model: almost Kähler manifolds. Main efforts of this work will be directed towards a studying of an interaction between the previous local cotangential model for $\mathfrak{L}$ and the following main geometric source of the symplectic structures.

1.5.1. Almost Hermitian manifolds. A smooth real $2n$-dimensional manifold M is called *almost Hermitian*, if its tangent bundle has a structure of complex Hermitian bundle of rank n. This means that TM admits a linear operator $TM \xrightarrow{I} TM$, a Rimannian metric $g : TM \xrightarrow{\sim} TM^*$, and a non degenerate skew symmetric form $\omega : TM \xrightarrow{\sim} TM^*$ such that $I^2 = -\text{Id}$, $\omega(v, Iw) = g(v, w)$, $\omega(v, Iw) = -\omega(Iv, w)$, and $g(v, Iw) = -g(Iv, w)$. Actually, the triple (I, g, ω) is recovered from any two of its elements.

Certainly, any holomorphic complex manifold is almost Hermitian as well. On the other side, one can develop a kind of "pseudo holomorphic" geometry on any almost Hermitian manifold. Since the complexified tangent bundle $\mathbb{C} \otimes TM = T_+M \oplus T_-M$ splits onto direct sum $\pm i$-eigensubspaces for I, we can define a *pseudo holomorphic function* $M \xrightarrow{f} \mathbb{C}$ as a one whose differential df annihilates T_-M. Similarly, we can consider pseudo holomorphic submanifolds, pseudo holomorphic complex vector bundles e. t. c.

1.5.2. PROPOSITION. *Let $V \subset M$ be a pseudo holomorphic (possible, singular) subvariety in an almost complex manifold M (that is, V is presented locally as a zero set of some system of pseudo holomorphic functions) and TC_xV be the tangent cone to V at an arbitrary point $x \in V$. Then ω has non degenerate restriction onto TC_xV, that is, for any non zero $v \in TC_xV$ there is $u \in TC_xV$ such that $\omega(v, u) \neq 0$.*

PROOF. Since V is pseudo holomorphic, the complex structure operator I preserves TC_xV and, hence, acts on TC_xV bijectively, because of $I^{-1} = -I$. So, since the symmetric form g is positive, for any non zero $v \in TC_xV$ we have $\omega(v, Iv) = g(v, v) > 0$ and can take $u = Iv \in TC_xV$. □

1.5.3. DEFINITION. *An almost Hermitian manifold M is called almost Kähler, if its skew symmetric form ω is closed, that is, defines the symplectic structure on M.*

In some sense, the Lagrangian geometry of almost Kähler manifolds is perpendicular to their pseudo holomorphic geometry.

1.5.4. COROLLARY. *A non zero section of a pseudo holomorphic line bundle can not vanish along a Lagrangian cycle.*

PROOF. The zero set of a pseudo holomorphic section is a pseudo holomorphic $(2n - 2)$-dimensional variety in M. If it contains an n-dimensional Lagrangian subvariety, then the restriction of ω should be degenerated along it. This contradicts with n° 1.5.2. □

1.5.5. Tame example: tangent bundles. Let X be an oriented smooth n-dimensional manifold. Then there is a canonical complex structure on its tangent bundle TX. Namely, the tangent space TTX splits canonically as $TX \oplus TX$, where the first summand is the 'vertical' kernel of the differential of the canonical projection $TX \twoheadrightarrow X$ and the second one is the 'horizontal' image of the differential of the zero section $X \hookrightarrow TX$. In terms of this splitting, the multiplication by $i \in \mathbb{C}$ at a point $(v, q) \in TX$, where $v \in T_xX$, $q \in X$, takes a tangent vector $w = (w_1, w_2) \in T_(v, q)TX$ to $(-w_2, w_1)$. This complex structure is, clearly, integrable: each local coordinate system $(y_1, y_2, \ldots, y_n)$ on X near $q \in X$ produces the holomorphic local coordinates $(z_1, z_2, \ldots, z_n)$ on TX, where $z_\nu \stackrel{\text{def}}{=} x_\nu + i\,y_\nu$ and x_ν is linear coordinate along $\partial/\partial y_\nu$ in the fiber T_qX. One can check immediately that smooth coordinate changes on X produce the holomorphic coordinate changes on TX and also that the transition functions for the tangent bundle produce the holomorphic gluing rules for the complex charts just described. In terms of these holomorphic coordinates, the multiplication by $i \in \mathbb{C}$ on $T_(v,q)TX$ sends

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \longmapsto \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)$$

On the other side, the cotangent bundle T^*X has the canonical symplectic structure $\omega = d\eta$ whose symplectic coordinates are of the form $(dy, \partial/\partial y)$ (see n° 1.3). So, if there is an isomorphism of bundles $TX \xrightarrow{\beta} T^*X$, then we get immediately an integrable *weak* Kähler structure on the both TX, T^*X, that is, a triple (I, g, ω) which satisfy all conditions from n° 1.5.1 except for positivity of the symmetric form $g(u_1, u_2) = \omega(u_1, Iu_2)$.

For example, if X is equipped with a Rimannian metric $TX \xrightarrow{G} T^*X$, then its cotangent bundle $M = T^*X$ admits an integrable almost complex structure

$$I : (\xi, v) \longmapsto (Gv, -G^{-1}\xi)$$

compatible with the canonical symplectic structure $\omega = d\eta$ from (1-3). In this case, the symmetric form $g(u_1, u_2) = \omega(u_1, Iu_2)$ is positive as well and we get an integrable almost Kähler structure on TX.

§ 2. Abelian preliminaries.

2.1. Complex line bundles. In this paragraph we would like to fix some basic notations and concepts concerning complex line bundles, which have an Abelian structure group: $U(1)$ or its complexified version $GL_1(\mathbb{C}) = \mathbb{C}^*$. Topologically, a complex line bundle $L \twoheadrightarrow X$ on a smooth manifold X is defined up to an isomorphisms by the cohomology class of the transition functions between local trivializations of L. These transition functions form the Čech 1-cocycle with coefficients in the sheaf of smooth $\mathbb{C}^*$-valued functions $\mathcal{C}^\infty(X, \mathbb{C}^*)$. Its cohomology class is called *a first Chern class* (see, say [GH]):

$$c_1(L) \in H^1(X, \mathcal{C}^\infty(X, \mathbb{C}^*)) \stackrel{\delta}{=\!=\!=} H^2(X, \mathbb{Z}) \,,$$

where the isomorphism δ is induced by the exponential exact triple of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}^\infty(X, \mathbb{C}) \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{C}^\infty(X, \mathbb{C}^*) \longrightarrow 0$$

because $\mathcal{C}^\infty(X, \mathbb{C})$ is flabby.

2.2. Hermitian line bundles. Each complex Hermitian line bundle $L \twoheadrightarrow X$ is associated via tautological representation of $U(1)$ with a principal $U(1)$-bundle $P \xrightarrow{p} X$, that is, $L = P \times_{U(1)} \mathbb{C}$. So, P has inverse transition functions and is naturally identified with the bundle of unit circles in L^* (in particular, P and L are uniquely recovered from each other).

We identify the Lie algebra $\mathfrak{su}(1)$ with $\mathbb{R}$ by sending $t \longmapsto 2\pi i t$. So, the exponential map looks like

$$\mathbb{R} \xrightarrow{t\mapsto e^{2\pi i t}} U(1)$$

and the tautological representation of $\mathfrak{su}(1)$ sends $t \in \mathbb{R}$ to the skew Hermitian operator $z \longmapsto 2\pi i t z$. In particular, the $\mathbb{R}$-valued differential forms may be considered as $\mathfrak{u}(1)$-valued via this representation. We denote by ν the canonical invariant 1-form on $U(1)$ such that $\displaystyle\int_{U(1)} \nu = 1$ and denote by u the dual invariant vector field satisfying $\nu(u) = 1$. Corresponding invariant vertical 1-form and vector field on P will be denoted by ν and u as well.

2.2.1. A Hermitian connection *a* on a principal $U(1)$-bundle $P \xrightarrow{p} X$ is a differential 1-form $\alpha = \alpha_a$ on P such that

$$\mathcal{L}ie_u \alpha = 0 \qquad \text{and} \qquad \alpha(u) = 1\,.$$

The first condition means that α is $U(1)$-invariant; the second says that $TP \xrightarrow{\alpha} TU(1)$ gives an $U(1)$-equivariant splitting of the following canonical exact triple of vector bundles on P:

$$0 \longrightarrow TU(1) \underset{\longrightarrow}{\overset{\alpha}{\longleftarrow}} TP \xrightarrow{dp} p^*TX \longrightarrow 0\,.$$

The kernel hyperplanes of α in TP are called *horizontal*. They are identified with the fibers of p^*TX by dp. So, the space of all Hermitian connections $\mathcal{A}_{\mathrm{H}}(L)$ is an affine space over $C^\infty(X, T^*X)$, because the difference between two splittings is a linear map $p^*TX \longrightarrow TU(1)$, that is, a pull back of a 1-form on X. The connection lifts any vector field v on X to a unique $U(1)$-invariant horizontal vector field $\widetilde{v}$ on P such that $dp(\widetilde{v}) = v$. Namely, if w is any $U(1)$-invariant vector field on X projected to v by dp, then $\widetilde{v} = w - \alpha(w)u$.

2.2.2. Covariant derivative. In terms of the associated Hermitian line bundle L, the connection a is presented by a covariant derivative $\nabla = \nabla_a$, which is a differential operator

$$\mathcal{C}^\infty(X, L) \xrightarrow{\nabla_a} \mathcal{C}^\infty(X, T^*X \otimes L)$$

sending a local unitary basic section σ over a small open disk $U \subset X$ to $\nabla\sigma = 2\pi i\,\xi_\sigma \otimes \sigma$, where $\xi_\sigma = \sigma_P^*\alpha$ is a pull back of the connection form along a local section $U \xhookrightarrow{\sigma_P} P$ induced[1] by σ. Then any other local section $f\sigma$, where $f \in C^\infty(U, \mathbb{C})$, is differentiated as

$$\nabla(f\sigma) = (df + 2\pi i\, f\xi_\sigma) \otimes \sigma\,. \tag{2-1}$$

and a local unitary base change $\sigma \longmapsto \sigma' = h\sigma$ with $h \in C^\infty(U, U(1))$ takes ξ_s to

$$\xi_{\sigma'} = \xi_\sigma - \frac{i}{2\pi}\,\frac{dh}{h} = \xi_\sigma - \frac{i}{2\pi}\, d\log h\,. \tag{2-2}$$

In particular, the local covariantly constant sections $f\sigma$ are given by the differential equation

$$d\log f = -2\pi i\,\xi_\sigma \tag{2-3}$$

Of course, it is locally solvable iff $d\xi_\sigma = 0$.

2.2.3. Curvature. It follows from (2-2) that the differentials $d\xi_\sigma$ do not depend on a local unitary base choice, that is, there is a global closed 2-form $\omega_a \in C^\infty(X, \Lambda^2 T^*X)$ such that

[1] actually, this section is defined only up a $U(1)$-action but it does not affect on α

$\omega_a|_U = d\xi_\sigma$ for any unitary section σ over any small open ball U. The base change formula (2-2) shows also that the cohomology class $[\omega_a] \in H^2(X,\mathbb{R})$ is integer and coincides with the first Chern class:

$$[\omega_a] = c_1(L) \in H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R}) \ .$$

A $\mathfrak{su}(1)$-valued form $F_a \stackrel{\text{def}}{=} 2\pi i\,\omega_a$ is called *a curvature form* of a. But we will work mostly with the original *real valued* 2-form ω_a and refer it as the *real 2-form associated with the curvature of* a. In terms of the principal bundle $P \xrightarrow{p} X$, the pull back $p^*\omega_a = d\alpha_a$ sends a pair of $U(1)$-invariant vector fields v_1, v_2 on P to

$$d\alpha(v_1,v_2) = \Big(\mathcal{L}ie_{v_1}\alpha - d\,(\,\alpha(v_1)\,)\Big)(v_2) = \mathcal{L}ie_{v_1}\alpha(v_2) = -\alpha([v_1,v_2]) \ .$$

In terms of the covariant derivative, the second order differentiation

$$C^\infty(X,L) \xrightarrow{\nabla_a^2} C^\infty(X,\Lambda^2T^*X\otimes L) \ ,$$

induced by ∇_a via Leibnitz rule is nothing but a $C^\infty(X)$-linear operator multiplying the sections by $2\pi i\,\omega_a = F_a$, because of $\nabla^2\sigma = 2\pi i\, d\xi_\sigma\otimes\sigma$.

