Max-Planck-Institut für Mathematik
Bonn

Genus two partition and correlation functions for fermionic vertex operator superalgebras I

by

Michael P. Tuite
Alexander Zuevsky

Max-Planck-Institut für Mathematik
Preprint Series 2010 (101)

Genus two partition and correlation functions for fermionic vertex operator superalgebras I

Michael P. Tuite
Alexander Zuevsky

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland Galway
University Road
Galway
Ireland

MPIM 10-101

Genus Two Partition and Correlation Functions for Fermionic Vertex Operator Superalgebras I

Michael P. Tuite and Alexander Zuevsky*
School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland Galway
University Road, Galway, Ireland.

November 9, 2010

Abstract

We define the partition and n-point correlation functions for a vertex operator superalgebra on a genus two Riemann surface formed by sewing two tori together. For the free fermion vertex operator superalgebra we obtain a closed formula for the genus two continuous orbifold partition function in terms of an infinite dimensional determinant with entries arising from torus Szegö kernels. We prove that the partition function is holomorphic in the sewing parameters on a given suitable domain and describe its modular properties. Using the bosonized formalism, a new genus two Jacobi product identity is described for the Riemann theta series. We compute and discuss the modular properties of the generating function for all n-point functions in terms of a genus two Szegö kernel determinant. We also show that the Virasoro vector one point function satisfies a genus two Ward identity.

*Supported by a Science Foundation Ireland Frontiers of Research Grant, and by Max–Planck Institut für Mathematik, Bonn

1 Introduction

Genus two (and higher) partition functions and correlation functions have been studied for some time in string and conformal field theory e.g. [EO], [FS], [DP], [Kn], [DVPFHLS]. Meanwhile, in the theory of Vertex Operator Algebras (VOAs) [B], [FHL], [FLM], [Ka], [MN], [MT5] higher genus approaches based on algebraic geometry have also been developed e.g. [TUY], [KNTY], [Z2], [U]. A more constructive VOA approach has recently been described whereby genus two partition and n-point correlation functions are defined in terms of genus one VOA data [T], [MT1], [MT2], [MT3], [MT4]. This approach is based solely on the properties of a VOA with no assumed analytic or modular properties for partition or correlation functions. A compact genus two Riemann surface can be obtained from tori by either sewing two separate tori together, which we refer to as the ϵ-formalism, or by self-sewing a torus, which we refer to as the ρ-formalism [MT2]. The theory of partition and n-point correlation functions in the ϵ-formalism is described in ref. [MT1] where these functions are explicitly computed for the Heisenberg VOA and its modules including lattice VOAs. The corresponding functions are considered in the ρ-formalism in ref. [MT3].

This paper extends these methods to the study of genus two partition and n-point functions in the ϵ-formalism for Vertex Operator Superalgebras (VOSA). In particular, we explicitly compute and prove convergence and modular properties of the genus two continuous orbifold partition and n-point functions for the rank two fermion VOSA $V(H, \mathbb{Z} + \frac{1}{2})^{\otimes 2}$. (The alternative ρ-formalism is considered elsewhere [TZ3]). These functions are computed in terms of appropriate torus n-point functions described in [MTZ]. We also make extensive use of the expression of the genus two Szegö kernel $S^{(2)}$ of (7) in terms of genus one Szegö kernel data described in [TZ1]. The partition function is then expressed as a certain infinite determinant whose components arise from genus one Szegö kernel data. Furthermore, the generating function of all n-point correlation functions is computed in terms of a genus two Szegö kernel determinant.

Section 2 consists of a review of aspects of the ϵ-formalism for constructing a genus two Riemann surface by sewing two separate tori with modular parameters τ_1, τ_2 respectively for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^\epsilon$, a specific domain for which the sewing is defined [MT2]. We also review the construction of the genus two Szegö kernel $S^{(2)}$ in terms of genus one Szegö kernel data [TZ1]. In particular we introduce an infinite block matrix

$$Q = \left(\begin{array}{cc} 0 & \xi F_1(\tau_1) \\ -\xi F_2(\tau_2) & 0 \end{array} \right),$$

where $\xi = \pm\sqrt{-1}$ and $F_a(\tau_a)$ for $a = 1, 2$ are certain infinite matrices whose entries involve twisted modular forms in τ_a associated with genus one Szegö kernels [MTZ]. Section 3 is a review of Vertex Operator Superalgebras (VOSA) and the Li–Zamolodchikov (Li–Z) metric on a VOSA [L], [Sche]. The free fermion rank one VOSA $V(H, \mathbb{Z} + \frac{1}{2})$ is also reviewed. In Section 4 we consider the orbifold partition and n-point function on a genus two surface in the ϵ-formalism for a VOSA with a Li–Z metric. These are defined in terms of genus one n-point orbifold functions associated with a pair of commuting VOSA automorphisms f_a, g_a on a torus with modular parameter τ_a for $a = 1, 2$.

Section 5 contains the main results of the paper wherein the partition function and the generating function for n-point functions are computed for the rank two fermion VOSA with continuous automorphisms generated by the Heisenberg vector. In particular we prove in Theorem 5.1 that the partition function is given by

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (\tau_1)\ Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\tau_2)\ \det(I - Q),$$

where $f = (f_1, g_1)$ and $g = (f_2, g_2)$ and $Z^{(1)} \left[\begin{smallmatrix} f_a \\ g_a \end{smallmatrix}\right] (\tau_a)$ is the orbifold partition function on the torus with modular parameter τ_a. The partition function is holomorphic for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^\epsilon$, a specific domain on which the ϵ-formalism can be carried out [MT2]. In Theorem 5.6 we find the generating function for all genus two n-point functions as a differential form which is expressed in terms of a finite dimensional determinant of genus two Szegö kernels $S^{(2)}$. We also discuss the bosonization of the fermion VOSA wherein the partition function can be expressed in terms of a genus two Riemann theta series and the Heisenberg genus two partition function. This leads to a new genus two version of the classical Jacobi product identity expressing the genus two Riemann theta series in terms of certain infinite products. We also discuss the genus two Ward identity satisfied by the Virasoro one point function in this bosonized setting.

In Section 6 we discuss modular invariance of the genus two partition and n-point generating form under a modular group preserving $\mathcal{D}^\epsilon$. The

Appendix describes some general aspects of Riemann surfaces such as the period matrix, the projective connection and the prime form. We also recall some facts from the classical and twisted elliptic function theory [MTZ].

We collect here notation for some of the more frequently occurring functions and symbols employed. $\mathbb{Z}$ is the set of integers, $\mathbb{C}$ the complex numbers, $\mathbb{H}$ the complex upper-half plane. We will always take τ to lie in $\mathbb{H}$, and z will lie in $\mathbb{C}$ unless otherwise noted. For a symbol z we set $q_z = \exp(z)$ and in particular $q = q_{2\pi i\tau} = \exp(2\pi i\tau)$.

2 The Szegö Kernel on a Genus Two Riemann Surface Formed from Two Sewn Tori

The central role played by the Szegö kernel $S^{(g)}$ for the fermion VOSA has been long known e.g. [RS], [R], [DVFHLS], [DVPFHLS]. In this Section we review the form of the Szegö kernel on a Riemann surface $\Sigma^{(2)}$ of genus two obtained by sewing together two tori described in [TZ1]. Some further details appear in Appendix 7.1.

2.1 The Szegö Kernel on a Riemann Surface

Consider a compact connected Riemann surface $\Sigma^{(g)}$ of genus g with canonical homology cycle basis a_i, b_i for $i = 1, \ldots g$. Let $\nu_i^{(g)}$ be a basis of holomorphic 1-forms with normalization $\oint_{a_i} \nu_j^{(g)} = 2\pi i\delta_{ij}$ and period matrix $\Omega_{ij}^{(g)} = \frac{1}{2\pi i} \oint_{b_i} \nu_j^{(g)} \in \mathbb{H}_g$, the Siegel upper half plane (e.g. [FK], [Sp]). Define the theta function with real characteristics [M], [F1], [FK]

$$\vartheta^{(g)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(z|\Omega^{(g)}\right) = \sum_{n\in\mathbb{Z}^g} e^{i\pi(n+\alpha).\Omega^{(g)}.(n+\alpha)+(n+\alpha).(z+2\pi i\beta)}, \tag{1}$$

for $\alpha = (\alpha_j), \beta = (\beta_j) \in \mathbb{R}^g$ and $z = (z_j) \in \mathbb{C}^g$ for $j = 1, \ldots, g$.

The Szegö Kernel [Schi], [HS], [F1], [F2] is defined for $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\Omega^{(g)}) \neq 0$ by

$$S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x, y) = \frac{\vartheta^{(g)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\int_y^x \nu^{(g)} \,|\Omega^{(g)}\right)}{\vartheta^{(g)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0|\Omega^{(g)}) E^{(g)}(x, y)}, \tag{2}$$

where $\theta = (\theta_j),\ \phi = (\phi_j) \in U(1)^n$ for

$$\theta_j = -e^{-2\pi i \beta_j}, \quad \phi_j = -e^{2\pi i \alpha_j}, \quad j = 1, \ldots, g. \tag{3}$$

and $E^{(g)}(x, y)$ is the prime form (see Appendix 7.1). The factors of -1 in (3) are included for later convenience. The Szegö kernel has multipliers along the a_i and b_j cycles in x given by $-\phi_i$ and $-\theta_j$ respectively and is a meromorphic $(\frac{1}{2}, \frac{1}{2})$-form satisfying

$$\begin{aligned} S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x, y) &\sim \frac{1}{x - y}\, dx^{\frac{1}{2}}\, dy^{\frac{1}{2}} \quad \text{for } x \sim y, \\ S^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x, y) &= -S \begin{bmatrix} \theta^{-1} \\ \phi^{-1} \end{bmatrix} (y, x), \end{aligned}$$

where $\theta^{-1} = (\theta_i^{-1})$ and $\phi^{-1} = (\phi_i^{-1})$.

2.2 Genus Two Riemann Surfaces Formed from Two Sewn Tori

Consider the genus two Riemann surface formed by sewing together two tori in the sewing scheme referred to as the ϵ-formalism in refs. [MT1], [MT2], [TZ1]. Let $\Sigma_a^{(1)} = \mathbb{C}/\Lambda_a$ for $a = 1, 2$ be oriented tori with lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ for $\tau_a \in \mathbb{H}$. Choose a local coordinate $z_a \in \mathbb{C}/\Lambda_a$ on $\Sigma_a^{(1)}$ in the neighborhood of a point $p_a \in \Sigma_a^{(1)}$ and consider the closed disk $|z_a| \le r_a$ for $r_a < \frac{1}{2}D(q_a)$ where [MT2]

$$D(q_a) = \min_{\lambda \in \Lambda_a, \lambda \neq 0} |\lambda|,$$

is the minimal lattice distance. Introduce a complex sewing parameter ϵ where $|\epsilon| \le r_1 r_2$, and excise the disk

$$\{z_a, |z_a| \le |\epsilon|/r_{\bar{a}}\} \subset \Sigma_a^{(1)},$$

to form a punctured torus

$$\widehat{\Sigma}_a^{(1)} = \Sigma_a^{(1)} \backslash \{z_a, |z_a| \le |\epsilon|/r_{\bar{a}}\}.$$

Here and below, we use the convention

$$\overline{1} = 2, \quad \overline{2} = 1.$$

Define the annulus

$$\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\} \subset \widehat{\Sigma}^{(1)}_a,$$

and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{4}$$

In this way we obtain a compact genus two Riemann surface $\Sigma^{(2)} = \{\widehat{\Sigma}^{(1)}_1 \backslash \mathcal{A}_1\} \cup \{\widehat{\Sigma}^{(1)}_2 \backslash \mathcal{A}_2\} \cup \mathcal{A}$, parameterized by the domain [MT2]

$$\mathcal{D}^{\epsilon} = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(q_1) D(q_2)\}. \tag{5}$$

2.3 The Genus Two Szegö Kernel in the ϵ-Formalism

On a torus the prime form is $E^{(1)}(x,y) = K^{(1)}(x-y,\tau)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}}$ where $K^{(1)}(z,\tau) = \frac{\vartheta_1(z,\tau)}{\partial_z \vartheta_1(0,\tau)}$ and $\vartheta_1(z,\tau) = \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right](z,\tau)$ for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$. For $(\theta, \phi) \neq (1,1)$ with $\theta = -\exp(-2\pi i \beta)$ and $\phi = -\exp(2\pi i \alpha)$ the genus one Szegö kernel is

$$S^{(1)} \left[\begin{array}{c} \theta \\ \phi \end{array}\right] (x, y|\, \tau) = P_1 \left[\begin{array}{c} \theta \\ \phi \end{array}\right] (x-y, \tau) dx^{\frac{1}{2}} dy^{\frac{1}{2}}, \tag{6}$$

where

$$\begin{aligned} P_1 \left[\begin{array}{c} \theta \\ \phi \end{array}\right] (z,\tau) &= \frac{\vartheta^{(1)} \left[\begin{array}{c} \alpha \\ \beta \end{array}\right] (z,\tau)}{\vartheta^{(1)} \left[\begin{array}{c} \alpha \\ \beta \end{array}\right] (0,\tau)} \frac{1}{K^{(1)}(z,\tau)} \\ &= -\sum_{k \in \mathbb{Z}} \frac{q_z^{k+\lambda}}{1 - \theta^{-1} q^{k+\lambda}}, \end{aligned}$$

is a 'twisted' Weierstrass function [MTZ] and where $q_z = e^z$ and $\phi = e^{2\pi i \lambda}$ for $0 \leq \lambda < 1$ (see Appendix 7.2 for details).

