ON THE DIMENSION OF SOME LINEAR SPACES OVER FINITE EXTENSION OF $\mathbb{Q}$

Part 2.

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Think of all the things that will
Really be within your reach !
V.Nabokov.

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§0. Foreword.

The first part of this paper is published here as 'On the rank over $\mathbb{Q}$ of some real matrices', Part 1, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, vol. 27, pages 1 – 32. When I wrote the Part 2, I have decided to change the name of the paper. The paragraph 1 is introductory. In the paragraph 2 I have collected a several working theorems for an estimation of dimension of some linear spaces over a finite extension of $\mathbb{Q}$. In the paragraph 3 I introduce and study the notion of measure of nondiscreteness for some free modules over the ring of integers of a finite extension of the field $\mathbb{Q}$. A survey of some results which are obtained by means of this method is given in the paragraph 4.

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§1. Introduction.

We consider a finite extension $K/\mathbb{Q}$ of the field $\mathbb{Q}$, and let

$$[K:\mathbb{Q}] = d.$$

Let K has r_1 real places and r_2 complex places. Each such place is the monomorphism of the field K in the field $\mathbb{R}$, if a place is real, or in the field $\mathbb{C}$, if a place is not real; we will denote these monomorphisms respectively by $\sigma_1, \dots \sigma_{r_1+r_2}$. Then $d = r_1 + 2r_2$. Let $\mathfrak{B}$ is the fixed integer basis

$$\omega_1, \dots, \omega_d \tag{2.1.1}$$

of the field K over the field $\mathbb{Q}$. With extension of the base field of the algebra $K/\mathbb{Q}$ from $\mathbb{Q}$ to $\mathbb{R}$ appears an isomorphism of the algebra $\mathfrak{K} = K \otimes \mathbb{R}$ onto direct sum

$$\underbrace{\mathbb{R} \oplus \dots \oplus \mathbb{R}}_{r_1 \text{ times}} \oplus \underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{r_2 \text{ times}}$$

of r_1 copies of the field $\mathbb{R}$ and of r_2 copies of the field $\mathbb{C}$.

We will identify the algebra $\mathfrak{K}$ with the specified direct sum by means of this isomorphism. Let π_j, where $j = 1, \dots, r_1+r_2$, denotes projection of the algebra $\mathfrak{K}$ onto $j-$th among its direct summands and let also π_j denotes the natural extension of this projection onto set of all the matrices having all elements in $\mathfrak{K}$. So, $\pi_j(\mathfrak{K}) = \mathbb{R}$, if $j = 1, \dots, r_1$ and $\pi_j(\mathfrak{K}) = \mathbb{C}$, if $j = r_1+1, \dots, r_1+r_2$. Let further $\mathfrak{i}_{\mathfrak{K}}$ denotes a embedding of the field $\mathbb{R}$ by the diagonal way in $\mathfrak{K}$ and let also $\mathfrak{i}_{\mathfrak{K}}$ denotes the extension of this embedding onto set of all the real matrices . So, $\mathbb{R}$ is imbedded by means of $\mathfrak{i}_{\mathfrak{K}}$ by diagonal way in $\mathfrak{K}$. Each element $Z \in \mathfrak{K}$ has a unique representation in the form:

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_{r_1+r_2} \\ \overline{z_{r_1+1}} \\ \vdots \\ \overline{z_{r_1+r_2}} \end{pmatrix}, \tag{2.1.2}$$

where $z_j = \pi_j(Z) \in \mathbb{R}$ for $j = 1, \dots, r_1$ and $z_j = \pi_j(Z) \in \mathbb{C}$ for $j = r_1+1, \dots, r_1+r_2$. Let $\mathrm{Tr}_{\mathfrak{K}}(Z)$ denotes the sum

$$\sum_{j=1}^{r_1} z_j + \sum_{j=r_1+1}^{r_1+r_2} 2\,\mathrm{Re}\,(z_j) =$$

$$\sum_{j=1}^{r_1}\pi_j(Z)+\sum_{j=r_1+1}^{r_1+r_2}2\,\mathrm{Re}\,(\pi_j(Z),$$

and let $q_\infty^{(\mathfrak{K})}(Z)$ denotes the value

$$\max(|z_1|,\ldots,|z_{r_1+r_2}|)=$$

$$\max(|\pi_1(Z)|,\ldots,|\pi_{r_1+r_2}(Z)|).$$

Clearly,

$$q_\infty^{(\mathfrak{K})}(Z_1Z_2)\leq q_\infty^{(\mathfrak{K})}(Z_1)q_\infty^{(\mathfrak{K})}(Z_2),$$

$$q_\infty^{(\mathfrak{K})}(Z_1+Z_2)\leq q_\infty^{(\mathfrak{K})}(Z_1)+q_\infty^{(\mathfrak{K})}(Z_2),$$

$$q_\infty^{(\mathfrak{K})}(\mathfrak{i}_{\mathfrak{K}}(\lambda)Z)=|\lambda|q_\infty^{(\mathfrak{K})}(Z)$$

for any $Z_1\in\mathfrak{K}$, $Z_2\in\mathfrak{K}$, $Z\in\mathfrak{K}$ and $\lambda\in\mathbb{R}$. Let $q_\infty^{(\mathfrak{K})}$ denotes the natural extension of the specified norm $q_\infty^{(\mathfrak{K})}$ onto set of all the matrices having all elements in $\mathfrak{K}$ (i.e. as the maximum of the norm $q_\infty^{(\mathfrak{K})}$ of the elements of the matrix) . If

$$Z=\begin{pmatrix}z_1\\ \vdots\\ z_d\end{pmatrix}\in K,$$

then

$$z_j=\sigma_j(Z),$$

where $j=1,\ldots,r_1+r_2$,

$$z_{r_1+r_2+j}=\overline{\sigma_{r_1+j}(Z)},$$

where $j=1,\ldots,r_2$. In particular,

$$\omega_k=\begin{pmatrix}\sigma_1(\omega_k)\\ \vdots\\ \sigma_{r_1+r_2}(\omega_k)\\ \overline{\sigma_{r_1+1}(\omega_k)}\\ \vdots\\ \overline{\sigma_{r_1+r_2}(\omega_k)}\end{pmatrix},\tag{2.1.3}$$

As usually, $\mathbb{Z}_K$ denotes the ring of all the integers of the field K. The ring $\mathbb{Z}_K$ is embedded in the ring $\mathfrak{K}$ as discrete lattice. Moreover, if $Z\in\mathbb{Z}_K\backslash\{0\}$, then

$$\left(\prod_{i=1}^{r_1}|\sigma_j(Z)|\right)\prod_{i=1}^{r_2}|\sigma_{r_1+i}(Z)|^2=|\,\mathrm{Nm}_{\,K/\mathbb{Q}}(Z)|\in\mathbb{N}$$

and therefore

$$q_\infty^{(\mathfrak{K})}(Z) \geq 1. \tag{2.1.4}$$

for any $Z \in \mathbb{Z}_K \backslash \{0\}$. We reserve the name K-integer for denoting of the elements of the ring Z_K.

On $\mathfrak{K}$ there is a involution transforming each element (2.1.2) in $\mathfrak{K}$ into

$$\overline{Z} = \begin{pmatrix} \overline{z_1} \\ \vdots \\ \overline{z_d} \end{pmatrix} \in \mathfrak{K}.$$

Every element Z of the algebra $\mathfrak{K}$ has a unique representation in the form

$$Z = \sum_{j=1}^{d} x_j \omega_j \tag{2.1.6}$$

with $x_j \in \mathbb{R}$ and ω_j in (2.1.1). Then

$$K = \left\{ \sum_{j=1}^{d} x_j \omega_j \colon x_j \in \mathbb{Q}, j = 1, \dots, d \right\},$$

$$\mathbb{Z}_K = \left\{ \sum_{j=1}^{d} x_j \omega_j \colon x_j \in \mathbb{Z}, j = 1, \dots, d \right\}.$$

For each element $Z \in \mathfrak{K}$, which is represented in the form (2.1.6), let $q_\infty^{(\mathfrak{B})}(Z)$ denotes the number

$$\max(|x_1|, \dots, |x_s|),$$

let $[Z]_{\mathfrak{B}}$ denotes the sum

$$\sum_{j=1}^{d} [x_j] \omega_j, \tag{2.1.7}$$

and, finally,let $X_Z^{(\mathfrak{B})}$ denotes the column

$$\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$

Clearly,

$$[Z]_{\mathfrak{B}} \in \mathbb{Z}_K. \tag{2.1.8}$$

Let

$$\{Z\}_{\mathfrak{B}} = Z - [Z]_{\mathfrak{B}}$$

for any $Z \in \mathfrak{K}$. Let $q_\infty^{(\mathfrak{B})}$ also denotes the natural extension of the specified norm $q_\infty^{(\mathfrak{B})}$ onto set of all the matrices, which have all the elements in $\mathfrak{K}$ (i.e. as the maximum of the norm $q_\infty^{(\mathfrak{B})}$ of elements of the matrix). So,

$$q_\infty^{(\mathfrak{B})}(Z) = q_\infty^{(\mathbb{R})}(X_Z^{(\mathfrak{B})}), \tag{2.1.9}$$

for $Z \in \mathfrak{K}$. Clearly, $q_\infty^{(\mathfrak{B})}(\{Z\}_{\mathfrak{B}}) < 1$ for any $Z \in \mathfrak{K}$. On a finite-dimensional linear space over the field $\mathbb{R}$ any two norms are equivalent. Therefore there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$q_\infty^{(\mathfrak{K})}(Z) \le C_1 q_\infty^{(\mathfrak{B})}(Z) \tag{2.1.10}$$

and

$$q_\infty^{(\mathfrak{B})}(Z) \le C_2 q_\infty^{(\mathfrak{K})}(Z) \tag{2.1.11}$$

for each element $Z \in \mathfrak{K}$. Let $e_j^{\vee}$, where $j = 1, \ldots, d$ denotes the $j-$th column of a unit $d \times d-$matrix.

Let further

$$e_k^{\wedge} = e_k^{\vee},$$

if $k = 1, \ldots, r_1$, and

$$e_k^{\wedge} = e_k^{\vee} + e_{k+r_2}^{\vee},$$

if $k = r_1 + 1, \ldots, r_1 + r_2$. Then

$$1_{\mathfrak{K}} = \sum_{k=1}^{r_1+r_2} e_k^{\wedge}, \tag{2.1.12}$$

$$e_k^{\wedge} e_k^{\wedge} = e_k^{\wedge}, e_k^{\wedge} e_l^{\wedge} = 0, \tag{2.1.13}$$

where $k \in \{1, \ldots, r_1 + r_2\}$ and $l \in \{1, \ldots, r_1 + r_2\} \backslash \{k\}$.

Let $\mathfrak{M}$ is the unitary $\mathfrak{K}$-module. The idempotent $e_k^{\wedge}$, where $k = 1, \ldots, r_1 + r_2$, determines a $\mathfrak{K}$-submodule $e_k^{\wedge}\mathfrak{M}$ of the module $\mathfrak{M}$. According to (2.1.13), the equality (2.1.12) determines splitting of the module $\mathfrak{M}$ into direct sum

$$e_1^{\wedge}\mathfrak{M} \oplus \ldots \oplus e_{r_1+r_2}^{\wedge}\mathfrak{M}.$$

Let $\psi_j^{(\mathfrak{M})}$, where $j = 1, \ldots, r_1 + r_2$, is the projection of the module $\mathfrak{M}$ onto its direct summand $e_j M$. Then

$$f = \sum_{j=1}^{r_1+r_2} \psi_j^{(\mathfrak{M})}(f) \tag{2.1.14}$$

for any $f \in \mathfrak{M}$.