Let us remark finally that the curvature presents a moment map for the natural action of the gauge group $C^\infty(X,U(1))$ on the space of Hermitian connections. Indeed, a gauge transformation $X \xhookrightarrow{h} U(1)$ takes α to $\alpha + p^*f^*\nu$ and preserves the curvature, because the basic unitary invariant vertical form ν on P is closed.

2.2.4. Flat Hermitian connections. A Hermitian connection is called *flat* if its curvature is the zero form. In this case, L is topologically trivial, that is has a global never vanishing section $X \xhookrightarrow{\sigma} L$. Moreover, the equation (2-3) becomes solvable over any simply connected open set U and its solutions are given by the formula

$$f = c\,e^{-2\pi i g} \ , \tag{2-4}$$

where $c\in\mathbb{C}$ is an arbitrary constant and g is a smooth function on U (defined up to an additive constant) such that $dg = \xi_\sigma$. So, L admits *a covariant* trivialization over any simply connected open set. Such the trivialization by some f from (2-4) is *unitary* iff $|c| = 1$. So, the unitary covariant trivializations form a principal $U(1)$-homogeneous space and these spaces over different U's are glued together precisely into the principal bundle P associated with L.

Let us take the universal covering $\widetilde{X} \xrightarrow{\pi} X$ and fix some covariant trivialization for the pulled back bundle $\pi^*L \longrightarrow \widetilde{X}$ by some global covariantly constant unitary section $\widetilde{X} \xhookrightarrow{\widetilde{\sigma}} \pi^*L$. Identifying $C^\infty(\widetilde{X},\mathbb{C})$ with $C^\infty(\widetilde{X},\pi^*L)$ by taking $f\mapsto f\widetilde{\sigma}$, we get a representation of $\pi_1(X)$ in the space of the constant unitary global sections of π^*L. The equivalence class of this representation does not depend on the choice of $\widetilde{\sigma}$ and is uniquely defined by its character $\pi_1(X) \xrightarrow{\chi} U(1)$. In terms of L, this character gives the holonomy of the connection a, that is, $\chi(\gamma)$ equals a fiber shift under the covariant parallel transport along γ. The cohomology class of χ in $H^1(X,U(1))$ is preserved by the gauge transformations and goes to $0 = c_1(L)$ in the long exact cohomology sequence

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathbb{R}) \longrightarrow H^1(X,U(1)) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

associated with the exact triple of the constant sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{t\mapsto e^{2\pi i t}} U(1) \longrightarrow 0 \ .$$

So, the cohomology classes of characters χ, that is the gauge classes of the flat Hermitian connections on L, are naturally parametrized by the real torus

$$J_S \overset{\text{def}}{=} H^1(S,\mathbb{R})/H^1(S,\mathbb{Z})$$

called *a Jacobian* of X.

2.3. Supercycles. A Lagrangian cycle $S \longrightarrow M$ equipped with a flat $U(1)$-connection a on the trivial complex line bundle $C = S \times \mathbb{C}$ is called *a supercycle* or *a brane* supported by S. We consider the sections of C as functions on S. There is a distinguished supercycle (S, a_0) whose connection form $\alpha_0 = \nu$ coincides with the unitary vertical form ν on P and the covariant differentiation $\nabla_0 = d$ coincides with the external differential. So, we can identify the affine space of flat connections $\mathcal{A}_{\text{FH}}(C)$ with the underlying vector space $Z^1_{\text{DR}}(S)$ by sending α_a and ∇_a to $\alpha_a - \alpha_0$ and $\nabla_a - \nabla_0$. We define *a space of Lagrangian supercycles* as an orbit space

$$\mathfrak{L}^{\text{s}} = \mathit{Map}_{\text{LA}}(S,M) \times \mathcal{A}_{\text{FH}}(C)/\mathit{Diff}_0(S)\ , \tag{2-5}$$

where $\mathcal{A}_{\text{FH}}(C)$ is the space of flat Hermitian connections and $\mathit{Diff}_0(S)$ acts diagonally. It is fibered over $\mathfrak{L}$ via forgetful map

$$\mathfrak{L}^{\text{s}} \xrightarrow{(S,a)\mapsto S} \mathfrak{L}\ .$$

Of course, this map is nothing but the tangent bundle $\mathfrak{L}^{\text{s}} \simeq T\mathfrak{L}$. So, the supercycle space $\mathfrak{L}^{\text{s}}$ has the natural integrable complex structure of a tangent bundle (see n° 1.5.5). If we identify a small neighborhood of $S \in \mathfrak{L}$ with the small tube around the zero section $S \hookrightarrow T^*S$ and represent Lagrangian cycles nearby S by the closed 1-forms (see n° 1.4.2), then the complex coordinate system on $\mathfrak{L}^{\text{s}}$ near (S,α) consists of $\xi_1 + i\,\xi_2$ that runs trough $Z^1_{\text{DR}}(S,\mathbb{C})$ and corresponds to a supercycle supported by the graph of ξ_1 and equipped with the flat connection $\alpha + \xi_2$, where ξ_1, ξ_2 are real closed 1-forms on S.

2.3.1. Isodrastic deformations of supercycles. The tangent space $T_{(S,a)} = Z^1_{\text{DR}}(S,\mathbb{C})$ contains a complex subspace $B^1_{\text{DR}}(S,\mathbb{C})$, of exact forms. We refer this distribution of complex planes as *a complex* isodrastic distribution. The same arguments as in n° 1.4.3 show that it is locally integrable near each smooth supercycle to some complex foliation.

2.4. Jacobian bundles. Let $C = S \times \mathbb{C}$ be the trivial complex line bundle, as before. A semidirect product $\mathcal{G} \overset{\text{def}}{=} \mathit{Map}(S,U(1)) \rtimes \mathit{Diff}_0(S)$, where $\mathit{Diff}_0(S)$ acts on $\mathit{Map}(S,U(1))$ as $h\gamma h^{-1} \overset{\text{def}}{=} \gamma \circ h^{-1}$, is the group of diffeomorphisms $L \xrightarrow{\sim} L$ sending fibers to fibers by linear unitary transformations. It acts naturally on $\mathit{Map}_{\text{LA}}(S,M) \times \mathcal{A}_{\text{FH}}(C)$ and the orbit space

$$\mathcal{J} \overset{\text{def}}{=} \mathit{Map}_{\text{LA}}(S,M) \times \mathcal{A}_{\text{FH}}(C)/\mathcal{G}$$

is called *a Jacobian bundle* over $\mathfrak{L}$. Its fiber over a given S consists of the gauge classes of flat $U(1)$-connections on C (see n° 2.2.4) and is naturally identified with the Jacobian

$$J_S = H^1(S,\mathbb{R})/H^1(S,\mathbb{Z})\ .$$

If we consider $\mathbb{C}^*$-connections on C instead of the Hermitian ones and factorize them by $\mathbb{C}^*$-gauge group $C^\infty(S,\mathbb{C})$, then we get *a complex Jacobian bundle*

$$\mathcal{J}^{\mathbb{C}} \longrightarrow \mathfrak{L}$$

whose fiber over a Lagrangian cycle S is a complex torus

$$J_S^{\mathbb{C}} \overset{\text{def}}{=} H^1(S,\mathbb{C})/H^1(S,\mathbb{Z})\ .$$

This torus parametrizes the $\mathbb{C}^*$-valuated characters $\pi_1(S) \xrightarrow{\widetilde{\chi}} \mathbb{C}^*$. It is called *a complex Jacobian* of S. There is a natural surjection $J_S^{\mathbb{C}} \overset{r}{\twoheadrightarrow} J_S$ sending

$$\widetilde{\chi} \longmapsto \widetilde{\chi}/|\widetilde{\chi}|\ .$$

2.4.1. Remark. Since $T_S\mathfrak{L} = Z^1_{\mathrm{DR}}(S)$ and $TJ_S = H^1(S,\mathbb{R})$, there is a canonical linear map from the base tangent space to the fiber tangent space of the real Jacobian bundle. This map sends a closed form to its cohomology class. The existence of such the map will play maybe hidden but important role in almost all our constructions.

On the other side, the Jacobian bundle may be equipped, certainly, with the Gauss - Manin connection, which comes from a local uniform base choice in the lattice $H^1(S,\mathbb{Z})$ via the exponential map. In terms of this connection, say the complex isodrastic deformations may be described as follows. Each infinitesimal deformation $\xi = (\xi_1 + i\,\xi_2) \in T_{(S,a)}\mathfrak{L}^{\mathrm{s}}$ of a supercycle (S,a) defines a flat $\mathbb{C}^*$-connection $\alpha + \xi_1 + i\,\xi_2$ whose $\mathbb{C}^*$-gauge class gives a point $\widetilde{\chi}_\xi \in J_S^{\mathbb{C}}$. A family of supercycles is complex isodrastic, if the real projection of this character $r(\widetilde{\chi}_\xi) = \widetilde{\chi}_\xi/|\widetilde{\chi}_\xi| \in J_S$ is a covariantly constant section of the real Jacobian bundle w. r. t. the Gauss - Manin connection. As soon as we will not study the global topology of $\mathfrak{L}$ in this paper, we will not need such type constructions and stop here their further development.

§ 3. Prequantization equipment.

3.1. Prequantization data. Suppose that the cohomology class of the symplectic form ω on M is *integer* $[\omega] \in H^2(M,\mathbb{Z}) \subset H^1(M,\mathbb{R})$, that is, $[\omega] = c_1(L)$ presents the first Chern class of some complex line bundle $L \twoheadrightarrow M$. We fix a Hermitian structure on L and equip it with a Hermitian connection a with the curvature $F_a = 2\pi i\,\omega$; so, the real 2-form associated with this curvature (in accordance with our agreements in n° 2.2.3) coincides with the symplectic structure: $\omega_a = \omega$. Let $P_k \overset{p_k}{\twoheadrightarrow} M$ be the principal $U(1)$-bundle associated with $L^{\otimes k}$ via tautological representation. Then for all $k,m \in \mathbb{N}$ we have the commutative diagram

$$\begin{tikzcd} P_k \arrow[rr, two heads, "\mu_{m,k}"] \arrow[dr, "p_k"'] & & P_{mk} \arrow[dl, two heads, "p_{mk}"] \\ & M & \end{tikzcd} \tag{3-1}$$

where $P_k \overset{\mu_{m,k}}{\twoheadrightarrow} P_{mk}$ is an unramifided covering of degree m that takes $z \mapsto z^m$ in each fiber. We denote by α_k the (real valued) connection form induced by a on P_k. The curvature of this connection equals

$$F_k = 2\pi i k\,\omega\ , \quad k \in \mathbb{N}\ , \tag{3-2}$$

that is, has associated real 2-form $\omega_k = k\omega$. We refer the quadruple $(M, \omega, P_k, \alpha_k)$ as *a level k prequantization data* on the phase space M. The corresponding complex Hermitian line bundle $L^{\otimes k}$ is called *a level k prequantization line bundle* on (M, ω).

3.2. Bohr - Sommerfeld cycles. Let $S \xrightarrow{\varphi} M$ be a Lagrangian cycle. It follows from (3-2) that $\varphi^* L^{\otimes k}$ is the trivial line bundle with a flat connection on S, that is a supercycle (see n° 2.3). Taking its gauge for each Lagrangian cycle $S \in \mathfrak{L}$, we get a section of the Jacobian bundle $\mathfrak{L} \xhookrightarrow{j_k} \mathcal{J}$ (see n° 2.4). The zero set of this section is denoted by $\mathcal{B}_k$ and is called a space of *level k Bohr - Sommerfeld cycles* (or *BS$_k$-cycles* for shortness).

