ON THE SPECTRUM OF HECKE TYPE OPERATORS RELATED TO SOME FRACTAL GROUPS

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ABSTRACT. We give the first example of a connected 4-regular graph whose Laplace operator's spectrum is a Cantor set, as well as several other computations of spectra following a common "finite approximation" method. These spectra are simple transforms of the Julia sets associated to some quadratic maps. The graphs involved are Schreier graphs of fractal groups of intermediate growth, and are also "substitutional graphs". We also formulate our results in terms of Hecke type operators related to some irreducible quasi-regular representations of fractal groups and in terms of the Markovian operator associated to noncommutative dynamical systems via which these fractal groups were originally defined in [Gri80].

In the computations we performed, the self-similarity of the groups is reflected in the self-similarity of some operators; they are approximated by finite counterparts whose spectrum is computed by an ad hoc factorization process.

1. INTRODUCTION

The Hecke, Markov, and Laplace operators occur in various guises throughout mathematics. We start by a review of their more common appearances.

1.1. **Discrete Laplacian and Hecke type Operators.** Let $\mathcal{G} = (V, E)$ be a locally finite graph: there are maps $\alpha, \omega : E \to V$ giving the extremities of edges, and every vertex $v \in V$ has finite degree $\deg v = |\{e \in E | \, \alpha(e) = v\}|$. Therefore all edges are oriented, and $\mathcal{G}$ may have loops ($\alpha(e) = \omega(e)$) and multiple edges. The *discrete Laplace operator* of $\mathcal{G}$ is the operator $\Delta = 1 - M$ on $\ell^2(V, \deg)$, where M is the "adjacency" or Markovian operator

$$(Mf)(v) = \frac{1}{\deg v} \sum_{e \in E:\, \alpha(e) = v} f(\omega(e)).$$

The theory of discrete Laplace operators Δ has a long history, and is a popular topic of contemporary mathematics [CDS79, Woe94]. In the context of random walks on graphs, one usually considers the Markovian operator M rather than Δ: if e_v be

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the Dirac delta function at the vertex v, then $\langle M^n e_v | e_w \rangle$ is the probability of a random walk starting at v to reach w in n steps. If the random walk is symmetric, then M is a self-adjoint operator and its spectrum lies in $[-1, 1]$. The spectral properties of M contain valuable information for the theory of random walks and discrete potential, graph theory, abstract harmonic analysis, the theory of operator algebras, etc. For instance, a theorem by Harry Kesten [Kes59], generalized by different mathematicians (see [Woe94] and [CGH99] with its bibliography) asserts that $\mathcal{G}$ is amenable if and only if 1 is in the spectrum of M. Note that the random walk need not be simple (probabilities of moving in different directions may be different); a Markovian operator can still be associated to the walk. Let us finally mention a more general setting, developed these last years [Nov97]: the (discrete) Schrödinger operators $\Delta + P$, where P is diagonal, and the coefficients of Δ may depend on the vertex they correspond to.

The theory of Hecke type operators was developed in parallel: if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a finitely generated group G given with a symmetric generating system $S = \{s_1, \dots, s_m\} = S^{-1}$ in a Hilbert space $\mathcal{H}$, then one associates to π a self-adjoint operator H on $\mathcal{H}$:

$$H = \sum_{i=1}^{m} p(i)\pi(s_i),$$

for some $p(i) \in \mathbb{C}$. The most important choice is $p(i) = \frac{1}{m}$ for all $i \in \{1, \dots, m\}$; we shall restrict to this choice in the sequel, and assume, when no weight is given, that this one is used.

Operators of Hecke type play an important role in mathematical physics [Con94], Arakelov theory in number theory (see [Li96], [Ser97] and [Ser95] for the connection between number theory and operators), and Ramanujan graphs [Lub94]. The group-theoretical content of H, mainly in the case of the regular representation, was studied by Pierre de la Harpe, A. Guyan Robertson and Alain Valette in [HRV93a, HRV93b], and in many other papers — see the bibliography in [Woe94].

1.2. **Spectra of Noncommutative Dynamical Systems.** Let T be an invertible measure-preserving transformation of a measure space (X, μ), and let A be the corresponding unitary operator in $L^2(X, \mu)$, given by $(Af)(x) = f(T^{-1}x)$. By the *spectrum* of the dynamical system one usually means the spectrum of the operator A, or, as is almost the same, the spectrum of the self-adjoint operator $A + A^{-1}$. These last spectra are in correspondence through the map $z \mapsto z + z^{-1}$. If T is aperiodic, then $\operatorname{spec}(A + A^{-1}) = [-1, 1]$.