Clearly, the unitary $\mathfrak{K}$-module $\mathfrak{M}$ is a free $\mathfrak{K}$-module with finite set of free $\mathfrak{K}$-generators if and only if there exists a finite set I and for each $k = 1, \ldots, r_1 + r_2$ there exists an injection α_k of the set I into $e_k^{\wedge}\mathfrak{M}$ with the following property:

$\alpha_k(I)$ is a basis of the linear space $e_k^{\wedge}\mathfrak{M}$ over the field $\pi_k(\mathfrak{K})$.

Let $\mathfrak{M}_0$ denotes the free unitary $\mathfrak{K}$-module with s free $\mathfrak{K}$-generators.

Let

$$e_1, \ldots, e_s, \tag{2.1.15}$$

are free generators of $\mathfrak{M}_0$ over the ring $\mathfrak{K}$. Let, as above, $d = [K : \mathbb{Q}]$ and (2.1.1) is the integer basis of K. Let M_0 is the $\mathbb{Z}_K$-module

$$M_0 = \{u_1 e_1 + \ldots u_s e_s \colon u_1 \in \mathbb{Z}_K, \ldots, u_s \in \mathbb{Z}_K\}. \tag{2.1.16}$$

Let, as above, $\mathfrak{i}_{\mathfrak{K}}$ is the embedding of $\mathbb{R}$ into $\mathfrak{K}$ by diagonal way. Let $\mathfrak{L}_0 = \mathfrak{i}_{\mathfrak{K}}(\mathbb{R})e_1 + \ldots + \mathfrak{i}_{\mathfrak{K}}(\mathbb{R})e_s$. We identify the module $\mathfrak{M}_0$ with a tensor product $\mathfrak{K} \otimes_{\mathbb{R}} \mathfrak{L}_0$. Then the system

$$\omega_k \otimes_{\mathbb{R}} e_l = \omega_k e_l \tag{2.1.17}$$

with $k = 1, \ldots, d$ and $l = 1, \ldots, d$ is a basis of $\mathfrak{M}_0$ over the field $\mathbb{R}$ and is a basis of M_0 over the ring $\mathbb{Z}$. Therefore we identify $\mathfrak{M}_0$ with

$$(\mathfrak{K})^s = \oplus_{k=1}^{r} \mathfrak{K} \tag{2.1.18}$$

and M_0 with

$$(\mathbb{Z}_K)^s = \oplus_{k=1}^{s} \mathbb{Z}_K.$$

Therefore every $\mathfrak{K}$-homomorphism ϕ of module $\mathfrak{M}_0$ into $\mathfrak{K}$-module $\mathfrak{M}$ is uniquely defined by s $\mathfrak{K}$-homomorphisms ϕ_k with $k = 1, \ldots, s$ of the ring $\mathfrak{K}$ into $\mathfrak{M}$ such that ϕ_k coincides with ϕ on the $k-$th direct summand in (2.1.18) and is equal to 0 on others.

Let τ_k, where $k = 1, \ldots, r_1$, denotes the embedding of the field $\mathbb{R}$ into ring $\mathfrak{K}$ with the following property:

$\tau_k(\alpha) = \alpha e_k^{\vee}$ for any $\alpha \in \mathbb{R}$.

Let further τ_k, where $k = r_1 + 1, \ldots, r_1 + r_2$, denotes the embedding of $\mathbb{C}$ into $\mathfrak{K}$ with the following property:

$\tau_k(\alpha) = \alpha e_k^\vee + \overline{\alpha} e_{k+r_2}^\vee$ for any $\alpha \in \mathbb{C}$.

Then

$$\tau_k(\alpha) 1_{\mathfrak{K}} = \tau_k(\alpha) e_k^\wedge = \tau_k(\alpha),$$

with $\alpha \in \mathbb{R}$, if $k = 1, \ldots, r_1$ and with $\alpha \in \mathbb{C}$, if $k = r_1 + 1, \ldots, r_1 + r_2$. The embedding τ_k defines on the module $e_k^\wedge \mathfrak{M}$ the structure of linear space over $\pi_k(\mathfrak{K}) = \mathbb{R}$ (respectively $\pi_k(\mathfrak{K}) = \mathbb{C}$) such that for any $g \in e_k^\wedge \mathfrak{M}$ and for any $\alpha \in \mathbb{R}$ (respectively $\alpha \in \mathbb{C}$) the result of the multiplication of the element g by the number α is equal to $\tau_k(\alpha) g$. Clearly,

$$\pi_j(\tau_k(\alpha)) = 0, \tag{2.1.19}$$

if

$$\{1, \ldots, r_1 + r_2\} \ni j \neq k \in \{1, \ldots, r_1 + r_2\},$$

but

$$\pi_k(\tau_k(\alpha)) = \alpha \tag{2.1.20}$$

with $\alpha \in \mathbb{R}$, if $k \in \{1, \ldots, r_1\}$, and with $\alpha \in \mathbb{C}$, if $k \in \{r_1 + 1, \ldots, r_1 + r_2\}$. Therefore

$$\psi_k^{(\mathfrak{K})}(Z) = \tau_k(\pi_k(Z)),$$

where $Z \in \mathfrak{K}$ and $k = 1, \ldots, r_1 + r_2$. Consequently,

$$Z = \sum_{k=1}^{r_1+r_2} \tau_k(\pi_k(Z))$$

for any $Z \in \mathfrak{K}$. Every $\mathfrak{K}$-homomorphism ϕ of $\mathfrak{M}$ into $\mathfrak{K}$ is uniquely defined as sum

$$\sum_{j=1}^{r_1+r_2} \phi_j, \tag{2.1.21}$$

where $\phi_j = \phi \psi_j^{(\mathfrak{M})}$ for any $j \in \{1, \ldots, r_1 + r_2\}$; moreover, if β_j are an arbitrary $\mathfrak{K}$-homomorphisms of the module $e_j^\wedge \mathfrak{M}$ into $\mathfrak{K}$ and we take

$$\phi_j(f) = \beta_j(f), \phi_k(f) = 0$$

for any $j \in \{1, \ldots, r_1 + r_2\}$, for any $k \in \{1, \ldots, r_1 + r_2\} \setminus \{j\}$ and for any $f \in e_j^\wedge M$,) then (2.1.21) defines $\mathfrak{K}$-homomorphism ϕ of the module $\mathfrak{M}$ into $\mathfrak{K}$ such that ϕ coincides with β_j on the module $e_j^\wedge \mathfrak{M}$.

The $\mathfrak{K}$-homomorphism ϕ_j with $j = 1, \dots, r_1$ is uniquely defined by means of the linear over $\mathbb{R}$ functions

$$\phi_{j,k} = \pi_k \phi_j, \tag{2.1.22}$$

where $k = 1, \dots, r_1$,

$$\phi_{j,k} = \mathrm{Re}\,(\pi_k \phi_j), \phi_{j,k+r_2} = \mathrm{Im}\,(\pi_k \phi_j), \tag{2.1.23}$$

where $k = r_1 + 1, \dots, r_1 + r_2$, such that

$$\phi_j(e_j^\wedge f) = \left(\sum_{k=1}^{r_1} \tau_k(\phi_{j,k}(e_j^\wedge f)) e_k^\wedge\right) +$$

$$\left(\sum_{k=r_1+1}^{r_1+r_2} \tau_k(\phi_{j,k}(e_j^\wedge f) + i\phi_{j,k+r_2}(e_j^\wedge f))\right) e_k^\wedge.$$

The $\mathfrak{K}$-homomorphism ϕ_j with $j = r_1 + 1, \dots, r_1 + r_2$ is uniquely defined by means of linear over $\mathbb{R}$ functions

$$\phi_{j,k} = \mathrm{Re}\,(\pi_k \phi_j), \tag{2.1.24}$$

where $k = 1, \dots, r_1 + r_2$,

$$\phi_j(e_j^\wedge f) = \left(\sum_{k=1}^{r_1} \tau_k(\phi_{j,k}(e_j^\wedge f)) e_k^\wedge\right) +$$

$$\left(\sum_{k=r_1+1}^{r_1+r_2} \tau_k(\phi_{j,k}(e_j^\wedge f) - i\phi_{j,k}(\tau_j(i) e_j^\wedge f))\right) e_k^\wedge.$$

So,

$$\phi(f) = \sum_{j=1}^{r_1+r_2} \phi(\pi_j^{(\mathfrak{M})}(f)) = \tag{2.1.25}$$

$$\left(\sum_{j=1}^{r_1+r_2} \sum_{k=1}^{r_1} \tau_k(\pi_k(\phi(\pi_j^{(\mathfrak{M})}(f))))\right) +$$

$$\left(\sum_{j=1}^{r_1} \sum_{k=r_1+1}^{r_1+r_2} \tau_k(\pi_k(\phi(\pi_j^{(\mathfrak{M})}(f))))\right) +$$

$$\left(\sum_{j=r_1+1}^{r_1+r_2}\sum_{k=r_1+1}^{r_1+r_2}\tau_k(\pi_k(\phi(\pi_j^{(\mathfrak{M})}(f))))\right)=$$

$$\left(\sum_{j=1}^{r_1+r_2}\sum_{k=1}^{r_1}\tau_k(\phi_{j,k}(f))\right)+$$

$$\left(\sum_{j=1}^{r_1}\sum_{k=r_1+1}^{r_1+r_2}\tau_k(\phi_{j,k}(f)+i\phi_{j,k+r_2}(f))\right)+$$

$$\left(\sum_{j=r_1+1}^{r_1+r_2}\sum_{k=r_1+1}^{r_1+r_2}\tau_k(\phi_{j,k}(f)-i\phi_{j,k}(\tau_j(i)f))\right)=$$

$$\left(\sum_{j=1}^{r_1+r_2}\sum_{k=1}^{r_1}\tau_k(\phi_{j,k}(\psi_j^{(\mathfrak{M})}(f)))\right)+$$

$$\left(\sum_{j=1}^{r_1}\sum_{k=r_1+1}^{r_1+r_2}\tau_k(\phi_{j,k}(\psi_j^{(\mathfrak{M})}(f))+i\phi_{j,k+r_2}(\psi_j^{(\mathfrak{M})}(f)))\right)+$$

$$\left(\sum_{j=r_1+1}^{r_1+r_2}\sum_{k=r_1+1}^{r_1+r_2}\tau_k(\phi_{j,k}(\psi_j^{(\mathfrak{M})}(f))-i\phi_{j,k}(\tau_j(i)\psi_j^{(\mathfrak{M})}(f)))\right)$$

for any $f\in\mathfrak{M}$.

Clearly, the elements

$$g_1\in\mathfrak{M}\,,\ldots,g_s\in\mathfrak{M} \tag{2.1.26}$$

compose a linearly independent system over over K if and only if the elements

$$\omega_k g_l \tag{2.1.27}$$

with $k=1\,,\ldots,d,\ l=1\,,\ldots,s$ and with basis (2.1.1) of the field K compose a linearly independent system over $\mathbb{Q}$.

Definition 2.1.1. *Let $\mathfrak{M}$ is a unitary $\mathfrak{K}$-module. We say that the function q on $\mathfrak{M}$ is a norm, when the following conditions are realized:*

(i) $q(f)\geq 0$ for any $f\in\mathfrak{M}$; the equalities $q(f)=0$ and $f=0$ are equivalent;

(ii) the inequality $q(f_1+f_2)\leq q(f_1)+q(f_2)$ (i.e. the inequality of triangle) is fulfilled for any $f_1\in\mathfrak{M}$ and any $f_2\in\mathfrak{M}$;

(iii) the equality

$$q(f)=\max_{j\in\{1,\dots,r_1+r_2\}} q(e_j^{\wedge}\lambda f)=q_\infty^{(\mathfrak{K})}(e_j^{\wedge}\lambda)q(e_jf)$$

holds for any $f\in\mathfrak{M}$ and for any $j\in\{1,\dots,r_1+r_2\}$;

(iv) the equality $q(e_j^{\wedge}(\lambda f))=|\pi_j(\lambda)|q(e_jf)$ is fulfilled for any $f\in\mathfrak{M}$, for any $\lambda\in\mathfrak{K}$ and for any $j\in\{1,\dots,r_1+r_2\}$.