In other words, a Lagrangian cycle $S \xrightarrow{\varphi} M$ is BS$_k$-cycle iff the connection $\varphi^*\alpha_k$ on the pull back $\varphi^* L^{\otimes k}$ has the trivial holonomy, that is, admits a *covariant* trivialization $\varphi^* L^{\otimes k} \simeq S \times \mathbb{C}$. Since the covariant trivialization of $\varphi^* L^{\otimes k}$ trivializes all the tensor powers $\varphi^* L^{\otimes mk}$ as well, we have the natural inclusions

$$\mathcal{B}_k \xhookrightarrow{\beta_{k,m}} \mathcal{B}_{mk} \quad \forall\, k, m \in \mathbb{Z} \tag{3-3}$$

3.3. Planckian cycles and Berry bundle. The connection 1-form α_k presents a *contact structure*[1] on P_k with the volume element

$$\alpha_k \wedge (d\alpha_k)^{\wedge n} = k^n\, \alpha_k \wedge (p^*\omega)^{\wedge n} \tag{3-4}$$

Let $Map_{\text{LE}}(S, P_k)$ be the space of all smooth maps $S \xrightarrow{\widetilde{\varphi}} P_k$ such that $\widetilde{\varphi}^*(\alpha_k) = 0$ (this means that $\varphi(S) \subset P$ is the *Legendrian* submanifold w. r. t. the contact structure α_k). The orbit space

$$\mathcal{P}_k \overset{\text{def}}{=} Map_{\text{LE}}(S, P_k)/Diff_0(S)$$

is called a space of level k *Planckian cycles*. We write $\varphi = p_k \circ \widetilde{\varphi} : S \longrightarrow M$ for the composition of a Legendrian map $S \xrightarrow{\widetilde{\varphi}} P_k$ with the projection $P_k \xrightarrow{p_k} M$. This composition has to be a BS$_k$-cycle, because of $\varphi^*\omega = d\widetilde{\varphi}^*\alpha_k/k = 0$. So, $S \xrightarrow{\widetilde{\varphi}} P_k$ is just a covariant constant section of $\varphi^* P_k$, or equivalently, a covariant constant unitary section of $\varphi^* L^{\otimes k}$ over S. Since such the sections form a principal homogeneous $U(1)$-space (see n° 2.2.4), we get the principal $U(1)$-bundle

$$\mathcal{P}_k \xrightarrow{\pi_k} \mathcal{B}_k \tag{3-5}$$

called *the principal Berry bundle* of level k. The Hermitian line bundle associated with $\mathcal{P}_k$ via tautological representation is called *the Berry line bundle* on $\mathcal{B}_k$ and is denoted by $\mathcal{L}_k$. The diagrams (3-1) and (3-3) show that we have the non ramified degree m covering, which takes each covariant section to its m-th power:

$$\mathcal{P}_k \xrightarrow{\widetilde{\mu}_{m,k}} \beta^*_{m,k}\mathcal{P}_{mk} \quad \forall\, k, m \in \mathbb{N}\,. \tag{3-6}$$

In other words, $\beta^*_{k,m}\, \mathcal{L}_{mk} \,=\, \mathcal{L}_k^{\otimes m}$.

3.3.1. PROPOSITION. *A tangent space $T_{\widetilde{\varphi}}\mathcal{P}_k$ at a smooth Planckian cycle $S \xhookrightarrow{\widetilde{\varphi}} P_k$ is naturally identified with $C^\infty(S)$. Under this identification, the differential $d\widetilde{\mu}_{k,m}$ acts on $C^\infty(S)$ as the scalar multiplication by m.*

[1] see [A, App. 4] about the contact structures

PROOF. The same arguments as in n° 1.4 show that the tangent space

$$T_{\widetilde{\varphi}}\big(Map_{\text{LE}}(S,P_k)/Diff_0(S)\big) = \Big\{\, v \in C^\infty\left(S,\, N_{P_k/S}\right) \mid \widetilde{\varphi}^*(\mathcal{L}ie_v\alpha_k) = 0\,\Big\}$$

that is, a tangent vector to $\mathcal{P}_k$ at $\widetilde{\varphi}$ is a class of vector fields (modulo the fields tangent to $\widetilde{S}$) defined near $\widetilde{S}$ on P_k such that $\mathcal{L}ie_v\alpha_k = 0$. Since α_k annihilates the fields tangent to $\widetilde{S}$, the contraction with α_k gives a well defined map $T_{\widetilde{\varphi}}\mathcal{P}_k \longrightarrow C^\infty(S)$ taking

$$v \longmapsto f_v \overset{\text{def}}{=} \widetilde{\varphi}^*\alpha_k(v) \in C^\infty(S)\ .$$

Moreover, the horizontal component of v, that is $dp(v - \alpha_k(v)\cdot u) \in T_{p\varphi}\mathfrak{L} \simeq Z^1_{\text{DR}}(S)$ (where u is the basic unitary invariant vector field on P_k) coincides with df_v/k, since for any local vector field w tangent to $\widetilde{S}$ on P_k we have:

$$\begin{aligned}\omega\big(\,dp(v - \alpha_k(v)\cdot u)\,,\,dp(w)\,\big) = p^*\omega(v,w) = \frac{1}{k}\,d\alpha_k(v,w) = \\ = \frac{1}{k}\Big[\mathcal{L}ie_v\alpha_k + d(\alpha_k(v))\Big]\,(w) = \frac{1}{k}\,df_v\,(w)\end{aligned}$$

Conversely, any $f \in C^\infty(S)$ produces (after an extension on a small neighborhood of $\widetilde{S}$) a local vector field v_f on P_k such that $f_{v_f} = \alpha_k(v_f) = f$: the vertical component of v_f equals fu and the horizontal component of v_f is the horizontal lifting of $\omega^{-1}(d\widehat{f})/k$ from M on P_k, where $\widehat{f}$ is an extension of f to a small neighborhood of $\varphi(S)$ in M. □

3.3.2. Local structure of $\mathcal{P}$ near a smooth cycle. Let a smooth Planckian cycle $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ be given by a covariant constant section $S \overset{\sigma}{\hookrightarrow} \varphi^*P_k$ over a smoothly immersed Bohr - Sommerfeld cycle $S \xrightarrow{\varphi} M$. Then the previous proof identifies the differential

$$T_{\widehat{\varphi}}\mathcal{P}_k \xrightarrow{d\pi|_{\widehat{\varphi}}} T_{\varphi}\mathcal{B}_k\ ,$$

of the Berry bundle $\mathcal{P}_k \xrightarrow{\pi} \mathcal{B}_k$, with the divided by k external differentiation of functions on S

$$C^\infty(S,\mathbb{R}) \xrightarrow{\frac{1}{k}\,d} B^1_{\text{DR}}(S)\ .$$

In particular, the space $\mathcal{B}_k \subset \mathfrak{L}$ is isodrastic, $T_S\mathcal{B}_k \simeq B^1_{\text{DR}}(S)$, and under this identification, the differential $d\beta_{k,m}$ acts on $B^1_{\text{DR}}(S)$ as the scalar multiplication by m.

Further, by the Darboux - Weinstein theorem (see n° 1.4.2), φ is continued to a symplectic immersion $U \overset{\widehat{\varphi}}{\hookrightarrow} M$, where U is some neighborhood of the zero section $S \hookrightarrow T^*S$. Let us denote by $S^{(k)}_f \subset T^*S$ the graph of a closed 1-form $df/k \in B^1_{\text{DR}}(S)$. A map

$$df \longmapsto \widehat{\varphi}\left(S^{(k)}_f\right)$$

coordinates some neighborhood of φ in $\mathcal{B}_k$ by a small ball around the zero in $B^1_{\text{DR}}(S)$ (see n° 1.4.2). Now we can lift this coordinatization on a small ball around the zero in $C^\infty(S,\mathbb{R}) = T_{\widetilde{\varphi}}\mathcal{P}_k$ and get the coordinates for a neighborhood of $\widetilde{\varphi}$ in $\mathcal{P}_k$. Namely, let $P^U_k = \widehat{\varphi}^*P_k$ be the pull back of P_k on U along the Darboux - Weinstein immersion $U \overset{\widehat{\varphi}}{\hookrightarrow} M$ and let $U \overset{\widehat{\sigma}}{\hookrightarrow} P^U_k$ be the covariant constant section whose restriction on $S \subset U$ is the initial covariant trivialization

$S \xhookrightarrow{\sigma} \varphi^* P_k$. We send each $f \in C^\infty(S, \mathbb{R})$ such that $S_f^{(k)} \subset U$ to BS$_k$-cycle $S_f^{(k)}$ over which P_k^U is trivialized by the section

$$\sigma_f \stackrel{\text{def}}{=} e^{if} \widehat{\sigma}|_{S_f^{(k)}} . \tag{3-7}$$

So, we have got the real smooth functions on S as the local coordinates near $\widetilde{\varphi}$ in $\mathcal{P}_k$. We call them *DW-coordinates*.

3.3.3. Complex DW-coordinates on $T\mathcal{P}_k$. Recall, that the tangent bundle $T\mathcal{P}_k$ admits the canonical integrable complex structure (see n° 1.5.5): for any smooth Planckian cycle $S \xhookrightarrow{\widetilde{\varphi}} P_k$ and any tangent vector $g \in C^\infty(S, \mathbb{R}) = T_{\widetilde{\varphi}} \mathcal{P}_k$ the multiplication by i on

$$T_{(\widetilde{\varphi}, g)}(T\mathcal{P}_k) = T\mathcal{P}_k \oplus T\mathcal{P}_k = C^\infty(S, \mathbb{R}) \oplus C^\infty(S, \mathbb{R})$$

sends $(\psi_1, \psi_2) \longmapsto (\psi_2, -\psi_1)$, where the first component runs through the 'vertical' linear fibers of the tangent bundle $T\mathcal{P}_k \twoheadrightarrow \mathcal{P}_k$ and the second one runs through the 'horizontal' summand $T\mathcal{P}_k$ included into $TT\mathcal{P}_k$ by the differential of the zero section $\mathcal{P}_k \hookrightarrow T\mathcal{P}_k$. In other words, complex coordinates on $T\mathcal{P}_k$ are the complex valued smooth functions $\psi = \psi_1 + i\psi_2 \in C^\infty(S, \mathbb{C})$ whose real part ψ_1 is interpreted as linear coordinate in the fiber of the tangent bundle and imaginary part ψ_2 is interpreted as DW-coordinate on $\mathcal{P}_k$. We will refer these coordinates as the *complex DW-coordinates* on $T\mathcal{P}_k$.

3.4. Half weighted Planckian cycles. Our next task is to combine the natural complex structure on $T\mathcal{P}$ and the natural symplectic structure on $T^*\mathcal{P}$ in order to get an integrable Kähler structure on a suitable vector bundle over $\mathcal{P}$.

3.4.1. Measures. Let $\mathfrak{M} = \mathfrak{M}(S)$ be an equivalence class of measures on the source manifold S induced by its manifold structure (see [M, § 2-6]). It contains a subclass $\mathfrak{M}_0 \subset \mathfrak{M}$ of *smooth* measures whose pairwise *Radon - Nikodym derivatives* are the smooth functions on S. Let us recall that the equivalence $\lambda \sim \mu$ of two measures λ, μ on S means that $\lambda(B) = 0 \Leftrightarrow \mu(B) = 0$ for any Borel subset $B \subset S$, or by the Radon - Nikodym theorem, that there are two real integrable positive functions called the Radon - Nikodym derivatives:

$$\varrho_{\mu\lambda} = \frac{d\mu}{d\lambda} \in L^1(S, \lambda) \qquad \text{and} \qquad \varrho_{\lambda\mu} = \frac{d\lambda}{d\mu} \in L^1(S, \mu)$$

such that $\int_B d\mu = \int_B \varrho_{\mu\lambda}\, d\lambda$ and $\int_B d\lambda = \int_B \varrho_{\lambda\mu}\, d\mu$ for any Borel subset $B \subset S$. Of course, $\varrho_{\mu\lambda}\varrho_{\lambda\zeta} = \varrho_{\mu\zeta}$ and $\varrho_{\mu\lambda}\varrho_{\lambda\mu} = 1$ almost everywhere w. r. t. any measure from $\mathfrak{M}$. One writes usually $\lambda = \varrho_{\lambda\mu}\mu$ for two equivalent measures.

3.4.2. Half weighings. By the definition, *half weight* $\vartheta = f\sqrt{\mu}$ on S is an equivalence class of pairs (f, μ), where $\mu \in \mathfrak{M}$, $f \in L^2_{\mathbb{R}}(S, \mu)$, and the equivalence $(f_1, \mu_1) \sim (f_2, \mu_2)$ means that $f_1\sqrt{\varrho_{\mu_1\lambda}} = f_2\sqrt{\varrho_{\mu_2\lambda}}$ almost everywhere w. r. t. any $\lambda \in \mathfrak{M}$. A half weight $f\sqrt{\mu}$ is called *smooth*, if $f \in C^\infty(S)$ and $\mu \in \mathfrak{M}_0$. The straightforward check[1] shows that the half weights on S form the real closed normed vector space $\mathfrak{H}(S)$ with the norm induced by a scalar product

$$(\, f_1 \sqrt{\mu_1}\, ,\, f_2 \sqrt{\mu_2}\,) \stackrel{\text{def}}{=} \int_S f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\, d\lambda \tag{3-8}$$

[1] see, for example, [M, § 2-6] or [Go, sec 4.1], where our half weights are referred as σ-functions

and the linear combinations defined as $t_1\cdot f_1\sqrt{\mu_1}+t_2\cdot f_2\sqrt{\mu_2}\overset{\text{def}}{=}\left(t_1 f_1\sqrt{\varrho_{\mu_1\lambda}}+t_2 f_2\sqrt{\varrho_{\mu_2\lambda}}\right)\sqrt{\lambda}$, where λ in the both formulas is an arbitrary measure from $\mathfrak{M}$. The smooth half weights form a dense subspace $\overset{\circ}{\mathfrak{H}}(S)\subset\mathfrak{H}(S)$.