In [TZ1] we determine the genus two Szegö kernel

$$S^{(2)}(x,y) = S^{(2)} \left[\begin{array}{c} \theta^{(2)} \\ \phi^{(2)} \end{array}\right] (x,y) \tag{7}$$

with periodicities $\left(\theta^{(2)}, \phi^{(2)}\right) = (\theta_a, \phi_a)$ for $a = 1, 2$ on the inherited homology basis on the genus two Riemann surface $\Sigma^{(2)}$ formed by sewing two tori $\Sigma_a^{(1)}$ in terms of genus one Szegö kernel data $S_a^{(1)}(x, y) = S^{(1)} \left[\begin{smallmatrix} \theta_a \\ \phi_a \end{smallmatrix}\right] (x, y)$. Note that we exclude those Riemann theta characteristics for which $S^{(2)}$ exists but where one of the lower genus theta functions vanishes i.e. $(\theta_a, \phi_a) \neq (1, 1)$ so that $S_a^{(1)}$ exists on the torus $\Sigma_a^{(1)}$ for $a = 1, 2$.

In [TZ1] we show how to reconstruct $S^{(2)}(x, y)$ from the Laurant expansions (68) of $P_1 \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right] (k, l, \tau)$ with coefficients $C \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right] (k, l, \tau)$ and $D \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right] (k, l, \tau, z)$ of (69) and (70) of Appendix 7.2. In particular, we define for $k, l \geq 1$

$$F_a \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, l, \tau_a, \epsilon) = \epsilon^{\frac{1}{2}(k+l-1)} C \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, l, \tau_a). \tag{8}$$

We let $F_a = (F_a \left[\begin{smallmatrix} \theta_a \\ \phi_a \end{smallmatrix}\right] (k, l, \epsilon))$ denote the infinite matrix indexed by $k, l \geq 1$. We also define holomorphic $\frac{1}{2}$-forms on $\widehat{\Sigma}_a^{(1)}$

$$\begin{aligned} h_a \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, x, \tau_a, \epsilon) &= \epsilon^{\frac{k}{2} - \frac{1}{4}} \, D \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (1, k, \tau_a, x) \, dx^{\frac{1}{2}}, \\ \bar{h}_a \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, y, \tau_a, \epsilon) &= \epsilon^{\frac{k}{2} - \frac{1}{4}} \, D \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, 1, \tau_a, -y) \, dy^{\frac{1}{2}}. \end{aligned} \tag{9}$$

We let $h_a(x) = (h_a \left[\begin{smallmatrix} \theta_a \\ \phi_a \end{smallmatrix}\right] (k, x, \tau_a, \epsilon))$ and $\bar{h}_a(y) = (\bar{h}_a(\left[\begin{smallmatrix} \theta_a \\ \phi_a \end{smallmatrix}\right] (k, y, \tau_a, \epsilon))$ denote infinite row vectors indexed by k.

Recalling the ϵ sewing relation (4) we note that

$$dz_a^{\frac{1}{2}} = (-1)^{\bar{a}} \, \xi \, \epsilon^{\frac{1}{2}} \, \frac{dz_{\bar{a}}^{\frac{1}{2}}}{z_{\bar{a}}}, \tag{10}$$

where $\xi \in \{\pm\sqrt{-1}\}$ depending on the branch of the double cover of $\Sigma_a^{(1)}$ chosen. It is useful to introduce the infinite block matrices

$$\Xi = \begin{pmatrix} 0 & \xi I \\ -\xi I & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \xi F_1 \\ -\xi F_2 & 0 \end{pmatrix}, \tag{11}$$

where I denotes the infinite identity matrix. Then Theorem 3.6 of [TZ1] states that

$$S^{(2)}(x, y) = \begin{cases} S_a^{(1)}(x, y) + h_a(x) \left(I - F_{\bar{a}} F_a\right)^{-1} F_{\bar{a}} \bar{h}_a^T(y), & x, y \in \widehat{\Sigma}_a^{(1)}, \\ \xi(-1)^{\bar{a}} h_a(x) \left(I - F_{\bar{a}} F_a\right)^{-1} \bar{h}_{\bar{a}}^T(y), & x \in \widehat{\Sigma}_a^{(1)}, \ y \in \widehat{\Sigma}_{\bar{a}}^{(1)}, \end{cases} \tag{12}$$

where T denotes the transpose. Equivalently, for $x, y \in \widehat{\Sigma}^{(1,1)} = \widehat{\Sigma}_1^{(1)} \cup \widehat{\Sigma}_2^{(1)}$, the disconnected union of punctured tori, we define the forms

$$\begin{aligned}
S^{(1,1)}(x,y) &= \begin{cases} S_a^{(1)}(x,y), & x,y \in \widehat{\Sigma}_a^{(1)} \\ 0, & x \in \widehat{\Sigma}_a^{(1)},\ y \in \widehat{\Sigma}_{\bar{a}}^{(1)} \end{cases} \\
h(x) &= \begin{cases} (h_1(x), 0), & x \in \widehat{\Sigma}_1^{(1)} \\ (0, h_2(x)), & x \in \widehat{\Sigma}_2^{(1)} \end{cases} \\
\overline{h}(x) &= \begin{cases} \left(\overline{h}_1(x), 0\right), & x \in \widehat{\Sigma}_1^{(1)} \\ \left(0, \overline{h}_2(x)\right), & x \in \widehat{\Sigma}_2^{(1)}. \end{cases}
\end{aligned} \tag{13}$$

Thus $h(x)$ describes an infinite row vector indexed by $k \geq 1$ and $a = 1, 2$ with $(h(x))\,(k,a) = \delta_{ab} h_b \begin{bmatrix} \theta_b \\ \phi_b \end{bmatrix} (k, x, \tau_b, \epsilon)$ for $x \in \widehat{\Sigma}_b^{(1)}$ and similarly for $\overline{h}(x)$. With these definitions (12) is equivalent to

$$S^{(2)}(x,y) = S^{(1,1)}(x,y) + h(x)\Xi(I-Q)^{-1}\overline{h}^T(y), \tag{14}$$

for $x, y \in \widehat{\Sigma}^{(1,1)}$.

Lastly, defining the determinant of $I - Q$ by the formal power series in ϵ

$$\log\det(I-Q) = Tr\log(I-Q) = -\sum_{n\geq 1}\frac{1}{n}Tr(Q^n),$$

it is shown in ref. [TZ1] that

$$\det(I-Q) = \det(I - F_1F_2), \tag{15}$$

is non-vanishing and holomorphic on $\mathcal{D}^\epsilon$.

3 Vertex Operator Superalgebras

3.1 General Definitions

We discuss some aspects of Vertex Operator Superalgebra theory to establish context and notation. For more details see [B], [FHL], [FLM], [Ka], [MN], [MT5]. A Vertex Operator Superalgebra (VOSA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ as follows: V is a superspace i.e. a complex vector space $V = V_{\bar{0}} \oplus V_{\bar{1}} = \oplus_\alpha V_\alpha$

with index label α in $\mathbb{Z}/2\mathbb{Z}$ so that each $a \in V$ has a parity (fermion number) $p(a) \in \mathbb{Z}/2\mathbb{Z}$. V has non-negative $\frac{1}{2}\mathbb{Z}$-grading with

$$V = \oplus_{r \in \frac{1}{2}\mathbb{Z}} V_r, \quad \dim V_r < \infty,$$

related to the superspace grading by

$$V_{\bar{0}} = \oplus_{r \in \mathbb{Z}} V_r, \quad V_{\bar{1}} = \oplus_{r \in \mathbb{Z}+\frac{1}{2}} V_r. \tag{16}$$

$\mathbf{1} \in V_0$ is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. Y is a linear map $Y : V \to (\operatorname{End} V)[[z, z^{-1}]]$, for formal variable z, so that for any vector $a \in V$

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}.$$

The component operators (modes) $a(n) \in \operatorname{End} V$ are such that

$$a(n)\mathbf{1} = \delta_{n,-1} a,$$

for $n \geq -1$. Furthermore, for $a \in V_\alpha$

$$a(n) : V_\beta \to V_{\beta+\alpha}. \tag{17}$$

The vertex operators satisfy locality:

$$(x - y)^N [Y(a, x), Y(b, y)] = 0,$$

for all $a, b \in V$ and $N \gg 0$, where the commutator is defined in the graded sense:

$$[Y(a, x), Y(b, y)] = Y(a, x)Y(b, y) - (-1)^{p(a)p(b)} Y(b, y)Y(a, x). \tag{18}$$

The vertex operator for the vacuum is $Y(\mathbf{1}, z) = Id_V$, whereas that for ω is

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},$$

where $L(n) = \omega(n+1)$ forms a Virasoro algebra for central charge c

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}.$$

$L(-1)$ generates translations with

$$Y(L(-1)a, z) = \frac{d}{dz} Y(a, z).$$

$L(0)$ determines the grading with $L(0)a = wt(a)a$ for $a \in V_r$ and $r = wt(a)$, the weight of a.

3.2 The Li–Zamolodchikov (Li–Z) Metric

The subalgebra $\{L(-1), L(0), L(1)\} \cong SL(2,\mathbb{C})$ associated with Möbius transformations on z naturally acts on a VOSA (e.g. [B], [Ka]). In particular,

$$\gamma_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : z \mapsto w = -\frac{\lambda^2}{z}, \tag{19}$$

is generated by $T_\lambda = \exp(\lambda L(-1)) \exp(\frac{1}{\lambda} L(1)) \exp(\lambda L(-1))$ where

$$T_\lambda Y(u,z) T_\lambda^{-1} = Y\left(\exp(-\frac{z}{\lambda^2} L(1)) \left(-\frac{z}{\lambda}\right)^{-2L(0)} u, -\frac{\lambda^2}{z}\right). \tag{20}$$

Later we will be particularly interested in the Möbius map $z \mapsto w = \epsilon/z$ associated with the sewing condition (4) with

$$\lambda = -\xi \epsilon^{\frac{1}{2}}, \tag{21}$$

with $\xi \in \{\pm\sqrt{-1}\}$ as previously introduced in (10).

For $u \in V$ of half-integral weight the action of $-\gamma_\lambda = \gamma_{-\lambda}$ is distinguished from that of γ_λ whereas for integral weight they are equivalent. In particular we must distinguish the choices $\lambda = \pm\sqrt{-1}$ in (19) corresponding to the inversion map $z \mapsto z^{-1}$ normally used to define the adjoint vertex operator. Following ref. [Sche] we therefore define

$$Y^\dagger(u,z) = \sum_n u^\dagger(n) z^{-n-1} = T_\lambda Y(u,z) T_\lambda^{-1}. \tag{22}$$

One can verify that $\left(Y^\dagger\right)^\dagger(u,z) = (-1)^{2wt(u)} Y(u,z)$ for u of weight $wt(u)$.

