Fixed points and Reducibles in Equivariant gauge theory

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Abstract

In this paper we prove two technical theorems about the equivariant moduli space of ASD connections on a SU_2 or SO_3 bundle over a smooth oriented four-manifold X which is equipped with a smooth and orientation preserving action of a finite group π. The first theorem relates, in the case $\pi = \mathbf{Z}/p$ and compact moduli spaces, the existence of a non empty fixed set in the moduli space to the value of a certain Donaldson polynomial invariant. The second theorem gives a criterion under which one can avoid fixed reducible ASD connections by slightly varying the metric on X within the class of equivariant metrics.

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1 Introduction

Since the ground breaking work of S. K. Donaldson [Don83] Yang-Mills gauge theory proved to be a powerful tool for understanding the geometry of smooth four-dimensional manifolds. Beside answering many of the mainstream questions in four-dimensional topology, like the existence and uniqueness of smooth structures on topological four-manifolds, a couple of authors, among others see [FS85, HL92, BM93], used the Donaldson moduli space to investigate smooth group actions on four-manifolds. By now all of the mainstream questions in four-dimensional topology have been reproved using the Seiberg-Witten moduli space. However, till now, the Seiberg-Witten moduli space has not been used to investigate group actions, and it seems that the Donaldson moduli space will continue to be the more appropriate tool for equivariant problems.

The aim of this paper is to prove two theorems about this equivariant moduli space. Theorem A deals with the question when the fixed set of the induced π action on $\mathcal{M}$ is non empty. The theorem relates, in the case of a compact moduli space and $\pi = \mathbb{Z}/p$, the existence of a non empty fixed set in $\mathcal{M}$ to the divisibility of the Donaldson polynomial invariant associated to $\mathcal{M}$. Theorem A is shown to apply to some algebraic surfaces. The second theorem deals with the question of singularities in $\mathcal{M}$ arising from reducible connections. In the non equivariant setting it is proved in [DK90] that one can avoid reducibles by slightly varying the metric as long as

X has indefinite intersection form. Theorem B gives a criterion, depending only on the representation of π on $H^*(X)$, that allows such a perturbation within the class of equivariant metrics.

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1.1 Equivariant gauge theory

Let X be a smooth, closed, oriented simply connected 4-manifold on which a finite group π acts smoothly preserving the orientation. Let $G = SU_2$ or $G = SO_3$ and let $G \to P \to X$ be a principal G-bundle such that $g^*P \cong P \quad \forall g \in \pi$. This is always fulfilled for $G = SU_2$, and since SO_3 bundles are classified by w_2 and p_1 this is always fulfilled as long as π preserves the orientation and fixes w_2.

Let $\mathcal{A}$ be the space of connections on P and $\mathcal{G}$ the space of all gauge transformations on P, i.e. G-equivariant self-maps of P which cover the identity on X. Since for all $g \in \pi$ $g^*P \cong P$ there is a bundle isomorphism $\hat{g} : P \to P$ which covers g. Any two such lifts differ by a gauge transformation, which means that there always is a well defined π action on $\mathcal{B} = \mathcal{A}/\mathcal{G}$, no matter if P carries a π action covering the one on X or not.

We refer to [HL92] and [BM93] as references for equivariant gauge theory. Here we'll follow the exposition in [BM93]. There is a group extension

$$1 \to \mathcal{G} \to \mathcal{G}(\pi) \to \pi \to 1$$

where we identify π with its image in $\mathrm{Diff}^+(X)$ and $\mathcal{G}(\pi)$ are the bundle isomorphisms of P covering the elements in π. If $[A] \in \mathcal{B}^\pi$ then for every $A_0 \in [A]$ there is an extension of finite dimensional groups

$$1 \to \mathcal{G}_{A_0} \to \mathcal{G}(\pi)_{A_0} \to \pi \to 1$$

where $\mathcal{G}_{A_0}$ and $\mathcal{G}(\pi)_{A_0}$ are the stabilizers of A_0 in $\mathcal{G}$ and $\mathcal{G}(\pi)$ respectively. If $[A] \in \mathcal{B}^*$ is fixed under π then $\mathcal{G}_{A_0} = 1$ for all $A_0 \in [A]$ and therefore there is a unique lift of π to $\mathcal{G}(\pi)$ which fixes A_0.

We call two lifts from π to $\mathcal{G}(\pi)$ equivalent if they are conjugate via an element of $\mathcal{G}$. Let I be the set of equivalence classes of these lifts, then the fixed set $(\mathcal{B}^*)^\pi$ decomposes as the disjoint union

$$(\mathcal{B}^*)^\pi = \coprod_{i \in I} (\mathcal{A}_i^*/\mathcal{G}_i) =: \coprod_{i \in I} \mathcal{B}_i^*,$$

where $\mathcal{A}_i^*$ is the space of i-invariant irreducible connections and $\mathcal{G}_i$ is the space of i-invariant gauge transformations. Note that for $i \neq j$ $\mathcal{A}_i/\mathcal{G}_i$ and $\mathcal{A}_j/\mathcal{G}_j$ might intersect at reducible connections.

Putting the C^∞-quotient topology on $\mathcal{B}^*$ turns them to Frechet manifolds, and we have

Lemma 1.1 *The fixed set $(\mathcal{B}^*)^\pi$ is a smooth, closed submanifold of $\mathcal{B}^*$ of infinite codimension.*

Except for the codimension part this is Proposition 1.3 from [BM93]. The rest can be seen by varying a fixed connection within a small chart contained in the free stratum of X. This yields an infinite dimensional complement to the space of fixed connections.