According to (iii) and (iiii), if $\lambda\in\mathbb{R}$ and $f\in\mathfrak{M}$, then $q(\mathrm{i}_{\mathfrak{K}}(\lambda)f)=|\lambda|q(f)$. So, $\mathfrak{M}$ is the real linear normed space relative to the norm q.

Let $\mathfrak{M}$ is a normed $\mathfrak{K}$-module. Let ϕ is a $\mathfrak{K}$-homomorphism of the module $\mathfrak{M}$ into $\mathfrak{K}$. Clearly, the homomorphism ϕ is continuous if and only if the linear functionals (2.1.22),(2.1.23) and (2.1.24) are continuous. Let

$$q_\infty^{\sim}(\phi)=\sup_{f\in\mathfrak{M},q(f)=1} q_\infty^{\mathfrak{K}}(\phi(f)). \tag{2.1.28}$$

$$q_0^{\sim}(\phi)=\sum_{j=1}^{r_1}\sum_{k=1}^{d} q_\infty^{\sim}(\phi_{j,k})+\sum_{j=r_1+1}^{r_1+r_2}\sum_{k=1}^{r_1+r_2} q_\infty^{\sim}(\phi_{j,k}). \tag{2.1.29}$$

Clearly,

$$q_\infty^{\sim}(\phi)\le q_0^{\sim}(\phi)\le d^2 q_\infty^{\sim}(\phi).$$

Therefore the $\mathfrak{K}$-homomorphism ϕ of the normed $\mathfrak{K}$-module into $\mathfrak{K}$ is continuous if and only if

$$q_0^{\sim}(\phi)<\infty; \tag{2.1.30}$$

if the inequality (2.1.30) is fulfilled, we say that ϕ is bounded. So, we have the following analog of Hahn-Banach Theorem:

Lemma 2.1.8. *Let $\mathfrak{M}$ is a normed $\mathfrak{K}$-module and let $\mathfrak{M}_1$ is $\mathfrak{K}-$submodule of the module $\mathfrak{M}$. Let further $\phi^{\vee}$ is a continuous $\mathfrak{K}$-homomorphism of the module $\mathfrak{M}_1$ into $\mathfrak{K}$.*

Then there exists the continuous $\mathfrak{K}-$-homomorphism ϕ of the module $\mathfrak{M}$ into $\mathfrak{K}$ such that ϕ coincides with $\phi^{\vee}$ on $\mathfrak{M}_1$ and

$$q_0^{\sim}(\phi)=q_0^{\sim}(\phi^{\vee}).$$

Proof. The assertion of the Lemma is the direct corollary of equalities (2.1.25), (2.1.29) and of the Hahn-Banach Theorem (more precisely, of real and complex form of this Theorem). ■

We shall continue the norm $q_\infty^{(\mathfrak{K})}$ by the usual way on the $\mathfrak{K}$-module $\mathfrak{M}_0$, if we shall put

$$q_\infty^{(\mathfrak{K})}(f) = \max(q_\infty^{(\mathfrak{K})}(a_1)\,, \dots\,, q_\infty^{(\mathfrak{K})}(a_s))$$

for any

$$f = a_1e_1 + \ldots + a_se_s \in \mathfrak{M}_0$$

with $a_1 \in \mathfrak{K}\,, \dots\,, a_s \in \mathfrak{K}$. If this continuation of the norm $q_\infty^{(\mathfrak{K})}$ on $\mathfrak{M}_0$ will be done, then each $\mathfrak{K}-$ homomorphism of module $\mathfrak{M}_0$ into $\mathfrak{K}$ becomes continuous.

§2. On the dimension of some linear spaces over finite extension of the field $\mathbb{Q}$.

Let, as above, $\mathfrak{i}_\mathfrak{K}$ denotes the embedding of $\mathbb{R}$ in $\mathfrak{K}$ by the diagonal way and also denotes the extension of this embedding onto the set of all the real matrices .

Definition 2.2.1. *Let $A = (a_{i,k})$ is the $m \times n$-matrix with $a_{i,k} \in \mathfrak{K}$, where $i = 1\,, \dots\,, m, k = 1\,, \dots\,, n$. Let further E_m denotes the unit $m \times m$-matrix. We say that the matrix A is K-quasi-irrational, if the system of columns of the matrix*

$$(A, \mathfrak{i}_\mathfrak{K}(E_m)) \tag{2.2.1}$$

is linearly independent over K.

The following examples illustrate this definition:

if $[K:\mathbb{Q}] = 2$, then the column $\begin{pmatrix}1\\0\end{pmatrix} \in \mathfrak{K}$ is K-quasi-irrational. In fact, among elements of K only $\mathfrak{i}_\mathfrak{K}(1) = 1_\mathfrak{K} = \begin{pmatrix}1\\1\end{pmatrix}$ has the first coordinate equal to 1; on the other hand the column $\begin{pmatrix}\mathfrak{i}_\mathfrak{K}(1)\\ \mathfrak{i}_\mathfrak{K}(0)\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix}\end{pmatrix}$ is not K-quasi-irrational. For the field $K = \mathbb{Q}[\sqrt{2}]$, the column $\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\end{pmatrix} \in \mathfrak{K}$ is quasi-irrational but not K-quasi-irrational; on the other hand the column $\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix} \in \mathfrak{K}$ is K-quasi-irrational.

The following simple Lemma specifies the domain of applications and of interest of definition 2.2.1.

Lemma 2.2.1. *Let A is the real $m \times n$-matrix, let $B = (A, E_m)$, let*

$$B_1 = (\omega_1 \otimes B\,, \dots\,, \omega_{r_1} \otimes B),$$

if $r_1 > 0$, let

$$B_2 = (\mathrm{Re}\,(\omega_{r_1+1}) \otimes B,\ \mathrm{Im}\,(\omega_{r_1+1}) \otimes B\,, \dots\,, \mathrm{Re}\,(\omega_{r_1+r_2}) \otimes B,\ \mathrm{Im}\,(\omega_{r_1+r_2}) \otimes A),$$

if $r_2 > 0$, *let* $B_3 = B_1$, *if* $r_2 = 0$, *let* $B_3 = B_2$, *if* $r_1 = 0$ *and, finally, let* $B_3 = (B_1,\ B_2)$, *if* $r_1 r_2 > 0$.

Then the matrix $\mathrm{i}_{\mathfrak{K}}(A)$ *is* K*-quasi-irrational if and only if the system of columns of matrix* B_3 *is linearly independent over* $\mathbb{Q}$. *In particular, if the matrix* $\mathrm{i}_{\mathfrak{K}}(A)$ *is* K*-quasi-irrational, then the matrix* A *is quasi-irrational.*

Proof. The assertion of the lemma is trivial. ■

Lemma 2.2.2. *Let* $p \in \mathbb{N}, q \in p + \mathbb{N}$. *Let further* $a_{i,k} \in \mathfrak{K}$ *for* $i = 1, \ldots, p$ *and* $k = 1, \ldots, q$. *Let, finally,*

$$L_i = a_{i,1}z_1 + \ldots + a_{i,q}z_q,$$

where $i = 1, \ldots, p$.

Then there exists a constant $C > 0$ *and, for each* $\nu \in \mathbb{N}$, *there are* $Z_1 \in \mathbb{Z}_K, \ldots, Z_q \in \mathbb{Z}_K$, *which satisfy to the following inequalities :*

$$q_\infty^{(\mathfrak{K})}(Z_1) + \ldots + q_\infty^{\mathfrak{K}}(Z_q) > 0,$$

$$q_\infty^{(\mathfrak{K})}(a_{i,1}Z_1 + \ldots + a_{i,q}Z_q) \leq \frac{1}{\nu}$$

for $i = 1, ..., p$ *and*

$$q_\infty^{(\mathfrak{K})}(Z_k) < C\nu^{\frac{p}{q-p}}$$

for $k = 1, ..., q$.

Proof. If $K = \mathbb{Q}$, then $\mathfrak{K} = \mathbb{R}$ and the Lemma coincides with well known Siegel Lemma. Representing Z_k in the form

$$Z_k = \sum_{l=1} \omega_l x_{l,k},$$

where ω_l with $l = 1, \ldots, d$ is the basis (2.1.1), we reduce the Lemma to the case of dp linear forms with real coefficients depending from dq variables. We obtain the assertion of the Lemma 2.2.2, applying Siegel's Lemma now . ■

Lemma 2.2.3. *The following assertions are equivalent:*

(i) if $X \in K^n \backslash \{0\}$, *then* $AX \notin K^m$;

(ii) there are

$$\alpha_l(\nu) \in \mathbb{Z}_K, \tag{2.2.2}$$

where $\nu \in \mathbb{N}, l = 1 \ldots m + n$, *which satisfy to the condition*

$$\lim_{\nu \to \infty} q_\infty^{(\mathfrak{K})} \left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k}\alpha_j(\nu) \right) = 0, \tag{2.2.3}$$

where $k = 1, \ldots, n$ *and, for each* $\kappa \in \mathbb{N}$, *the system, composed by the infinite sequences*

$$\alpha_l(\kappa+1), \ldots, \alpha_l(\kappa+\mu), \ldots, \tag{2.2.4}$$

where μ *runs through* $\mathbb{N}$ *and* $l = 1, \ldots, m+n$, *is linearly independent system over* $\mathbb{K}$

(iii) the matrix A *is* K*-quasi-irrational.*

Moreover if the assertion (iii) is true, then $\alpha_l(\nu)$ *may be chosen in such a way that the following conditions will be fulfilled:*

$$q_\infty^{(\mathfrak{K})}(\alpha_l(\nu)) = O(\nu^n),$$

where $\nu \in \mathbb{N}, l = 1, \ldots, m+n$, *and*

$$q_\infty^{(\mathfrak{K})}\left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k}\alpha_j(\nu)\right) = O(1/\nu),$$

where $\nu \in \mathbb{N}, k = 1, \ldots, n$.

Proof. Clearly,condition (i) is equivalent to (iii).

Let the assertion (ii) is true. Then there exists a column X such that $X \in K^n\backslash\{0\}$ and $AX \in K^m$. On choosing a suitable $d \in \mathbb{N}$ and replacing of column X by the column dX, if necessary, we may assume that

$$X \in \mathbb{Z}_K^n\backslash\{0\}\ , AX \in \mathbb{Z}_K^m.$$

Let x_k, for $k = 1, \ldots, n$,is the $k-$th element of the column X. According to (2.2.3),

$$q_\infty^{(\mathfrak{K})}\left(\left(-\sum_{k=1}^{n} x_k\alpha_{m+k}(\nu)\right) + \sum_{j=1}^{m}\alpha_j(\nu)\sum_{k=1}^{n} a_{j,k}x_k\right) = \tag{2.2.5}$$

$$q_\infty^{(\mathfrak{K})}\left(\sum_{k=1}^{n} x_k\left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k}\alpha_j(\nu)\right)\right) \leq$$

$$\sum_{k=1}^{n} q_\infty^{(\mathfrak{K})}(x_k) q_\infty^{(\mathfrak{K})}\left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m}\alpha_j(\nu)a_{j,k}\right).$$

But

$$\lim_{\nu\to\infty}\left(\sum_{k=1}^{n} q_\infty^{(\mathfrak{K})}(x_k) q_\infty^{(\mathfrak{K})}\left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m}\alpha_j(\nu)a_{j,k}\right)\right) = 0.$$

Therefore

$$\lim_{\nu\to\infty}\left(q_\infty^{(\mathfrak{K})}\left(\left(-\sum_{k=1}^{n}x_k\alpha_{m+k}(\nu)\right)+\sum_{j=1}^{m}\alpha_j(\nu)\sum_{k=1}^{n}a_{j,k}x_k\right)\right)=0.$$

Consequently, there exists a number $\kappa\in\mathbb{N}$ with the following property:

for any $\mu\in\mathbb{N}$, the value of the norm $q_\infty^{(\mathfrak{K})}$ of the element

$$\left(-\sum_{k=1}^{n}x_k\alpha_{m+k}(\kappa+\mu)\right)+\sum_{j=1}^{m}\alpha_j(\kappa+\mu)\sum_{k=1}^{n}a_{j,k}x_k\in\mathbb{Z}_K \tag{2.2.6}$$

is less than 1; therefore, according to (2.1.4), this element of the ring $\mathbb{Z}_K$ is equal to 0; however this is impossible for every μ in $\mathbb{N}$ because the sequences (2.2.4) compose a linearly independent system over K and $x_k\neq 0$ for some $k=1,\dots,n$.