By a *noncommutative dynamical system* we mean a collection S of invertible measure-class-preserving transformations on a measure space (X, μ), that do not necessarily commute. Let G be the group generated by S. It has a natural unitary representation π in $(L^2(X, \mu))$ given by

$$(\pi(g)f)(x) = \sqrt{\mathfrak{g}(x)} f(g^{-1}x),$$

where $\mathfrak{g}(x) = dg\mu(x)/d\mu(x)$ is the Radon-Nikodým derivative. The *spectrum* of the dynamical system S is the spectrum of the Hecke type operator associated to G, $S \cup S^{-1}$ and π. In case $|S| = 1$, this definition reduces to the previous, classical one.

1.3. **Examples.** One of the most famous operators of Hecke type is the Harper-Mathieu-Peierls operator H_λ on $\ell^2(\mathbb{Z})$ acting on infinite sequences $f : \mathbb{Z} \to \mathbb{C}$ by

$$(H_\lambda(f))(n) = f(n-1) + f(n+1) + (2\cos\lambda n)f(n),$$

for any $\lambda \in \mathbb{R}$. These H_λ are the Hecke type operators associated to the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & m & p \\ & 1 & n \\ & & 1 \end{pmatrix} \middle|\, m,n,p \in \mathbb{Z} \right\} \cong \langle a,b,c |\, [a,b] = c, [a,c] = [b,c] = 1 \rangle$$

and its representation π_λ in $\ell^2(\mathbb{Z})$, where $\pi_\lambda(a)$ acts by translation: $[\pi_\lambda(a)](f)(n) = f(n-1)$, and $\pi_\lambda(b)$ acts by pointwise multiplication with the function $e^{i\lambda n}$, namely $[\pi_\lambda(b)](f)(n) = e^{i\lambda n} f(n)$.

The Harper operator is the operator related to the Quantum Hall effect, and originally arose in connection with the two-dimensional lattice. One can start from any Cayley graph, and construct a corresponding Harper operator, which would be the discrete analogue of the magnetic Laplacian [CHMM98].

The spectral properties of this operator were thoroughly investigated; if λ is a Liouville number, the spectrum of H_λ is a Cantor set [BS82]. We note that H_λ is a Schrödinger operator on the one-dimensional lattice. By Fourier transform, it can be realized as an element of the cross product C^*-algebra $R_\lambda \ltimes \mathcal{C}(\mathbb{S}^1)$, where R_λ is the dynamical system generated by an angle-λ rotation on $\mathbb{S}^1$ and $\mathcal{C}(\mathbb{S}^1)$ denotes the algebra of continuous functions on the circle.

Another example was studied by David Kazhdan [Kaž65]. Let α and β be two noncommuting rotations in the plane $\mathbb{R}^2$. They generate a group G with a unitary action π on $L^2(\mathbb{R}^2)$. Kazhdan studies the operator $M = \pi(\alpha) + \pi(\alpha^{-1}) + \pi(\beta) + \pi(\beta^{-1})$ and shows that its Fourier transform decomposes as a direct integral of operators acting on functions on the circle. Each of these is an element of $R_\lambda \ltimes \mathcal{B}(\mathbb{S}^1)$, where $\mathcal{B}(\mathbb{S}^1)$ denotes the bounded functions on the circle, and happens to be a Schrödinger operator; spectral properties of these operators are then used to show that the orbits of G in $\mathbb{R}^2$ are uniformly distributed.

1.4. **Main Results.** We produce examples of operators of Hecke type with Cantor set spectrum, but where additionally the representations π involved are quasi-regular. This produces graphs whose Laplace operators have totally discontinuous spectrum, namely the associated Schreier graphs. Our main results read:

Theorem 1.1. 1. *There is a connected* 4*-regular graph of polynomial growth, which is a Schreier graph of a group of intermediate growth, and whose Laplacian's spectrum is a Cantor set.*
2. *There is a connected* 4*-regular graph of polynomial growth, which is a Schreier graph of a group of intermediate growth, and whose Laplacian's spectrum is the union of a Cantor set K and a countable set P of isolated points whose accumulation set is K.*
3. *There are noncommutative dynamical systems generated by* 2 *transformations whose spectrum are the same as in the above two points.*
4. *The above spectra are calculated explicitly. The Cantor set K is of the form $F(J)$ where F is a simple algebraic function and J is the Julia set of a quadratic map $z \mapsto z^2 - \lambda$, where $\lambda = 45/16$ in the first case and $\lambda = 6$ in the*

second case. J is the set of points of the form

$$\pm\sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}}.$$

To the best of our knowledge, these are the first examples of graphs of constant vertex degree whose spectrum is totally disconnected. There are, however, examples of Schrödinger operators on $\mathbb{Z}$ (or $\mathbb{R}$) with nowhere dense spectrum; they are obtained as Harper operators (as mentioned above) or following a result by Jürgen Moser [Mos81].

There are also examples of random walks on non-regular graphs, but with vertex degrees 1 or 3, whose spectrum is the union of a countable set and a Cantor set of null Lebesgue measure [Mal95].