There is a natural action of $\mathit{Diff}_0(S)$ on $\mathfrak{H}(S)$ via $(f,\mu)\overset{h}{\longmapsto}\left(f\circ h^{-1},\,(h^{-1})^*\mu\right)$ and $\overset{\circ}{\mathfrak{H}}$, clearly, goes to itself under this action. The orbit spaces w.r.t. the corresponding diagonal actions:

$$\mathfrak{L}^{\text{hw}}\overset{\text{def}}{=}\mathit{Map}_{\text{LA}}(S,M)\times\mathfrak{H}(S)/\mathit{Diff}_0(S)\qquad\mathcal{P}_k^{\text{hw}}\overset{\text{def}}{=}\mathit{Map}_{\text{LE}}(S,P_k)\times\mathfrak{H}(S)/\mathit{Diff}_0(S)\,.$$

are called spaces of *half weighted* Lagrangian and Planckian cycles (of level k) respectively. If we replace $\mathfrak{H}$ by $\overset{\circ}{\mathfrak{H}}$ in these definitions, then we get the dense subspaces $\overset{\circ}{\mathfrak{L}}{}_k^{\text{hw}}\subset\mathfrak{L}_k^{\text{hw}}$ and $\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}\subset\mathcal{P}_k^{\text{hw}}$ called the spaces of *smoothly* half weighted cycles. The forgetful maps $\mathfrak{L}^{\text{hw}}\overset{F}{\longrightarrow}\mathfrak{L}$ and $\mathcal{P}_k^{\text{hw}}\overset{\widetilde{F}}{\longrightarrow}\mathcal{P}_k$ (and their smooth versions) provide these spaces with the vector bundle structures. These bundles may be arranged in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_k^{\text{hw}} \arrow[r, two heads] \arrow[d, "\widetilde{F}=\pi^*F|_{\mathcal{B}_k^{\text{hw}}}"'] & \mathcal{B}_k^{\text{hw}} \arrow[r, hook] \arrow[d, "F|_{\mathcal{B}_k^{\text{hw}}}"] & \mathfrak{L}^{\text{hw}} \arrow[d, "F"] \\
\mathcal{P} \arrow[r, two heads, "\pi"] & \mathcal{B}_k \arrow[r, hook] & \mathfrak{L}\,.
\end{tikzcd}\tag{3-9}$$

3.5. Kähler structure on $\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}$. For any fixed half weight $f\sqrt{\mu}$ the tangent space $T_{\widetilde{S}}\mathcal{P}_k=C^\infty(S)$ is included into $T_{f\sqrt{\mu}}\mathfrak{H}\simeq\mathfrak{H}$ by sending

$$f_1\longmapsto f f_1\sqrt{\mu}\,.\tag{3-10}$$

All together these maps produce an inclusion $T\mathcal{P}_k\hookrightarrow\mathcal{P}_k^{\text{hw}}$ as a dense set. Restricting ourselves by the smooth half weights, we get the identification $T\mathcal{P}_k\overset{\sim}{\longrightarrow}\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}$, which allows to push down onto $\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}$ the canonical integrable complex structure on $T\mathcal{P}_k$ described in n° 3.3.3. Namely, consider a smooth Planckian cycle $S\overset{\widetilde{\varphi}}{\hookrightarrow}P$ laying over a BS$_k$-cycle $S\overset{\varphi}{\hookrightarrow}M$ (like in n° 3.3.2) and half weighted by smooth $\vartheta=f\sqrt{\mu}$. Then we have the splitting:

$$T_{(\widetilde{\varphi},\vartheta)}\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}\simeq T_\vartheta\overset{\circ}{\mathfrak{H}}(S)\oplus T_{\widetilde{\varphi}}\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}\simeq C^\infty(S)\oplus C^\infty(S)\tag{3-11}$$

and can coordinate a small neighborhood of $(\widetilde{\varphi},\vartheta)$ in $\overset{\circ}{\mathcal{P}}{}_k^{\text{hw}}$ by smooth complex valued functions $\psi=\psi_1+i\psi_2\in C^\infty(S,\mathbb{C})$ as follows. The imaginary part ψ_2 produces the Planckian cycle with the DW-coordinate ψ_2 as it was explained in n° 3.3.2: to produce it we should continue the section $S\overset{\sigma}{\hookrightarrow}\varphi^*Pk$, which defines the Planckian cycle $\widetilde{\psi}$, to the covariant constant section $U\overset{\widehat{\sigma}}{\hookrightarrow}P_k^U$, where $U\subset T^*S$ is the Darboux - Weinstein neighborhood of the zero section immersed in M via $U\overset{\widehat{\varphi}}{\hookrightarrow}M$ and $P_k^U=\widehat{\varphi}^*P_k$; then the Planckian cycle being constructed is the graph $S_{\psi_2}^{(k)}\subset U$, of closed 1-form $d\psi_2/k$, and P_k^U is trivialized over this graph by the section

$$\sigma_{\psi_2}\overset{\text{def}}{=}e^{i\psi_2}\widehat{\sigma}|_{S_{\psi_2}^{(k)}}\,.$$

The real part ψ_1 gives the half weight on $S_{\psi_2}^{(k)}$ as follows. Let $S_{\psi_2}^{(k)}\overset{pr}{\longrightarrow}S$ be the restriction of the canonical projection $T^*S\longrightarrow S$. Then

$$pr^*\vartheta=(pr\circ f)\cdot\sqrt{pr^*\mu}$$

is a well defined half weight on $S^{(k)}_{\psi_2}$. So, in accordance with (3-10) we have to equip $S^{(k)}_{\psi_2}$ with the half weight $\psi_1 pr^*\vartheta$. We refer just described complex coordinates $\psi_1 + i\psi_2 \in C^\infty(S,\mathbb{C})$ on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ as the *the complex DW-coordinates*. In fact, they correspond to some Kähler structure on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$. To define it, let us describe now two its remaining ingredients.

3.5.1. The symplectic and Riemannian structures on $\mathcal{P}^{\mathrm{hw}}_k$. The product of two half weights $\vartheta_1 = f_1\sqrt{\mu_1}$ and $\vartheta_2 = f_2\sqrt{\mu_2}$ is nothing but *a measure* multiplied by $f_1 f_2$, that is

$$\vartheta_1\vartheta_2 \overset{\mathrm{def}}{=} f_1 f_2 \sqrt{\varrho_{\mu_1}\varrho_{\mu_2\lambda}}\,\lambda\ ,$$

which does not depend on the choice of $\lambda \in \mathfrak{M}$ (see [M, § 2-6]). So, we have a well defined integration w. r. t. the product of two half weights

$$\int_S f\, d\vartheta_1\vartheta_2 \overset{\mathrm{def}}{=} \int_S f f_1 f_2 \sqrt{\varrho_{\mu_1\lambda}\varrho_{\mu_2\lambda}}\, d\lambda \tag{3-12}$$

for any $f \in C^\infty(S)$ and any half weights $\vartheta_1 = f_1\sqrt{\mu_1}$, $\vartheta_2 = f_2\sqrt{\mu_2}$. In particular, each half weight ϑ produces a linear form

$$\varkappa(\vartheta)\ :\quad f \longmapsto \int_S f\, d\vartheta^2 \tag{3-13}$$

on the tangent space $T_{\widetilde{S}}\mathcal{P}_k \simeq C^\infty(S)$.

In other words, $\mathfrak{H}(S)$ double covers $T^*_{\widetilde{S}}\mathcal{P}_k$, because $\varkappa$ is quadratic in ϑ. The differential of this quadratic covering sends $\vartheta_1 \in \mathfrak{H}(S)$ to the linear form

$$d\varkappa_\vartheta(\vartheta_1)\ :\quad f \longmapsto 2\int_S f\, d\vartheta\vartheta_1\ .$$

on $C^\infty(S) = T_{\widetilde{S}}\mathcal{P}_k$. So, $d\varkappa$ identifies $T^*\mathcal{P}_k$ with $\mathcal{P}^{\mathrm{hw}}_k$. Now, consider an action form $\mathrm{H} \overset{\mathrm{def}}{=} \varkappa^*\eta$, where η is the canonical action 1-form on the cotangent bundle $T^*\mathcal{P}_k$ (see n° 1.3). In terms of the splitting (3-11), H sends $v = (\vartheta_1, f_1) \in T_{(S,\vartheta_0)}\mathcal{P}^{\mathrm{hw}} \simeq \mathfrak{H}(S) \oplus C^\infty(S)$ to

$$\mathrm{H}(v) = 2\int_S f_1\, d\vartheta_0\vartheta_1\ , \tag{3-14}$$

and its differential $\Omega = d\mathrm{H}$ acts on $T_{(S,\vartheta_0)}\mathcal{P}^{\mathrm{hw}}$ by the rule

$$\Omega\big(\,(\,\vartheta_1, f_1\,)\ ,\,(\,\vartheta_2, f_2\,)\,\big) = \int_S f_2\, d\vartheta_0\vartheta_1 - \int_S f_1\, d\vartheta_0\vartheta_2\ , \tag{3-15}$$

which is a straightforward infinite dimensional generalization[1] of (1-3) with the summation replaced by the integration over S. In terms of the complex DW-coordinates from n° 3.5, the real tangent vector, along $(\psi_1, 0)$, and an imaginary tangent vector $(0, \psi_2)$ are paired by Ω as

$$\Omega\big(\,(\,\psi_1, 0\,)\ ,\,(\,0, \psi_2\,)\,\big) = \int_S \psi_1\psi_2\, d\vartheta_0^2$$

[1] We would like to remark here that this infinite dimensional symplectic form was commonly used for $S = S^1$ in the context of the Korteweg - de Vries equation (see [DKN] and also our ‘toy example’ in § 5). Such the symplectic form has appeared also in the Hamiltonian formulation of the motion of vortex patches for planar incompressible fluids.

that coincide with L^2-product of ψ_1 and ψ_2 w. r. t. the measure ϑ_0^2. So, Ω is compatible with the complex structure on $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$, that is, produces the Kähler triple whose Riemannian metric G acts on $T_{(S,\vartheta_0)}\mathcal{P}^{\mathrm{hw}} = C^\infty(S) \oplus \mathfrak{H}(S)$ by the rule

$$G\big(\,(\vartheta_1, f_1)\,,\,(\vartheta_2, f_2)\,\big) = \int_S f_2\, d\vartheta_1\vartheta_0 + \int_S f_1\, d\vartheta_2\vartheta_0\ , \tag{3-16}$$

So, we have proved the following:

3.5.2. THEOREM. *The space $\mathcal{P}_k^{hw}$, of half weighted Planckian cycles of level k, has the natural structure of an infinite dimensional Kähler variety; the symplectic form and the Riemannian metric induced by this structure are given by the formulas* (3-15) *and* (3-16) *respectively; the complex coordinates near a given $\widetilde{S} \in \mathcal{P}_k^{hw}$ are the smooth complex valued functions on S.* □

3.6. Half weighted Bohr - Sommerfeld cycles of fixed volume. The natural $U(1)$-action on $\mathcal{P}_k$, which changes a covariant trivialization of L over a given $S \subset M$, has the standard lifting on $T\mathcal{P}_k$ and on $\mathcal{P}_k^{\mathrm{hw}}$ as well. This action preserves the Kähler structure: $z \in U(1) \subset \mathbb{C}$ just multiplies by z the complex DW-coordinates. In particular, the both forms (3-15), (3-16) are $U(1)$-invariant (in fact, it is clear without the coordinates, because our identification $T_S\mathcal{P}_k \simeq C^\infty(S)$ did not depend on the choice of a trivialization of P over S, see n° 3.3.1). Since the volume function $\mathcal{P}_k^{\mathrm{hw}} \xrightarrow{vol} \mathbb{R}$, which sends $(\widetilde{S}, \vartheta)$ to $\displaystyle\int_S \vartheta^2$, is $U(1)$-invariant, we can separately factorize by $U(1)$-action each level hypersurface $\mathcal{P}_{k,t}^{\mathrm{hw}}$ that consists of weighted Planckian cycles whose underlying BS_k-cycle has the fixed volume t. The corresponding orbit space $\mathcal{B}_{k,t}^{\mathrm{hw}} \overset{\mathrm{def}}{=} \mathcal{P}_{k,t}^{\mathrm{hw}}/U(1)$ is called a space of *half weighted* BS_k-cycles of volume t.