If μ_0 is a π-equivariant metric on X then the moduli space $\mathcal{M}$ of anti self dual connections on P is π-invariant.

2 A fixed point theorem

In this section we restrict our attention to the case $G = SO_3$, $w_2P \neq 0$ and $-4 \leq p_1P < 0$. By the Uhlenbeck compactification theorem this forces the moduli space to be compact for all metrics on X.

For the case of a compact moduli space we give a brief reminder for the definition of the Donaldson polynomial invariants: The free SO_3 action on P induces a free SO_3 action on $\mathcal{B}^*(P) \times_{\mathcal{G}} P$ with quotient $\mathcal{B}^* \times X$. The associated principal bundle is called $\mathbb{P}^{ad}$ and we call $\tilde{p_1} \in H^4(\mathcal{B}^* \times X; \mathbb{Z})$ its first Pontrjagin class. The μ map is defined by

$$\mu : H_2(X;\mathbb{Z}) \to H^2(\mathcal{B}^*;\mathbb{Q}) : \alpha \mapsto \frac{1}{4}\tilde{p_1}/\alpha.$$

If $w_2(P) = 0$ then $\mu(\Sigma)$ is integral, if $w_2(P) \neq 0$ then $2\mu(\Sigma)$ is integral, see [Kot91, 2.8-9].

Now let $k = -1/4p_1(P)$ and let $\dim\mathcal{M} = 8k - 3(1 + b_2^+) =: 2d$ be even. The Freed-Uhlenbeck generic metric theorem allows us to choose a metric on X such that $\mathcal{M}$ is a smooth manifold. Note that it might not be possible to choose this metric out of the equivariant ones. Let Ω be an orientation of $H^0(X;\mathbb{Z}) \oplus H^+(X;\mathbb{Z})$ and $\hat{\alpha}$ be an integral lift of $w_2(P)$. Then Ω and $\hat{\alpha}$ determine an orientation of $\mathcal{M}$ (see [Kot91, 2.16-17]), and formally the value of the Donaldson polynomial associated to $\mathcal{M}$ -with this orientation- on classes $\Sigma_1, \ldots, \Sigma_d \in H^2(X;\mathbb{Z})$ is

$$q^X_{d,\Omega,\hat{\alpha}}(\Sigma_1, \ldots, \Sigma_d) = \langle \mu\Sigma_1 \cup \cdots \cup \mu\Sigma_d | [\mathcal{M}] \rangle.$$

This is independent of the good metric chosen as long as $b^+ \geq 2$. In the case $b^+ = 1$ the invariant depends on the metric chosen in the following way: The classes $e \in H^2(X;\mathbb{Z})$ satisfying $e \equiv w_2(P) \mod 2$ and $p_1(P) \leq e^2 < 0$ give rise to topological S^1-reductions of P and the bundles of lower charge. They determine a chamber structure on $H^2_{deR}(X)$. A metric on X determines a up-to-sign unique normalised self dual two form $\omega \in H^2_{deR}(X)$. The Donaldson Polynomial in the $b^+ = 1$ case depends on the metric only through the chamber of ω. (See [Kot91, KM94].)

The choice of orientation Ω of $H^0(X) \oplus H^+(X)$ and the particular choice of the integral lift $\hat{\alpha}$ of $\alpha = w_2(P)$ only affect the sign of the Donaldson polynomial. Since we will be interested only in the divisibility of it, we will drop this from our notation and will write $q^X_{d,\alpha}$ for the degree d Polynomial associated to the SO_3 bundle with $w_2(P) = \alpha$.

If $w_2(P) = 0$ then $q(\Sigma_1, \ldots, \Sigma_d)$ is an integer. If $w_2(P) \neq 0$ then q takes values in $\frac{1}{2^d}\mathbb{Z}$. This justifies

Definition 2.1 *We say that an odd prime p divides the Donaldson Polynomial $q^X_{d,\alpha}$ iff p divides the numerator of $q_{d,\alpha}(\Sigma_1, \ldots, \Sigma_d)$ for every combination of $\Sigma_1, \ldots, \Sigma_n$ in $H_2(X;\mathbb{Z})$.*

We can now state

Theorem A *Let X be a smooth, closed, simply connected four-manifold with $b^+ = 1$ on which $\mathbb{Z}/p$, with p an odd prime number, acts preserving the orientation on X and on $H^+(X;\mathbb{Z})$. Let $G \to P \to X$ be a principal SO_3 bundle with nonzero $\alpha := w_2(P) \in H^2(X;\mathbb{Z}/2)^{\mathbb{Z}/p}$ and $-4 \leq p_1(P) < 0$. Moreover if $\widehat{\alpha}$ is some integral lift of α we assume that $\widehat{\alpha}$ is fixed by $\mathbb{Z}/p$.*

Let μ_0 be an equivariant metric on X s.t. $\mathcal{M}(X, \mu_0)$ does not contain any reducible connections. The Donaldson invariant $q^X_{d,\alpha}$ is understood to be associated to the chamber of μ_0.

If the induced $\mathbb{Z}/p$ action on $\mathcal{M}_P(X, \mu_0)$ is free then p divides $q^X_{d,\alpha}$ when restricted to $H_2(X;\mathbb{Z})^{\mathbb{Z}/p}$.