Similarly we produce Hecke type operators of quasi-regular representations that have the same spectra as above. These are probably the first examples of quasi-regular representations of virtually torsion-free groups with totally discontinuous spectrum; at least, the representations π_λ of the Heisenberg group are not quasi-regular, but come from the cross-product construction, which is often used to produce interesting examples. In the sequel we produce interesting examples of spectra using purely non-commutative dynamical systems and associated methods.

The graphs mentioned in Theorem 1.1 are Schreier graphs of some fractal groups. They are of polynomial growth and have a clear "fractal" appearance; see Figure 10. By a fractal group we mean a group which acts on a regular rooted tree $\mathcal{T}$, such that this action has some self-similar properties; this notion is very much related to that of branch group introduced in [Gri98]. The Schreier graphs $\mathcal{S}(G,P,S)$ are defined in 3.1; in our examples we take for P the stabilizer $\mathrm{st}_G(e)$ of an infinite ray starting at the root of $\mathcal{T}$, i.e. an element of the boundary $\partial\mathcal{T}$.

The parabolic subgroups $P = \mathrm{st}_G(e)$ have the remarkable property of being *weakly maximal*: $[G : P] = \infty$ but $[G : L] < \infty$ for all $L \gneqq P$. The corresponding quasi-regular representations $\rho_{G/P}$ are irreducible, and $\bigcap_{g\in G} P^g = 1$. We thus have an important family of faithful irreducible representations of G, that deserves further investigation.

The first example of group of fractal type was constructed in [Gri80] as an example of infinite torsion 2-group; it was described as a set of measure-preserving transformations of the interval $[0,1]$, but can equivalently be described by its action on a rooted tree (the binary expansion of a real in $[0,1]$ giving a path in the rooted binary tree). Later many new examples of this sort appeared [GS83b, Gri83, BG98]. It then became clear that the study of these groups via their tree action was most fruitful and led to interesting ideas and results; for a survey see [Gri98]; and for an introduction to the first example, G, see [Har, CMS98].

Among the five groups we consider, two (Γ and $\overline{\Gamma}$) are virtually torsion-free, and have a totally disconnected spectrum. The existence of such groups lends some hope to the existence of torsion-free fractal groups whose Laplace operator has a totally disconnected spectrum, or at least a gap in the spectrum. Such an example would provide a counterexample to the Kaplansky-Kadison conjecture on idempotents (that implies that the spectrum of the Laplace operator related to the regular representation is connected), and to the Baum-Connes conjecture [Val89]. Note that if such a group existed, it would be non-amenable [HK97].

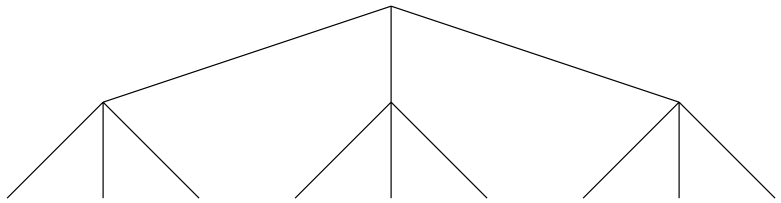
The method used in the computations is the following: we compute explicitly the spectrum of the finite graph $\mathcal{S}(G, P_n, S)$, where P_n is the stabilizer of the rightmost vertex in the nth row of the tree $\mathcal{T}$. Then some arguments, coming from [Lub95, GŻ97, MV98] but adapted to our goals, are used to obtain the spectrum of the infinite graphs from the finite spectra.

1.5. **Notation.** We assume all groups act on the left on sets, and write $g^h = hgh^{-1}$ and $[g,h] = ghg^{-1}h^{-1}$. We also write $\langle S\rangle$ and $\langle S\rangle^G$ for the subgroup and normal subgroup of G generated by S. The regular representation of G in $\ell^2(G)$ is written ρ_G, and the quasi-regular representation of G in $\ell^2(G/H)$ is written $\rho_{G/H}$. The symmetric group on a set S of cardinality n is written $\mathfrak{S}_S$ or $\mathfrak{S}_n$.

1.6. **Guide to Quick Reading.** We present in this paper five computations of spectra related to groups; however, the reader interested solely in examples of graphs with Cantor-set spectrum may wish to skip the group-theoretic discussion. In this case the sections 4.3 and 5 should describe the construction in a fairly self-contained manner.

2. Groups acting on rooted trees

The groups we shall consider will all be subgroups of the group $\mathrm{Aut}(\mathcal{T})$ of automorphisms of a regular rooted tree $\mathcal{T}$. Let Σ be a finite alphabet. The vertex set of the tree $\mathcal{T}_\Sigma$ is the set of finite sequences over Σ; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in Σ. The top node is the empty sequence $\emptyset$, and the children of σ are all the σs, for $s \in \Sigma$. We suppose $\Sigma = \mathbb{Z}/d\mathbb{Z}$, with the operation $\overline{s} = s + 1 \mod d$. Let a, called the *rooted automorphism* of $\mathcal{T}_\Sigma$, be the automorphism of $\mathcal{T}_\Sigma$ defined by $a(s\sigma) = \overline{s}\sigma$: it acts nontrivially on the first symbol only, and geometrically is realized as a cyclic permutation of the d subtrees just below the root.