For a quasi-primary vector u (i.e. $L(1)u = 0$) of weight $wt(u)$

$$u^\dagger(n) = \lambda^{-2wt(u)} (-\lambda^2)^{n+1} u(2wt(u) - n - 2). \tag{23}$$

e.g. $L^\dagger(n) = (-\lambda^2)^n L(-n)$. Furthermore

$$Y^\dagger(u,w) dw^{wt(u)} = Y(u,z) dz^{wt(u)}, \tag{24}$$

where for half-integral $wt(u)$ we choose the branch covering for which

$$\left(\frac{dw}{dz}\right)^{wt(u)} = \left(\frac{\lambda}{z}\right)^{2wt(u)}. \tag{25}$$

We say a bilinear form $\langle\ ,\rangle_\lambda$ on V is invariant if for all $a, b, u \in V$ [Sche]

$$\langle Y(u,z)a, b\rangle_\lambda = (-1)^{p(u)p(a)}\langle a, Y^\dagger(u,z)b\rangle_\lambda, \tag{26}$$

i.e. $\langle u(n)a, b\rangle_\lambda = (-1)^{p(u)p(a)}\langle a, u^\dagger(n)b\rangle_\lambda$. Thus it follows that $\langle L(0)a, b\rangle_\lambda = \langle a, L(0)b\rangle_\lambda$ so that $\langle a, b\rangle_\lambda = 0$ if $wt(a) \neq wt(b)$ for homogeneous a, b. One also finds $\langle a, b\rangle_\lambda = \langle b, a\rangle_\lambda$ [FHL], [Sche].

$\langle\ ,\rangle_\lambda$ is unique up to normalization if $L(1)V_1 = V_0$ (we choose the normalization $\langle \mathbf{1}, \mathbf{1}\rangle_\lambda = 1$ throughout) and is non-degenerate if and only if V is simple [L]. We call such a unique non-degenerate symmetric bilinear form the Li–Zamolodchikov (Li–Z) metric. Given any V basis $\{u^\alpha\}$ we define the Li–Z dual V basis $\{\overline{u}^\beta\}$ where $\langle u^\alpha, \overline{u}^\beta\rangle_\lambda = \delta^{\alpha\beta}$.

3.3 Free Fermion VOSA

Consider the rank one free fermion VOSA $V(H, \mathbb{Z} + \frac{1}{2})$ with $H = \mathbb{C}\psi$ for a (fermion) vector ψ of parity 1 [FFR], [Ka] with modes obeying

$$[\psi(m), \psi(n)] = \psi(m)\psi(n) + \psi(n)\psi(m) = \delta_{m+n+1,0}. \tag{27}$$

The superspace is spanned by Fock vectors we denote by[1]

$$\Psi(\mathbf{k}) \equiv \psi(-k_1)\psi(-k_2)\ldots\psi(-k_s)\mathbf{1}, \tag{28}$$

for distinct ordered integers $1 \leq k_1 < \ldots < k_s$ and where $\psi(k)\mathbf{1} = 0$ for $k \geq 0$. The VOSA is generated by $Y(\psi, z)$ with conformal vector $\omega = \frac{1}{2}\psi(-2)\psi(-1)\mathbf{1}$ of central charge $c = \frac{1}{2}$ for which $\Psi(\mathbf{k})$ has $L(0)$ weight $wt(\Psi(\mathbf{k})) = \sum_{1\leq i\leq m}(k_i - \frac{1}{2}) \in \frac{1}{2}\mathbb{Z}$. In particular $wt(\psi) = \frac{1}{2}$.

Since $\psi^\dagger(n) = \lambda^{-1}(-\lambda^2)^{n+1}\psi(-n-1)$ it follows from (23) that the Fock vectors form an orthogonal basis with respect to the Li–Z metric $\langle\ ,\rangle_\lambda$ with

$$\overline{\Psi}(\mathbf{k}) = (-1)^{[wt(\Psi)]}\,\lambda^{2wt(\Psi)}\Psi(\mathbf{k}), \tag{29}$$

for $\Psi(\mathbf{k})$ of weight $wt(\Psi)$ and where $[x]$ denotes the integral part of x.

We next consider the rank two fermion VOSA $V(H, \mathbb{Z} + \frac{1}{2})^{\otimes 2}$, the tensor product of two copies of the rank one fermion VOSA. We employ the off-diagonal basis $\psi^\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$ for fermions $\psi_1 = \psi \otimes \mathbf{1}$ and $\psi_2 = \mathbf{1} \otimes \psi$.

[1] Denoted by $\Psi(-\mathbf{k})$ in ref. [MTZ]

The VOSA is generated by $Y(\psi^{\pm},z)=\sum_{n\in\mathbb{Z}}\psi^{\pm}(n)z^{-n-1}$ where the modes obey the commutation relations

$$[\psi^+(m),\psi^-(n)]=\delta_{m,-n-1},\quad [\psi^+(m),\psi^+(n)]=0,\quad [\psi^-(m),\psi^-(n)]=0.$$

The VOSA vector space V is a Fock space spanned by[2]

$$\Psi(\mathbf{k},\mathbf{l})\equiv\psi^+(-k_1)\ldots\psi^+(-k_s)\psi^-(-l_1)\ldots\psi^-(-l_t)\mathbf{1}, \tag{30}$$

for distinct positive integers $k_1,\ldots,k_s$ and distinct $l_1,\ldots,l_t$ with $\psi^{\pm}(k)\mathbf{1}=0$ for all $k\geq 0$. We define the conformal vector to be

$$\omega=\frac{1}{2}[\psi^+(-2)\psi^-(-1)+\psi^-(-2)\psi^+(-1)]\mathbf{1}, \tag{31}$$

whose modes generate a Virasoro algebra of central charge 1. Then $\psi^{\pm}$ has $L(0)$-weight $\frac{1}{2}$ and $\Psi(\mathbf{k},\mathbf{l})$ has $L(0)$-weight $wt(\Psi)=\sum_{1\leq i\leq s}(k_i-\frac{1}{2})+\sum_{1\leq j\leq t}(l_j-\frac{1}{2})$. Similarly to (29), the Li–Z dual of $\Psi(\mathbf{k},\mathbf{l})$ is

$$\overline{\Psi}(\mathbf{k},\mathbf{l})=(-1)^{st}\,(-1)^{[wt(\Psi)]}\,\lambda^{2wt(\Psi)}\Psi(\mathbf{l},\mathbf{k}),$$

where the $(-1)^{st}$ factor arises from the ordering chosen in (30). For the parameter choice (21) we find for $\Psi(\mathbf{k},\mathbf{l})$) of parity p_Ψ that

$$\overline{\Psi}(\mathbf{k},\mathbf{l})=(-1)^{st}\,(-\xi)^{p_\Psi}\epsilon^{wt(\Psi)}\Psi(\mathbf{l},\mathbf{k}). \tag{32}$$

The weight 1 space is $V_1=\mathbb{C}a$ for Heisenberg vector

$$a=\psi^+(-1)\psi^-(-1)\mathbf{1}, \tag{33}$$

with modes obeying

$$[a(m),a(n)]=m\delta_{m,-n}.$$

Then $\omega=\frac{1}{2}a(-1)^2\mathbf{1}$ is the standard conformal vector for the Heisenberg VOA M. Thus V can be decomposed into irreducible M-modules $M\otimes e^m$ for $a(0)$ eigenvalue $m\in\mathbb{Z}$ e.g. [FFR], [Ka]. Furthermore, $a(0)$ is a generator of continuous V automorphisms $e^{2\pi i\gamma a(0)}$ for real γ.

[2] Denoted by $\Psi(-\mathbf{k},-\mathbf{l})$ in ref. [MTZ]

4 Partition Functions and Correlation Functions on a Genus Two Riemann Surface

In this section we consider the partition and n-point correlation functions for a VOSA on a Riemann surface of genus two formed by sewing two tori. In the next section we will compute these quantities in the case of a rank two fermion VOSA with arbitrary automorphisms generated by $a(0)$.

4.1 Torus n-Point Correlation Functions

We first review aspects of genus one orbifold n-point (correlation) functions for twisted VOSA modules. For more details see refs. [Z1], [DLM], [MT4], [DZ], [MTZ].

Let $\sigma \in \mathrm{Aut}(V)$ denote the parity (fermion number) automorphism

$$\sigma a = (-1)^{p(a)} a, \tag{34}$$

for all $a \in V$. Let $f, g \in \mathrm{Aut}(V)$ denote two commuting automorphisms that also commute with σ. Consider a σg-twisted V-module $M_{\sigma g}$ with vertex operators $Y_{\sigma g}$ [DLM], [DZ], [MTZ]. We assume that $M_{\sigma g}$ is stable under σ and f i.e. both σ and f act on $M_{\sigma g}$. Then for vectors $v_1, \ldots, v_n \in V$ we define the torus orbifold n-point function by [Z1], [MTZ]

$$\begin{aligned} &Z^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, z_1; \ldots; v_n, z_n; \tau) \\ &\equiv \mathrm{STr}_{M_{\sigma g}} \left(f\, Y_{\sigma g}(q_1^{L(0)} v_1, q_1) \ldots Y_{\sigma g}(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24} \right). \end{aligned} \tag{35}$$

where $q = \exp(2\pi i \tau)$, $q_i = \exp(z_i)$, $i = 1, \ldots, n$, for variables $z_1, \ldots, z_n$ and where STr_M denotes the supertrace defined by

$$\mathrm{STr}_M(X) = Tr_M(\sigma X).$$

It follows from (17) that the n-point function (35) is non-vanishing provided

$$p_1 + \ldots + p_n = 0 \mod 2, \tag{36}$$

for parity $p_i = p(v_i)$.

Taking all $v_i = \mathbf{1}$ in (35) yields the genus one orbifold partition function which we denote by $Z^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau)$. Taking $n = 1$ in (35) gives the genus one 1-point function which we denote by $Z^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (v; \tau)$ and is independent of z.

In order to consider modular-invariance of n-point functions at genus 1, Zhu [Z1] introduced a second isomorphic 'square-bracket' VOSA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ associated to a given VOSA $(V, Y(,), \mathbf{1}, \omega)$. The new vertex operators are defined by a change of coordinates

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1),$$

while the new conformal vector $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. We set $Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n] z^{-n-2}$ and write $wt[v] = k$ if $L[0]v = kv$, $V_{[k]} = \{v \in V | wt[v] = k\}$. Only primary vectors are homogeneous with respect to both $L(0)$ and $L[0]$, in which case $wt(v) = wt[v]$. One can show that n-point functions can be expressed in terms of 1-point functions to find [MT4]

$$\begin{aligned} & Z^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, z_1; \ldots; v_n, z_n; \tau) \\ &= Z^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (Y[v_1, z_1 - z_n] \ldots Y[v_{n-1}, z_{n-1} - z_n] v_n; \tau). \end{aligned} \tag{37}$$

4.2 Genus Two n-Point Correlation Functions

In the ϵ-sewing scheme we sew two tori $\Sigma_a^{(1)}$, $a = 1, 2$ with modular parameters τ_a via the sewing relation (4). Similarly to ref. [MT1] for VOAs, we define the genus two orbifold n-point correlation function in the ϵ-sewing scheme for a VOSA V with a Li–Z metric as follows. Let f_a, g_a be V automorphisms and let $M_{\sigma g_a}$ be σg_a-twisted V-modules stable under σ and f_a for commuting f_a, g_a and σ. We combine f_1, g_1 orbifold correlation functions on $\Sigma_1^{(1)}$ with f_2, g_2 orbifold correlation functions on $\Sigma_2^{(1)}$. For $x_1, \ldots, x_k \in \Sigma_1^{(1)}$ with $|x_i| \geq |\epsilon|/r_2$ and $y_{k+1}, \ldots, y_n \in \Sigma_2^{(1)}$ with $|y_i| \geq |\epsilon|/r_1$, define the genus two orbifold n-point function as the following formal series in ϵ

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(v_1, x_1; \ldots; v_k, x_k | v_{k+1}, y_{k+1}; \ldots, v_n, y_n; \tau_1, \tau_2, \epsilon)$$

$$= \sum_{u \in V} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(Y[v_1, x_1] \ldots Y[v_k, x_k]u; \tau_1)$$

$$\cdot Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(Y[v_{k+1}, y_{k+1}] \ldots Y[v_n, y_n]\bar{u}; \tau_2), \quad (38)$$

where f (respectively g) denotes the pair f_1, f_2 (respectively g_1, g_2). The sum is taken over any V-basis where $\bar{u}$ is the dual of u with respect to the Li–Z metric $\langle\ ,\ \rangle_\lambda^{\mathrm{sq}}$ of (26) as defined by the square bracket Virasoro operators $\{L[n]\}$ and with λ of (21).