Proof of Theorem A

We start with the main topological ingredient of our fixed point theorem:

Lemma 2.2 *Let π be a finite cyclic group and Z a free π-space with $b_1 Z = 0$. Let $pr : Z \to Z/\pi$ be the projection.*

If $x \in H^2(Z;\mathbb{Z})^\pi$ then $x = pr^*(x')$ for some $x' \in H^2(Z/\pi;\mathbb{Z})$.

Proof: We follow [HL92, Theorem 6.1]. Recall that the Borel cohomology of a π-space Z is defined by

$$H^*_\pi(Z;\mathbb{Z}) := H^*(Z \times_\pi E\pi;\mathbb{Z}),$$

which can be calculated using the Leray-Serre spectral sequence for the fibration $Z \to Z \times_\pi E\pi \to B\pi$ with E_2-term given by $E_2^{p,q} = H^p(\pi, H^q(Z))$. In the case when Z is a free π-space $E \times_\pi E\pi$ is homotopy equivalent to Z/π and the fiber inclusion $Z \hookrightarrow Z \times_\pi E\pi$ is homotopic to the quotient map $pr : Z \to Z/\pi$. Combining this with the fact that the fiber inclusion induces the edge homomorphism

$$H^i(Z \times_\pi E\pi) \twoheadrightarrow E_\infty^{0,i} \subseteq \cdots \subseteq E_2^{0,i} = H^i(Z;\mathbb{Z})^\pi,$$

we see that the lemma follows if we show that the $E_2^{0,2}$-term survives to $E_\infty^{0,2}$.

All of this was true for an arbitrary finite group π and we now specialize to π being finite cyclic. Since $H^j(\pi;\mathbb{Z})$ is infinite cyclic for $j=0$, π for positive even j and zero otherwise, we see that $b_1Z=0$ implies that all differentials starting at $E_*^{0,2}$ are zero. Therefore $H^2(Z)^\pi = E_2^{0,2} = E_\infty^{0,2}$ which proves the claim. $\square$

In our case Z will be $\mathcal{B}^* \setminus (\mathcal{B}^*)^{\mathbf{Z}/p}$ and the classes x will be $\mu(\Sigma)$ for some $\Sigma \in H_2(X;\mathbf{Z})^{\mathbf{Z}/p}$. To see that $\mu\Sigma$ is invariant we need the following remark:

Note that any $f \in \mathrm{Diff}(X)$ satisfying $f^*P \cong P$ induces a self map f^* of $\mathcal{B}^*(P)$ given by pull back. Therefore $(f^* \times f)$ is a self map of $\mathcal{B}^* \times X$. For any such f we will write in abuse of notation f^* for the induced map on $\mathcal{B}^*$ and for the map induced by f^* on $H^*(\mathcal{B})$. With this simplifying notation we have

Proposition 2.3 *With the notation as above the relation*

$$f^*\mu(\Sigma) = \mu(f_*^{-1}\Sigma)$$

holds for any $\Sigma \in H_2(X;\mathbb{Z})$.

Proof: First note that the relation $f^*P \cong P$ implies that $(f^* \times f)^*(\mathbb{P}^{ad}) \cong \mathbb{P}^{ad}$ over $\mathcal{B}^* \times X$. The proposition follows from this and the naturality properties of the slant product. $\square$

Now turn to the second technical problem: Recall that in order to define the Donaldson invariants for a moduli space $\mathcal{M}$ one chooses a metric μ with the property that $\mathcal{M}$ and all moduli spaces involved in the compactification of $\mathcal{M}$ are smooth. The Freed-Uhlenbeck theorem enables one to pick such a metric out of a generic set but this generic set might not contain an equivariant metric.

Because of this we'll have to work in a slightly more complicated setting, namely a combination of the two perturbations: variation of the metric and perturbation of the ASD equation.

Fix a SO_3-bundle P satisfying the assumptions of Theorem A and choose a metric μ on X. Recall that the moduli space $\mathcal{M}(X,\mu)$ is the zero set of the section $F^+ : \mathcal{B} \to \Omega_2^+$, where $\Omega_2^+ = \mathcal{A} \times_{\mathcal{G}} L_2^2\Omega^+(\mathrm{ad}P) \to \mathcal{B}$ and F^+ is the plus part of the curvature with respect to μ. F^+ can be perturbed in two ways: Let $\sigma : \mathcal{B} \to \mathcal{A} \times_{\mathcal{G}} L_3^2\Omega^+$ be a section, and let $i : L_3^2\Omega^+ \hookrightarrow L_2^2\Omega^+$ be the compact inclusion. Then $i \circ \sigma$ is a compact perturbation of F^+ and for a generic choice of σ the zero set $\mathcal{M}^\sigma(X,\mu) := (F^+ + i\sigma)^{-1}(0)$ is a smooth manifold of the expected dimension. On the other hand we can vary the metric as described in [DK90, p.148] or the next section.