Fix some Σ and let $\mathcal{A} = \mathrm{Aut}(\mathcal{T}_\Sigma)$. For any subgroup $G < \mathcal{A}$, let $\mathrm{st}_G(\sigma)$ denote the subgroup of G consisting of the automorphisms that fix the sequence σ, and $\mathrm{st}_G(n)$ denote the subgroup of G consisting of the automorphisms that fix all sequences of length n:

$$\mathrm{st}_G(\sigma) = \{g \in G |\, g\sigma = \sigma\}, \qquad \mathrm{st}_G(n) = \bigcap_{\sigma \in \Sigma^n} \mathrm{st}_G(\sigma).$$

The $\mathrm{st}_G(n)$ are normal subgroups of finite index of G; in particular $\mathrm{st}_G(1)$ is of index at most $d!$. Let G_n be the quotient $G/\mathrm{st}_G(n)$. If $g \in \mathcal{A}$ is an automorphism fixing the sequence σ, we denote by $g_{|\sigma}$ the element of $\mathcal{A}$ corresponding to the restriction to sequences starting by σ:

$$\sigma g_{|\sigma}(\tau) = g(\sigma\tau).$$

As the subtree starting from any vertex is isomorphic to the initial tree $\mathcal{T}_\Sigma$, we obtain this way a map

$$\phi : \begin{cases} \mathrm{st}_{\mathcal{A}}(1) \to \mathcal{A}^\Sigma \\ h \mapsto (h_{|0}, \dots, h_{|d-1}) \end{cases} \tag{1}$$

which is an embedding.

Definition 2.1. *A subgroup $G < \mathcal{A}$ is* level-transitive *if the action of G on Σ^n is transitive for all $n \in \mathbb{N}$. We shall always implicitly make that assumption.*

G is fractal *if for every vertex σ of $\mathcal{T}_\Sigma$ one has $\mathrm{st}_G(\sigma)_{|\sigma} \cong G$, where the isomorphism is given by identification of $\mathcal{T}_\Sigma$ with its subtree rooted at σ.*

For a sequence σ and an automorphism $g \in \mathcal{A}$, we denote by g^σ the element of $\mathcal{A}$ acting as g on the sequences starting by σ, and trivially on the others:

$$g^\sigma(\sigma\tau) = \sigma g(\tau), \qquad g^\sigma(\tau) = \tau \text{ if } \tau \text{ doesn't start by } \sigma.$$

Let $G < \mathcal{A}$ be a group acting faithfully, and transitively on each level, on a rooted tree $\mathcal{T}_\Sigma$. The *rigid stabilizer* of σ is $\mathrm{rist}_G(\sigma) = \{g^\sigma | \, g \in G\} \cap G$. We say G has *infinite rigid stabilizers* if all the $\mathrm{rist}_G(\sigma)$ are infinite.

Definition 2.2.
1. *G is a* regular branch *group if it has a finite-index subgroup $K < \mathrm{st}_G(1)$ such that*
$$K^\Sigma < \phi(K).$$
2. *A subgroup $G < \mathcal{A}$ is a* branch group *if for every $n \geq 1$ there exists a subgroup $L_n < \mathcal{A}$ and an embedding*
$$L_n \times \cdots \times L_n \hookrightarrow \mathrm{st}_G(n),$$
where the direct product is indexed by Σ^n, the injection is given on each factor by $(\ell, \sigma) \mapsto \ell^\sigma$, and the image is normal of finite index in $\mathrm{st}_G(n)$.
3. *G is a* weak branch *group if all of its rigid stabilizers $\mathrm{rist}_G(\sigma)$ are infinite.*

If G is fractal, one has for all n an embedding $\mathrm{st}_G(n) < G^{d^n}$. Note that the definition of a branch group admits an even more general setting — see [Gri98]. Four of our examples will be regular branch groups, and the last one will not be a branch, but rather a weak branch group. The following lemma shows that, for fractal groups, 1 implies 2 implies 3 in Definition 2.2.

Lemma 2.3. *If G is a fractal, regular branch group, then it is a branch group. If G is a branch group, then it is a weak branch group.*

Proof. Assume G is a regular branch group on its subgroup K. Define $L_n = K$ for all n. Clearly $\mathrm{st}_G(n)$ contains the direct product $L_n^{\Sigma^n}$, and it is of finite index in G^{Σ^n}, so all the more of finite index in $\mathrm{st}_G(n)$. The second implication holds because branch groups are infinite, and 'finite index in infinite group' is stronger than 'infinite'. □

In the sequel we shall be concerned with subgroups G of $\mathcal{A}$ that are finitely generated, fractal, and contain the rooted automorphism a. These groups will be naturally equipped with the restriction of the map ϕ defined in (1), a descending sequence of normal subgroups $H_n = \mathrm{st}_G(n)$ and an approximating sequence of finite quotients $G_n = G/H_n$. These quotients can be seen as subgroups of the symmetric group $\mathfrak{S}_{\Sigma^n}$ on Σ^n. More details on all of these groups and their subgroups appear in [BG98].