3.6.1. THEOREM. *The space $\mathcal{B}_{k,t}^{hw}$, of half weighted BS_k-cycles of volume t, also has a natural structure of an infinite dimensional Kähler variety. This structure is induced by the Kählerian structure on $\mathcal{P}_k^{hw}$; in particular, its symplectic form and the Riemannian metric remain to be given by the formulas* (3-15) *and* (3-16) *respectively.*

PROOF. In terms of the decomposition (3-11), the tangent space $T_{(\tilde{S},\vartheta_0)}\mathcal{P}_{k,t}^{\mathrm{hw}}$ consists of $v = (f, \vartheta)$ such that $\displaystyle\int_S d\vartheta\vartheta_0 = 0$, because in the DW-coordinates an infinitesimal volume increment along v is measured by this integral. Since the $U(1)$-factorization takes (f, ϑ) to $(df, \vartheta) \in T_{(S,\vartheta_0)}\mathcal{B}_{k,t}^{\mathrm{hw}}$, the both forms (3-15), (3-16) remain to be well defined, because the conditions $\displaystyle\int_S d\vartheta_1\vartheta_0 = \int_S d\vartheta_2\vartheta_0 = 0$ annihilate an arbitrariness in lifting df to f. □

3.6.2. Remark towards the symplectic reduction. Of course, the above construction is nothing but taking the *symplectic quotient* $\mathcal{P}_k^{\mathrm{hw}}//U(1)$. Indeed, in our commutative case, the $U(1)$-invariant function *vol* gives a moment map for the $U(1)$-action on $\mathcal{P}_k^{\mathrm{hw}}$; factorizing its level hypersurfaces, we just follow the general machinery of producing the symplectic quotient from the moment map (see [At] and [AB2] for a detailed explanations of what's it in general setup).

3.7. Prequantization equipment of $\mathcal{B}_{k,t}^{\mathrm{hw}}$. The Kählerian manifold $\mathcal{B}_{k,t}^{\mathrm{hw}}$ is quipped with his own prequantization data. Namely, there is the principal $U(1)$-bundle

$$\mathcal{P}_{k,t}^{\mathrm{hw}} \xrightarrow{\pi_t^{\mathrm{hw}}} \mathcal{B}_{k,t}^{\mathrm{hw}}\ , \tag{3-17}$$

obtained by the restriction of $\mathcal{P}_k^{\mathrm{hw}} = \widetilde{F}^*\mathcal{P} \longrightarrow \mathcal{B}_k^{\mathrm{hw}}$ (upper left horizontal row on the diagram (3-9)) on $\mathcal{B}_{k,t}^{\mathrm{hw}}$ and equipped the $U(1)$-connection $\mathfrak{a} \overset{\mathrm{def}}{=} \mathrm{H}\Big|_{\mathcal{P}_{k,t}^{\mathrm{hw}}}$, which is the action form (3-14) restricted on the Planckian cycles of volume t. By the construction, its curvature $F_{\mathfrak{a}} = 2\pi i\,\Omega$, that is $\omega_{\mathfrak{a}} = \Omega$ as required in n° 3.1. Moreover, this prequantization data has the canonical Kähler polarization described in n° 3.6.1. So, it is ready for applying the Kählerian quantization procedure described in the next paragraph.

3.7.1. Remark on weighting rules. The prequantization data $(\mathcal{B}_{k,t}^{\mathrm{hw}}, \Omega, \mathcal{P}_{k,t}^{\mathrm{hw}}, \mathfrak{a})$ can be used for producing a prequantization equipment for the (non weighed) Berry bundle $\mathcal{P}_{k,t} \xrightarrow{\pi_t} \mathcal{B}_{k,t}$. Although there is no canonical $U(1)$-connection here, one can fix some one by pulling back the 'universal' connection $\mathfrak{a}$ from $\mathcal{P}_{k,t}^{\mathrm{hw}}$ along a suitable section $\mathcal{P}_k \hookrightarrow \mathcal{P}_k^{\mathrm{hw}}$. A choice of such a section is called *a Planckian half weighting rule.* A Planckian half weighting may be lifted in turn from a *Bohr - Sommerfeld* one, which is just a section

$$\mathcal{B}_k \hookrightarrow \mathcal{B}_k^{\mathrm{hw}} \ , \tag{3-18}$$

For example, if M is an almost Kählerian manifold, then the Riemannian metric g on M has non degenerate restriction on each Lagrangian cycle S and gives there a canonical measure $\mu_{L,g}$. Taking $\widetilde{S}$ with the half weight class $\vartheta_{L,g} = (1, \mu_{L,g})$, we get the natural half weighting rule called *the Kähler half weighting.*

§ 4. Holomorphic quantization.

4.1. Kähler polarization. To quantize a prequantization data $(M, \omega, L^{\otimes k}, \alpha_k)$ (see n° 3.1) we have *to polarize* it (see [T2]). By the definition, a *complex* (or *Kähler*) polarization is a choice of a complex structure I on M such that M_I turns to a Kähler manifold with the Kähler form ω. Since the curvature of the Hermitian connection α_k has the Hodge type $(1,1)$ w. r. t. I (see [GH, ch. 3]), the line bundle L turns automatically to a *holomorphic* line bundle on M_I.

4.1.1. Wave function spaces. As soon a complex polarization is fixed, *a complex quantization procedure* provides the collection of *wave function spaces*:

$$Q_k(M_I) = H^0(M_I, L^{\otimes k}) \ , \tag{4-1}$$

where $H^0(M_I, L^{\otimes k})$ is the space of I-holomorphic sections of the line bundle $L^{\otimes k}$. The integer index k is called *a level* of the quantization procedure (see the survey [K]). We will consider in this paper only the case of compact M where the wave function spaces (4-1) are finite dimensional. But we would like to outline here that all our constructions remains correct even in a non compact case, where the spaces of wave functions became infinite dimensional, for example, when $M = T^*X$ is the cotangent bundle of a configurations space X.

4.1.2. Hardy space and Szegö projector. Consider the initial principal $U(1)$-bundle $P = P_1$ on M_I as the boundary for the bundle of unit disks inside $L_{-1} = L^*$, the Hermitian dual to the prequantization line bundle L:

$$P = \partial D \ , \quad \text{where } D \subset L^* \longrightarrow M_I \ . \tag{4-2}$$

The total space D is a strictly pseudoconvex domain and its complex structure depends strongly on a choice of a complex structure I on M, that is, on a choice of a complex polarization. Let $L^2(P)$ be the L^2-completion of the space $C^\infty(P,\mathbb{C})$, of smooth $\mathbb{C}$-valued functions on P w. r. t. the Hermitian scalar product defined by the contact structure volume form (3-4):

$$(f,g) \stackrel{\text{def}}{=} \int_P f\overline{g}\,\alpha_k \wedge (d\alpha_k)^{\wedge n} \ . \tag{4-3}$$

A closed subspace $\mathcal{H} \subset L^2(P)$ that consists of functions admitting a holomorphic continuation onto D is called a *Hardy space* on P. The orthogonal projector

$$\Pi\colon L^2(P) \twoheadrightarrow \mathcal{H} \tag{4-4}$$

is called *the Szegö projector*.

4.1.3. $U(1)$ weight decomposition of the Hardy space. The action of $U(1)$ on the principal bundle P commutes with the Szegö projector and produces the weight decomposition $\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k$, where k-th isotype component $\mathcal{H}_k$ consists of fiberwise homogeneous polynomials of degree k on L^*, that is, can be naturally identified with $H^0(M, L^{\otimes k})$. So, the weight decomposition of $\mathcal{H}$ w. r. t. $U(1)$-action looks like

$$\mathcal{H} = \bigoplus_k H^0(L^{\otimes k}) \ . \tag{4-5}$$

Under this identification, the Hermitian structure on $\mathcal{H}$ coming from $L^2(P)$ goes to the Hermitian structure on $H^0(M, L^{\otimes k})$ induced by the Hermitian scalar product on $L^{\otimes k}$ and the integration through M:

$$(f,g)_k \stackrel{\text{def}}{=} \int_M (f,g)_{L^{\otimes k}}\,\omega^{\wedge n} \ , \tag{4-6}$$

because of the volume form $\omega^{\wedge n}$ on M is induced by the volume form $\alpha_k \wedge (d\alpha_k)^{\wedge n} = \alpha_k \wedge \omega^{\wedge n}$ on P. We write Π_k for the orthogonal Szegö projectors $L^2(P) \xrightarrow{\Pi_k} H^0(L^{\otimes k})$.

4.1.4. Toeplitz operators. Each function f on M produces a series of *Toeplitz operators* $H^0(L^{\otimes k}) \xrightarrow{Q_k(f)} H^0(L^{\otimes k})$ as follows. To calculate $Q_k(f)\psi$ for $\psi \in H^0(L^{\otimes k})$, we consider ψ as an element of $\mathcal{H} \subset L^2(P)$, then multiply it by f, and then apply the Szegö orthogonal projector onto $H^0(L^{\otimes k})$. Briefly,

$$Q_k(f)(\psi) = \Pi_k(f\psi) \in H^0(L^{\otimes k}). \tag{4-7}$$

All Toeplitz operators on $\mathcal{H}$ form a Poisson algebra, which is *a quantum deformation* of the Poisson algebra of smooth functions on the original symplectic manifold M in the sense of n° 0.4.

4.1.5. Projective embedding of M_I. The holomorphic line bundle $L^{\otimes k}$ defines a rational (that is, meromorphic) map $M_I \longrightarrow \mathbb{P}\left(H^0(L^{\otimes k})^*\right)$ by sending $x \in M_I$ to the hyperplane

$$Ann\,(x) = \{\,\psi \in H^0(L^{\otimes k}) \mid \psi(x) = 0\,\} \ .$$

More precisely, there is an exact sequence of sheaves on M

$$0 \longrightarrow \mathcal{I}_x \otimes L^{\otimes k} \longrightarrow L^{\otimes k} \xrightarrow{res} \mathcal{O}_M\big|_x \longrightarrow 0$$

where *res* is the restriction epimorphism and $\mathcal{I}_x$ is the ideal sheaf of x. Its long exact cohomology sequence

$$\cdots \longrightarrow H^0(\mathcal{I}_x \otimes L^{\otimes k}) \longrightarrow H^0(L^{\otimes k}) \longrightarrow \mathbb{C} \longrightarrow H^1(\mathcal{I}_x \otimes L^{\otimes k}) \longrightarrow \cdots$$

shows that $H^0(\mathcal{I}_x \otimes L^{\otimes k})$ is a hyperplane if $H^1(\mathcal{I}_x \otimes L^{\otimes k}) = 0$. Since $H^1(\mathcal{I}_x \otimes L^{\otimes k}) = 0$ for $k \gg 0$ by Serre's classical A & B Theorems, the map

$$\varphi_k \colon M_I \xrightarrow{x \mapsto Ann\,(x)} \mathbb{P}\left(H^0(L^{\otimes k})^* \right) \tag{4-8}$$

is an embedding for $k \gg 0$. This standard construction reduces the study of M_I to the projective algebraic geometry.

4.1.6. The space of states. The Hermitian structure (4-6) gives an antiholomorphic isomorphism

$$H^0(L^{\otimes k}) \xrightarrow{\sim} \overline{H^0(L^{\otimes k})}^{\,*} . \tag{4-9}$$

Applying its projectivization after the holomorphic embedding (4-8), we get the complex conjugate (antiholomorphic) embedding

$$\overline{\varphi}_k \colon M_I \hookrightarrow \mathbb{P}H^0(L^{\otimes k}) \tag{4-10}$$

In the complex quantization context, vectors in $H^0(L^{\otimes k})$ are called *states* and vectors $\overline{\varphi}(x)$, where $x \in M$, are called *coherent states*.

4.1.7. Important warning. Although the principal prequantization bundle $P_k \xrightarrow{p_k} M$ of level k is nothing but the pull back of the degree k Hopf circle bundle over the projective space $\mathbb{P}H^0(M_I, L^{\otimes k})$ along (4-10), the curvature 1-form α_k on P_k does not coincide, in general, with the pull back of the Fubini - Study invariant 1-form on the Hopf bundle.