Remark 4.1 (38) *reduces to the definition given in ref. [MT1] as follows. For u, v of equal square bracket weight we have*

$$\langle u, v\rangle_\lambda^{\mathrm{sq}} = \epsilon^{-wt[u]}\langle u, v\rangle^{\mathrm{sq}}, \quad (39)$$

where $\langle u, v\rangle^{\mathrm{sq}}$ denotes the standard Li–Z metric corresponding to the choice $\lambda = \pm\sqrt{-1}$. Then (38) *can be rewritten as*

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(v_1, x_1; \ldots; v_k, x_k | v_{k+1}, y_{k+1}; \ldots, v_n, y_n; \tau_1, \tau_2, \epsilon)$$

$$= \sum_{r \in \mathbb{Z}/2} \epsilon^r \sum_{u \in V_{[r]}} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(Y[v_1, x_1] \ldots Y[v_k, x_k]u; \tau_1)$$

$$\cdot Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(Y[v_{k+1}, y_{k+1}] \ldots Y[v_n, y_n]\bar{u}; \tau_2),$$

where here u ranges over any $V_{[r]}$-basis and $\bar{u}$ is the dual of u with respect to the standard Li–Z metric $\langle u, v\rangle^{\mathrm{sq}}$.

In the case where no states v_i are inserted then (38) defines the genus two partition (or 0-point) function

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(u; \tau_1) Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\bar{u}; \tau_2). \quad (40)$$

The definition (38) depends on the choice of insertion points $x_i \in \widehat{\Sigma}_1^{(1)}$ and $y_j \in \widehat{\Sigma}_2^{(1)}$. However, similarly to the situation for a VOA discussed in ref. [MT1], we may define an associated formal differential form for quasi-primary vectors as follows:

Proposition 4.2 *Let $v_i \in V$ be quasi-primary vectors of square bracket weight $wt[v_i]$ for $i = 1, \ldots, n$. Let $x_i \in \widehat{\Sigma}_1^{(1)}$ and $y_i \in \widehat{\Sigma}_2^{(1)}$ be related by the sewing relation*

$$x_i y_i = \epsilon = -\lambda^2.$$

Then the formal differential form

$$\begin{aligned}
&\mathcal{F}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, \ldots, v_n; \tau_1, \tau_2, \epsilon) \\
&\equiv (-1)^{N_k} Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, x_1; \ldots; v_k, x_k | v_{k+1}, y_{k+1}; \ldots; v_n, y_n; \tau_1, \tau_2, \epsilon) \\
&\quad \cdot \prod_{i=1}^{k} dx_i^{wt[v_i]} \prod_{j=k+1}^{n} dy_j^{wt[v_j]},
\end{aligned} \tag{41}$$

is independent of the choice of $k = 0, \ldots, n$ where N_k is the number of odd parity vectors in the set $\{v_1, \ldots, v_k\}$ and where the branch covering (25) is chosen with

$$\left(\frac{dy_i}{dx_i}\right)^{wt[v_i]} = \left(\frac{\lambda}{x_i}\right)^{2wt[v_i]}.$$

Proof. For $k \in \{1, \ldots, n\}$ consider

$$\begin{aligned}
&Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (v_1, x_1; \ldots; v_k, x_k | v_{k+1}, y_{k+1}; \ldots, v_n, y_n) \prod_{i=1}^{k} dx_i^{wt[v_i]} \prod_{j=k+1}^{n} dy_j^{wt[v_j]} \\
&= \sum_{u \in V} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (Y[v_1, x_1] \ldots Y[v_k, x_k] u; \tau_1) \prod_{i=1}^{k} dx_i^{wt[v_i]} \\
&\quad \cdot Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (Y[v_{k+1}, y_{k+1}] \ldots Y[v_n, y_n] \bar{u}; \tau_2) \prod_{j=k+1}^{n} dy_j^{wt[v_j]},
\end{aligned} \tag{42}$$

We have $Y[v_k,x_k]u=\sum_{v\in V}\langle \bar{v},Y[v_k,x_k]u\rangle_\lambda^{\mathrm{sq}}v$ where v is summed over any V-basis. Since v_k is quasi-primary, (24) implies

$$\begin{aligned}\langle \bar{v},Y[v_k,x_k]u\rangle_\lambda^{\mathrm{sq}} &= \langle \bar{v},Y^{\dagger}[v_k,y_k]u\rangle_\lambda^{\mathrm{sq}}\left(\frac{dy_k}{dx_k}\right)^{wt[v_k]}\\ &= (-1)^{p_kp(v)}\langle Y[v_k,y_k]\bar{v},u\rangle_\lambda^{\mathrm{sq}}\left(\frac{dy_k}{dx_k}\right)^{wt[v_k]},\end{aligned}$$

using invariance (26) and where $p_k=p(v_k)$. Hence (42) becomes

$$\sum_{v\in V}(-1)^{p_kp(v)}Z^{(1)}\begin{bmatrix} f_1\\ g_1\end{bmatrix}(Y[v_1,x_1]\ldots Y[v_{k-1},x_{k-1}]v;\tau_1)\prod_{i=1}^{k-1}dx_i^{wt[v_i]}$$
$$(-1)^{p_k(p_{k+1}+\ldots+p_n)}Z^{(1)}\begin{bmatrix} f_2\\ g_2\end{bmatrix}(Y[v_k,y_k]Y[v_{k+1},y_{k+1}]\ldots Y[v_n,y_n]\bar{v};\tau_2)\prod_{j=k}^{n}dy_j^{wt[v_j]},$$

using $\sum_{u\in V}\langle Y[v_k,y_k]\bar{v},u\rangle_\lambda^{\mathrm{sq}}\bar{u}=Y[v_k,y_k]\bar{v}$ and locality. Finally, (36) implies non-vanishing contributions arise only if $p(v)=p_1+\ldots+p_{k-1}$ so that $(-1)^{p_kp(v)}(-1)^{p_k(p_{k+1}+\ldots+p_n)}=(-1)^{p_k}$. But $N_k=p_k+N_{k-1}$ where N_{k-1} is the number of odd parity vectors in the set $\{v_1,\ldots v_{k-1}\}$. Hence $(-1)^{p_k}=(-1)^{N_{k-1}-N_k}$ and the result follows. □

5 The Free Fermion VOSA

5.1 Genus One

Consider the rank 2 free fermion VOSA $V(H,\mathbb{Z}+\frac{1}{2})^{\otimes 2}$ generated by $\psi^{\pm}$. In this case, the parity automorphism (34) is described by $\sigma=e^{i\pi a(0)}$ for Heisenberg vector a. We also define two commuting automorphisms f,g by[3]

$$\sigma f=e^{2\pi i\beta a(0)},\quad \sigma g=e^{-2\pi i\alpha a(0)},$$

for real α,β. It is also convenient to define $\theta=-e^{-2\pi i\beta}$, $\phi=-e^{2\pi i\alpha}$, in accordance with (3). The twisted partition function is then e.g. [Ka], [MTZ]

$$Z^{(1)}\begin{bmatrix} f\\ g\end{bmatrix}(\tau) = q^{\alpha^2/2-1/24}\prod_{l\geq 1}\left(1-\theta^{-1}q^{l-\frac{1}{2}+\alpha}\right)\left(1-\theta q^{l-\frac{1}{2}-\alpha}\right). \quad (43)$$

[3]Note some notational changes from ref. [MTZ]

(43) vanishes for $(\alpha,\beta)=(\frac{1}{2},\frac{1}{2})$ i.e. $(\theta,\phi)=(1,1)$. We will assume that $(\theta,\phi)\neq(1,1)$ for the remainder of this discussion.

In ref. [MTZ] it is shown by using associativity how to compute all twisted genus one n-point functions from a generating function which is the $2n$-point function for n ψ^+ and n ψ^- vectors:

$$Z^{(1)}\begin{bmatrix} f \\ g \end{bmatrix}(\psi^+,x_1;\psi^-,y_1;\ldots;\psi^+,x_n;\psi^-,y_n;\tau)=\det\mathbf{P}\cdot Z^{(1)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau), \quad (44)$$

where $\mathbf{P}$ is the $n\times n$ matrix:

$$\mathbf{P}=\left(P_1\begin{bmatrix} \theta \\ \phi \end{bmatrix}(x_i-y_j,\tau)\right), \quad (45)$$

for $1\leq i,j\leq n$ and where $P_1\begin{bmatrix} \theta \\ \phi \end{bmatrix}(z,\tau)$ is the twisted Weierstrass function defined in (67). Thus, in particular, for a homogeneous square bracket weight Fock vector

$$\Psi[\mathbf{k},\mathbf{l}]\equiv\psi^+[-k_1]\ldots\psi^+[-k_s]\psi^-[-l_1]\ldots\psi^-[-l_t]\mathbf{1}, \quad (46)$$

we find that the genus one 1-point function is given by [MTZ]

$$Z^{(1)}\begin{bmatrix} f \\ g \end{bmatrix}(\Psi[\mathbf{k},\mathbf{l}],\tau)=\delta_{st}(-1)^{s(s-1)/2}Z^{(1)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau)\det C\begin{bmatrix} \theta \\ \phi \end{bmatrix}(\mathbf{k},\mathbf{l},\tau), \quad (47)$$

where $C\begin{bmatrix} \theta \\ \phi \end{bmatrix}(\mathbf{k},\mathbf{l},\tau)$ is the $s\times s$ matrix:

$$C\begin{bmatrix} \theta \\ \phi \end{bmatrix}(\mathbf{k},\mathbf{l},\tau)=\left(C\begin{bmatrix} \theta \\ \phi \end{bmatrix}(k_i,l_j,\tau)\right), \quad (48)$$

for $1\leq i,j\leq s$ as defined by (69). Note that (47) is non-vanishing for $\Psi[\mathbf{k},\mathbf{l}]$ of even parity (integer weight) in agreement with (36).

5.2 The Genus Two Partition Function

We now come to the main results of this paper where for the rank two fermion VOSA we compute the genus two partition function and the generating form on the genus two Riemann surface formed by sewing together two tori as defined by (38). Consider commuting automorphisms f_a, g_a for $a = 1, 2$ parameterized by

$$\sigma f_a = e^{2\pi i \beta_a a(0)}, \quad \sigma g_a = e^{-2\pi i \alpha_a a(0)},$$

and define $\theta_a = -e^{-2\pi i \beta_a}$, $\phi_a = -e^{2\pi i \alpha_a}$ where $(\theta_a, \phi_a) \neq (1, 1)$. (The case where $(\theta_a, \phi_a) = (1, 1)$ will be considered elsewhere [TZ4]). Recall the infinite matrices F_a, Q of (8) and (11)

$$F_a \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} = \left(\epsilon^{\frac{1}{2}(k+l-1)} C \begin{bmatrix} \theta_a \\ \phi_a \end{bmatrix} (k, l, \tau_a) \right), \quad Q = \left(\begin{array}{cc} 0 & \xi F_1 \left[\begin{smallmatrix} \theta_1 \\ \phi_1 \end{smallmatrix} \right] \\ -\xi F_2 \left[\begin{smallmatrix} \theta_2 \\ \phi_2 \end{smallmatrix} \right] & 0 \end{array} \right).$$

We find the partition function (40) is as follows:

Theorem 5.1 *The genus two partition function for the rank two fermion VOSA is a non-vanishing holomorphic function on $\mathcal{D}^\epsilon$ given by*

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (\tau_1) \; Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\tau_2) \; \det (I - Q). \tag{49}$$

To prove this result we first note some determinant formulas for finite matrices. Let R be an $N \times N$ matrix and let $\mathbf{k} = (k_1, \ldots, k_n)$ denote n ordered subindices with $1 \leq k_1 < \ldots < k_n \leq N$. We refer to $\mathbf{k}$ as an *N-subindex of length n*. Let

$$R(\mathbf{k}, \mathbf{l}) = (R_{k_r l_s}) \quad r, s = 1 \ldots n, \tag{50}$$

denote the $n \times n$ submatrix of R indexed by a pair $\mathbf{k}, \mathbf{l}$ of N-subindices of length n. We define $R(\mathbf{k}, \mathbf{l}) = 1$ in the degenerate case $n = 0$.