Combining these two we set $\mathcal{P} := \mathcal{C} \times \mathcal{S}$, where $\mathcal{C}$ is the space of conformal classes of metrics on X and $\mathcal{S}$ is the space of sections in $L_3^2(\Omega^+)$, and set $\mathcal{B} \times \mathcal{P} \xrightarrow{F^+} \Omega^+$, the parametrized self dual curvature map. Since the partial derivatives in one of the $\mathcal{P}$ directions alone suffices to achieve a surjective differential of F^+, we see that the 'universal moduli space' $(F^+)^{-1}(0) \subset \mathcal{B} \times \mathcal{P}$ is smooth. Recall that the regular values of the projection $pr : (F^+)^{-1}(0) \to \mathcal{P}$ are exactly the perturbation parameters which yield smooth moduli spaces. Any two good parameters (μ_i, σ_i), $i = 1, 2$, can be joined by a good path γ, i.e. $pr^{-1}(\gamma)$ is a smooth cobordism between $\mathcal{M}^{\sigma_1}(X, g\mu_1)$ and $\mathcal{M}^{\sigma_2}(X, \mu_2)$. This cobordism is also compact for the proof of the 'parametrized compactness theorem' [DK90, 9.1.2] is the same for families of perturbed ASD connections. The point is that Uhlenbecks compactness theorem requires bounded energy, and this is still fulfilled as long as the perturbation σ is bounded. In particular if one end of the cobordism is compact so is the other, and they define the same homology class in $\mathcal{B}$.

Now let μ_0 be the equivariant metric from Theorem A. We assume that $\mathcal{M}(X, \mu_0)$ is compact and that the $\mathbb{Z}/p$ action on $\mathcal{M}(X, \mu_0)$ is free. Therefore $\mathcal{M}(X, \mu_0)$ has a positive distance to the fixed set $\mathcal{B}^{\mathbb{Z}/p}$ and so we can find an equivariant perturbation σ_0 s.t. (μ_0, σ_0) yields a smooth moduli space. Now choose a good metric g_1 close to μ_0. By the preceeding remark we see that the homology classes of $\mathcal{M}^{\sigma_0}(X, \mu_0)$ and $\mathcal{M}(X, g_1)$ in $H_*(\mathcal{B})$ agree. This means that the Donaldson Polynomial associated to P can be calculated by evaluating the cohomology classes $\mu(\Sigma)$ on the fundamental class of $\mathcal{M}^{\sigma_0}(X, \mu_0)$. We summarize this paragraph in

Lemma 2.4 *Let $P \to X$ be a SO_3 principal bundle satisfying the conditions of Theorem A. When the $\mathbb{Z}/p$ action on $\mathcal{M}(X, \mu_0)$ is free then there is an equivariant perturbation σ_0 of the ASD equation s.t. the Donaldson invariant associated to P can be calculated by evaluating against the fundamental class of $\mathcal{M}^{\sigma_0}$, which is a smooth compact manifold with free $\mathbb{Z}/p$ action.*

Proof of Theorem A: To complete the proof of our fixed point theorem we now collect the pieces:

Assume that the $\mathbb{Z}/p$ action on $\mathcal{M}(X, \mu_0)$ is free. Choose an equivariant perturbation σ_0 like in Lemma 2.4. Note that the restrictions on $p_1(P)$ and $b_1(X)$ imply that $\mathcal{M}^{\sigma_0}(X, \mu_0)$ is zero or two dimensional. In both cases $\mathcal{M}^{\sigma_0}(X, \mu_0)$ is a smooth compact manifold on which $\mathbb{Z}/p$ acts freely preserving the orientation.

In the zero dimensional case this just means that the points in $\mathcal{M}^{\sigma_0}(X, \mu_0)$ come in free p-orbits, which means that their count with orientation is divisible by p.

For the two dimensional case take $\Sigma \in H_2(X; \mathbb{Z})^{\mathbb{Z}/p}$. Then 2.3 implies that $\mu\Sigma \in H^2(\mathcal{B}^*)^{\mathbb{Z}/p}$. Now [AMR95] calculated

$$\pi_1(\mathcal{B}^*) = \begin{cases} \mathbb{Z}/2 & p_1(P) \equiv \operatorname{sign}(X) \bmod 8 \quad \text{and } w_2(P) = w_2(X) \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with the fact that $(\mathcal{B}^*)^{\mathbb{Z}/p} \hookrightarrow \mathcal{B}^*$ has infinite codimension we see that $b_1(\mathcal{B}^* \setminus (\mathcal{B}^*)^{\mathbb{Z}/p}) = 0$. Set $\mathcal{B}^*_e := \mathcal{B}^* \setminus (\mathcal{B}^*)^{\mathbb{Z}/p}$, $i : \mathcal{B}^*_e \hookrightarrow \mathcal{B}^*$ and $pr : \mathcal{B}^*_e \to \mathcal{B}^*_e/_{\mathbb{Z}/p}$. Lemma 2.2 implies that $i^*\mu(\Sigma) = pr^*(x')$ for some $x' \in H^2(\mathcal{B}^*_e/(\mathbb{Z}/p); \mathbb{Z})$. Now $pr_* : H_2(\mathcal{M}^{\sigma_0}) \to H_2(\mathcal{M}^{\sigma_0}/(\mathbb{Z}/p); \mathbb{Z})$ has degree p and therefore

$$q^X_{1,\alpha}(\Sigma) = \langle i^*\mu(\Sigma) \mid [\mathcal{M}^{\sigma_0}] \rangle = \langle x' \mid pr_*[\mathcal{M}^{\sigma_0}] \rangle$$

is divisible by p.