4.2. Borthwick - Paul - Uribe map. Now, let $S \overset{\widetilde{\varphi}}{\hookrightarrow} P_k$ be a Planckian cycle, that is a BS_k-cycle $S \overset{\varphi}{\hookrightarrow} M$ together with a fixed covariant constant unitary section $S \overset{\sigma}{\hookrightarrow} \varphi^* L^{\otimes k}$. For any holomorphic section $s \in H^0(M_I, L^{\otimes k})$ consider a smooth complex valued function $\psi_{\widetilde{S}} \in C^\infty(S, \mathbb{C})$ such that $\varphi^* s = \psi_{\widetilde{S}} \cdot \sigma$ in $C^\infty(S, \varphi^* L^{\otimes k})$. Note that the linear map

$$H^0(M, L^{\otimes k}) \overset{s \mapsto \psi_{\widetilde{S}}}{\hookrightarrow} C^\infty(S, \mathbb{C}) \tag{4-11}$$

is *injective*, because of n° 1.5.4, and holomorphic w. r. t. the standard complex structure on $C^\infty(S, \mathbb{C})$, which is nothing but the fiber of the *holomorphic* tangent bundle $T_{(\widetilde{\varphi}, \vartheta)} \overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$, where $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ is considered with the intrinsic holomorphic structure defined in n° 3.5. So, we have an inclusion of coherent sheaves on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$:

$$0 \longrightarrow \mathcal{H}_k \otimes \mathcal{O}_{\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k} \longrightarrow \mathcal{T}_{\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k} , \tag{4-12}$$

where the first is the sheaf of holomorphic sections of the trivial vector bundle with the fiber $\mathcal{H}_k = H^0(M_I, L^{\otimes k})$ and the second is the sheaf of holomorphic vector fields on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$.

4.2.1. THEOREM. *A map $\overset{\circ}{\mathcal{P}}{}^{hw} \xrightarrow{\overline{\mathfrak{m}}_k} H^0(M, L^{\otimes k})^*$ that sends a half weighted Planckian cycle $(\widetilde{\varphi}, \vartheta)$ to $\mathbb{C}$-linear form taking $s \in H^0(M, L^{\otimes k})$ to $\displaystyle\int_S \psi_{\widetilde{S}}\, d\vartheta^2 = \int_S \psi_{\widetilde{S}}\, f^2\, d\mu \in \mathbb{C}$ is antiholomorphic with the surjective differential.*

PROOF. In terms of the complex DW-coordinates on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}$ near $(\widetilde{\varphi},\vartheta)$, a linear functional $\overline{\mathfrak{m}}_k(\psi_1+i\psi_2)$ on $H^0(M,L^{\otimes k})$ takes s to $\displaystyle\int_S \psi_{\widetilde{s}}\psi_1 e^{-i\psi_2}\,d\vartheta^2$, because P_k is trivialized by $e^{i\psi_2}\widehat{\sigma}$ over $S^{(k)}_{\psi_2}$ (see n° 3.5) and s is holomorphic, that is, $\alpha_k(s)=0$. So, the differential $d\overline{\mathfrak{m}}_k$ sends $\psi_1+i\psi_2$ to the functional

$$s \longmapsto \int_S \psi_{\widetilde{s}}(\psi_1-i\psi_2)\,d\vartheta^2\ , \tag{4-13}$$

which $\mathbb{C}$-antilinearly depends on $\psi_1+i\psi_2$ as required. Moreover, the right hand site is nothing but the standard Hermitean pairing on the Hilbert space $L^2(S,\mathbb{C})$ w. r. t. the measure ϑ^2, that is, any $\mathbb{C}$-linear form on the finite dimensional subspace $H^0(M,L^{\otimes k})\subset L^2(S,\mathbb{C})$ has the form (4-13) for suitable $\psi_1+i\psi_2$. So, $d\overline{\mathfrak{m}}_k$ is surjective at every point. □

4.2.2. Holomorphic BPU$_k$-map. The map $\overline{\mathfrak{m}}_k$ from the theorem n° 4.2.1 is called the *antiholomorphic BPU$_k$-map.* Its composition with the Hermitian structure isomorphism (4-9), that is, the holomorphic map $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}\xrightarrow{\mathfrak{m}_k}H^0(M,L^{\otimes k})$, is called a *holomorphic* BPU$_k$-map. It sends a half weighted Planckian cycle $(\widetilde{\varphi},\vartheta)$ to the holomorphic section $\mathfrak{m}_k(\widetilde{\varphi},\vartheta)\in H^0(M,L^{\otimes k})$ such that

$$\int_M (\,s\,,\,\mathfrak{m}_k(\widetilde{\varphi},\vartheta)\,)_{L^{\otimes k}}\,d\varrho=\int_S \psi_{\widetilde{s}}\,d\vartheta^2$$

for any $s\in H^0(M,L^{\otimes k})$. It follows from (4-13) that the differential $T_{(\widetilde{\varphi},\vartheta)}\xrightarrow{d\mathfrak{m}_k}H^0(M,L^{\otimes k})$ sends $\psi\in C^\infty(S,\mathbb{C})$ to the section $s_\psi\in H^0(M,L^{\otimes k})$ such that

$$\int_M (\,s,s_\psi\,)_{L^{\otimes k}}\,d\varrho=\int_S \psi_{\widetilde{s}}\overline{\psi}\,d\vartheta^2\ .$$

So, the Hermitean dual map $\mathcal{H}_k\xrightarrow{d\mathfrak{m}_k^*}C^\infty(S,\mathbb{C})$ (w. r. t. the standard Hermitean structures on the both spaces) is nothing but the inclusion (4-11).

4.2.3. Symplectic reduction of the BPU$_k$-map w. r. t. $U(1)$-action. Recall (see [At]) that the projectivization $\mathbb{P}V=V/\mathbb{C}^*$, of a Hermitean vector space V, may be treated as symplectic reduction $\mathbb{P}V=V//U(1)$ w. r. t. the moment map $v\mapsto\|v\|$. So, the symplectic quotient $\mathcal{H}_k//U(1)$ w. r. t. the moment map $s\mapsto\displaystyle\int_M (s,s)_{L^{\otimes k}}d\varrho$ is the space of states $\mathbb{P}\mathcal{H}_k$. Recall also that in n° 3.6 we have considered the natural $U(1)$-action on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}$ with the moment map $(\widetilde{\varphi},\vartheta)\mapsto\displaystyle\int_S d\vartheta^2$. The corresponding symplectic quotient $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}//U(1)$ is the space of half weighted BS$_k$-cycles of fixed volume $\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}$. Now, the holomorphic BPU$_k$-map $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}\xrightarrow{\mathfrak{m}_k}H^0(M,L^{\otimes k})$ is $U(1)$-equivariant, because its differential is. Hence, it is factorizible to the holomorphic map

$$\mathfrak{m}_{k,t}\ :\ \overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}\longrightarrow\mathbb{P}\mathcal{H}_k$$

with the surjective differential. This map assigns holomorphically a state to each half weighted BS$_k$-cycle of a fixed volume t. We call it *BPU$_{k,t}$-map.* By the construction, this map is nothing but the projective map induced by the holomorphic Bery line bundle $\mathcal{L}_{k,t}$ associated with the principal Bery bundle $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k,t}\xrightarrow{\pi_t}\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}$, that is, $\mathcal{L}_{k,t}=\mathfrak{m}_{k,t}^*\big(\mathcal{O}_{\mathbb{P}\mathcal{H}_k}(1)\big)$.

4.3. Action of symplectomorphisms of M. The intrinsic infinite dimensional prequantization data $(\mathcal{B}^{\mathrm{hw}}_{k,t},\Omega,\mathcal{P}^{\mathrm{hw}}_{k,t},\mathfrak{a})$ described in n° 3.7 is invariant with respect to the natural action of the identity component Sym_0 in the symplectomorphism group $Sym = Sym(M,\omega)$. Recall that two half weighted BS_k-cycles $(S_1,\vartheta_1),(S_2,\vartheta_2)\in\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}$ are called *isotopic* if there is a one parameter family of symplectomorphisms $g_t\in Sym$ such that $g_0=\mathrm{Id}_M$ and $g_1\Big((S_1,\vartheta_1)\Big)=(S_2,\vartheta_2)$. The BPU_k-map is, clearly, invariant under the symplectomorphisms, that is, produces the family of invariants for the isotopy classes. We would like to make here some remarks towards more strong factorization that is an infinite dimensional analog of the *symplectic reduction* w. r. t. to the Sym_0-action. Since the BPU_k-map, which is quite similar to what we have in mind, produces a finite dimensional factor, it is reasonable to wait that $\dim\Big(\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k}//Sym_0\Big)<\infty$.

The Lie algebra $\mathfrak{sym}$, of the symplectomorphism group, is the algebra of Hamiltonian vector fields on M. Its Lie coalgebra may be treated as

$$\mathfrak{sym}^* = C_{2n-1}(M)/dC_{2n}(M)\ ,$$

where $C_j(M)$ is the space of degree $2n-j$ currents with a compact support (it is dual of the space of differential forms of degree j on M). The moment map $\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}\xrightarrow{\mu}\mathfrak{sym}^*$ is defined by sending a half weighted BS_k-cycle (S,ϑ) to the functional $\mu_{(S,\vartheta)}$ that takes $v\in\mathfrak{sym}$ to

$$\langle\,\mu_{(S,\vartheta)}\,,\,v\,\rangle\overset{\mathrm{def}}{=}\int_S\xi_v\,d\vartheta^2\ ,$$

where $\xi_v=\omega\xi$ is a 1-form corresponding to the vector field v on M. Clearly, μ is equivariant w. r. t. the coadjoint action of Sym on $\mathfrak{sym}^*$ and $\{\mu_{(S,\vartheta)}(v),\mu_{(S,\vartheta)}(w)\}=\mu_{(S,\vartheta)}([v,w])$ for any pair $v,w\in\mathfrak{sym}$. So, we have:

4.3.1. PROPOSITION. *The orbits of Sym_0 in $\overset{\circ}{\mathcal{B}}{}^{hw}_{k,t}$ are the isotopy components of $\overset{\circ}{\mathcal{B}}{}^{hw}_{k,t}$ and any such a component is diffeomorphic to the homogeneous space*

$$\mathbb{P}\left(H^0(L^k)\right)\times\left(Sym_0/Stab_{(S,\vartheta)}\right)\ ,$$

where (S,ϑ) is any point of the component. □

In other words, there exists a smooth bijective map from $S\times\mathbb{P}\left(H^0(L^k)\right)$ to the coadjoint orbit of $\mu_{(S,\vartheta)}$. Is it a diffeomorphism? A quite fruitful application of this construction will be considered in § 5 where we will deal with the loop space LS^2 and the space of Legendrian knots in S^3.

4.4. Analytical viewpoint on the BPU_k-map. Originally, the BPU_k-map was discovered in a pure analytical framework, see [BPU]. To allow the reader to compare our geometrical approach with the original one, we trace out very briefly some of the analytical key points in this section, although we will not use it in what follows.

4.4.1. Analytical definition of the BPU_k-map The analytical background of the BPU_k-map construction is based on the remarkable possibility to continue the Szegö projector $L^2(P,\mathbb{C})\xrightarrow{\Pi_k}H^0(M,L^{\otimes k})$ to some space of distributions $\mathcal{D}(\mathcal{H})\supset L^2(P,\mathbb{C})$, which includes the delta functions δ_p of all points $p\in P$. This continuation uses the Toeplitz structure concept developed in [BG] and sketched in n° 4.4.2 below.

Applying the continued Szegö projector to δ_p with $p\in\widetilde{S}\subset P$, we get a family of holomorphic sections $\{\Pi_k\left(\delta_p\right)\}_{p\in P_k}\subset H^0(M_I,L^{\otimes k})$ and the BPU_k-map $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}\xrightarrow{\mathfrak{m}_k}H^0(M_I,L^{\otimes k})$ just takes their overage along S, that is, sends each $(\widetilde{S},\vartheta)\in\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_{k}$ to $\mathfrak{m}_k(S,\vartheta)=\displaystyle\int_S\Pi_k(\delta_p)d\vartheta\,.$

Now, if we write $K_k(p_1,p_2)$ for the Schwartz kernel of the continued Π_k, then we have $K_k(p_1,p_2) = \Pi_k\,(\,\delta_{p_2}\,)\,(p_1)$, where $s(p_k(p)) \stackrel{\text{def}}{=} \langle\, s\,,\, \Pi_k(\delta_p)\,\rangle$ for $s \in H^0(M_I, L^{\otimes k})$. Taking here $s = \Pi_k(\delta_p)$, we get $K_k(p,p) = \|\Pi_k\,(\,\delta_p\,)\,\|^2$, which is a function of $p \in P_k$ with the L^2-norm

$$\|K_k(p,p)\|^2 = \int_{P_k} K_k(p,p)\,\alpha_k \wedge (d\alpha_k)^{\wedge n} = \operatorname{rk} H^0(M_I, L^{\otimes k}) \,\sim\, \left(\frac{k}{2\pi}\right)^n$$

for k large enough by the Riemann - Roch theorem.