2.1. **Dynamical Systems.** Assume as above that a group G generated by a set S acts on the d-regular rooted tree $\mathcal{T} = \{0, \dots, d-1\}^*$. Then G acts naturally on the boundary $\partial\mathcal{T} = \{0, \dots, d-1\}^{\mathbb{N}}$, and this action preserves the uniform Bernoulli measure ν on the compact space $\partial\mathcal{T}$. We associate thus a dynamical system $(G, S, \partial\mathcal{T}, \nu)$ to the group G.

This dynamical system is naturally isomorphic to a dynamical system $(G, S, [0,1], m)$, where m is the Lebesgue measure, and G (generated by S) acts on $[0,1]$ by measure-preserving transformations in the following way: let $g \in G$, and $\gamma \in [0,1]$ a d-adic irrational with base-d expansion $0.\gamma_1\gamma_2\dots$. Then $g(\gamma) = 0.\delta_1\delta_2\dots$, where the infinite sequence $(\gamma_1, \gamma_2, \dots)$ is mapped by g to $(\delta_1, \delta_2, \dots)$. This defines the action of G on a subset of full measure of $[0,1]$.

The orbits of G on $\partial\mathcal{T}$ can be made explicit as follows:

Definition 2.4. *Two infinite sequences $\sigma, \tau : \mathbb{N} \to \Sigma$ are* confinal *if there is an $N \in \mathbb{N}$ such that $\sigma_n = \tau_n$ for all $n \geq N$.*

Confinality is an equivalence relation, and equivalence classes are called confinality classes.

Proposition 2.5. *Let G be a group acting on a regular rooted tree $\mathcal{T}$, and assume that for any generator $g \in G$ and infinite sequence σ, the sequences σ and $g\sigma$ differ only in finitely many places. Then the confinality classes of the action of G on $\partial\mathcal{T}$ are unions of orbits. If moreover $\mathrm{st}_G(\sigma)$ contains the rooted automorphism a for all $\sigma \in \mathcal{T}$, the orbits of the action are confinality classes.*

The dynamics of the actions of a group on the boundary of a tree from the point of view of confinality are investigated in more detail in in [NS99]. We remark that the five example groups—$G, \tilde{G}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$—we shall consider satisfy the conditions of the proposition above.

2.2. **Growth of Groups and Parabolic Subgroups.** We recall some facts about word-growth of groups and sets on which they act.

Definition 2.6. *Let G be a group generated by a finite set S, let X be a set upon which G acts transitively, and choose $x \in X$. The* growth *of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by*

$$\gamma(n) = |\{gx \in X \mid |g| \leq n\}|,$$

where $|g|$ denotes the minimal length of g when written as a word over S. By the growth of G *we mean the growth of the action of G on itself by left-multiplication.*

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there is a constant $C \in \mathbb{N}$ such that $f(n) < Cg(Cn + C) + C$ for all $n \in \mathbb{N}$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$.

The equivalence class of the growth of X is independent of the choice of S and of x.

X is of polynomial growth *if $\gamma(n) \preceq n^d$ for some d. It is of* exponential growth *if $\gamma(n) \succeq e^n$. It is of* intermediate growth *in the remaining cases. This trichotomy does not depend on the choice of x.*

Assume now that G is a group acting on the tree Σ^*, and that a subset $S \subset G$ is given.

Definition 2.7. *The* portrait *of $g \in G$ with respect to S is a subtree of Σ^*, with inner vertices labeled by $\mathfrak{S}_\Sigma$ and leaf vertices labeled by $S \cup \{1\}$. It is defined recursively as follows: if $g \in S \cup \{1\}$, the portrait of g is the subtree reduced to the root vertex, labeled by g itself. Otherwise, let $\alpha \in \mathfrak{S}_\Sigma$ be the permutation of the top branches of Σ^* such that $g\alpha^{-1} \in \mathrm{st}_G(1)$; let $(g_0, \dots, g_{d-1}) = \phi(g\alpha^{-1})$ and let $\mathcal{T}_i$ be the portrait of g_i. Then the portrait of g is the subtree of Σ^* with α labeling the root vertex and subtrees $\mathcal{T}_0, \dots, \mathcal{T}_{d-1}$ connected to the root.*

The depth *of $g \in G$ is the height (length of a maximal path starting at the root vertex) $\partial(g) \in \mathbb{N} \cup \{\infty\}$ of the portrait of g.*

Therefore the depth of g is finite if and only if the portrait of G is finite. Both are finite for all groups we consider in this paper, and the depth may be estimated using the following lemma:

Lemma 2.8. *Assume S generates G and $\phi : g \mapsto (g_1, \dots, g_d)$ defined in (1) has the property that $|g_i| < |g|$ for all i. Then every $g \in G$ has a finite portrait. If moreover there are constants λ, μ with $\mu/(1-\lambda) < 2$ such that $|g_i| \le \lambda|g| + \mu$ for all i, then asymptotically when $|g| \to \infty$*

$$\partial(g) \le \log_{1/\lambda} |g|.$$

Proof. Consider the 'level' function

$$F(n) = \max_{|g|<n} \partial(g).$$

Then F is increasing, and by assumption $F(2) = 0$, $F(n) \le F(\lambda n + \mu) + 1$. It then follows that

$$F(n) \le F(\lambda n + \mu) + 1 \le \cdots \le F(\lambda^k n + \lambda^{k-1}\mu + \cdots + \lambda\mu + \mu) + k;$$

let us take for k a natural number satisfying $\lambda^k n + \lambda^{k-1}\mu + \cdots + \lambda\mu + \mu \le 2$, for instance

$$k = \left\lceil \log_\lambda \frac{2 - \mu/(1-\lambda)}{n - mu/(1-\lambda)} \right\rceil,$$

where $\lceil x \rceil$ is the least integer greater than x. The result follows, because then $F(n) \le k$ and $k \sim \log_{1/\lambda} n$. $\square$

Scholium 2.9. *For all groups considered in this paper, we have $|g_i| \le \frac{1}{2}|g| + \frac{1}{2}$ for their natural generating systems, as can be checked on the tables describing ϕ. As a consequence, they all satisfy $\partial(g) \precsim \log_2 |g|$.*

Definition 2.10. *Let $\mathcal{T} = \Sigma^*$ be a rooted tree. A* ray *e in $\mathcal{T}$ is an infinite geodesic starting at the root of $\mathcal{T}$, or equivalently an element of $\partial\mathcal{T} = \Sigma^{\mathbb{N}}$.*

Let $G < \mathcal{A}$ and e be a ray. The associated parabolic subgroup *is $\operatorname{st}_G(e) = \bigcap_{n\geq 0} \operatorname{st}_G(e_n)$, where e_n is the length-n prefix of e.*

Assume that G satisfies the conditions of Lemma 2.8. Then we have the

Proposition 2.11. *Let $G < \mathcal{A}$ satisfy the conditions of Proposition 2.5 and Lemma 2.8 (for the constant λ), and let P be a parabolic subgroup. Then G/P, as a G-set, is of polynomial growth of degree at most $\log_{1/\lambda}(d)$. If moreover G is level-transitive, then G/P's asymptotical growth is polynomial of degree $\log_{1/\lambda}(d)$.*

Proof. Suppose that $P = \operatorname{st}_G(e)$. Then G/P can be identified with the G-orbit of e, and, by Proposition 2.5, with the set of all infinite sequences over Σ that eventually coincide with e. If $\partial(g) < k$, it sends the infinite sequence e to one of the d^k sequences in $\Sigma^k e_k \dots$; thus the image of e under the set of elements of depth at most k is of cardinality bounded by d^k. The image of e under the set of elements of length at most n is then asymptotically bounded by $d^{\log_{1/\lambda}(n)} = n^{\log_{1/\lambda}(d)}$, by Lemma 2.8. □

We now recall some facts on commensurators:

Definition 2.12. *The* commensurator *of a subgroup H of G is*

$$\operatorname{comm}_G(H) = \{g \in G |\, H \cap H^g \text{ is of finite index in } H \text{ and } H^g\}.$$

Equivalently, letting H act on the left on the cosets $\{gH\}$,

$$\operatorname{comm}_G(H) = \{g \in G |\, H \cdot (gH) \text{ and } H \cdot (g^{-1}H) \text{ are finite orbits}\}.$$

Proposition 2.13 ([BG98])**.** *Let G be a weak branch group and let P be a parabolic subgroup. Then $\operatorname{comm}_G(P) = P$.*

Theorem 2.14 (Mackey [Mac76, BH97])**.** *Let $P < G$ be any subgroup inclusion. Then the quasi-regular representation $\rho_{G/P}$ is irreducible if and only if $\operatorname{comm}_G(P) = P$.*

Therefore, for all weak branch groups $\rho_{G/P}$ is irreducible.

The following lemma is well known:

Lemma 2.15. *Let $H < G$ be a subgroup of finite index. Then the orbits of H on G/H, the double cosets $HgH < G$ and the irreducible components of the G-space $\ell^2(G/H)$ are all in bijection.*

Therefore, the orbits of $P_n = P \cdot \operatorname{st}_G(n)$ on Σ^n are in bijection with the decomposition of ρ_{G/P_n} in irreducible subrepresentations.