Examples

In the following let $P_{\alpha,p}$ stand for the SO_3 bundle over X with $w_2(P) = \alpha$ and $p_1(P) = p$. Note that (w_2, p_1) classifies P and every pair (w_2, p_1) which satisfies $\mathcal{P}w_2 \equiv p_1 \mod 4$, $\mathcal{P}$ being the Pontrjagin square, is realized, (see [DW59]). Let $\mathcal{M}_{\alpha,p}(X, \mu_0)$ be the moduli space of ASD connections on $P_{\alpha,p}$ with respect to the metric μ_0. We assume that X and P satisfy the assumptions of Theorem A. Here μ_0 will always be a $\mathbb{Z}/p$ equivariant metric on X. In addition we will always assume that the induced $\mathbb{Z}/p$ action on $H^*(X;\mathbb{Z})$ is trivial.

Suppose $b^+ = 1$, $w_2X \neq 0$ and $\mathrm{sign}(X) \equiv 0, 1, 2$ or $3 \mod 8$. Then there are no reductions on a SO_3 bundle $P_{w_2(X),k}$ for $-4 \leq k < 0$. This follows since a class $e \in H^2(X)$ associated to a reduction of P to a S^1 bundle is characteristic by the w_2 assumption. Therefore $e^2 \equiv \mathrm{sign}(X) \mod 8$ and these were the cases excluded. This means that in this case the invariant does not depend on the metric.

• The blown up complex projective plane

Here $H^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}; \mathbb{Z}) =< H, E >$ where H and E stand for the hyperplane class and the class of the exceptional divisor. We have that $H^2 = 1$, $E^2 = -1$ and H and E are perpendicular. Let $\alpha := \overline{H - E}$ be the $\mathbb{Z}/2$ reduction of $H - E$. Then $\alpha = w_2(P)$, and by the remark above there is no chamber dependence of the Donaldson polynomial. In [Kot91, Proposition 7.1(4)] it is calculated that $q_{1,\alpha}^{\mathbb{C}P^2\#\overline{\mathbb{C}P^2}} = -2E$, and therefore $\mathcal{M}_{\alpha,-4}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \mu_0)$ has a $\mathbb{Z}/p$ fixed point.

The bundles $P_{\overline{H},-3}$ and $P_{\overline{E},-5}$ also yield compact moduli spaces, but unfortunately we do get a chamber structure in these cases, and in [Kot91, Proposition 7.1(3)] it is calculated that there are chambers for which $q_{0,\overline{H}}^{\mathbb{C}P^2\#\overline{\mathbb{C}P^2}}$ vanishes.

• The Barlow surface and its blow up

The Barlow surface B is defined to be the minimal desingularisation of a quotient of a certain Hilbert modular surface. It is a simply connected algebraic surface homeomorphic to $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$, (see [Kot89, §5]).

Let $\alpha = w_2(B)$. Then $\alpha^2 \equiv 1 \mod 8$ and the moduli space for the SO_3 bundle $P_{\alpha,-3}$ is zero dimensional. By the remark above the Donaldson invariant does not depend on the metric chosen, and in [Kot91, 7.8] it is calculated that $q_{0,\alpha}^B = -8$. Therefore every $\mathbb{Z}/p$ action on B induces a non free action on $\mathcal{M}_{\alpha,-3}(B, \mu_0)$.

Let $X = B \# \overline{\mathbb{C}P^2}$ be the blow up of the Barlow surface at one point. Then $\alpha = w_2(X)$ and $p_1 = -4$ give rise to a 2 dimensional moduli space, which again yields a Donaldson invariant which is independent of the metric. In [Kot91, 7.10] it is calculated that $q_{1,\alpha}^X = -16E$ where E is the class of the exceptional divisor. Hence every $\mathbb{Z}/p$ action on X induces a non free action on $\mathcal{M}_{\alpha,-4}(X, \mu_0)$.

• Dolgachev surfaces

The Dolgachev surface $X(a, b)$ is a minimal elliptic surface obtained from the rational surface $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ by two logarithmic transforms of multiplicities $a, b > 1$. Then

$Y := X(2,3)$ is homeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ [Don87, Proposition 3.16]. For $\alpha = w_2(Z)$ it is shown in [Kot91, 7.10] that $q^Z_{\alpha,1}$ -which again does not depend on the metric- is equal to 8κ where κ is primitive. This shows that every $\mathbb{Z}/p$ action on Y induces one on $\mathcal{M}_{\alpha,-4}(Y,\mu_0)$ which has a fixed point.

3 Avoiding reducibles by equivariant perturbations of the metric

Let μ_0 be a π-equivariant metric and $\Lambda_0^\pm$ be the ± 1 eigenbundles of $*_0 = *_{\mu_0}$ on $\Lambda^2(T^*X)$. All conformal classes of metrics on X can be parametrized by the bundle maps from Λ_0^- to Λ_0^+ with pointwise norm less than 1: $\mathcal{C} := \underline{\mathrm{HOM}}^{<1}(\Lambda_0^-, \Lambda_0^+)$. $\mathcal{C}$ carries a π action by setting $(g\cdot\mu)(\alpha) := g^*\left(\mu((g^{-1})^*\alpha)\right)$ for $\mu \in \mathcal{C}$ and $\alpha \in \Lambda_0^-$. The equivariant conformal classes correspond to the fixed set of π on $\mathcal{C}$: $\mathcal{C}^\pi$. Note that the π action naturally extends to a π action on $T_{\mu_0}\mathcal{C} = \underline{\mathrm{HOM}}(\Lambda_0^-, \Lambda_0^+)$ and that $T_{\mu_0}(\mathcal{C}^\pi) = (T_{\mu_0}\mathcal{C})^\pi$. Let μ be a metric such that $[\mu] \in \mathcal{C}^\pi$. The average of μ is equivariant and in the same conformal class as μ. Therefore, if we pick a metric out of an equivariant conformal class, we will assume that the metric itself is equivariant.