Proposition 5.2 *Let R be an $N \times N$ matrix and I the identity matrix. Then*

$$\det (I + R) = \sum_{n=0}^{N} \sum_{\mathbf{j}} \det R(\mathbf{j}, \mathbf{j}), \tag{51}$$

where the inner sum runs over all N-subindices of length n.

Proof. Consider $\det(I + xR) = \sum_{\sigma \in S_N} \epsilon_\sigma \prod_{i=1}^{N} (\delta_{i\sigma(i)} + x R_{i\sigma(i)})$ for parameter x where ϵ_σ is the signature of $\sigma \in S_N$ the permutation group. Consider the subset of S_N consisting of all permutations ρ fixing *at least* $N - n$ indices.

Each ρ is a permutation on some $\mathbf{j} = (j_1, \ldots, j_n)$, an N-subindex of length n, where the remaining $N-n$ indices are fixed. Then $\det(I+xR) = \sum_{0\leq n\leq N} a_n x^n$ for

$$a_n = \sum_{\mathbf{j}} \sum_{\rho} \epsilon_\rho \prod_{i=1}^{n} R_{j_i\rho(j_i)} = \sum_{\mathbf{j}} \det R(\mathbf{j}, \mathbf{j}).$$

□

Corollary 5.3 *Let A, B be $M \times M$ matrices and let $R = \begin{bmatrix} 0 & tA \\ t^{-1}B & 0 \end{bmatrix}$ be a $2M \times 2M$ block matrix for parameter $t \neq 0$. Then $\det(I+R)$ is t independent and is given by*

$$\det(I+R) = \sum_{m=0}^{M} (-1)^m \sum_{\mathbf{k},\mathbf{l}} \det A(\mathbf{k}, \mathbf{l}) \det B(\mathbf{l}, \mathbf{k}), \tag{52}$$

where the inner sum runs over all pairs $\mathbf{k}, \mathbf{l}$ of M-subindices of length m.

Proof. Clearly $I+R = \begin{bmatrix} tI_M & 0 \\ 0 & I_M \end{bmatrix} \begin{bmatrix} I_M & A \\ B & I_M \end{bmatrix} \begin{bmatrix} t^{-1}I_M & 0 \\ 0 & I_M \end{bmatrix}$ for $M \times M$ identity matrix I_M so that $\det(I+R)$ is independent of t. Next apply (51) to the block matrix R. The block structure of R and the t independence of $\det(I+R)$ imply that the inner sum of (51) runs over $2M$-indices of length $2m$ of the form $\mathbf{j} = (k_1, \ldots, k_m, M+l_1, \ldots, M+l_m)$. The pair $\mathbf{k}, \mathbf{l}$ are M-subindices of length m so that

$$\det(I+R) = \sum_{m=0}^{M} \sum_{\mathbf{k},\mathbf{l}} \det \begin{bmatrix} 0 & A(\mathbf{k}, \mathbf{l}) \\ B(\mathbf{l}, \mathbf{k}) & 0 \end{bmatrix}.$$

The result then follows. □

Proof of Theorem 5.1. We wish to compute the genus two partition function of (40) for the rank two fermion VOSA:

$$Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau_1, \tau_2, \epsilon) = \sum_{u\in V} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (u, \tau_1) Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\bar{u}, \tau_2),$$

where u is summed over any V-basis and $\bar{u}$ is the square bracket Li-Z dual. We choose the Fock basis $\{\Psi[\mathbf{k}, \mathbf{l}]\}$ with $1 \leq k_1 < \ldots < k_s$ and $1 \leq l_1 < \ldots < l_m$ of (46) with square-bracket dual from (32)

$$\overline{\Psi}[\mathbf{k}, \mathbf{l}] = (-1)^{sm} (-\xi)^{p_\Psi} \epsilon^{wt[\Psi]} \Psi[\mathbf{l}, \mathbf{k}]. \tag{53}$$

Furthermore, (47) implies the corresponding torus one point functions are non-vanishing for $m = s$ with even parity $p_\Psi = 0$ where

$$\begin{aligned}
\frac{Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\Psi[\mathbf{k},\mathbf{l}],\tau_1)}{Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\tau_1)} &= (-1)^{m(m-1)/2}\det C\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix}(\mathbf{k},\mathbf{l},\tau_1) \\
\frac{Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\overline{\Psi}[\mathbf{k},\mathbf{l}],\tau_2)}{Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\tau_2)} &= (-1)^{m(m-1)/2}(-1)^m\epsilon^{wt[\Psi]}\det C\begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix}(\mathbf{l},\mathbf{k},\tau_2).
\end{aligned}$$

Hence (suppressing the τ_1, τ_2, ϵ dependence) it follows that

$$\frac{Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}}{Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}} = \sum_{m\geq 0}(-1)^m\sum_{\mathbf{k},\mathbf{l}}\epsilon^{wt[\Psi]}\det C\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix}(\mathbf{k},\mathbf{l})\det C\begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix}(\mathbf{l},\mathbf{k}).$$

But $wt[\Psi] = \sum_{i=1}^m (k_i + l_i - 1)$ so that the $\epsilon^{k_i+l_j-\frac{1}{2}}$ factors may be absorbed into the above $m \times m$ determinants to find

$$\frac{Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}}{Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}} = \sum_{m\geq 0}(-1)^m\sum_{\mathbf{k},\mathbf{l}}\det F_1\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix}(\mathbf{k},\mathbf{l})\det F_2\begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix}(\mathbf{l},\mathbf{k}),$$

with F_a of (8). Let A and B denote the finite matrices found by truncating F_1 and F_2 to an arbitrary order in ϵ. Thus applying (52) to A and B with $t = -\xi$ it follows that

$$\frac{Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}}{Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}} = \det(I - Q),$$

as an identity between two formal series in ϵ. However, it is shown in ref. [TZ1] that $\det(I-Q)$ is non-vanishing and holomorphic on $\mathcal{D}^\epsilon$ and hence the Theorem holds. $\square$

We may similarly compute the genus two partition function in the ϵ-formalism for the original rank one fermion VOSA $V(H, \mathbb{Z}+\frac{1}{2})$ where, in this case, we may only construct a σ-twisted module. Then one finds:

Corollary 5.4 *For the rank one free fermion VOSA $V(H,\mathbb{Z}+\frac{1}{2})$ the genus two partition function in the ϵ-formalism for $f_a, g_a \in \{1,\sigma\}$ is given by*

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau_1,\tau_2,\epsilon) = Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\tau_1)\, Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\tau_2)\ \det(I-Q)^{1/2}, \qquad (54)$$

where $Z^{(1)}\begin{bmatrix} f_a \\ g_a \end{bmatrix}(\tau_a)$ is the rank one torus partition function. □

5.3 The Genus Two Generating Function

In this section we compute the genus two generating form for all n-point functions for the rank two free fermion VOSA. This is the genus two analogue of (44) and is defined by

$$\mathcal{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(w_1,\ldots,w_n,z_1,\ldots,z_n) = \mathcal{F}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\psi^+,\psi^-,\ldots,\psi^+,\psi^-;\tau_1,\tau_2,\epsilon), \qquad (55)$$

the formal $2n$-form of (41) found by alternatively inserting ψ^+ at $w_i \in \widehat{\Sigma}^{(1,1)}$ and ψ^- at $z_i \in \widehat{\Sigma}^{(1,1)}$ for $i=1,\ldots n$ where $\widehat{\Sigma}^{(1,1)}$ denotes the disconnected union of the two punctured tori. In order to describe $\mathcal{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}$ we recall the Szegö kernels and half-forms of (7) and (13) and define matrices

$$\begin{aligned} S^{(2)} &= \left(S^{(2)}(w_i,z_j)\right), \quad S^{(1,1)} = \left(S^{(1,1)}(w_i,z_j)\right), \\ H^+ &= \left((h(w_i))(k,a)\right), \quad H^- = \left((\overline{h}(z_i))(l,b)\right)^T. \end{aligned}$$

$S^{(2)}$ and $S^{(1,1)}$ are finite matrices indexed by w_i, z_j for $i,j=1,\ldots,n$; H^+ is semi-infinite with n rows indexed by w_i and columns indexed by $k \geq 1$ and $a=1,2$ and H^- is semi-infinite with rows indexed by $l \geq 1$ and $b=1,2$ and with n columns indexed by z_j. We then find

Proposition 5.5

$$\det\begin{bmatrix} S^{(1,1)} & H^+\Xi \\ H^- & I-Q \end{bmatrix} = \det S^{(2)}\det(I-Q).$$

with Q, Ξ of (11).

Proof. Consider the matrix identity

$$\left[\begin{array}{cc} S^{(1,1)} & H^{+}\Xi \\ H^{-} & I-Q \end{array}\right] =$$

$$\left[\begin{array}{cc} I_n & H^{+}\Xi(I-Q)^{-1} \\ 0 & I \end{array}\right]\left[\begin{array}{cc} S^{(1,1)}-H^{+}\Xi(I-Q)^{-1}H^{-} & 0 \\ H^{-} & I \end{array}\right]\left[\begin{array}{cc} I_n & 0 \\ 0 & I-Q \end{array}\right],$$

where I_n is the $n\times n$ identity matrix. But the genus two Szegö kernel of (14) implies

$$\left(S^{(1,1)}-H^{+}\Xi(I-Q)^{-1}H^{-}\right)(w_i,z_j) = S^{(2)}(w_i,z_j).$$

The result follows on taking the determinant. $\square$

We may next describe the generating form:

Theorem 5.6 *The generating form for the rank two free fermion VOSA is given by*

$$\mathcal{G}_n^{(2)}\left[\begin{array}{c} f \\ g \end{array}\right](w_1,\ldots,w_n,z_1,\ldots,z_n) \quad = \quad Z^{(2)}\left[\begin{array}{c} f \\ g \end{array}\right](\tau_1,\tau_2,\epsilon)\ \det S^{(2)}. \quad (56)$$

Remark 5.7 *Relative to the genus two partition function, the normalized 2-point for ψ^{+} and ψ^{-} is given by the Szegö kernel and more generally, the $2n$-point function is given by a Szegö kernel determinant. This agrees with the assumed form of the higher genus fermion $2n$-point function in [R] or as found by string theory methods using a Schottky parameterisation in [DVPFHLS].*

In order to prove Theorem 5.6 we require an extension of Proposition 5.2.

Proposition 5.8 *Let R and $J_p = \left(\begin{array}{cc} 0 & 0 \\ 0 & I_{N-p} \end{array}\right)$ be $N\times N$ matrices where I_{N-p} is the identity $(N-p)\times(N-p)$ matrix for $0\le p\le N$. Then*

$$\det(J_p+R) = \sum_{n=0}^{N-p}\sum_{\mathbf{j}_p}\det R(\mathbf{j}_p,\mathbf{j}_p), \quad (57)$$

where the inner sum runs over all N-subindices of length $n+p$ of the form $\mathbf{j}_p=(1,\ldots,p,j_1,\ldots,j_n)$.