Let the assertion (iii) is true. For each $i\in\{1,\dots,m\}$, we consider a system of n linear forms

$$L_{i,k}=y_0a_{i,k}-y_k,\ k=1,\dots,n, \tag{2.2.7}$$

which depend from variables $y_0,\dots,y_n$.

Since the number $n+1$ of the variables exceeds the number n of the linear forms, in view of the Lemma 2.2.2, there exists a constant $C>0$ which has the following properties:

for each $\mu\in\mathbb{N}$ and for each $k=1,\dots,n$ there are elements $\theta_k(i+m(\mu-1))\in\mathfrak{K}$ and elements $y_0,\dots,y_n$ in the ring $\mathbb{Z}_K$, which satisfy to the conditions

$$q_\infty^{(\mathfrak{K})}(\theta_k(i+m(\mu-1))\le 1, \tag{2.2.8}$$

$$q_\infty^{(\mathfrak{K})}(y_0)\le C(i+m(\mu-1))^n,\dots,q_\infty^{(\mathfrak{K})}(y_n)\le C(i+m(\mu-1))^n$$

$$L_{i,k}=\frac{1}{i+m(\mu-1)}\theta_k(i+m(\mu-1),$$

and $y_i\neq 0$ for some i; therefore $y_0\neq 0$. For these values of the variables $y_0,\dots,y_n$, we take

$$\alpha_i(i+m(\mu-1))=y_0,\alpha_{m+k}(i+m(\mu-1))=y_k,$$

with $k=1,\dots,n$, and

$$\alpha_l(i+m(\mu-1))=0$$

for $l\in\{1,\dots,m\}\backslash\{i\}$. We have thus constructed sequences of elements of the ring $\mathbb{Z}_K$

$$\alpha_l(\nu),\ \nu\in\mathbb{N},$$

with $l = 1, \dots, m+n$, which satisfy to the following conditions:

$$\alpha_i(\nu) \neq 0$$

for $\nu \equiv i \ (mod\ m)$ and $i \in \{1, \dots, m\}$,

$$\alpha_j(\nu) = 0$$

for $\nu \equiv i \ (mod\ m)$ and $j \in \{1, \dots, m\} \backslash \{i\}$,

$$q_\infty^{(\mathfrak{K})}(\alpha_l(\nu)) \leq C\nu^n$$

and

$$-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k}\alpha_j(\nu) = \tag{2.2.9}$$

$$-\alpha_{m+k}(\nu) + a_{i,k}\alpha_i(\nu) = \frac{\theta_k(\nu)}{\nu},$$

for $\nu \equiv i \ (mod\ m)$, $k = 1, \dots, n$. Let for some $\kappa \in \mathbb{N}$ the sequences (2.2.4) compose the linearly dependent system over K. Then one can find elements $t_l \in \mathbb{Z}_K$, $l = 1, \dots, m+n$, with property

$$\sum_{j=1}^{m} t_j\alpha_j(\kappa+\mu) + \sum_{k=1}^{n} t_{k+m}\alpha_{m+k}(\kappa+\mu)) = 0 \tag{2.2.10}$$

and $t_j \neq 0$ for some j. According to(2.2.9) and (2.2.10),

$$\sum_{j=1}^{m} t_j\alpha_j(\kappa+\mu) + \sum_{k=1}^{n} t_{k+m}\left(\sum_{j=1}^{m} a_{j,k}\alpha_j(\kappa+\mu)\frac{1}{\nu}\theta_k\right) = 0,$$

$$\sum_{j=1}^{m} t_j\alpha_j(\kappa+\mu)\left(t_j + \sum_{k=1}^{n} t_{k+m}a_{j,k}\right) = \sum_{k=1}^{n} t_{k+m}\frac{1}{\kappa+\mu}\theta_k \tag{2.1.11}$$

For $i \in [1, m] \cap \mathbb{N}$, let

$$\mu \equiv -\kappa + i \ (mod\ m). \tag{2.2.12}$$

Then $\alpha_i(\kappa+\mu) \neq 0, q^{(\mathfrak{K})}(\alpha_i(\kappa+\mu)) \geq 1, \alpha_j(\kappa+\mu) = 0$ for $j \neq i$ and, in view of (2.2.11),

$$q^{(\mathfrak{K})}\left(t_i + \sum_{k=1}^{n} t_{k+m} a_{i,k}\right) \leq$$

$$(\kappa+\mu)^{-1} \sum_{k=1}^{n} q^{(\mathfrak{K})}(t_{k+m}). \tag{2.2.13}$$

On passing to the limit $\mu \to \infty$, we obtain from (2.2.12) and (2.2.13) the following equality

$$t_i + \sum_{k=1}^{n} t_{k+m} a_{i,k} = 0,$$

with $i = 1, \ldots, m$. But this contradicts to (iii). ∎.

Lemma 2.2.4. *Let $\mathfrak{M}_1$ is a normed $\mathfrak{K}$-module with s $\mathfrak{K}$-generators*

$$f_1, \ldots, f_s. \tag{2.2.14}$$

If $\mathfrak{M}_1$ is free $\mathfrak{K}$-module with m free generators, then $\mathfrak{M}_1$ has a system of $\mathfrak{K}$-generators

$$g_1, \ldots, g_s \tag{2.2.15}$$

with following property:

the systems (2.2.14) and (2.2.15) are equivalent over $\mathbb{Q}$ and subsystem, consisting of last m elements of system (2.2.15) is the system of free $\mathfrak{K} -$ generators of the module $\mathfrak{M}_1$.

Proof. For each $i = 1, \ldots, r_1 + r_2$, let is chosen a basis of the system

$$e_i^{\wedge} f_1, \ldots, e_i^{\wedge} f_s, \tag{2.2.16}$$

where, as above, $e_i^{\wedge} = \pi_i(1_{\mathfrak{K}})$ and A_i is the $m \times s -$ matrix, composed from columns of coordinates of elements (2.2.16) in this basis. Let further X is the $s \times m -$ matrix, consisting of sm independent variables $x_{k,l}$, where $k = 1, \ldots, s$ and $l = 1, \ldots m$. Let further $P_i(X) = \det(AX)$. For each $i = 1, \ldots, r_1 + r_2$, there is a specialization $X = X_i \in (\pi_i(\mathfrak{K}))^{sm}$ with property $P_i(X_i) \neq 0$. Therefore $P_i(X) \neq 0$ and $P_0(X) = \prod_{i=1}^{m} P_i(X) \neq 0$. Therefore there is specialization $X = X_0 \in (\mathbb{Q})^{sm}$ with property $P_0(X_0) \neq 0$. Clearly, the columns of the $s \times m -$ matrix X_0 compose a linear independent system over $\mathbb{Q}$. But $s \geq m$. Therefore there exists a matrix $Y \in Mat_s(\mathbb{Q})$, with the matrix X on the last columns. ∎

Theorem 2.2.1 *Let $\mathfrak{M}$ is the normed $\mathfrak{K}$-module and $\mathfrak{M}_0$ is the free $\mathfrak{K}$-module with free generators (2.1.15). Let (2.2.14) are some elements of $\mathfrak{M}$. In order that the elements (2.2.14) compose a linearly independent system over K it is sufficient the existence of $\mathfrak{K}$-homomorphism α_0 of the module $\mathfrak{M}_0$ into $\mathfrak{M}$,*

the existence,for $\nu \in \mathbb{N}$, of continuous $\mathfrak{K}$-homomorphism $\phi_\nu^\vee$ of $\mathfrak{K}$-module $\mathfrak{M}$ into $\mathfrak{K}$

and existence, for $\nu \in \mathbb{N}$, of $\mathfrak{K}$-homomorphism ψ_ν of $\mathfrak{K}$-module $\mathfrak{M}_0$ into $\mathfrak{K}$ with the following properties:

(a) the system composed by the elements

$$\alpha_0(e_1)\,,\dots,\alpha_0(e_s) \tag{2.2.17}$$

is a system equivalent over K to the system (2.2.14);

(b)

$$\psi_\nu(f) \in \mathbb{Z}_K \tag{2.2.18}$$

for any $f \in M_0$ and $\nu \in \mathbb{N}$;

(c)

$$\lim_{\nu\to\infty} q_\infty^{(\mathfrak{K})}(\phi_\nu(\alpha_0(f)) - \psi_\nu(f)) = 0 \tag{2.2.19}$$

for any $f \in \mathfrak{M}_0$;

(d)

$$\bigcap_{\nu=\mu}^{\infty} \ker(\psi_\nu) = \{0\} \tag{2.2.20}$$

for any $\mu \in \mathbb{N}$.

If $\mathfrak{K}$-module $\mathfrak{M}_1$ generated by system (2.2.14) is a free $\mathfrak{K}$-module, then the conditions specified above are also necessary. Moreover, if $\mathfrak{M}_1$ is a free $\mathfrak{K}$-module with m generators, then the homomorphisms α_0, ϕ_ν and ψ_ν may be chosen such that

$$q_\infty^\sim(\phi_\nu) = O(\nu^{s-m}), q_\infty^\sim(\psi_\nu) = O(\nu^{s-m)}), q_\infty^\sim(\phi_\nu - \psi_\nu) = O(\nu^{-1}). \tag{2.2.21}$$

Proof.

Sufficiency. Let the equality

$$u_1\alpha_0(e_1) + \,\dots\, u_s\alpha_0(e_s) = 0. \tag{2.2.22}$$

is fulfilled for some elements

$$u_1 \in \mathbb{Z}_K\,,\dots,u_s \in \mathbb{Z}_K.$$

In view of (2.1.16),

$$f = u_1 e_1 + \ldots + u_s e_s \in M_0$$

and, according to (2.2.18),

$$\psi_\nu(f) \in \mathbb{Z}_K, \tag{2.2.23}$$

where $\nu \in \mathbb{N}$. In view of (2.2.22),

$$\alpha_0(f) = 0. \tag{2.2.24}$$

In view of (2.2.19) and (2.2.22),

$$\lim_{\nu \to \infty} (q_\infty^{(\mathfrak{K})}(\psi_\nu(f))) = 0. \tag{2.2.25}$$

Therefore there exists $\mu \in \mathbb{N}$ with the following property:

$$q_\infty^{(\mathfrak{K})}(\psi_\nu(f)) < 1 \tag{2.2.26}$$

for all the integers $\nu \in \mu - 1 + \mathbb{N}$. According to (2.1.4) and (2.2.24),

$$\psi_\nu(f) = 0 \tag{2.2.26}$$

for all the integers $\nu \in \mu - 1 + \mathbb{N}$. According to (2.2.26) and (2.2.20), $f = 0$.

Necessity. According to the Lemma 2.2.4, the module $\mathfrak{M}_1$ has a system of $\mathfrak{K}$-generators 2.2.15 with following property:

the systems (2.2.14) and (2.2.15) are equivalent over $\mathbb{Q}$ and subsystem,consisting of last m elements of system (2.2.15) is the system of free $\mathfrak{K}$-generators of the module $\mathfrak{M}_1$.