Let $H^2_\mu(X)$ be the μ-harmonic 2-forms on X, and recall that $\alpha \mapsto [\alpha]$ is a canonical isomorphism from $H^2_\mu(X)$ to $H^2_{deR}(X)$. We will write $H^2(X)$ for $H^2_{deR}(X)$ in the following. Let $\mathrm{Neg}_{b^-}(H^2) \subset \mathrm{Gr}_{b^-}(H^2)$ be the open submanifold of $\mathrm{Gr}_{b^-}(H^2)$ consisting of negative definite subspaces with respect to the intersection form on H^2. Note that under $H^2_\mu \xrightarrow{\cong} H^2$ the subspace $H^-_\mu := H^2_\mu \cap \Gamma\Lambda^-_\mu$ is mapped to an element of $\mathrm{Neg}_{b^-}(H^2)$. This defines the period map

$$P : \mathrm{Metrics} \to \mathrm{Neg}_{b^-}(H^2).$$

Since on Ω^2 the star operator is conformally invariant, we have that $H^\pm_\mu$ is invariant under conformal changes of the metric, which means that the period map factors through the conformal classes:

$$P : \mathcal{C} \to \mathrm{Neg}_{b^-}(H^2).$$

Now take $x \in H^2(X)$ with $x^2 < 0$, and set $N_x := \{V \in \mathrm{Neg}_{b^-}(H^2) | x \in V\}$. Recall the following

Lemma 3.1 *The tangent space of* $\mathrm{Neg}_{b^-}(H^2)$ *at a subspace* A *is given as*

$$T_A(\mathrm{Neg}_{b^-}(H^2)) = \mathrm{HOM}(A, A^\perp).$$

N_x is a submanifold of $\mathrm{Neg}_{b^-}(H^2)$ *of codimension b^+. The tangent space of N_x at an element $A \in N_x$ is equal to*

$$Ker(ev_x) \leq \mathrm{HOM}(A, A^\perp), \quad \text{where} \quad ev_x : \mathrm{HOM}(A, A^\perp) : \alpha \mapsto \alpha(x).$$

In [DK90, 4.3.24] the differential of P at μ_0 is calculated:

$$\begin{array}{rccl} D_{\mu_0}P: & \underline{\mathrm{HOM}}(\Lambda_0^-,\Lambda_0^+) & \to & \mathrm{HOM}(H_0^-,H_0^+) \\ & m & \mapsto & \left(\alpha \mapsto \Pi_{H_0^+} m(\alpha)\right), \end{array}$$

Where $\Pi_{H_0^+}$ is the L^2-projection to the space of μ_0-harmonic and μ_0-SD forms. We define a π action on $\mathrm{HOM}(H_0^-,H_0^+)$ by setting $g\cdot\phi := g^*\circ\phi\circ(g^{-1})^*$.

Proposition 3.2 *$D_{\mu_0}P$ is π-equivariant.*

Proof: First note that for any $\beta\in\Gamma\Lambda_0^+$ and $g\in\mathrm{Diff}^+(X)$ a μ_0 isometry the relation

$$\left(\Pi_{H_0^+}\circ g^*\right)\beta = \left(g^*\circ\Pi_{H_0^+}\right)\beta$$

holds. This follows from the Hodge decomposition theorem and the fact that pullback via g, being a μ_0-isometry, commutes with d^*.The proposition is now a direct consequence of the definition of the two π actions on $\underline{\mathrm{HOM}}(\Lambda_0^-,\Lambda_0^+)$ and $\mathrm{HOM}(H_0^-,H_0^+)$. □

Let $P\to X$ be a SU_2 principal bundle with $c_2(P)=k>0$. Since μ_0 is π-equivariant the moduli space of gauge equivalence classes of $*_0$ ASD connections on P, $\mathcal{M}(P,\mu_0)$, carries a π action, induced from the one on X. A singularity in the moduli space coming from a reducible connection corresponds to a class $\pm x\in H^2(X;\mathbb{Z})$ with $x^2=-k$, and $\pm x$ gives rise to a singularity in $\mathcal{M}(P,\mu_0)$ iff $x\in H_{\mu_0}^-$. Here we also wrote x for the image of x in real cohomology. Moreover, the singularity.is in the fixed set $\mathcal{M}^\pi$ iff $\{x,-x\}$ is π-invariant.

Let $P\to X$ be a SO_3 principal bundle with $w_2(P)=w$ and $p_1(P)=p<0$. Here a singularity corresponds to a class $\pm x\in H^2(X;\mathbb{Z})$ with $x\equiv w \mod 2$ and $x^2=p$. Any such class gives rise to a singularity in the moduli space $\mathcal{M}(P,\mu_0)$ iff $x\in H_{\mu_0}^-$. Again, the singularity is in the fixed set $\mathcal{M}^\pi$ iff $\{x,-x\}$ is π-invariant.

Definition 3.3 *Let μ_0 be an equivariant metric.*

If $P\to X$ is a principal SU_2 bundle we call $x\in H^2(X;\mathbb{Z})$ a fixed reducible if $x^2=-c_2(P)$, $\{x,-x\}$ is π invariant and $x\in H_{\mu_0}^-$.