Proof. The proof follows along the same lines as Proposition 5.2 where here we consider $\det(J_p + xR) = x^p \sum_{\sigma\in S_N} \epsilon_\sigma \prod_{i=1}^{p} R_{i\sigma(i)} \prod_{i=p+1}^{N}(\delta_{i\sigma(i)} + xR_{i\sigma(i)})$. Then $\det(J_p + xR) = x^p \sum\limits_{0\leq n\leq N-p} a_n x^n$ for

$$a_n = \sum_{\mathbf{j_p}} \sum_{\rho} \epsilon_\rho \prod_{i=1}^{p} R_{i\rho(i)} \prod_{r=1}^{n} R_{j_r\rho(j_r)} = \sum_{\mathbf{j_p}} \det R(\mathbf{j}_p\,, \mathbf{j}_p),$$

where ρ is a permutation of $\mathbf{j}_p = (1, \ldots, p, j_1, \ldots, j_n)$. The result then follows as before. $\square$

Corollary 5.9 *Let A, B be $M \times M$ matrices and let U be a $p \times M$ matrix and W be a $M \times p$ matrix with $p \leq M$. Define the $(p + 2M) \times (p + 2M)$ block matrix*

$$R = \begin{bmatrix} 0 & 0 & U \\ 0 & 0 & tA \\ W & t^{-1}B & 0 \end{bmatrix},$$

where t is a non-zero scalar parameter. Then $\det(J_p + R)$ is independent of t and is given by

$$\det(J_p + R) = \sum_{m=p}^{M} (-1)^m \sum_{\mathbf{k},\mathbf{l}} \det U_A(\mathbf{k}, \mathbf{l}) \det W_B(\mathbf{l}, \mathbf{k}), \tag{58}$$

where $\mathbf{k}$ and $\mathbf{l}$ are M-subindices of length $m - p$ and m respectively. $U_A(\mathbf{k}, \mathbf{l})$ and $W_B(\mathbf{l}, \mathbf{k})$ are the $m \times m$ submatrices with components

$$\begin{aligned} U_A(\mathbf{k}, \mathbf{l})_{ij} &= \begin{cases} U_{i\,l_j} & i = 1, \ldots, p \\ A_{k_{i-p}\,l_j} & i = p + 1, \ldots, m \end{cases} \\ W_B(\mathbf{l}, \mathbf{k})_{ij} &= \begin{cases} W_{l_i\,j} & j = 1, \ldots, p \\ B_{l_i\,k_{j-p}} & j = p + 1, \ldots, m. \end{cases} \end{aligned}$$

Proof. $\det(J_p + R)$ is t invariant since

$$(J_p + R)|_{t=1} = \mathrm{diag}(I_p, t^{-1}I_M, I_M)\ (J_p + R)\ \mathrm{diag}(I_p, tI_M, I_M),$$

for identity matrices I_p and I_M. t invariance and the off-diagonal structure of R imply that the inner sum in (57) is taken over $(p + 2M)$-subindices of length $2m$ described by

$$\mathbf{j}_p = (1, \ldots, p, p + k_1, \ldots, k_{m-p}, p + M + l_1, \ldots, p + M + l_m),$$

for $1 \leq k_1 < \ldots < k_{m-p} \leq M$ and $1 \leq l_1 < \ldots < l_m \leq M$ i.e. $\mathbf{k}$ and $\mathbf{l}$ are M-subindices of length $m-p$ and m respectively. Hence

$$\det(J_p + R) = \sum_{m=p}^{M} \sum_{\mathbf{k},\mathbf{l}} \det \begin{bmatrix} 0 & U_A(\mathbf{k},\mathbf{l}) \\ W_B(\mathbf{l},\mathbf{k}) & 0 \end{bmatrix}.$$

The result then follows. $\square$

Proof of Theorem 5.6. Following Proposition 4.2 we may evaluate $\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix}$ by inserting the quasi-primary vectors $\psi^{\pm}$ in any way on the disconnected union of punctured tori $\widehat{\Sigma}^{(1,1)}$. In particular, we choose ψ^+ at $w_i \in \widehat{\Sigma}_1^{(1)}$ and ψ^- at $z_i \in \widehat{\Sigma}_2^{(1)}$ for $i = 1, \ldots, n$. Thus, reordering operators and using (38) and (41) we find

$$\begin{aligned}
&\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} = \mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (w_1, \ldots, w_n, z_1, \ldots, z_n) \\
&= (-1)^{n(n-1)/2}(-1)^n \sum_{u \in V} Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (Y[\psi^+, w_1] \ldots Y[\psi^+, w_n]u, \tau_1) \\
&\quad \cdot Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (Y[\psi^-, z_1] \ldots Y[\psi^-, z_n]\bar{u}, \tau_2) \prod_{i=1}^{n} dw_i^{\frac{1}{2}} dz_i^{\frac{1}{2}}.
\end{aligned} \tag{59}$$

Choose the Fock basis $\{\Psi[\mathbf{k},\mathbf{l}]\}$ with $1 \leq k_1 < \ldots < k_s$ and $1 \leq l_1 < \ldots < l_m$ of (46) with square bracket dual (53). The corresponding torus one point functions are non-vanishing for $n + s = m$ with parity $p_\Psi = n \mod 2$ from (36). Expanding (45) using (68) one finds (see Proposition 15 of ref. [MTZ] for details)

$$\begin{aligned}
\frac{Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (Y[\psi^+, w_1] \ldots \Psi[\mathbf{k},\mathbf{l}], \tau_1)}{Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} (\tau_1)} &= (-1)^{m(m-1)/2} \det E_1(\mathbf{k},\mathbf{l}), \\
\frac{Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (Y[\psi^-, z_1] \ldots \overline{\Psi}[\mathbf{k},\mathbf{l}], \tau_2)}{Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} (\tau_2)} &= (-1)^{m(m+1)/2}(-\xi)^{p_\Psi} \epsilon^{wt[\Psi]} \det E_2(\mathbf{l},\mathbf{k}),
\end{aligned}$$

for $m \times m$ matrices with components

$$\begin{aligned}
(E_1(\mathbf{k},\mathbf{l}))_{ij} &= \left\{ \begin{array}{ll} D\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(1,l_j,\tau_1,w_i) & i=1,\ldots,n \\ C\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(k_i,l_j,\tau_1) & i=n+1,\ldots,m \end{array}\right. \\
(E_2(\mathbf{l},\mathbf{k}))_{ij} &= \left\{ \begin{array}{ll} D\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(l_i,1,\tau_2,-z_j) & j=1,\ldots,n \\ C\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}(l_i,k_j,\tau_2) & j=n+1,\ldots,m, \end{array}\right.
\end{aligned}$$

for $C\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}, D\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}$ of (69) and (70). Since $p_\Psi = n \mod 2$ one finds $\xi^{p_\Psi} = (-1)^{n(n+1)/2}\xi^n$ so that altogether

$$\frac{\mathcal{G}_n^{(2)}\begin{bmatrix}f\\ g\end{bmatrix}}{Z^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}Z^{(1)}\begin{bmatrix}f_2\\ g_2\end{bmatrix}} = \sum_{m\geq 0}(-1)^m\sum_{\mathbf{k},\mathbf{l}}\epsilon^{wt[\Psi]}\xi^n \det E_1(\mathbf{k},\mathbf{l})\det E_2(\mathbf{l},\mathbf{k}).$$

But $wt[\Psi] = \sum_{i=1}^{m-n}(k_i-\frac{1}{2}) + \sum_{j=1}^{m}(l_j-\frac{1}{2})$ so that factors of $\epsilon^{\frac{1}{2}l_j-\frac{1}{4}}$ and $\epsilon^{\frac{1}{2}k_i-\frac{1}{4}}$ may be absorbed into the rows and columns of the above determinants. Furthermore, factors of $dw_i^{\frac{1}{2}}$ and $dz_i^{\frac{1}{2}}$ can be absorbed into the first n rows and columns of $\det E_1$ and $\det E_2$ repectively. Lastly, a factor of ξ can be absorbed into the first n rows of $\det E_1(\mathbf{k},\mathbf{l})$ to find

$$\frac{\mathcal{G}_n^{(2)}\begin{bmatrix}f\\ g\end{bmatrix}}{Z^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}Z^{(1)}\begin{bmatrix}f_2\\ g_2\end{bmatrix}} = \sum_{m\geq 0}(-1)^m\sum_{\mathbf{k},\mathbf{l}}\det G_1(\mathbf{k},\mathbf{l})\det G_2(\mathbf{l},\mathbf{k}),$$

for $m \times m$ matrices

$$\begin{aligned}
(G_1(\mathbf{k},\mathbf{l}))_{ij} &= \left\{ \begin{array}{ll} \xi h_1(l_j,\tau_1,w_i) & i=1,\ldots,n \\ F_1(k_i,l_j,\tau_1) & i=n+1,\ldots,m \end{array}\right. \\
(G_2(\mathbf{l},\mathbf{k}))_{ij} &= \left\{ \begin{array}{ll} \overline{h}_2(l_i,\tau_2,z_j) & j=1,\ldots,n \\ F_2(l_i,k_j,\tau_2) & j=n+1,\ldots,m, \end{array}\right.
\end{aligned}$$

with F_a, h_a of (8) and (9). Finally, let A, B, U and W denote the finite matrices found by truncating F_1, F_2, $h_1(w_i)$ and $\overline{h}_2(z_j)$ respectively to an arbitrary order in ϵ. Thus applying Corollary 5.9 to A, B, U and W with $n = p$ and $t = -\xi$ it follows that as a formal series in ϵ we have

$$\frac{\mathcal{G}_n^{(2)}\begin{bmatrix}f\\ g\end{bmatrix}}{Z^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}Z^{(1)}\begin{bmatrix}f_2\\ g_2\end{bmatrix}} = \det\begin{bmatrix} 0 & H^+\Xi \\ H^- & I-Q \end{bmatrix},$$

where $H^+\Xi = (0, h_1(l_j, w_i))$ and $H^- = \left(0, (\overline{h}_2(l_i, z_j)\right)^T$. Finally, using Proposition 5.5 for $w_i \in \widehat{\Sigma}_1^{(1)}$ and $z_i \in \widehat{\Sigma}_2^{(1)}$ we find a convergent series in ϵ

$$\frac{\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix}}{Z^{(1)} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} Z^{(1)} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix}} = \det S^{(2)} \det(I - Q),$$

and hence the Theorem follows on applying Theorem 5.1. □

Remark 5.10 *The other choices of the insertion points for $\psi^\pm$ give rise to corresponding $H^\pm$ and $S^{(1,1)}$ terms in Proposition 5.5 leading to the same result* (56)*.*

As an illustration of the use of the generating form we compute the one-point function for the Virasoro vector $\widetilde{\omega} = \frac{1}{2}(\psi^+[-2]\psi^- + \psi^-[-2]\psi^+)$. Let $w, z \in \widehat{\Sigma}_1^{(1)}$ and consider the generating form $\mathcal{G}_1^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (w, z) = S^{(2)}(w, z) Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix}$ (where we suppress the τ_1, τ_2, ϵ dependence). Using (37) we find

$$\begin{aligned} &\partial_w Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\psi^+, w; \psi^-, z) = \partial_w Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (Y[\psi^+, w - z]\psi^-, z) \\ &= -\frac{1}{(w-z)^2} Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} + Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\psi^+[-2]\psi^-, z) + \ldots \end{aligned}$$

and similarly for ∂_z. Letting $S^{(2)}(w, z) = K^{(2)}(w, z) dw^{\frac{1}{2}} dz^{\frac{1}{2}}$ it follows that the Virasoro 1-point form is given by

$$\mathcal{F}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\widetilde{\omega}, z) = dz^2 \lim_{w \to z} \left[\frac{1}{2} \left(\partial_w - \partial_z\right) K^{(2)}(w, z) + \frac{1}{(w-z)^2} \right] Z^{(2)} \begin{bmatrix} f \\ g \end{bmatrix}. \tag{60}$$

An alternative expression for this is shown below in Proposition 5.14.