2.3. **Groups and Finite Automata.** There are various uses of automata in group theory, most notably as *word acceptors*, where the automata recognize some words as group elements and perform operations on these words [ECH+92]; and as *transducers* or *sequential machines* (see [Eil74, Chapter XI] or [GC71]), where the automata themselves are the elements of the group, and are distinguished by the transformation they perform on their input. The former use gives rise to the theory of *automatic groups*; we propose to call the latter *automata groups*. The automata they are built with are called *Mealy machines* or *Moore machines* (see [Glu61] or [Bra84, page 109]).

We present a restricted definition of *finite transducers*. In the standard terminology, they would be called *invertible transducers*.

Definition 2.16. *Let Σ be a finite alphabet. A* finite transducer *on Σ is a finite directed graph $\mathcal{G} = (V, E)$, a labeling $\lambda : E \to \Sigma$ of the edges such that for each vertex $v \in V$ the restriction of λ is a bijection between $\{e \in E | \alpha(e) = v\}$ and Σ, and a labeling $\tau : V \to \mathfrak{S}_\Sigma$ of the nodes (called* states*) by the symmetric group on Σ.*

An initial transducer *$\mathcal{G}_q$ is a finite transducer $\mathcal{G}$ with a distinguished initial state $q \in V$.*

Let $\mathcal{G}$ be a finite transducer, and $\{\mathcal{G}_q\}_{q\in V}$ be the set of its initial transducers. Each $\mathcal{G}_q$ defines an automorphism $\overline{\mathcal{G}_q}$ of the rooted tree $\mathcal{T}_\Sigma$ as follows: let $\sigma = \sigma_0 \dots \sigma_n$ be a vertex of $\mathcal{T}$. Let e be the edge of $\mathcal{G}_q$ labeled σ_0. Define recursively

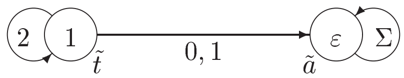
$$\overline{\mathcal{G}_q}(\sigma_0 \dots \sigma_n) = \tau(q)(\sigma_0)\overline{\mathcal{G}_{\omega(e)}}(\sigma_1 \dots \sigma_n).$$

We shall call two initial transducers $\mathcal{G}_q$ and $\mathcal{G}'_{q'}$ *equivalent* if their actions $\overline{\mathcal{G}_q}$ and $\overline{\mathcal{G}'_{q'}}$ on Σ^* are the same. Every initial transducer is equivalent to a unique transducer that is minimal with respect to its number of nodes [Eil74, Chapter XII, Theorem 4.1].

Define now $G(\mathcal{G})$ as the group generated by the $\overline{\mathcal{G}_q}$, where q ranges over the set of states of $\mathcal{G}$. We call such a group an *automata group*. The following fact is well known, and dates back to Jiří Hořejš in the early 60's [Hoř63]:

Proposition 2.17. *Let $\mathcal{G}_q$ and $\mathcal{G}'_{q'}$ be finite initial transducers on the same alphabet Σ. Then $\overline{\mathcal{G}_q}^{-1}$ and $\overline{\mathcal{G}_q} \circ \overline{\mathcal{G}'_{q'}}$ can be represented as finite initial transducers.*

In general different transducers can generate isomorphic groups. For instance, consider the three-vertex transducer in the middle of Figure 2. The group it generates is isomorphic (and even conjugate in $\mathcal{A}$) to the one generated by the following two-state transducer $\mathcal{G}$ on $\Sigma = \{0, 1, 2\}$, because the elements $\tilde{t}, \tilde{a}$ satisfy the same recursions as t, a (see Subsection 3.3):



Here ε is the cycle $(0, 1, 2) \in \mathfrak{S}_3$. The actions of $\mathcal{G}_{\tilde{t}}$ and $\mathcal{G}_{\tilde{a}}$ are as follows:

$$\mathcal{G}_{\tilde{t}}(2 \dots 2\sigma_m \dots \sigma_n) = 2 \dots 2\varepsilon(\sigma_m) \dots \varepsilon(\sigma_n) \qquad \text{when } \sigma_m \neq 2,$$
$$\mathcal{G}_{\tilde{a}}(\sigma_0 \dots \sigma_n) = \varepsilon(\sigma_0) \dots \varepsilon(\sigma_n).$$

2.4. **The Group G.** We give here some basic facts about the first of our examples, the group G [Gri80, Gri84]. We take $\Sigma = \{0, 1\}$. Recall a is the automorphism permuting the top two branches of $\mathcal{T}_2$. Let recursively b be the automorphism