If $P\to X$ is a principal SO_3 bundle we call $x\in H^2(X;\mathbb{Z})$ a fixed reducible if $x\equiv p_1(P) \mod 2$, $x^2=p_1$, $\{x,-x\}$ is π-invariant and $x\in H_{\mu_0}^-$.

In both cases we say that x is of type I iff $x\in H^2(X)^\pi$, otherwise of type II.

Note that a fixed reducible of type II determines a subgroup $\pi_x\le\pi$ of index two which fixes x. π_x determines a representation $\chi_x:\pi\to\{\pm1\}$, and for $g\in\pi$ the relation $g^*x=\chi_x(g)x$ holds.

We want to apply the calculations above to show that under a certain cohomological condition there are no fixed reducible connections in $\mathcal{M}$. Note that the π representation $H_\mu^\pm$ is independent of the equivariant metric chosen, and therefore a cohomological invariant of the group action on X. In the proof of our theorem we'll need the following

Lemma 3.4 *Let $f_1, f_2, \dots$ be countably many non zero continuous homomorphisms between two real vector spaces V and W.*

Then there is a Baire set $V_{\neq 0} \subset V$ s.t. $f_i(v) \neq 0$ $\forall i$ and all $v \in V_{\neq 0}$. If there are only finitely many f_i then $V_{\neq 0}$ is also open.

Proof: We look for these $v \in V$ which are in the complement of the union of all kernels of the f_i. Set

$$V_{\neq 0} := V \setminus \cup_i \operatorname{Ker}(f_i) = \cap_i \left(V \setminus \operatorname{Ker}(f_i)\right) = \cap_i f_i^{-1}(W \setminus \{0\}).$$

Since $f_i^{-1}(W\setminus\{0\})$ is open and dense in V, we see that $V_{\neq 0}$ is a countable intersection of open and dense sets in V, and therefore a Baire set. $\square$

Definition 3.5 *A fixed reducible of type I is called removable iff $H_0^+(X)^\pi \neq 0$. A fixed reducible x of type II is called removable iff $H_0^+(X)^\pi < H_0^+(X)^{\pi_x}$.*

Theorem B *Fix an equivariant metric μ_0.*

If all fixed reducibles are removable then the tangent space $T_{\mu_0}\mathcal{C}^\pi$ contains a Baire set Γ of 'good' tangent directions s.t. for all $\mu_{t,\gamma} := \mu_0 + t\gamma$, $0 \neq t \in \mathbb{R}$ small and $\gamma \in \Gamma$, the moduli space $\mathcal{M}(P, \mu_{t,\gamma})$ does not contain any fixed reducibles.

On the other hand: If there is a fixed reducible x which is not removable then $\mathcal{M}(P, \mu)$ will contain a reducible fixed connection for all equivariant metrics μ.

Proof: Let $x_1, x_2, \dots$ be integral classes in $H^-_{\mu_0}(X)$ which are fixed reducibles. We will show that $\mathrm{ev}_x \circ DP \not\equiv 0$ iff x is removable.

Assume that all fixed reducibles are removable. Therefore $\mathrm{ev}_{x_i} \circ DP \not\equiv 0$ for all i and Lemma 3.4 yields a Baire set $\Gamma \subset T_{\mu_0}\mathcal{C}^\pi$ s.t. $\mathrm{ev}_{x_i} \circ DP(\gamma) \neq 0$ $\forall \gamma \in \Gamma$ and all i. This means that $DP(\gamma)$ is not tangential to N_{x_i} for all i, and therefore $x_i \notin H^-_{\mu_{t,\gamma}}(X)$ for all i, $\gamma \in \Gamma$ and $t \in \mathbb{R}$ small. This is equivalent to saying that $P(\mu_{t,\gamma}) \notin N_{x_i}$ for all i.

On the other hand let $\mathrm{ev}_x \circ DP \equiv 0$. This means that DP maps $T_{\mu_0}\mathcal{C}^\pi$ into the tangent space of N_x. So P will map all equivariant metrics close to μ_0 into N_x. Since the cohomological condition doesn't change in the passage from μ_0 to μ close by and the equivariant metrics are path connected it follows that $P(\mu) \in N_x$ for all μ.

Case I: The fixed reducible x is of type I. So $x \in (H^2)^\pi$, and therefore the evaluation map

$$\mathrm{ev}_x : \mathrm{HOM}(H_0^-, H_0^+) \to H_0^+ \; : \; \alpha \mapsto \alpha(x)$$

is π-equivariant. This means that the composition

$$\mathrm{ev}_x \circ D_{\mu_0}P : \underline{\mathrm{HOM}}(\Lambda_0^-, \Lambda_0^+) \to H_0^+$$

is π-equivariant. In [DK90, p.153] it is shown that $\mathrm{ev}_x \circ D_{\mu_0}P$ maps surjectively onto H_0^+. Therefore the restriction to the π-invariant part is also surjective:

$$\mathrm{ev}_x \circ D_{\mu_0}P : \underline{\mathrm{HOM}}(\Lambda_0^-, \Lambda_0^+)^\pi \twoheadrightarrow (H_0^+)^\pi.$$

This means in particular that $\mathrm{ev}_x \circ D_{\mu_0}P$ restricted to $T_{\mu_0}\mathcal{C}^\pi$ is not zero iff $(H_0^+)^\pi \neq 0$.