5.4 Bosonization and a Genus Two Jacobi Product Identity

Consider the decomposition of the rank two fermion VOSA into irreducible modules $M \otimes e^m$ modules (for $m \in \mathbb{Z}$) of the Heisenberg subVOA M generated by the Heisenberg state a. The genus one partition function (43) can thus also be expressed as (e.g. [Ka], [MTZ])

$$Z^{(1)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau) = \frac{e^{-2\pi i\alpha\beta}}{\eta(\tau)}\vartheta^{(1)}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau),$$

for theta function (1) and Dedekind eta-function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. All n-point functions can be similarly computed in terms of Heisenberg module traces [MTZ] so that the genus two partition function (49) can also be computed in this bosonized formalism to obtain [MT1]

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau_1,\tau_2,\epsilon) = e^{-2\pi i\alpha\cdot\beta} Z_M^{(2)}(\tau_1,\tau_2,\epsilon)\ \vartheta^{(2)}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\Omega^{(2)}), \qquad (61)$$

for genus two Riemann theta function with characteristics $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2)$ and where

$$Z_M^{(2)}(\tau_1,\tau_2,\epsilon) = \frac{1}{\eta(\tau_1)\eta(\tau_2)\det(I - A_1A_2)^{1/2}},$$

is the genus two partition function for the rank one free Heisenberg VOA M. A_a for $a = 1,2$ is an infinite matrix with components indexed by $k,l \geq 1$ [MT1], [MT2]

$$A_a(k,l,\tau_a,\epsilon) = \epsilon^{(k+l)/2}\frac{(-1)^{k+1}(k+l-1)!}{\sqrt{kl}(k-1)!(l-1)!}E_{k+l}(\tau_a),$$

for standard Eisenstein series $E_n(\tau) = E_n\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\tau)$. Comparing with Theorem 5.1 we find a new identity relating the genus two theta function to determinants on $\mathcal{D}^\epsilon$ as follows

Theorem 5.11

$$\frac{\vartheta^{(2)}\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right](\Omega^{(2)})}{\vartheta^{(1)}\left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix}\right](\tau_1)\ \vartheta^{(1)}\left[\begin{smallmatrix} \alpha_2 \\ \beta_2 \end{smallmatrix}\right](\tau_2)} = \det(I - A_1A_2)^{1/2}\ \det(I - Q).$$

$\square$

It is shown in ref. [MT1] that $\det(I - A_1A_2)$ can be expressed as an infinite product as follows. Let $\sigma_{2n} = (k_1 \ldots k_{2n})$ denote a cycle permutation on $2n$ positive integers. We may canonically associate each σ with an oriented graph N consisting of $2n$ valence 2 nodes labelled by $k_1, \ldots, k_{2n}$. N is said to be *rotationless* when it admits no non-trivial rotations (a rotation being an orientation-preserving automorphism of N which preserves node labels). Lastly, we define a weight function ζ_A on N by

$$\zeta_A(N) = \prod_{i=1}^{n} A_1(k_{2i-1}, k_{2i})A_2(k_{2i}, k_{2i+1}),$$

where $k_{2n+1} \equiv k_1$. We then find [MT1]

$$\det(I - A_1A_2) = \prod_{N\in\mathcal{R}} (1 - \zeta_A(N)),$$

where $\mathcal{R}$ denotes the set of rotationless oriented cycle graphs with an even number of nodes. This expansion can be similarly applied to $\det(I - Q) = \det(I - F_1F_2)$ with corresponding weight function ζ_F. Hence Theorem 5.11 implies a genus two Jacobi product-like formula

Proposition 5.12

$$\frac{\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\Omega^{(2)})}{\vartheta^{(1)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\tau_1)\,\vartheta^{(1)}\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}(\tau_2)} = \prod_{N\in\mathcal{R}} (1 - \zeta_A(N))^{1/2}\ (1 - \zeta_F(N)).$$

□

Remark 5.13 *The bosonization procedure can be applied to obtain an alternative expression for the genus two generating form of Theorem 5.6 to obtain Fay's tresecant identity relating* $\det S^{(2)}$ *to a product of prime forms [TZ4].*

5.5 A Genus Two Ward Indentity

We may also recompute the 1-point function (60) for the Virasoro vector $\tilde{\omega} = \frac{1}{2}a[-1]a$ in the bosonized version of the rank two free fermion VOSA. We introduce the differential operator [F1], [U], [MT1]

$$\mathcal{D} = \frac{1}{2\pi i} \sum_{1\le i\le j\le 2} \nu_i^{(2)}(x)\nu_j^{(2)}(x)\frac{\partial}{\partial\Omega_{ij}^{(2)}}, \tag{62}$$

for holomorphic 1-forms $\nu_i^{(2)}$. We also recall the genus two projective connection $s^{(2)}(x)$ of Appendix 7.1. Using (61) and results of [MT1] we find

Proposition 5.14 *The Virasoro 1-point form for the rank two fermion VOSA satisfies a genus two Ward identity*

$$\mathcal{F}^{(2)}(\tilde{\omega}, x; \tau_1, \tau_2, \epsilon) = e^{-2\pi i\alpha\cdot\beta} Z_M^{(2)}(\tau_1, \tau_2, \epsilon)\big(\mathcal{D} + \frac{1}{12}s^{(2)}(x)\big)\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\Omega^{(2)}). \tag{63}$$

The Ward identity (63) is similar to previous results in physics and mathematics e.g. [EO], [KNTY].

6 Modular Invariance Properties

We next consider the automorphic properties of the genus two partition function for the rank two fermion VOSA. In [MTZ] we define the action of $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ on a genus one orbifold partition function $Z^{(1)}\begin{bmatrix} f\\ g\end{bmatrix}(\tau)$ as follows:

$$Z^{(1)}\begin{bmatrix} f\\ g\end{bmatrix}\Bigg|\gamma(\tau) = Z^{(1)}\left(\gamma.\begin{bmatrix} f\\ g\end{bmatrix}\right)(\gamma.\tau), \tag{64}$$

where $\gamma.\tau = \frac{a\tau+b}{c\tau+d}$ and $\gamma.\begin{bmatrix} f\\ g\end{bmatrix} = \begin{bmatrix} f^a g^b\\ f^c g^d\end{bmatrix}$. For the rank two fermion VOSA we find modular invariance with [MTZ]

$$Z^{(1)}\begin{bmatrix} f\\ g\end{bmatrix}\Bigg|\gamma(\tau) = e_\gamma^{(1)}\begin{bmatrix} f\\ g\end{bmatrix} Z^{(1)}\begin{bmatrix} f\\ g\end{bmatrix}(\tau), \tag{65}$$

where $e_\gamma^{(1)}\begin{bmatrix} f\\ g\end{bmatrix} \in U(1)$ is a specific multiplier system.[4]

In Theorem 5.1 we showed that the genus two partition function is holomorphic on the domain $\mathcal{D}^\epsilon$ of (5). $\mathcal{D}^\epsilon$ is preserved under the action of $G \simeq (SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})) \rtimes \mathbb{Z}_2$, the direct product of the left and right

[4]Note a notational change for the multiplier from that of ref. [MTZ]

torus modular groups, which are interchanged upon conjugation by an involution β defined as follows [MT2]

$$\begin{aligned}
\gamma_1(\tau_1,\tau_2,\epsilon) &= \left(\gamma_1.\tau_1,\tau_2,\frac{\epsilon}{c_1\tau_1+d_1}\right),\\
\gamma_2(\tau_1,\tau_2,\epsilon) &= \left(\tau_1,\gamma_2.\tau_2,\frac{\epsilon}{c_2\tau_2+d_2}\right),\\
\beta(\tau_1,\tau_2,\epsilon) &= (\tau_2,\tau_1,\epsilon),
\end{aligned}$$

for $(\gamma_1,\gamma_2)\in SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ with $\gamma_i=\begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}$. There is a natural injection $G\to Sp(4,\mathbb{Z})$ in which the two $SL(2,\mathbb{Z})$ subgroups are mapped to

$$\Gamma_1=\left\{\begin{bmatrix} a_1 & 0 & b_1 & 0\\ 0 & 1 & 0 & 0\\ c_1 & 0 & d_1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right\},\quad \Gamma_2=\left\{\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & a_2 & 0 & b_2\\ 0 & 0 & 1 & 0\\ 0 & c_2 & 0 & d_2\end{bmatrix}\right\},$$

and the involution is mapped to

$$\beta=\begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

In a similar way to (64) we define an action of $\gamma\in G$ on the genus two orbifold twisted partition function (40) by

$$Z^{(2)}\begin{bmatrix} f\\ g\end{bmatrix}\Bigg|\gamma(\tau_1,\tau_2,\epsilon)=Z^{(2)}\left(\gamma.\begin{bmatrix} f\\ g\end{bmatrix}\right)\gamma.\left(\tau_1,\tau_2,\epsilon\right),$$

generated by $\gamma_i\in\Gamma_i$ and β with

$$\gamma_1.\begin{bmatrix} f_1\\ f_2\\ g_1\\ g_2\end{bmatrix}=\begin{bmatrix} f_1^{a_1}g_1^{b_1}\\ f_2\\ f_1^{c_1}g_1^{d_1}\\ g_2\end{bmatrix},\quad \gamma_2.\begin{bmatrix} f_1\\ f_2\\ g_1\\ g_2\end{bmatrix}=\begin{bmatrix} f_1\\ f_2^{a_2}g_2^{b_2}\\ g_1\\ f_2^{c_2}g_2^{d_2}\end{bmatrix},\quad \beta.\begin{bmatrix} f_1\\ f_2\\ g_1\\ g_2\end{bmatrix}=\begin{bmatrix} f_2\\ f_1\\ g_2\\ g_1\end{bmatrix}.$$

We may now describe the modular invariance of the genus two partition function for the rank two VOSA of Theorem 5.1 under the action of G. Define a genus two multiplier system $e_{\gamma}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix} \in U(1)$ for $\gamma \in G$ in terms of the genus one multiplier system as follows

$$e_{\gamma_i}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix} = e_{\gamma_i}^{(1)}\begin{bmatrix} f_i \\ g_i \end{bmatrix}, \quad e_{\beta}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix} = 1, \tag{66}$$

for G generators $\gamma_i \in \Gamma_i$ and β. We then find

Theorem 6.1 *The genus two orbifold partition function for the rank two VOSA is modular invariant with respect to $G = (SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})) \rtimes \mathbb{Z}_2$ with multiplier system* (66) *i.e.*

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}\Bigg|\gamma\,(\tau_1,\tau_2,\epsilon) = e_{\gamma}^{(2)}\begin{bmatrix} f \\ g \end{bmatrix} Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau_1,\tau_2,\epsilon)\,.$$

Proof. We recall from Theorem 5.1 that the genus two partition function can be expressed as

$$Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}(\tau_1,\tau_2,\epsilon) = \sum_{\mathbf{k},\mathbf{l}} (-1)^m\, \epsilon^{wt[\Psi]} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\Psi[\mathbf{k},\mathbf{l}],\tau_1) Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\Psi[\mathbf{l},\mathbf{k}],\tau_2),$$

for $1 \leq k_1 < \ldots < k_m$ and $1 \leq l_1 < \ldots < l_m$ with Fock basis $\{\Psi[\mathbf{k},\mathbf{l}]\}$ of square bracket weight $wt[\Psi] = \sum_{i=1}^{m}(k_i + l_i - 1)$. Let us consider the action of $\gamma_1 \in \Gamma_1$. It follows from (71) (see also Proposition 21. of [MTZ]) that

$$Z^{(1)}\left(\gamma_1.\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\right)(\Psi[\mathbf{k},\mathbf{l}],\gamma_1.\tau_1) = e_{\gamma_1}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(c_1\tau_1+d_1)^{wt[\Psi]} Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\Psi[\mathbf{k},\mathbf{l}],\tau_1).$$

Hence from (66) we find

$$\begin{aligned} Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}\Bigg|\gamma_1 &= e_{\gamma_1}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \sum_{\mathbf{k},\mathbf{l}} (-1)^m \left(\frac{\epsilon}{c_1\tau_1+d_1}\right)^{wt[\Psi]} (c_1\tau_1+d_1)^{wt[\Psi]} \\ &\quad \cdot Z^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\Psi[\mathbf{k},\mathbf{l}],\tau_1) Z^{(1)}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}(\Psi[\mathbf{l},\mathbf{k}],\tau_2) \\ &= e_{\gamma_1}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} Z^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}. \end{aligned}$$

A similar result holds for $\gamma_2 \in \Gamma_2$ whereas invariance under β is obvious. The result follows. $\square$

Remark 6.2 *Modular invariance can also inferred from Theorem 5.11 using modular properties of the Riemann theta function together with those for the Heisenberg genus two partition function described in [MT1].*

Finally, we can also obtain modular invariance for the generating form $\mathcal{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}$ described in Theorem 5.6. In particular, as is described in [TZ1], the genus two Szegö kernel of (12) is invariant under the action of G. Hence it follows that

Theorem 6.3 $\mathcal{G}_n^{(2)}\begin{bmatrix} f \\ g \end{bmatrix}$ *is modular invariant with respect to* G *with multiplier system* (66). $\square$

7 Appendix

7.1 Some Riemann Surface Theory

Consider a compact Riemann surface $\Sigma^{(g)}$ of genus g with canonical homology cycle basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. In general there exists g holomorphic 1-forms $\nu_i^{(g)}$, $i = 1, \ldots, g$ which we may normalize by e.g. [FK]

$$\oint_{a_i} \nu_j^{(g)} = 2\pi i \delta_{ij}.$$

The genus g period matrix $\Omega^{(g)}$ is defined by

$$\Omega_{ij}^{(g)} = \frac{1}{2\pi i} \oint_{b_i} \nu_j^{(g)},$$

for $i, j = 1, \ldots, g$. $\Omega^{(g)}$ is symmetric with positive imaginary part i.e. $\Omega^{(g)} \in \mathbb{H}_g$, the Siegel upper half plane. It is useful to introduce the *normalized differential of the second kind* defined by [Sp], [M], [F1]:

$$\omega^{(g)}(x, y) \sim \frac{dxdy}{(x-y)^2} \quad \text{for } x \sim y,$$

for local coordinates x, y, with normalization $\int_{a_i} \omega^{(g)}(x, \cdot) = 0$ for $i = 1, \ldots g$. Using the Riemann bilinear relations, one finds that

$$\nu_i^{(g)}(x) = \oint_{b_i} \omega^{(g)}(x, \cdot).$$

The projective connection $s^{(g)}$ is defined by [G]

$$s^{(g)}(x) = 6 \lim_{x \to y} \left(\omega^{(g)}(x,y) - \frac{dxdy}{(x-y)^2} \right).$$

$s^{(g)}(x)$ is not a global 2-form but rather transforms under a general conformal transformation $x \to \phi(x)$ as

$$s^{(g)}(\phi(x)) = s^{(g)}(x) - \{\phi; x\} dx^2,$$

where $\{\phi; x\} = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$ is the Schwarzian derivative.