Since the elements (2.2.14) are linearly independent over K, the elements (2.2.15) are also linearly independent over K. Let

$$n = s - m. \tag{2.2.27}$$

Then the system

$$g_{n+k}, \tag{2.2.28}$$

where $k = 1, \ldots, m$, is a system of free generators of $\mathfrak{M}_1$. Clearly, there exists a unique $\mathfrak{K}$-homomorphism α_0 of $\mathfrak{M}_0$ into the module $\mathfrak{M}_1$ with following property:

$$\alpha_0(e_l) = g_l, \tag{2.2.29}$$

where $l = 1, \dots, s$.

Clearly, for any $j = 1, \dots, m$, there exists a unique $\mathfrak{K}$-homomorphism θ_j of the $\mathfrak{K}$-module $\mathfrak{M}_1$ into $\mathfrak{K}$ with the following property:

$\theta_j(g_{n+k}) = 0_{\mathfrak{K}}$ for any $k \in \{1, \dots, m\}\backslash\{j\}$ and $\theta_k(g_{n+k}) = 1_{\mathfrak{K}}$ for each $k \in \{1, \dots, m\}$.

Also for $j = 1, \dots, s$, there exists a unique $\mathfrak{K}$-homomorphism η_j of the $\mathfrak{K}-$ module $\mathfrak{M}_0$ into $\mathfrak{K}$ with the following property:

$\eta_j^{\vee}(e_k) = 0_{\mathfrak{K}}$ for $k \in \{1, \dots, s\}\backslash\{j\}$ and $\eta_k(e_k) = 1_{\mathfrak{K}}$ for $k \in \{1, \dots, s\}$.

Then $\eta_k = \theta_k^{\vee}\alpha_0$ for $k \in \{n+1, \dots, s\}$. Clearly, the homomorphism α_0 is continuous, the homomorphism η_k with $k \in \{1, \dots, s\}$ is continuous and homomorphism θ_k with $k \in \{1, \dots, m\}$ is continuous. According to the lemma 2.1.8, for $k \in \{1, \dots, m\}$, there exists a continuous $\mathfrak{K}$-homomorphism $\theta_k^{\wedge}$ of the module $\mathfrak{M}$ into $\mathfrak{K}$ with the following property:

$\theta_k^{\wedge}$ coincides with θ_k on the module $\mathfrak{M}_1$.

Then

$$\eta_k = \theta_k^{\wedge}\alpha_0$$

for $k \in \{n+1, \dots, s\}$. Clearly, $\eta_k(M_0) \subset \mathbb{Z}_K$ for $k \in \{1, \dots, s\}$. Clearly,

$$\bigcap_{k=1}^{s} \ker(\eta_k) = \{0_{\mathfrak{K}}\}.$$

If $n = 0$, then, for $k \in \{1, \dots, s\}, \nu \in \mathbb{N}, \nu \equiv k \mod s$, we can put

$$\phi_\nu = \theta_k^{\wedge},\ \psi_\nu = \eta_k.$$

Then (2.2.18), (2.2.19) and (2.2.21) are fulfilled.

If $m < s$, then according to (2.2.27) and (2.2.28),

$$g_{n+1}, \dots, g_{n+m} \tag{2.2.30}$$

is system of free $\mathfrak{K}$-generators of $\mathfrak{M}_1$ and therefore

$$g_k = \sum_{i=1}^{m} a_{i,k} g_{n+i}, \tag{2.2.31}$$

where $a_{i,k} \in \mathfrak{K}$ for $k = 1, \dots, n, i = 1, \dots, m$. According to the Lemma 2.2.3, there are

$$\alpha_l(\nu) \in \mathbb{Z}_K, \tag{2.2.32}$$

where $\nu \in \mathbb{N}, l = 1 \dots m+n$, which satisfy to the following conditions:

(i)

$$q_\infty^{(\mathfrak{K})}(\alpha_l(\nu)) = O(\nu^n)$$

where $l = 1, \dots, m+n$ and $\nu \in \mathbb{N}$;

(ii)

$$q_\infty^{(\mathfrak{K})}\left(-\alpha_{m+k}(\nu) + \sum_{j=1}^{m} a_{j,k}\alpha_j(\nu)\right) = O(\nu^{-1}),$$

where $k = 1, \dots, n$ and $\nu \in \mathbb{N}$;

(iii) for each $\kappa \in \mathbb{N}$, the system, composed by the infinite sequences

$$\alpha_l(\kappa+1), \dots, \alpha_l(\kappa+\mu), \dots,$$

where μ runs through $\mathbb{N}$ and $l = 1, \dots, m+n$, is linearly independent system over the field $\mathbb{K}$.

Let

$$\psi_\nu = \left(\sum_{i=1}^{m} \alpha_i(\nu)\eta_{n+i}\right) + \sum_{k=1}^{n} \alpha_{m+k}(\nu)\eta_k, \tag{2.2.33}$$

$$\phi_\nu = \sum_{i=1}^{m} \alpha_i(\nu)\theta_{n+i}.$$

Then

$$\phi_\nu(\alpha_0(e_{n+i})) = \phi_\nu(g_{n+i}) = \psi_\nu(e_{n+i}) = \alpha_i(\nu) = O(\nu^n), \tag{2.2.34}$$

where $i = 1, \dots, m$,

$$\psi_\nu(e_k) = \alpha_{m+k}(\nu) = O(\nu^n), \tag{2.2.35}$$

where $k = 1, \dots, n$,

$$\phi_\nu(g_k) = \phi_\nu\left(\sum_{i=1}^{m} a_{i,k}g_{n+i}\right) = \sum_{i=1}^{m} a_{i,k}\phi_\nu(g_{n+i}) = \tag{2.2.36}$$

$$\sum_{i=1}^{m} a_{i,k}\alpha_i(\nu) = O(\nu^n),$$

where $k = 1, \dots, n$,

$$\phi_\nu(\alpha_0(e_k)) - \psi_\nu(e_k) = \phi_\nu(g_k) - \psi_\nu(e_k) = \tag{2.2.37}$$

$$\left(\sum_{i=1}^{m} a_{i,k}\alpha_i(\nu)\right) - \alpha_{m+k}(\nu) = O(1/\nu).$$

We see that (2.2.18), (2.2.19) and (2.2.21) follow from
(2.2.33), (2.2.34), (2.2.35),(2.2.36) and (2.2.37) .
Finally, (2.2.20) follows from the (2.2.33), (2.2.32) and (iii) . ■

Let $\mathfrak{L}$ is a linear normed space over the field $\mathbb{R}$ and $\mathfrak{L}'$ is the linear space conjugate over the field $\mathbb{R}$ to the space $\mathfrak{L}$. Let q denotes the norm on the space $\mathfrak{L}$ and q' denotes the norm which is defined on the space $\mathfrak{L}'$ and is the norm associated with q. Let $\mathfrak{L}_0$ is a linear space over the field $\mathbb{R}$. Let further $s = \dim_{\mathbb{R}}(\mathfrak{L}_0)$ and

$$e_1 \, , \dots , e_s \tag{2.2.38}$$

is the basis of the space $\mathfrak{L}_0$ over the field $\mathbb{R}$. Let

$$L_0 = \{x_1 e_1 + \, \dots \, x_s e_s \colon x_1 \in \mathbb{Z}\, , \dots , x_s \in \mathbb{Z}\}. \tag{2.2.39}$$

Let $\mathfrak{L}_0'$ is the linear space conjugate over the field $\mathbb{R}$ to the space $\mathfrak{L}_0$ and

$$\eta_1 \, , \dots , \eta_s \tag{2.2.40}$$

is the basis of the space $\mathfrak{L}_0'$ biorthonormal with the basis (2.2.38). Then

$$\eta_i(x_1 e_1 + \, \dots \, x_s e_s) = x_i, \tag{2.2.41}$$

where $x_1 \in \mathbb{R}, \dots , x_s \in \mathbb{R}$ and

$$L_0^{\wedge} = \{x_1 \eta_1 + \, \dots \, x_s \eta_s \colon x_1 \in \mathbb{Z}\, , \dots , x_s \in \mathbb{Z}\}. \tag{2.2.42}$$

is the lattice dual to the lattice L_0. Let, finally, q_0 denotes a some norm on $\mathfrak{L}_0$ and q_0' denotes the norm which is defined on $\mathfrak{L}_0'$ and is the norm associated with q_0.

Theorem 2.2.2. *Let the space $\mathfrak{L}$ contains all the elements of the system* 2.2.14. *Let further $\mathfrak{L}_1$ is the linear envelope over the field $\mathbb{R}$ of the system* 2.2.14. *In order that the elements (2.2.14) compose a linearly independent system over the field $\mathbb{Q}$ it is necessary and sufficient the existence of a linear map α_0 of the space $\mathfrak{L}_0$ into $\mathfrak{L}$, the existence of a sequence of a linear over the field $\mathbb{R}$ continuous functionals $\phi_\nu^\vee$ (where $\nu \in \mathbb{N}$) on the linear normed space $\mathfrak{L}$ and the existence of sequence of a linear over $\mathbb{R}$ functionals $\psi_\nu^\vee$ (where $\nu \in \mathbb{N}$) on the linear normed space $\mathfrak{L}_0$ such that the system*

$$\alpha_0(e_1) \, , \dots , \alpha_0(e_s)$$

would be equivalent over the field $\mathbb{Q}$ to the system (2.2.14) and the following conditions would be fulfilled:

$$\psi_\nu(f) \in \mathbb{Z}$$

for any $f \in L_0$;

$$\lim_{\nu\to\infty}(\phi_\nu(\alpha_0(f)) - \psi_\nu(f)) = 0 \tag{2.2.43}$$

for any $f \in \mathfrak{L}_0$; *and*

$$\bigcap_{\nu=\mu}^{\infty} \mathrm{Ker}\,(\psi_\nu) = \{0\} \tag{2.2.44}$$

for any $\mu \in \mathbb{N}$. *Moreover, if (2.2.43) and (2.2.44) are fulfilled, then*

$$\dim_{\mathbb{Q}}(\alpha_0(\mathfrak{L}_0)) = s; \tag{2.2.45}$$

if the equality (2.2.39) is fulfilled and

$$m = \dim_{\mathbb{R}}(\mathfrak{L}_1),$$

then there exists a linear over $\mathbb{R}$ *map* α_0 *of space* $\mathfrak{L}_0$ *into space* $\mathfrak{L}$, *there exists a sequence of linear functions* $\phi_\nu \in \mathfrak{L}'$ *(where* $\nu \in \mathbb{N}$ *) and there exists a sequence of linear functions* $\psi_\nu \in L_0^\wedge$ *(where* $\nu \in \mathbb{N}$ *) such that the equality (2.2.44) holds and the following conditions are fulfilled:*

$$q'(\phi_\nu) = O(\nu^{s-m+1}),\ q_0'(\psi_\nu) = O(\nu^{s-m}), \tag{2.2.46}$$

$$|\phi_\nu(\alpha_0(f)) - \psi_\nu(f)| = q_0(f)O(1/\nu) \tag{2.2.47}$$

for any $f \in \mathfrak{L}_0$.