4.4.2. Generalized Toeplitz operators. Let us cut the zero section out the cotangent bundle T^*P_k and consider a section $P_k \xhookrightarrow{\alpha_k} T^*P_k \setminus \{0\}$ given by the curvature form. An open real positive cone over its image

$$\Sigma \stackrel{\text{def}}{=} \{\, t\ \alpha_k|_p \subset T^*P_k\ \setminus\ \{0\} \mid p \in P_k\,,\ t \in \mathbb{R},\ t > 0\,\} \tag{4-14}$$

is a *symplectic* submanifold w. r. t. the standard symplectic structure on the cotangent bundle. The pair (Π, Σ), where Π is the Szegö projector (4-4), is called a *Toeplitz structure* of Boutet de Monvel and Guillemin (see [BG]). For any pseudodifferetial operator A of order n on P_k an operator $T_A = \Pi \circ A \circ \Pi\, :\, L^2(P) \longrightarrow \mathcal{H}$ is called *a generalized Toeplitz operator*. A *symbol* of the generalized Toeplitz operator T_A is the principal symbol $p(A)$ restricted on the symplectic submanifold (4-14) : $\sigma(T_A) \stackrel{\text{def}}{=} p(A)|_\Sigma$. It is well defined, although A is not uniquely recovered from $p(A)$. Moreover, the symbols of the generalized Toeplitz operators obey the standard rules for pseudodifferetial operator symbols:

$$\begin{gathered}
\sigma\,(\,T_{A_1}T_{A_2}\,) = \sigma\,(\,T_{A_1}\,)\,\sigma\,(\,T_{A_2}\,) \\
\sigma\,(\,[T_{A_1}, T_{A_2}]\,) = i\,\{\sigma\,(\,T_{A_1}\,)\,, \sigma\,(\,T_{A_2}\,)\}\ ,
\end{gathered}$$

where the last Poisson bracket is with respect to the restriction of the standard symplectic form to Σ. For example, a usual (non generalized) zero order Toeplitz operator T_f coming from the multiplication by $f \in L^2(P_k)$ has the symbol $\sigma(T_f) = p_\Sigma^*(f)$, where p_Σ is the composition of the projections $\Sigma \twoheadrightarrow P_k \xrightarrow{p_k} M$.

4.4.3. Extended Szegö projector. Of course, the conormal bundle of a smooth Planckian cycle $N^*\widetilde{S} = Ann\,(T\widetilde{S}) \subset T^*P_k$ is Lagrangian w. r. t. the standard symplectic structure on T^*P_k. Hence, it defines a series of Hermitian distributions spaces $I^m(P_k, N^*S) \subset D(P_k)$ of half integer order $m \in \frac{1}{2}\mathbb{Z}$ (see [BG]) and $L^2(P_k) \subset I^m(P_k, N^*S)$, certainly. Each Hermitean distribution also has *a symbol*, which is a symplectic spinor on N^*S. Using composition theorems from [BG] for Hermitian distributions, the Szegö projector (4-4) can be continued to the map

$$I^m(P_k, N^*\widetilde{S}) \xrightarrow{\ \Pi\ } \mathcal{D}(\mathcal{H})$$

as follows. The Schwartz kernel of Π lays in $I^{\frac{1}{2}}(P_k \times P_k, \Sigma_2)$, where

$$\Sigma_2 = \{\,(p_1, t\alpha(p_1))\,;\ (p_2, -t\alpha(p_2))\,\} \subset T^*P_k \times P_k\ .$$

is an isotropic submanifold, which is *composable* with N^*S (see [BG, Th. 9.4]), that is, one can apply the Fourier integral operator associated with Σ_2 to the Lagrangian distributions on N^*S. This gives the map $I^m(P_k, N^*\widetilde{S}) \hookrightarrow \mathcal{D}(\mathcal{H})$ whose image $J^{m-\frac{n}{2}}(P_k, S) \subset \mathcal{D}(\mathcal{H})$ is called the space of *Legendrian distributions* of order $m - \frac{1}{4}\dim M$. Finally, $J^{\frac{1}{2}}(P_k, S)$ may be identified with the space of half weights: each element of $J^{\frac{1}{2}}(P_k, S)$ produces a half weight on S associated with its symbol, which is a symplectic spinor on the Legendrian submanifold $\widetilde{S} \subset P_k$ (see [BG] and [BPU]), and wise versa, a Hermitean distribution is recovered from a half weight ϑ on S as

$$s(\vartheta) = T^{m-\frac{1}{2}}_{\mathcal{L}ie_{u_k}} \Pi(\delta_S) ,$$

where $\delta_{(S,\vartheta)}$ is the delta function defined by a pair (S, ϑ) and $T_{\mathcal{L}ie_{u_k}}$ is the generalized first order Toeplitz operator induced by the Lie differentiation along the standard $U(1)$-invariant vertical unitary vector field u_k on P_k (its symbol is the real function t, the coordinate along the rulers on the conic submanifold (4-14)).

§ 5. A 'toy' example: the Legendrian knots in S^3.

5.1. Standard prequantization data on Riemannian sphere. We are going to illustrate the constructions from the last two paragraphs in the simplest non trivial case, where (M, ω) is the Riemannian sphere $S^2 = \mathbb{CP}_1 = \mathbb{P}(\mathbb{C}^2)$ equipped with the standard Kähler structure on $\mathbb{CP}_1$ and the standard very ample line bundle of homogeneous degree 1 forms $L = \mathcal{O}(1)$ as the prequantization line bundle. So, the principal prequantization bundle $P \xrightarrow{p} \mathbb{CP}_1$, that is the bundle of unit circles in the tautological line bundle $L^* = \mathcal{O}(-1)$, is nothing but the Hopf bundle $S^3 \xrightarrow{p} \mathbb{CP}_1$ whose total space is the unit sphere $S^3 \subset \mathbb{C}^2$ w.r.t. the standard Hermitean structure: its fiber over $\ell \subset \mathbb{C}^2$ is the unit circle $\ell \cap S^3$. The homotopy class of this bundle gives the standard negative generator $\{-1\} \in \pi_3(S^2) \simeq \mathbb{Z}$ and the other negative elements of this group are exhausted by our level k principal prequantization bundles P_k, which are the twisted Hopf bundles $S^3 \xrightarrow{p_k} S^2$, of the unit circles in $\mathcal{O}(-k)$. The standard $\mathfrak{u}(1)$-valued connection form on $P = P_1$ is usually written as of the $U(2)$-invariant 1-form

$$2\pi i\, \alpha = \left(\partial - \overline{\partial}\right) \log \varrho ,$$

restricted on $S^3 \subset \mathbb{C}^2$, where $\mathbb{C}^2 \xrightarrow{\varrho} \mathbb{R}$ is the Hermitean length $\varrho(v) = \|v\| = \sqrt{(v,v)}$. The differential

$$d\alpha = \frac{i}{2\pi}\, \partial\overline{\partial} \log \varrho = p^*\omega$$

is the pull back of the standard Kähler form on $\mathbb{CP}_1$ (see, for example, [GH, ch. 0 , § 7]). The cohomology class $[\omega]$ coincides with $c_1(L)$ and gives the standard 'positive' generator for the group $H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$.

5.2. Lagrangian cycles, or unicoursal curves on S^2. We take the simplest source manifold for the Lagrangian cycles, namely, the circle $S = S^1$. Since any smooth map $S^1 \xrightarrow{\varphi} S^2$ is Lagrangian just because of the dimension reasons, the space of smooth Lagrangian immersions $Map_{\mathrm{LA}}(S^1, S^2)$ turns to the space of loops LS^2 and the space of Lagrangian cycles $\mathfrak{L} = Map_{\mathrm{LA}}(S^1, S^2)/Diff_0(S^1)$ is just the space of (unparameterized) *unicoursal curves* $\varphi(S^1) \subset S^2$. Further, the tangent space to the loop space is the space of smooth sections of the pulled back tangent bundle: $T_\varphi\,(LS^2) = C^\infty(S^1, \varphi^*TS^2)$. This 2-bundle is equipped with the non degenerate skew form $\varphi^*\omega$, which gives the natural symplectic structure

$$\Omega_{LS^2}(\sigma_1, \sigma_2) = \int_{S^1} \varphi^*\omega^{-1}(\sigma_1) \wedge \varphi^*\omega^{-1}(\sigma_2) \qquad (5\text{-}1)$$

on $T\,(\,LS^2\,)$. It is, clearly, $\mathit{Diff}_0(S^1)$-invariant and gives the symplectic structure on $\mathfrak{L}$. It is closed and weakly non degenerated but *a priori* does not define the isomorphism between the tangent and cotangent bundles, in particular, only the *supersmooth* functions (see [Br, def. 3.1.5]) on $\mathfrak{L}$ will define the Hamilton vector fields.

5.2.1. Darboux - Weinstein coordinates. Following n° 1.4, we identify the normal bundle to S^1 along φ with the cotangent bundle on S^1, which *is trivial*:

$$N_\varphi S^1 = \varphi^* TS^2/TS^1 \simeq \varphi^* T^* S^2/\mathit{Ann}\,(TS^1) = T^* S^1 = S^1 \times \mathbb{R}\,. \tag{5-2}$$

Let us fix this trivialization by taking the rotation invariant 1-form ν_{S^1} with $\displaystyle\int_{S^1} \nu_{S^1} = 1$ as the base of T^*S^1. Note that this form produces the smooth measure μ_0, whose class will be used below for our half weighings.

The Darboux - Weinstein construction (see n° 1.4.2) continues any smooth immersion $S^1 \longrightarrow S^2$ to an *symplectic* immersion $U \xrightarrow{\widehat{\varphi}} S^2$, where $U \subset T^*S^1$ is some neighborhood around the zero section considered with the canonical symplectic structure $\widehat{\varphi}^*\omega = d\eta$ on the cotangent bundle T^*S^1. Thus, the graphs of closed 1-forms laying in U form the standard chart near φ on $\mathfrak{L}$. A local coordinatization of $\mathfrak{L}$ by such the charts from $Z^1_{\mathrm{DR}}(S^1)$ is called *the DW-coordinatization.*

5.2.2. Remark on extra symmetries. The case we are deal with now has some special extra symmetries that are lacking in the general setup. They are:

- the triviality of the cotangent bundle (5-2);
- the natural inclusion $U(1) \subset \mathit{Diff}_0(S^1)$, which allows to consider the source manifold S^1 as the principal $U(1)$-space;
- the natural coincidence between $U(1)$ and the Jacobian $J(S^1) = H^1(S^1,\mathbb{R})/H^1(S^1,\mathbb{R})$.

We will outline in special remarks the additional effects added in the general picture by these coincidences.

5.3. Planckian cycles, or Legendrian knots in S^3. For each k the pull back $\varphi^*L^{\otimes k}$ along any loop $S^1 \xrightarrow{\varphi} S^2$ is the trivial complex line bundle on S^1 with the flat $U(1)$-connection $\varphi^*\alpha_k$. Its gauge class is described by a character $\pi_1(S^1) = \mathbb{Z} \xrightarrow{\chi_{\alpha_k}} U(1)$ such that $\chi_{\alpha_k}(1) \in U(1)$ is the fiber twist in $\varphi^*L^{\otimes k}$ under the one round parallel transport along S^1 (in particular, $\chi_{\alpha_k} = k\cdot\chi_{\alpha_1}$). The map $j_k\ :\quad LS^2 \xhookrightarrow{\varphi\mapsto k\cdot\chi_\alpha(1)} \mathcal{J}$ gives a section of the Jacobian bundle[1] (see n° 2.4). The zero set of this section is the codimension 1 subspace $\mathcal{B}_k \subset \mathfrak{L}$ of level k Bohr - Sommerfeld cycles. Clearly, $\forall\, k,m \in \mathbb{Z}$ we have the natural inclusions (3-3) : $\mathcal{B}_k \xhookrightarrow{\beta_{k,m}} \mathcal{B}_{mk}$.

A Planckian cycle is a class of Legendrian maps $S^1 \xrightarrow{\widetilde{\varphi}} S^3$ w. r. t. the contact structure α_k on $S^3 \simeq P_k$ modulo $\mathit{Diff}_0(S^1)$. Usually, a smooth map of this kind is called *a Legendrian knot* in S^3. In our terms, such a knot is nothing but a covariant constant global section $S^1 \xhookrightarrow{\sigma} \varphi^*P_k$ over a BS$_k$-cycle $S^1 \xrightarrow{\varphi} S^2$, that is, a covariant constant global unitary section of $\varphi^*L^{\otimes k}$ over S^1. So, Legendrian knots have got *the levels* and the knots of level k form a

[1]in our case $\mathcal{J}$ is the principal $U(1)$-bundle over LS^2, because of the third coincidence in n° 5.2.2: $H^1(S^1,\mathbb{R})/H^1(S^1,\mathbb{Z}) = U(1)$

principal $U(1)$-bundle $\mathcal{P}_k \xrightarrow{\pi_k} \mathcal{B}_k$. called *the Berry bundle*. The corresponding Hermitian line bundles $\mathcal{L}_k$ over $\mathcal{B}_k$ satisfy the conditions $\beta^*_{k,m}\,\mathcal{L}_{mk} = \mathcal{L}_k^{\otimes m}$.