There exists a (nonsingular and odd) character $\left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]$ such that [M], [F1]

$$\vartheta^{(g)} \begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0) = 0, \qquad \partial_{z_i}\vartheta^{(g)} \begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0) \neq 0,$$

for the theta function with real characteristics (1). Define

$$\zeta(x) = \sum_{i=1}^{g} \partial_{z_i}\vartheta^{(g)} \begin{bmatrix}\gamma\\ \delta\end{bmatrix}(0)\nu_i^{(g)}(x),$$

a holomorphic 1-form, and let $\zeta(x)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\widetilde{\Sigma}$ of Σ. We also refer to $\zeta(x)^{\frac{1}{2}}$ as a (double-valued) $\frac{1}{2}$-form on Σ. We define the prime form $E(x,y)$ by

$$E(x,y) = \frac{\vartheta^{(g)} \left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]\left(\int_y^x \nu^{(g)}\right)}{\zeta(x)^{\frac{1}{2}}\zeta(y)^{\frac{1}{2}}} \sim (x-y)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}} \quad \text{for } x \sim y,$$

where $\int_y^x \nu^{(g)} = (\int_y^x \nu_i^{(g)}) \in \mathbb{C}^g$. $E(x,y) = -E(y,x)$ is a holomorphic differential form of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\widetilde{\Sigma} \times \widetilde{\Sigma}$. $E(x,y)$ has multipliers along the a_i and b_j cycles in x given by 1 and $e^{-i\pi\Omega_{jj}^{(g)} - \int_y^x \nu_j^{(g)}}$ respectively [F1].

7.2 Twisted Elliptic Functions

Let $(\theta, \phi) \in U(1) \times U(1)$ denote a pair of modulus one complex parameters with $\phi = \exp(2\pi i\lambda)$ for $0 \leq \lambda < 1$. For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we define 'twisted' Weierstrass functions for $k \geq 1$ as follows [MTZ]

$$P_k \begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau) = \frac{(-1)^k}{(k-1)!} \sum_{n \in \mathbb{Z}+\lambda}' \frac{n^{k-1}q_z^n}{1-\theta^{-1}q^n},$$

for $q = q_{2\pi i\tau}$ where $\sum'$ means we omit $n = 0$ if $(\theta, \phi) = (1, 1)$. We have a Laurant expansion

$$P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = \frac{1}{z} - \sum_{n \geq 1} E_n \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) z^{n-1}, \tag{67}$$

in terms of twisted Eisenstein series for $n \geq 1$, defined by

$$\begin{aligned} E_n \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) &= -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!} {\sum_{r \geq 0}}' \frac{(r+\lambda)^{n-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} \\ &\quad + \frac{(-1)^n}{(n-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{n-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}, \end{aligned}$$

where $\sum'$ means we omit $r = 0$ if $(\theta, \phi) = (1, 1)$ and where $B_n(\lambda)$ is the Bernoulli polynomial defined by

$$\frac{q_z^\lambda}{q_z - 1} = \frac{1}{z} + \sum_{n \geq 1} \frac{B_n(\lambda)}{n!} z^{n-1}.$$

We also have Laurant expansions

$$\begin{aligned} P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x - y, \tau) &= \frac{1}{x - y} + \sum_{k,l \geq 1} C \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l)\, x^{k-1}\, y^{l-1}, \\ P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z + x - y, \tau) &= \sum_{k,l \geq 1} D \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l, z)\, x^{k-1}\, y^{l-1}, \end{aligned} \tag{68}$$

where for $k, l \geq 1$ we define

$$C \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l, \tau) = (-1)^l \binom{k+l-2}{k-1} E_{k+l-1} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau), \tag{69}$$

$$D \begin{bmatrix} \theta \\ \phi \end{bmatrix} (k, l, \tau, z) = (-1)^{k+1} \binom{k+l-2}{k-1} P_{k+l-1} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau). \tag{70}$$

In [MTZ] we show that for $(\theta, \phi) \neq (1, 1)$, $E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix}$ is a twisted modular form of weight k i.e.

$$E_k \left(\gamma . \begin{bmatrix} \theta \\ \phi \end{bmatrix} \right) (\gamma . \tau) = (c\tau + d)^k E_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau), \tag{71}$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ we have $\gamma . \tau = \frac{a\tau + b}{c\tau + d}$ and $\gamma . \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \theta^a \phi^b \\ \theta^c \phi^d \end{bmatrix}$.

References

[B] Borcherds, R.E.: Vertex algebras, Kac-Moody algebras and the Monster, Proc.Nat.Acad.Sc. **83** (1986) 3068–3071.

[DLM] Dong, C., Li, H. and Mason, G.: Modular-invariance of trace functions in orbifold theory and generalized moonshine. Comm.Math.Phys. **214** (2000) 1–56.

[DP] D'Hoker, E. and Phong, D.H., The geometry of string perturbation theory, Rev.Mod.Phys. **60** (1988) 917–1065.

[DVFHLS] di Vecchia, P., Hornfeck, K., Frau, M., Lerda, A. and Sciuto, S.: N-string, g-loop vertex for the fermionic string, Phys.Lett. **B211** (1988) 301–307.

[DVPFHLS] di Vecchia, P., Pezzella, F., Frau, M., Hornfeck, K., Lerda, A. and Sciuto, S.: N-point g-loop vertex for a free fermionic theory with arbitrary spin, Nucl.Phys. **B333** (1990) 635–700.

[DZ] Dong, C. and Zhao, Z.: Modularity in orbifold theory for vertex operator superalgebras. Comm.Math.Phys. **260** (2005) 227–256.

[EO] Eguchi, T. and Ooguri, H.: Conformal and current algebras on a general Riemann surface, Nucl.Phys. **B282** (1987), 308–328

[F1] Fay, J.D.: Theta functions on Riemann surfaces. Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, (Berlin-New York, 1973).

[F2] Fay, J.D.: Kernel functions, analytic torsion, and moduli spaces. Mem.Amer.Math.Soc. **96** (1992) no. 464.

[FFR] Feingold, A.J., Frenkel, I.B. and Reis, J.F.X: Spinor construction of vertex operator algebras and $E_8^{(1)}$, Contemp.Math. **121** (1991).

[FHL] Frenkel, I., Huang, Y. and Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules, Mem.Amer.Math.Soc. **104** (1993).

[FK] Farkas, H.M. and Kra, I.: *Riemann Surfaces*, Springer-Verlag (New York, 1980).

[FLM] Frenkel, I., Lepowsky, J. and Meurman A.: *Vertex operator algebras and the Monster*, Academic Press, (New York, 1988).

[FS] Freidan, D. and Shenker, S.: The analytic geometry of two dimensional conformal field theory, Nucl.Phys. **B281** (1987), 509–545.

[G] Gunning, R.C.: *Lectures on Riemann surfaces* , Princeton Univ. Press, (Princeton, 1966).

[HS] Hawley, N.S. and Schiffer, M.: Half-order differentials on Riemann surfaces. ActaMath. **115** (1966) 199–236.

[Ka] Kac, V.: *Vertex Operator Algebras for Beginners*, University Lecture Series, Vol. 10, (AMS 1998).

[Kn] Knizhnik, V.G.: Multiloop amplitudes in the theory of quantum strings and complex geometry, Sov.Phys.Usp. **32** (1989) 945–971.

[KNTY] Kawamoto, N., Namikawa, Y., Tsuchiya, A. and Yamada, Y.: Geometric realization of conformal field theory on Riemann surfaces, Commun.Math.Phys. **116**(1988), 247–308.

[L] Li, H.: Symmetric invariant bilinear forms on vertex operator algebras, J.Pure.Appl.Alg. **96** (1994) 2790–297.

[MN] Matsuo, A. and Nagatomo, K,: *Axioms for a vertex algebra and the locality of quantum fields*, Math.Soc.Jap.Mem. **4** (1999).

[MT1] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces I, arXiv:0912.0117. To appear in Commun.Math.Phys.

[MT2] Mason, G. and Tuite, M.P.: On genus two Riemann surfaces formed from sewn tori, Commun.Math.Phys. **270** (2007) 587–634.

[MT3] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces II, to appear.

[MT4] Mason, G. and Tuite, M.P.: Torus chiral n-point functions for free boson and lattice vertex operator algebras, Commun.Math.Phys. **235** (2003) 47–68.

[MT5] Mason, G. and Tuite, M.P.: Vertex operators and modular forms, *A Window into Zeta and Modular Physics* eds. K. Kirsten and F. Williams, MSRI Publications **57** 183–278, Cambridge University Press, (Cambridge, 2010).

[MTZ] Mason, G., Tuite, M.P. and Zuevsky, A.: Torus n-point functions for $\mathbb{R}$-graded vertex operator superalgebras and continuous fermion orbifolds, Commun.Math.Phys. **283** (2008) 305–342.

[M] Mumford, D.: *Tata Lectures on Theta I and II*, Birkhäuser, (Boston, 1983).

[R] Raina, A.K.: Fay's trisecant identity and conformal field theory, Comm.Math.Phys. **122** (1989) 625–641.

[RS] Raina, A.K. and Sen, S.: Grassmannians, multiplicative Ward identities and theta-function identities, Phys.Lett. **B203** (1988) 256–262.

[Sche] Scheithauer, N.: Vertex algebras, Lie algebras and superstrings, J.Alg. **200** (1998) 363–403.

[Schi] Schiffer, M.: Half-order differentials on Riemann surfaces. SIAMJ.Appl.Math. **14** (1966) 922–934.

[Sp] Springer, G.: *Introduction to Riemann Surfaces*. Addison-Wesley (Reading, Mass. 1957).

[T] Tuite, M.P: Genus two meromorphic conformal field theory, CRM Proceedings and Lecture Notes **30** (2001) 231-251.

[TZ1] Tuite, M.P. and Zuevsky A.: The Szegö Kernel on a Sewn Riemann Surface. arXiv:1002.4114v1[math.QA], Submitted to Commun.Math.Phys.,2010.

[TZ2] Tuite, M.P. and Zuevsky, A.: Shifting, Twisting, and Intertwining Heisenberg Modules. To appear.

[TZ3] Tuite, M.P. and Zuevsky, A.: Genus two partition and correlation functions for fermionic vertex operator superalgebras II, to appear.

[TZ4] Tuite, M.P. and Zuevsky, A.: To appear.

[TUY] Tsuchiya, A, Ueno, K. and Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries, Adv.Stud.Pure.Math. **19** (1989) 459–566.

[U] Ueno, K.: Introduction to conformal field theory with gauge symmetries, *Geometry and Physics - Proceedings of the conference at Aarhus Univeristy, Aaarhus, Denmark*, Marcel Dekker, (New York 1997).

[Z1] Zhu, Y.: Modular invariance of characters of vertex operator algebras. J.Amer.Math.Soc. **9** (1996) 237–302.

[Z2] Zhu, Y.: Global vertex operators on Riemann surfaces, Commun.Math.Phys. **165** (1994), 485–531.