Proof. Let $K = \mathbb{Q}$. Then $\mathfrak{K} = \mathbb{R}$. Since each $\mathbb{R}$-module is free, it follows that the Theorem 2.2.2 is direct corollary of the Theorem 2.2.1. ■

Corollary 1. *Let* $B = (b_{i,k}), i = 1, \dots, p, k = 1, \dots, s$ *is a* $p \times s$*-matrix,all the elements of which are real numbers. In order that the columns of matrix* B *compose a linearly independent system over the field* $\mathbb{Q}$ *it is necessary and sufficient the existence of* p *sequences of real numbers* $\beta_1(\nu), \dots, \beta_p(\nu)$ *and the existence of* s *sequences of integer numbers* $\xi_1(\nu), \dots, \xi_s(\nu)$ *such that the following conditions are fulfilled:*

(I)

$$\lim_{\nu\to\infty}\left(-\xi_k(\nu) + \sum_{j=1}^{p} b_{j,k}\beta_j(\nu)\right) = 0,$$

where $k = 1, \dots, s$;

(II)
for each $\kappa \in \mathbb{N}$, the system, composed by the infinite sequences

$$\xi_k(\kappa+1), \dots, \xi_k(\kappa+\mu), \dots,$$

where $k = 1, \dots, s$, is linearly independent system over $\mathbb{R}$.

Proof. Let $h_1, \dots, h_p$ is a standard basis of $\mathbb{R}^p$ (i.e. $h_1, \dots, h_p$ are columns of a unit $p \times p -$ matrix). Let further

$$g_k = \sum_{j=1}^{p} b_{j,k} h_j. \tag{2.2.48}$$

Then

$$g_1, \dots, g_s \tag{2.2.49}$$

are columns of the matrix B. Let $\mathfrak{L}^{\sim}$ is the linear envelope over the field $\mathbb{R}$ of system (2.1.47). We consider $\mathbb{R}^p$ in the role of the space $\mathfrak{L}$ of the Theorem 2.2.2.

Sufficiency. There exists a unique linear over $\mathbb{R}$ function ϕ_ν on $\mathfrak{L}$ with the following property:

$$\phi_\nu(h_j) = \beta_j(\nu), \tag{2.2.50}$$

where $j = 1, \dots, p$. According to (2.1.48) and (2.2.50),

$$\phi_\nu(g_k) = \sum_{j=1}^{p} b_{j,k} \phi_(\nu)(h_j) = \sum_{j=1}^{p} b_{j,k} \beta_j(\nu) \tag{2.2.51}$$

There exists a unique linear over $\mathbb{R}$ function ψ_ν on $\mathfrak{L}_0$ with the following property:

$$\psi_\nu(e_k) = \xi_k(\nu), \tag{2.2.52}$$

where $k = 1, \dots, s$. Since $\xi_k(\nu) \in \mathbb{Z}$ for $k = 1, \dots, s$ and $\nu \in \mathbb{N}$, it follows that $\phi_\nu \in L_0^\wedge$. The condition (I) implies the equality (2.2.43), the condition (II) implies the equality (2.2.44).

Therefore (2.2.45) holds.

Necessity. According to assertion of the Theorem 2.2.2, there exists a linear over $\mathbb{R}$ map α_0 of the space $\mathfrak{L}_0$ into the space $\mathfrak{L}$, there exists a sequence of a linear functions $\phi_\nu \in \mathfrak{L}'$ (where $\nu \in \mathbb{N}$) and there exists a sequence of the linear functions $\psi_\nu \in L_0^\wedge$ (where $\nu \in \mathbb{N}$) such that the equalities (2.2.43) and (2.2.44) hold. Let

$$\beta_j(\nu) = \phi_\nu(h_j), \tag{2.2.53}$$

where $j = 1, \dots, p$, and $\nu \in \mathbb{N}$,

$$\xi_k(\nu) = \psi_\nu(e_k), \tag{2.2.54}$$

where $k = 1, \dots, s$ and $\nu \in \mathbb{N}$. Since $\psi_\nu \in L_0^\wedge$, it follows that

$$\xi_k(\nu) \in \mathbb{Z}, \tag{2.2.55}$$

where $k = 1, \dots, s$ and $\nu \in \mathbb{N}$. Then

$$\phi_\nu(g_k) = \phi_\nu(g_k) = \phi_\nu(\sum_{i=1}^{p} b_{j,k}h_i) \tag{2.2.56}$$

$$\sum_{i=1}^{p} b_{i,k}\phi_\nu(h_i) = \sum_{j=1}^{p} b_{i,k}\beta_i(\nu).$$

The assertion (I) follows from (2.2.43) for $f = e_k$ with $k = 1, \dots, s$, (2.2.53), (2.2.54), (2.2.55) and(2.2.56). The equality (2.2.44) is equivalent to the assertion (II). ■

Corollary 2. *Let $C = (c_{j,k}), j = 1, \dots, r, k = 1, \dots, s$ is a $r \times s -$ matrix, all the elements of which are complex numbers. In order that the columns of matrix B are linear independent over the field $\mathbb{Q}$ it is necessary and sufficient the existence of r sequences of complex numbers $\gamma_1(\nu), \dots, \gamma_r(\nu)$ and existence of s sequences of integer numbers $\xi_1(\nu), \dots, \xi_s(\nu)$ such that the following conditions are fulfilled:*

(A)

$$\lim_{\nu\to\infty}(-\xi_k(\nu) + \sum_{j=1}^{r} \mathrm{Re}\,(c_{j,k}\overline{\gamma_j(\nu)})) = 0,$$

where $k = 1, \dots, s$;

(B) for each $\kappa \in \mathbb{N}$, the system, composed by the infinite sequences

$$\xi_k(\kappa + 1), \dots, \xi_k(\kappa + \mu), \dots,$$

where μ runs through $\mathbb{N}$ and $k = 1, \dots, s$, is linearly independent system over $\mathbb{R}$.

Proof. Let $C = U + iV$, where $U = (u_{j,k})$ is a real $r \times s -$ matrix and $V = (v_{j,k})$ is a real $r \times s -$ matrix. Let

$$C^* = \begin{pmatrix} U \\ V \end{pmatrix}.$$

Clearly, the columns of the matrix C compose a linearly independent system over the field $\mathbb{Q}$ if and only if the columns of the matrix C^* compose a linearly independent system over the field $\mathbb{Q}$. According to Corollary 1, the columns of the matrix

C^* compose a linear independent system over the field $\mathbb{Q}$ if and only if there exist a sequences of real numbers

$$\alpha_1(\nu)\,,\dots\,,\alpha_r(\nu),$$

$$\beta_1(\nu)\,,\dots\,,\beta_r(\nu)$$

and there exist sequences of integer numbers

$$\xi_1(\nu)\,,\dots\,,\xi_s(\nu)$$

(where $\nu \in \mathbb{N}$) such that the condition (II) of the corollary 1 is fulfilled and, for $k = 1\,,\dots\,,s$, the following equality holds:

$$\lim_{\nu\to\infty}(-\xi_k(\nu) + \sum_{j=1}^{r}(\alpha_j(\nu)u_{j,k} + \beta_j(\nu)v_{j,k})) = 0.$$

Let $\gamma_{j,k} = \alpha_{j,k} + i\beta_{j,k}$ for $j = 1\,,\dots\,,r$, $k = 1\,,\dots\,,s$. Then

$$\mathrm{Re}\,(c_{j,k}\overline{\gamma_j(\nu)} = \alpha_j(\nu)u_{j,k} + \beta_j(\nu)v_{j,k}$$

where $j = 1\,,\dots\,,r$, $k = 1\,,\dots\,,s$. ■

§3. Non-discrete free modules over the ring $\mathbb{Z}_K$.

Let, as in paragraph 1, $\mathfrak{M}_0$ is a free $\mathfrak{K}$ – module with s free generators (2.1.15) and M_0 is the free $\mathbb{Z}_K$-module (2.1.16).

Let further M is the free $\mathbb{Z}_K$-submodule of normed $\mathfrak{K}$-module $\mathfrak{M}$ (which, of course, we may regard as $\mathbb{Z}_K$-module).

Let β_0 is the $\mathbb{Z}_K$-isomorphism of M_0 onto M. If for any $C > 0$ and for any $\alpha > 0$ there exists an element $f \in M_0\backslash\{0\}$ satisfying to the inequality

$$q(\beta_0(f)) < C(q_\infty^{(\mathfrak{K})}(f))^{-\alpha},$$

then we name the module M by Liouvillie's module. In contrary case there are $C > 0$ and $\alpha > 0$ such that

$$q(\beta_0(f)) \geq C(q_\infty^{(\mathfrak{K})}(f))^{-\alpha}$$

for any $f \in M_0\backslash\{0\}$; we name the lower bound of such numbers α by the measure of nondiscreteness of module M relative to the norm q and we denote this lower bound by $\mu_q(M)$.

These definitions are independent from the choice of the isomorphism β_0. Let β_1 another such isomorphism. Let $M_0^{\sim}$ and $M^{\sim}$ are the linear over K spaces generated by M_0 and M respectively. Clearly,isomorphisms β_0 and β_1 can be continued uniquely to the isomorphisms respectively $\beta_0^{\sim}$ and $\beta_1^{\sim}$ of linear over K space $M_0^{\sim}$ onto linear over K space $M^{\sim}$. The maps $\beta_0^{\sim}(\beta_1^{\sim})^{-1}$ and $\beta_1^{\sim}(\beta_0^{\sim})^{-1}$ (respectively $(\beta_0^{\sim})^{-1}\beta_1^{\sim}$ and $(\beta_1^{\sim})^{-1}\beta_0^{\sim}$) are automorphisms of a linear over K spaces $M_0^{\sim}$ (respectively $M^{\sim}$), which are finite-dimensional over the field $\mathbb{Q}$; of course, the specified automorphisms are automorphisms also over the field $\mathbb{Q}$. Consequently, there exists a constant $C > 0$ satisfying to the inequalities

$$q_{\infty}^{\mathfrak{K}}((\beta_j^{\sim}(\beta_{1-j}^{\sim})^{-1})(f) \leq Cq_{\infty}^{\mathfrak{K}}(f),$$

$$q((((\beta_{1-j}^{\sim})^{-1}\beta_j^{\sim})(g)) \leq Cq(g)$$

for any $g \in M^{\sim}$, for any $f \in M_0^{\sim}$ and for any $j = 0, 1$. Then

$$q(\beta_j^{\sim}(f)) = q((\beta_j^{\sim}((\beta_{1-j}^{\sim})^{-1})\beta_{1-j}^{\sim}(f)) \leq$$

$$Cq(\beta_{1-j}^{\sim}(f))$$

for any $f \in M_0^{\sim}$ and for any $j = 0, 1$.

If $K = \mathbb{Q}, \alpha$ is not Liouvillie's irrational number and $M = \mathbb{Z} + \mathbb{Z}\alpha$, then for standard norm q on $\mathbb{R}$ the number $\mu_q(M)$ coincides with the measure of irrationality of the number α. Also, clearly, α is Liouvillie's number if and only if the $\mathbb{Z}$-module $M = \mathbb{Z} + \mathbb{Z}\alpha$, is Liouvillie's module.

I prove the following Lemma with aim to emphasize dependence of the measure of nondiscreteness from the norm q and to show that the name of this notion is natural.

Lemma 2.3.1. *Let M is a normed $\mathfrak{K}$-module $\mathfrak{M}$, having finite set of generators and let q is its norm. For the discreteness of $\mathbb{Z}_K$-submodule M of the module $\mathfrak{M}$ it is necessary and sufficient the realizability of the following condition:*

the existence in M of some free $\mathbb{Z}_K$-submodule $M^{\vee}$, which has a finite index in M and $\mu_q(M^{\vee}) = 0$.

Proof.

Necessity

Clearly, M contains some basis

$$f_1, , \dots , f_s$$

of a linear over K envelope $M^{\sim}$ of the module M. Let

$$M^{\vee} = \mathbb{Z}_K g_1 + \dots + \mathbb{Z}_K g_s.$$

Clearly,$M^{\vee}$ has a finite index in M. Clearly, there exists the unique the $\mathbb{Z}_K$-isomorphism β_0 of module M_0 in (2.1.16) onto $M^{\vee}$ with the following property:

$\beta_0(e_k) = f_k$ for $k = 1, \dots, s$.