The tangent space $T_{\widetilde{\varphi}}\mathcal{P}_k$ at a smooth Legendrian knot $S^1 \xhookrightarrow{\widetilde{\varphi}} P_k \simeq S^3$ of level k is naturally identified with $C^\infty(S^1)$. Extra symmetries listed in n° 5.2.2 make this to be almost trivial. Indeed, the tangent space $T_{\widetilde{\varphi}}\mathcal{P}_k$ is the space of smooth sections of the normal bundle

$$N_{\widetilde{\varphi}} = \widetilde{\varphi}^* T^* P_k / TS^1 = T^* S^1 \oplus T^* U(1) = S^1 \times \mathbb{R} \times \mathbb{R} \;,$$

where the right equality comes from (5-2) and the left one is induced by the connection α_k (in general case it holds only locally). Any $f \in C^\infty(S^1)$ gives the section $\tau_f \in C^\infty(S^1, N_{\widetilde{\varphi}})$ as follows. The vertical component of τ_f equals $kf \cdot \nu_{U(1)}$, where $\nu_{U(1)}$ is the basic 1-form $u_{U(1)}$ on $U(1)$ with $\displaystyle\int_{U(1)} \nu_{U(1)} = 1$. The horizontal component of τ_f is $k\,df \in C^\infty(T^*S^1)$ (and equals the angle derivative of kf multiplied by the standard 1-form ν_{S^1} on S^1 fixed in (5-2)).

5.4. Half weights. Let μ_0 be the rotation invariant measure on S^1 induced by ν_{S^1}. Then a smooth half weight $\vartheta = f\sqrt{\mu}$ on S^1 is an equivalence class of pairs (f, μ), where $f \in C^\infty(S^1)$, μ is a measure obtained from μ_0 by multiplication by a smooth function, and the equivalence $(f_1, \mu_1) \sim (f_2, \mu_2)$ means that

$$f_1\sqrt{\frac{d\mu_1}{d\lambda}} = f_2\sqrt{\frac{d\mu_2}{d\lambda}}$$

almost everywhere w. r. t. any measure λ equivalent to μ_0. The space of smooth half weights is denoted by $\overset{\circ}{\mathfrak{H}} = \overset{\circ}{\mathfrak{H}}(S^1)$ and the orbit spaces

$$\overset{\circ}{\mathfrak{L}}{}^{\mathrm{hw}} \overset{\mathrm{def}}{=} Map_{\mathrm{LA}}(S^1, S^2) \times \overset{\circ}{\mathfrak{H}}(S^1)/Diff_0(S^1) \qquad \overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k \overset{\mathrm{def}}{=} Map_{\mathrm{LE}}(S^1, S^3) \times \overset{\circ}{\mathfrak{H}}(S^1)/Diff_0(S^1) \;.$$

(w. r. t. the natural diagonal actions) are called spaces of *half weighted* unicoursal curves and Legendrian knots respectively.

5.4.1. Complex structure. For any fixed half weight $f\sqrt{\mu}$ the tangent space $T_{\widetilde{\varphi}}\mathcal{P}_k = C^\infty(S^1)$ is identified with $T_{f\sqrt{\mu}}\,\overset{\circ}{\mathfrak{H}} \simeq \overset{\circ}{\mathfrak{H}}$ by sending a function f_1 to the half weight $ff_1\sqrt{\mu}$. This gives the canonical integrable complex structure on $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$. Namely, consider a smooth knot $S^1 \xhookrightarrow{\widetilde{\varphi}} S^3$ laying over a BS$_k$-cycle $S^1 \xhookrightarrow{\varphi} S^3$ and half weighted by $\vartheta = f\sqrt{\mu}$. Then we have the splitting (3-11):

$$T_{(\widetilde{\varphi},\vartheta)}\,\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k \simeq T_\vartheta\,\overset{\circ}{\mathfrak{H}}(S^1) \oplus T_{\widetilde{\varphi}}\,\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k \simeq C^\infty(S^1) \oplus C^\infty(S^1) \;.$$

Now DW-construction identifies a small neighborhood of $(\widetilde{\varphi}, \vartheta)$ in $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ with a ball around the zero in the space of smooth complex valued functions $\psi = \psi_1 + i\psi_2 \in C^\infty(S^1, \mathbb{C})$. Indeed, the imaginary part ψ_2 produces the Legendrian knot, which is the section $(d\psi_2, e^{i\pi\psi_2})$ of the bundle

$$\widehat{\varphi}^* P_k|_U \subset T^*S^1 \times U(1)$$

The real part equips this knot with a half weight $\psi_1\vartheta = \psi_1 f\sqrt{\mu}$.

5.4.2. Kählerian structure on the space of half weighted Legendrian knots is given by the skew symmetric form whose action on $T_{(\widetilde{\varphi},\vartheta_0)}\,\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}}_k$ is given by

$$\Omega\big(\,(\,\vartheta_1, f_1\,)\;,\;(\,\vartheta_2, f_2\,)\,\big) = \int_{S^1} f_2\,d\vartheta_0\vartheta_1 - \int_{S^1} f_1\,d\vartheta_0\vartheta_2 \;.$$

This form is the differential of the action 1-form $\mathrm{H}\,(\,(\vartheta_1, f_1)\,) = 2\int_{S^1} f_1\, d\vartheta_0\vartheta_1$. The symmetric form corresponding to the above complex and symplectic structures acts on $T_{(S^1,\vartheta_0)}\mathcal{P}^{\mathrm{hw}} = C^\infty(S) \oplus \mathfrak{H}(S)$ as

$$G\big(\,(\vartheta_1, f_1)\,,\,(\vartheta_2, f_2)\,\big) = \int_{S^1} f_2\, d\vartheta_1\vartheta_0 + \int_{S^1} f_1\, d\vartheta_2\vartheta_0\ .$$

5.4.3. Half weighted Bohr - Sommerfeld unicoursal curves of fixed length are obtained by the symplectic reduction of the half weighted Legendrian knots w. r. t. the natural $U(1)$-action on the above data. The moment map for this action is given by the length functional

$$(\widetilde{\varphi}, \vartheta) \longmapsto \int_{S^1} d\vartheta^2$$

Its t-th level hypersurface factorized by $U(1)$ is denoted by $\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t}$. The previous complex structure and the forms Ω and G remains to be well defined on $T\,\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k}$ whose fiber over a given $(\widetilde{\varphi}, \vartheta_0)$ consists of all pairs (ϑ, df) such that $f \in C^\infty(S^1)$ and $\displaystyle\int_{S^1} d\vartheta\vartheta_0 = 0$.

To compare this Kählerian structure with the classical results, let us note that the fiber of the weight forgetful projection

$$\overset{\circ}{\mathcal{B}}{}^{\mathrm{hw}}_{k,t} \longrightarrow \mathcal{B}_{k,t} \tag{5-3}$$

is naturally identified with $\mathit{Diff}_0(S^1)/\,U(1)$, because each half weighting $\vartheta = f\sqrt{\mu}$ produces a well defined up to rotations parametrization of the curve by the arc length w. r. t. the measure $\vartheta^2 = f^2\mu$. The complex structure on $\mathit{Diff}_0(S^1)/\,U(1)$ is well known (see [K], for example). More generally, let $\mathcal{F}_{2m} \subset C^\infty(S^1)$ consist of functions having precisely $2m$ zeros (this space is a vector bundle over the $2m$-th symmetric power of S^1). These F_{2n} give the orbits for the natural action of $\mathit{Diff}(S^1)$ on $C^\infty(S^1)$ (see for example [K]). So, the space of, say, *positively* half weighted unicoursal curves is nothing but $LS^2/\,U(1) = \mathcal{F}_0$; in particular, it forms precisely one $\mathit{Diff}_0(S^1)$-orbit.

On the other hand, there is an almost complex structure on $\mathcal{B}_{k,t}$ defined as follows. Identifying S^1 with $U(1)$ like in n° 5.2.2, we get the complex structure on the loop space LS^3, just because the normal bundle of any loop $S^1 \xrightarrow{\widetilde{\varphi}} S^3$ splits as $N_{\widetilde{\varphi}} = T^*S^1 \oplus T^*U(1)$ and $T_{\widetilde{\varphi}}LS^2$ coincides with the space of smooth sections $C^\infty(S^1, N_{\widetilde{\varphi}})$. Restricting this complex structure on the space of Legendrian knots $\mathcal{P}_k \subset LS^3$ we get there an almost complex structure, which is invariant w. r. t. the diagonal action of$U(1)$ on $N_{\widetilde{\varphi}} = T^*S^1 \oplus T^*U(1)$. Its symplectic reduction produces the required complex structure on $\mathcal{B}_{k,t} = \mathcal{P}_{k,t}/\,U(1)$. Now, the point is that the forgetful map (5-3) is pseudo holomorphic (and has pseudo holomorphic fibers) w. r. t. the complex structures just described.

5.5. BPU$_k$-map. Let $V_k = H^0(\mathbb{CP}_1, L^{\otimes k})$ be the space of homogeneous binary forms of degree k. It is equipped with the standard Hermitean structure (actually, if we fix some basis $\{\,0\,,\,1\,,\,\infty\,\}$ on $\mathbb{CP}_1$, then V_k turns to the space of usual polynomials of degree k in one complex variable). Then, each level k Legendrian knot $S^1 \xhookrightarrow{\widetilde{\varphi}} S^3 = P_k$ defines the inclusion $V_k \hookrightarrow C^\infty(S^1, \mathbb{C})$ by sending a holomorphic section $s \in H^0(\mathbb{CP}_1, L^{\otimes k})$ to the function $\psi_{\widetilde{\varphi}}$ such that $\varphi^*s = \psi_{\widetilde{\varphi}} \cdot \sigma$, where $S^1 \xhookrightarrow{\sigma} \varphi^*L^{\otimes k}$ is the trivialization defined by the Legendrian knot. The map $\overset{\circ}{\mathcal{P}}{}^{\mathrm{hw}} \xrightarrow{\overline{\mathfrak{m}}_k} V_k^*$ sending a half weighted Legendrian knot $(\widetilde{\varphi}, \vartheta)$ to the form $\overline{\mathfrak{m}}_k(\widetilde{\varphi}, \vartheta)$that takes $s \in H^0(M, L^{\otimes k})$ to

$$s \longmapsto \int_S \psi_{\widetilde{\varphi}}\, d\vartheta^2 = \int_S \psi_{\widetilde{\varphi}}\, f^2\, d\mu$$

is antiholomorphic and has the surjective differential. Its composition with the Hermitian dualization $V_k^* \xrightarrow{\sim} \overline{V}_k$ gives the holomorphic map $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}} \xrightarrow{\mathfrak{m}_k} V_k$ called the BPU$_k$-map. It sends a half weighted Legendrian knot $(\widetilde{\varphi}, \vartheta)$ to $\mathfrak{m}_k(\widetilde{\varphi}, \vartheta) \in V_k$ defined by the identity

$$\langle\, s \ ,\ \mathfrak{m}_k(\widetilde{\varphi}, \vartheta)\,\rangle_{V_k} = \int_S \psi_{\widetilde{\varphi}}\, d\vartheta^2$$

which has to hold for any $s \in V_k$. Of course, this map is invariant w. r. t. the natural action of the symplectomorphism group $Sym_0 = Sym_0(S^2)$ and is equivariant w. r. t. the standard $U(1)$-action on $\overset{\circ}{\mathcal{P}}{}_k^{\mathrm{hw}}$ and V_k. The symplectic reduction of the BPU$_k$-map w. r. t. the $U(1)$-action gives the holomorphic map

$$\mathfrak{m}_{k,t}\ :\ \overset{\circ}{\mathcal{B}}{}_{k,t}^{\mathrm{hw}} \longrightarrow \mathbb{P}(V_k)\ ,$$

which is Sym_0-invariant as well and assigns holomorphically an algebraic cycle of degree k on $\mathbb{CP}_1$ to each isotopy class of the half weighted Bohr - Sommerfeld unicoursal curves of fixed volume. So, these isotopy classes, that is, the orbits of Sym_0 in $\overset{\circ}{\mathcal{B}}{}_{k,t}^{\mathrm{hw}}$, are diffeomorphic to homogeneous spaces

$$\mathbb{P}\,(\,V_k\,) \times \big(Sym_0/Stab_{(\varphi,\vartheta)}\big)\ ,$$

where (φ, ϑ) is any representative for a component. Let us recall, finally, that each isotopy component of the Legendrian knots space produces a series of well known polynomials: Alexander, V. Jones, Kaufmann, HOMFLY, e. t. c. The BPU$_k$-map holomorphically takes each Legendrian knot to a polynomial from V_k. How it interacts with the polynomials just mentioned?

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