Case II: The fixed reducible x is of type II. We start with an elementary remark: $H_0^+(X)^\pi < H_0^+(X)^{\pi_x}$ iff $\exists 0 \neq y \in H_0^+(X)$ s.t. $g^*y = \chi_x(g)y$. It is clear that the existence of such a y implies strict inclusion of $H_0^+(X)^\pi$ in $H_0^+(X)^{\pi_x}$. Now let the inclusion be strict. Since π_x is a normal subgroup of π it follows that $H_0^+(X)^{\pi_x}$ is π invariant. The induced π action on $H_0^+(X)^{\pi_x}$ determines a $\pi/\pi_x \cong \{\pm 1\}$ action on $H_0^+(X)^{\pi_x}$, which is not trivial since the inclusion is strict. This means that there is a $0 \neq y \in H_0^+(X)^{\pi_x}$ with $g^*y = \chi_x(g)y$.

Now assume that $x \in H_0^-$ is removable and take $0 \neq y \in H_0^+(X)^{\pi^+}$ with $g^*y = \chi_x(g)y$. Since $\mathrm{ev}_x \circ DP$ is surjective it follows that there is a $m_0 \in \underline{\mathrm{HOM}}(\Lambda_0^-, \Lambda_0^+)$ s.t. $DP(m_0)(x) = y$. Set $\overline{m_0} := |\pi|^{-1} \sum_{g\in\pi} g \circ m_0 \circ g^{-1}$. Then $\overline{m_0} \in \underline{\mathrm{HOM}}(\Lambda_0^-, \Lambda_0^+)^\pi$. Since DP is equivariant we calculate

$$\begin{aligned} DP(\overline{m_0})(x) &= \frac{1}{|\pi|}\left[\sum_{g\in\pi^+} g^*DP(m)(x) + \sum_{g\in\pi\setminus\pi^+} g^*DP(m)(-x)\right] \\ &= \frac{1}{|\pi|}\left[\sum_{g\in\pi^+} g^*y + \sum_{g\in\pi\setminus\pi^+} g^*(-y)\right] \\ &= y \neq 0. \end{aligned}$$

And therefore $\mathrm{ev}_x \circ DP_{|T_{\mu_0}\mathcal{C}^\pi} \neq 0$.

On the other hand let $(H_0^+)^\pi = (H_0^+)^{\pi^+}$. Take $g_0 \in (\pi \setminus \pi^+)$. For any $m \in \mathrm{HOM}(\Lambda_0^-, \Lambda_0^+)$ we calculate

$$\begin{aligned} \sum_{g\in\pi} DP(g\cdot m) &= \sum_{g\in\pi^+} g^*DP(m)(g^{-1})^*(x) + \sum_{g\in\pi^+} (g_0g)^*DP(m)(g^{-1})^*(-x) \\ &= \underbrace{\sum_{g\in\pi^+} g^*DP(m)(x)}_{\in(H_0^+)^{\pi^+}} - g_0^* \underbrace{\sum_{g\in\pi^+} g^*DP(m)(x)}_{\in(H_0^+)^{\pi^+}} \\ &= 0, \end{aligned}$$

which implies that $DP(\bar{m})(x) = 0$ for all $\bar{m} \in T_{\mu_0}(\mathcal{C}^\pi)$. □

Remarks:

i) Suppose there is an class $x \in H^2(X;\mathbb{Z})$ with $x^2 = -c_2(P)$, $\{x,-x\}$ π-invariant and the cohomological condition of removability on H^+ is not fulfilled. The theorem does not imply that there will be a fixed reducible connection for all equivariant metrics, since x might not lie in H_μ^- for any equivariant μ. However the theorem says that if $x \in H_\mu^-$ for one μ then there will be a fixed reducible connection for all μ.

ii) One can not hope for the stronger statement that $P_{|C^\pi}$ is transversal to N_x in general. In fact the proof shows that for $x \in H^2(X)^\pi$ transversality is equivalent to the much stronger cohomological condition $H^+(X)^\pi = H^+(X)$.

iii) The first results about avoiding reducibles by equivariant metrics known to the author were proved by M. Klemm in his thesis [Kle95]. The results are written up for the case $X = S^2 \times S^2$, but almost literally generalize to the case when the induced π action on cohomology is trivial.

iv) The above theorem sheds some light on R. Fintushel's standard example for the failure of good properties in equivariant gauge theory as written up in [HL92, Example 2.15]. After reversing the orientation to stay in our setting of ASD moduli spaces the example describes a $\mathbb{Z}/2$ action on a K3 surface with negative orientation and quotient $-\mathbb{C}P^2$. Pulling back connections from $\mathcal{M}^{ASD}_{-\mathbb{C}P^2}(P_1,g)$ produces a 5 parameter family of in $\mathcal{M}^{ASD}_{-K3}(P_2,\pi^*g)$ which will always contain the unique reducible from the moduli space over $-\mathbb{C}P^2$. Therefore one should not expect that one can perturb away all reducibles by an equivariant metric.

This is consistent with our theorem for $H^2(-K3;\mathbb{R})^{\mathbb{Z}/2} = H^2(-\mathbb{C}P^2;\mathbb{R})$ implies that $H^-(-\mathrm{K3}) = \mathbb{R} \oplus 2\mathbb{R}^-$ and $H^+(-\mathrm{K3}) = \oplus 19\mathbb{R}^-$. I.e. for any equivariant metric there is at most one fixed reducible of type I which is not perturbable since $H^+(-\mathrm{K3})^{\mathbb{Z}/2} = 0$. However all fixed reducibles of type II are perturbable, and therefore the theorem predicts one fixed reducible connection for a good equivariant metric.

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