Since M is discrete relative to the norm q, it follows that $M^{\vee}$ is discrete relative to the norm q. Since the module $M^{\vee}$ is discrete relative to the norm q, it follows that there exists the number $\delta > 0$ with the following property:

$q(g) = q(g - 0) \geq \delta$ for any $g \in M^{\vee} \setminus \{0\}$.

Then

$$q(\beta_0(f)) \geq \delta = \delta(q_{\infty}^{(\mathfrak{K})}(f))^{-0}$$

for any $f \in M_0^{\vee} \setminus \{0\}$.

Sufficiency.

Clearly, for the discreteness of module M it is necessary and sufficient the realizability of the following condition:

the discreteness of some submodule $M^{\vee}$ of module M and the finiteness of index of this submodule $M^{\vee}$ in M.

Therefore we may regard that $\mu_q(M) = 0$.

We consider first the case, when the generated by M the $\mathfrak{K}$-submodule $\mathfrak{M}_1$ of $\mathfrak{K}$-module $\mathfrak{M}$ is free $\mathfrak{K}$ – module. In this case, according to the Lemma 2.4.4, the isomorphism β_0 can be chosen such that

$$g_1 = \beta_0(e_1), \dots, g_s = \beta_0(e_s) \tag{2.3.1}$$

and m of last elements of system (2.3.1) are free generators of $\mathfrak{M}_1$. Let A is the $m \times s$-matrix composed by columns of coordinates of elements of system (2.3.1) in the basis

$$g_{n+1}, \dots, g_s. \tag{2.3.2}$$

If $m < s$, then, according to the Lemma 2.2.2, there are $C_1 > 0$ and for each $\nu \in \mathbb{N}$ a column

$$X_\nu = \begin{pmatrix} x_1(\nu) \\ \vdots \\ x_s(\nu) \end{pmatrix} \in (\mathbb{Z}_K)^s \setminus \{0\}$$

such that $q_{\infty}^{(\mathfrak{K})}(X) \leq C_1(\nu)^{m/(s-m)}$ and for

$$Y_\nu = \begin{pmatrix} y_1(\nu) \\ \vdots \\ y_m(\nu) \end{pmatrix} = AX$$

the following inequality holds:

$$q_{\infty}^{(\mathfrak{K})}(Y_\nu) \leq C_1/nu. \tag{2.3.3}$$

Therefore

$$q_{\infty}^{(\mathfrak{K})}(Y_{\nu}) \le (C_1)^{s/m}(q_{\infty}^{(\mathfrak{K})}(X_{\nu}))^{-(s-m)/m}.$$

Let

$$f_{\nu} = \sum_{k=1}^{s} x_k(\nu)e_k,$$

$$h_{\nu} = \sum_{i=1}^{m} y_i(\nu)g_{n+i}.$$

Then $f_{\nu} \neq 0, h_{\nu} = \beta_0(f_{\nu}) \neq 0$ and

$$0 < q(\beta_0(f_{\nu})) = q(h_{\nu}) = q\left(\sum_{i=1}^{m} y_i(\nu)g_{n+i}\right) \le \sum_{i=1}^{m} q(y_i(\nu)g_{n+i}) \le$$

$$\sum_{i=1}^{m} q_{\infty}^{(\mathfrak{K})}(y_i(\nu))q(g_{n+i}) \le$$

$$(C_1)^{s/m}(q_{\infty}^{(\mathfrak{K})}(f_{\nu}))^{-(s-m)/m}\sum_{i=1}^{m} q(g_{n+i}).$$

Let $\varepsilon \in (0, (s-m)/m)$. Since $\mu_q(M) = 0$, it follows that there exists $C_2 > 0$ with the following property:

$$C_2(q_{\infty}^{(\mathfrak{K})}(f_{\nu}))^{-\varepsilon} \le q(\beta_0(f_{\nu})).$$

Then

$$(q_{\infty}^{(\mathfrak{K})}(f_{\nu}))^{((s-m)/m)-\varepsilon} \le ((C_1)^{s/m}/C_2)\sum_{i=1}^{m} q(g_{n+i}). \tag{2.3.4}$$

But this is incompatible with (2.3.3) because M_0 is discrete $\mathbb{Z}_K$-module and the set of all its elements satisfying (2.3.4) is finite. Therefore the case $s > m$ is impossible. Let $s = m$. Then $\mathfrak{M}_1$ is a free $\mathfrak{K}$-module with the s free generators (2.3.2), where $n = 0$. There exists a unique $\mathfrak{K}$-homomorphism β_0^* of module $\mathfrak{M}_0$ into $\mathfrak{M}_1$ with the following property:

$\beta_0^*(e_k) = g_k$ for $k = 1, \dots, s$.

Since (2.3.2) with $n = 0$ is free system of $\mathfrak{K}$-generators of $\mathfrak{M}_1$, it follows that homomorphism β_0^* is isomorphism, which on M coincides with β_0 and on $M^{\sim}$ - with $\beta_0^{\sim}$. Let $q^{\sim}$ is the function defined on the module $\mathfrak{M}_1$ by means of the equality:

$$q^{\sim}(g) = q_{\infty}^{(\mathfrak{K})}((\beta_0^*)^{-1}(g))$$

for any $g \in \mathfrak{M}_1$. Then $q^{\sim}$ is the norm on $\mathfrak{M}_1$ and, clearly, M is discrete relative to $q^{\sim}$. But the norms q and $q^{\sim}$ are equivalent because they are defined by means their restrictions on finite-dimensional real or complex linear spaces, which are direct summands of the module $\mathfrak{M}_1$.

Therefore M is discrete also relative to the norm q.

We consider, finally, the general case. The discreteness (respectively nondiscreteness) of the set M, as a topological notion, is independent from algebraical structure on this set. Therefore we can consider M_0 (respectively M) not only as a free $\mathbb{Z}_K$-submodule of $\mathfrak{K}$-module $\mathfrak{M}_0$ (respectively $\mathfrak{M}$) with the free $\mathbb{Z}_K$-generators (2.1.15) (respectively (2.3.1)) but also as a free $\mathbb{Z}$-submodule of $\mathbb{R}$-module $\mathfrak{M}_0$ (respectively $\mathfrak{M}$) with free $\mathbb{Z}$ − generators (2.1.17)

(respectively

$$\omega_k g_l, \tag{2.3.5}$$

where $k = 1, \ldots, d,\ l = 1, \ldots, d.$) The system (2.1.17) is also the system of free $\mathbb{R}$-generators of free $\mathbb{R}$-module $\mathfrak{M}_0$ (i.e. basis of linear space $\mathfrak{M}_0$ over the field $\mathbb{R}$). The $\mathbb{Z}_K$-isomorphism β_0 maps the basis (2.1.17) onto the basis (2.3.5). Since the norms $q_\infty^{(\mathfrak{K})}$ and $q_\infty^{(\mathfrak{B})}$ are equivalent, as the norms of free $\mathfrak{K}$-module $\mathfrak{M}_0$ with finite set of a free $\mathfrak{K}$-generators, the equality

$$\mu_q^{(\mathfrak{K})}(M) = \mu_q^{(\mathbb{R})}(M)$$

follows from the equality (2.1.9) and, if $\mu_q^{(\mathfrak{K})}(M) = 0$, it follows that

$$\mu_q^{(\mathbb{R})}(M) = 0.$$

Since generated by means of the basis (2.3.5) $\mathbb{R}$-module, as linear space over the field $\mathbb{R}$, is free and coincides with $\mathfrak{M}_1$, it follows that, according to the first part of proof, the set M is discrete set in $\mathfrak{M}$ relative to the norm q. ■

For each $Z \in \mathfrak{K}$, let

$$\|Z\|_K = \inf_{W \in \mathbb{Z}_K} \{q_\infty^{(\mathfrak{K})}(Z - W)\}$$

Lemma 2.3.2 *If $Z \in \mathfrak{K}$ and $q_\infty^{(\mathfrak{K})}(Z) \leq \frac{1}{2}$, then*

$$\|Z\|_K = q_\infty^{(\mathfrak{K})}(Z).$$

Proof. If, for any $W \in \mathbb{Z}_K$, the inequality

$$q_\infty^{(\mathfrak{K})}(Z - W) < q_\infty^{(\mathfrak{K})}(Z)$$

holds, then

$$q_\infty^{(\mathfrak{K})}(W) \le q_\infty^{(\mathfrak{K})}(Z) + q_\infty^{(\mathfrak{K})}(Z-W) < 2q_\infty^{(\mathfrak{K})}(Z) \le 1$$

and, according to (2.1.4), $W = 0$. *blacksqare*

Let $m \in \mathbb{N},\ n \in \mathbb{N}$,

$$a_{i,k} \in \mathfrak{K} \tag{2.3.6}$$

for $i = 1, \dots, m,\ k = 1, \dots, n$,

$$\alpha_l^\wedge(\nu) \in \mathbb{Z}_K, \tag{2.3.7}$$

where $l = 1, \dots, m+n$ and $\nu \in \mathbb{N}$. Let further there are $\gamma_0, r_1^\wedge \ge 1, \dots, r_m^\wedge \ge 1$ such that

$$q_\infty^{(\mathfrak{K})}(\alpha_i(\nu)) < \gamma_0 (r_i^\wedge)^\nu, \tag{2.3.8}$$

where $i = 1, \dots, m$ and $\nu \in \mathbb{N}$. Let

$$y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum_{i=1}^m a_{i,k}\alpha_i^\wedge(\nu), \tag{2.3.9}$$

where $k = 1, \dots, n$ and $\nu \in \mathbb{N}$. If $X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in (\mathfrak{K})^n$, then let

$$y^\wedge(X) = y^\wedge(X, \nu) = \sum_{k=1}^n y_k^\wedge(\nu) Z_k \tag{2.3.10}$$

for $\nu \in \mathbb{N}$, let

$$\phi_i(X) = \sum_{k=1}^n a_{i,k} Z_k \tag{2.3.11}$$

for $i = 1, \dots, m$ and $\nu \in \mathbb{N}$ and let

$$\alpha_0^\wedge(X, \nu) = \sum_{k=1}^n \alpha_{m+k}^\wedge(\nu) Z_k \tag{2.3.12}$$

for $\nu \in \mathbb{N}$. Clearly,

$$y^\wedge(X, \nu) = -\alpha_0^\wedge(X, \nu) + \sum_{i=1}^m \alpha_i^\wedge(\nu)\phi_i(X) \tag{2.3.13}$$

for $X \in (\mathfrak{K})^n$ and $\nu \in \mathbb{N}$,

$$\alpha_0^{\wedge}(X,\nu) =\in \mathbb{Z}_K \tag{2.3.14}$$

for $X \in (\mathbb{Z}_K)^n$ and $\nu \in \mathbb{N}$.

Now I can proceed to main result of this part:

Theorem 2.3.1. *Let $l \in \mathbb{N}, \gamma_1 > 0$, let*

$$\gamma_2 > \frac{1}{2}, R_1 \geq R_2 > 1, \tag{2.3.15}$$

let

$$\alpha_i = (\log(r_i^{\wedge} R_1))/\log(R_2), \tag{2.3.16}$$

where $i = 1, \dots, m$, let

$$\gamma_3 = \gamma_1 (R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \tag{2.3.17}$$

$$\gamma_4 = \gamma_3 \left(\sum_{i=1}^{m} \gamma_0 (r_i^{\wedge})^{(log(2\gamma_2))/\log(R_2)+l} \right)^{-1}, \tag{2.3.18}$$

let

$$X \in (\mathbb{Z}_K)^n \backslash \{(0)\}, \tag{2.3.19}$$

and let for each $\nu \in \mathbb{N} - 1$ the following inequalities hold:

$$\gamma_1 (R_1)^{-\nu} q_\infty^{(\mathfrak{K})}(X) \leq \sup\{ q_\infty^{(\mathfrak{K})}(y^{\wedge}(X,\kappa)) : \kappa = \nu, \dots, \nu + l - 1))\}, \tag{2.3.20}$$

$$q_\infty^{(\mathfrak{K})}(y^{\wedge}(X,\nu)) \leq \gamma_1 (R_2)^{-\nu} q_\infty^{(\mathfrak{K})}(X). \tag{2.3.21}$$

Then

$$\sup\{ \|\phi_i(X)\|_K (q_\infty^{(\mathfrak{K})}(X))^{\alpha_i} : i = 1, \dots, m\} \geq \gamma_4. \tag{2.3.22}$$

Proof. According to (2.3.15), (2.3.19) and (2.1.4),

$$2\gamma_2 q_\infty^{(\mathfrak{K})}(X) > 1 \tag{2.3.23}$$

Let

$$\nu_1 = \inf\{\nu \in \mathbb{N} - 1 \colon \gamma_2 q_\infty^{(\mathfrak{K})}(X)(R_2)^{-\nu} \leq 1/2\} = \tag{2.3.24}$$

$$\inf\{\nu \in \mathbb{N} - 1 \colon \nu \geq \log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2)$$

Clearly,

$$\nu_1 = \log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2) + \theta,$$

where $\theta \in [0, 1)$,

$$\gamma_1(R_1)^{-\nu_1} q_\infty^{(\mathfrak{K})}(X) = \tag{2.3.25}$$

$$\gamma_1 \exp(-(\log(R_1))(log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2) + \theta)) q_\infty^{(\mathfrak{K})}(X) =$$

$$\gamma_1 (q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)} \times$$

$$\exp(-(\log(R_1))((log(2\gamma_2))/\log(R_2) + \theta)) \geq$$

$$\gamma_1(\exp(-(\log(R_1))(log(2\gamma_2 R_2))/\log(R_2))) \times$$

$$(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)} =$$

$$\gamma_3 (q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)}.$$

In view of (2.3.24),

$$\gamma_2 q_\infty^{(\mathfrak{K})}(X)(R_2)^{-\nu} \leq 1/2 \tag{2.3.26}$$

for $\nu \in \nu_1 + \mathbb{N} - 1$. Let $\nu_2 \in \{\nu_1, \ldots, \nu_1 + l - 1\}$ and

$$q_\infty^{(\mathfrak{K})}(y^\wedge(X, \nu_2)) = \sup\{q_\infty^{(\mathfrak{K})}(y^\wedge(X, \kappa)) \colon \kappa = \nu_1, \ldots, \nu + l - 1\}.$$

Then, in view of (2.3.20), (2.3.25), (2.3.19), (2.3.24) and (2.3.8),

$$\frac{1}{2} \geq q_\infty^{(\mathfrak{K})}(y^\wedge(X, \nu_2)) \geq \gamma_1(R_1)^{-\nu_1} q_\infty^{(\mathfrak{K})}(X) \geq$$

$$\gamma_3 (q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/(R_2)/\log(R_2)}.$$

Let $f_1 \in \mathbb{Z}_K, \ldots, f_m \in \mathbb{Z}_K$ are chosen such that

$$q_\infty^{(\mathfrak{K})}(\phi_i(X) - f_i) = \|\phi_i(X)\|_K, \tag{2.3.27}$$

and let

$$g = -\alpha_0^\wedge(X, \nu_2) + \sum_{i=1}^{m} \alpha_i^\wedge(\nu_2) f_i. \tag{2.3.28}$$

Then, in view of(2.3.12), (2.3.27), and, according to the Lemma 2.3.2, $g \in \mathbb{Z}_K$ and

$$\gamma_3(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)} \leq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2) =$$

$$\|q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2))\|_K \leq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2) - g) \leq$$

$$\sum_{i=1}^{m} q_\infty^{(\mathfrak{K})}(\alpha_i^\wedge(\nu_2))q_\infty^{(\mathfrak{K})}(\phi_i(X) - f_i) =$$

$$\sum_{i=1}^{m} q_\infty^{(\mathfrak{K})}(\alpha_i^\wedge(\nu_2))\|\phi_i(X)\|_K \leq$$

$$\sum_{i=1}^{m} \|\phi_i(X)\|_K \gamma_0(r_i^\wedge)^{\nu_1+l-1}.$$

But

$$(r_i^\wedge)^{\nu_1} = \exp((\log(r_i^\wedge))((log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X)))/\log(R_2) + \theta)) =$$

$$(q_\infty^{(\mathfrak{K})}(X))^{(\log(r_i^\wedge))/\log(R_2)}(r_i^\wedge)^{(log(2\gamma_2))/\log(R_2))+\theta))},$$

where $i = 1, \ldots, m$. Therefore

$$\gamma_3(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/(R_2)/\log(R_2)} \leq q_\infty^{(\mathfrak{K})}(y^\wedge k(X,\nu_2) \leq$$

$$\sup\{\|\phi_i(X)\|_K(q_\infty^{(\mathfrak{K})}(X))^{(\log(r_i^\wedge))/\log(R_2)} : i = 1, \ldots, m\}\times$$

$$\sum_{i=1}^{m} \gamma_0(r_i^\wedge)^{(log(2\gamma_2))/\log(R_2)+l} \tag{2.3.29}$$

The inequality (2.3.22) follows from (2.3.29) . ■ This Theorem was obtained in [28] (Lemma 25) and, if the number field coincides with $\mathbb{Q}$, in [26] (Theorem 8).

§4. Survey of some results.

It is known that for Riemann's $\zeta-$ function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

the following equality holds:

$$\zeta(2k) = B_{2k}(2\pi)^{2k}(-1)^{k+1}/(2(2k)!),$$

where $k \in \mathbb{N}$ an $B_{2k} \in \mathbb{Q}(k \in \mathbb{N})$ are the Bernoulli's Numbers . But about the numbers $\zeta(2k+1)(k \in \mathbb{N})$ it was known nothing till 1978, when R.Apéry [1] has proved that the number $\zeta(3)$ is irrational.

In 1979, [21], [22] author by means of Meijer's functions proved the following Theorem:

Theorem 2.4.1. *Let M is the $\mathbb{Z}-$ module with generators*

$$\begin{pmatrix} log(4) \\ \zeta(2) \end{pmatrix}, \begin{pmatrix} \zeta(2) \\ 3\zeta(3) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4.1.}$$

Then the columns (2.4.1) are $\mathbb{Z}-$ free generators of module M, the module M is not Liouvillie's module and has the measure of nondiscreteness not bigger than $106,1019$.

Later [27] the estimation specified in the Theorem 2.4.1 was decreased up to $22,4271$.

Let

$$a_1 = d_1 = \log(2), c_1 = -b_1 = -\frac{\pi}{2}, \gamma^{(1)} = 3,3887,$$

$$a_2 = d_2 = \log(4/3), c_2 = -3b_2 = -\frac{\pi}{\sqrt{3}}, \gamma^{(2)} = 2,6161,$$

$$a_3 = d_3 = \log(3), c_3 = -3b_3 = -\frac{\pi}{2}, \gamma^{(3)} = 3,0736,$$

Let further

$$f_{i,1} = \begin{pmatrix} a_i \\ c_i \end{pmatrix}, f_{i,2} = \begin{pmatrix} b_i \\ d_i \end{pmatrix}, f_{i,3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_{i,4} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4.2}$$

where $i = 1,2,3,$ and M_i, is the $\mathbb{Z}-$ module with four generators (2.4.2). From Baker's Theorem follows that the generators (2.4.2) are a free $\mathbb{Z}$-generators of $\mathbb{Z}$-module M_i. The following Theorem was proved in [28] :

Theorem 2.4.2. *For $i = 1,2,3$ the measure of nondiscreteness of M_i is not bigger than $\gamma^{(i)}$.*

The following Theorem, which is proved in [29], may be mentioned in connection with Theorem 2.3.1:

Theorem 2.4.3. *Let $m = 2,3,\dots,37$, let $K = \mathbb{Q}[\exp(2\pi i/m]$ let $\mathrm{Gal}\,(K/\mathbb{Q})$ is the Galois group of the extension $K/\mathbb{Q}$, let $\log(z)$, where $z \in \mathbb{C}\backslash(\infty,0]$ is the branch of $\mathrm{Log}\,(z)$ with $|\arg(z)| < \pi$ and let α_m is the $(m-1)-th$ element of the following sequence of real numbers (the numbers in this sequence are separated each from other by means of semicolon):*

$$2,8913997710; 3,0735255571; 3,3886633478; 4,7846293625;$$

$$4,0874625966; 6,7393917326; 7,6354250465; 9,0606062958;$$
$$9,5089288012; 11,890912329; 13,408522861; 15,302385909;$$
$$16,312767157; 18,878054599; 20,789878818; 23,159619505;$$
$$25,015235814; 28,383873057; 31,232455499; 34,903849079;$$
$$38,193794800; 43,271329176; 48,284195791; 54,400694383;$$
$$60,675050525; 69,926378752; 80,085043105; 93,086523748;$$
$$108,61532969; 131,33065226; 161,36496124; 206,46682440;$$
$$278,01614475; 421,43377055; 808,33395542; 6355,0702897.$$

Then there exists $C > 0$ with the following property:

$$\max_{\sigma \in \mathrm{Gal}\,(K\mathbb{Q})} (|q^\sigma \log(1 - 1/z^\sigma) - p^\sigma|) \geq$$

$$C(\max_{\sigma \in \mathrm{Gal}\,(K\mathbb{Q})} (|q^\sigma|))^{-\alpha_m} \tag{2.4.2}$$

The estimates obtained by Hata [16] for $|qlog2 - p|$ and $\left|q\frac{\pi}{\sqrt{3}} - p\right|$ by means of another method follow from (2.4.2) for $m = 2$ and $m = 6$. The cases $m = 3$ and $m = 4$ are considered in [28]. These problems are a part of a more general problem of Diophantine Approximations of polylogarithms, i.e. of functions of a kind

$$L_s(z) = \sum_{n=1}^{\infty} n^{-s} z^n.$$

Nikishin [5] has proved the following Theorem (I follow in the formulation the paper [15]) :

Nikishin's Theorem. *Let $(a, b) \in \mathbb{Z} \times \mathbb{N}$, $-a > (4es)^{s(s-1)} b^{s+1})$ for some $s \in \mathbb{N}$. Then the numbers*

$$1, L_1(b/a), L_2(b/a), \ldots, L_s(b/a)$$

compose a linearly independent system over the field $\mathbb{Q}$.

This result was improved and generalized in [23], [24], [25].

Let $m \in \mathbb{N}, n \in \mathbb{N}, a \in \mathbb{Z} \backslash \{0\}, b \in \mathbb{N}$ and let $\Lambda_{m,n}(b/a) = (a_{i,j}(b/a))$ denotes the $m \times n-$ matrix, whose element at the intersection of its i-th row and its j-th column is equal to

$$a_{i,j}(b/a) = \binom{i+j-2}{i-1} L_{i+j-1}(b/a),$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Let further E_m is the unit $m \times m-$ matrix. I have proved in [24], [25] that, if

$$|a| > \left(\frac{m}{n}\right)^m \exp\left(\frac{n}{m}\left(n + (m-1)\max\left(\frac{m}{n}, 1\right)\right)\right) b^{\frac{m+n}{m}},$$

then the $\mathbb{Z}-$ module M generated by columns of the matrix

$$(\Lambda_{m,n}(b/a), E_m)$$

is $\mathbb{Z}-$ module with $m+n$ free $\mathbb{Z}-$ generators. I have proved in [26] that it is not a Liouville module and have found a upper estimate of the measure of nondiscreteness of this $\mathbb{Z}$-module.

I hope to give, in the next parts, the proofs of certain results mentioned above and also of some other results .

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