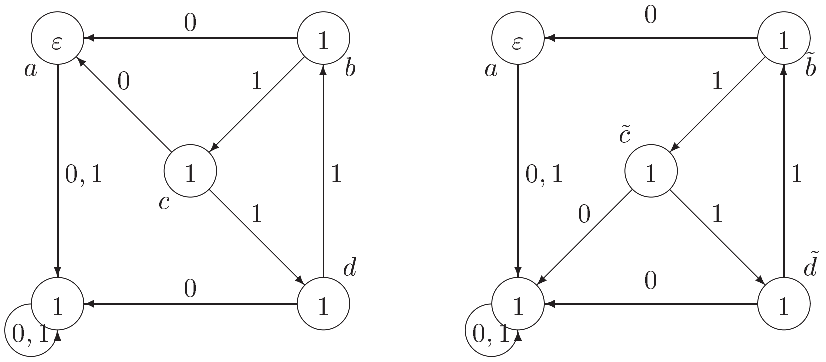


FIGURE 1. The Finite Transducers for G and $\tilde{G}$

acting as a on the right branch and c on the left, c be the automorphism acting as a on the right branch and d on the left, and d be the automorphism acting as 1 on the right branch and b on the left. In formulæ,

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma),\\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma),\\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

G is the group generated by $\{a, b, c, d\}$. It is readily checked that these generators are of order 2 and that $\{1, b, c, d\}$ constitutes a Klein group; one of the generators $\{b, c, d\}$ can thus be omitted.

G was originally defined in [Gri80] as the following dynamical system acting on the interval $[0, 1]$ from which rational dyadic points are removed:

$$a(z) = \begin{cases} z + \frac{1}{2} & \text{if } z < \frac{1}{2} \\ z - \frac{1}{2} & \text{if } z \geq \frac{1}{2}, \end{cases}$$

$$\begin{array}{rl} b(z) = & \underset{0}{|}\quad a \quad \underset{\frac{1}{2}}{|}\quad a \quad \underset{\frac{3}{4}}{|}\ 1\ \underset{\frac{7}{8}}{|}\ a..\ \underset{1}{|} \\ c(z) = & |\quad a \quad |\quad 1 \quad |\ a\ |\ a..\ | \\ d(z) = & |\quad 1 \quad |\quad a \quad |\ a\ |\ 1..\ | \end{array}$$

Here the intervals represent $[0, 1]$, with either a or 1 (the identity transformation) acting on the described subintervals in a similar way as a or 1 act on $[0, 1]$. Finally, G is also an automata group, see the left graph in Figure 1 (the trivial and non-trivial elements of $\mathfrak{S}_2$ are represented as 1 and ϵ and are used to label states).

Recall the map ϕ defined in (1); it restricts to an embedding $\phi : H \to G \times G$ given by

$$\phi : \begin{cases} b \to (a, c), & b^a \to (c, a) \\ c \to (a, d), & c^a \to (d, a) \\ d \to (1, b), & d^a \to (b, 1), \end{cases} \tag{2}$$

where $H = \mathrm{st}_G(1) = \langle b, c, d\rangle^G$ is an index-2 subgroup. Consider also $K = \langle (ab)^2\rangle^G$. Let e be the infinite sequence 0^∞; set $P_n = \mathrm{st}_G(0^n)$ and $P = \mathrm{st}_G(e) = \cap_{n\geq 0} P_n$ as above. Clearly P_n has index 2^n in G, as G acts transitively on Σ^n, and P has infinite index.

We note the following facts about G: it

- is an infinite torsion 2-group;
- is of intermediate growth, and therefore amenable (see Definition 3.3);
- is fractal, and regular branch on its subgroup K;
- is just infinite;
- has a recursive presentation with infinitely many relators; these relators are obtained as iterates of a substitution on a finite set of words [Lys85];
- is residually finite, and more precisely has a natural sequence of finite approximating quotients $G_n = G/\mathrm{st}_G(n)$, of order $2^{5\cdot 2^{n-3}+2}$ for $n \geq 3$ (and order 2^{2^n-1} for $n \leq 3$);
- has a faithful action on the set G/P, of linear growth by Proposition 2.11;
- has a faithful action an $\partial\mathcal{T}$ whose orbits are confinality classes.

The decomposition of ρ_{G/P_n} in irreducibles is given by the following lemma, combined with Lemma 2.15:

Lemma 2.18 ([BG98]). *P_n has $n+1$ orbits in Σ^n; they are 0^n and the $0^i1\Sigma^{n-1-i}$ for $0 \leq i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $0^i\Sigma^*$ for all $i \in \mathbb{N}$.*

2.5. **The Group $\tilde{G}$.** We describe briefly another fractal group, acting on the same tree $\mathcal{T}_2$ as G. More details appear in [BG98]. We denote again by a the automorphism permuting the top two branches, and let recursively $\tilde{b}$ be the automorphism acting as a on the right branch and $\tilde{c}$ on the left, $\tilde{c}$ be the automorphism acting as 1 on the right branch and $\tilde{d}$ on the left, and $\tilde{d}$ be the automorphism acting as 1 on the right branch and $\tilde{b}$ on the left. In formulæ,

$$\begin{aligned} \tilde{b}(0x\sigma) &= 0\overline{x}\sigma, & \tilde{b}(1\sigma) &= 1\tilde{c}(\sigma),\\ \tilde{c}(0\sigma) &= 0\sigma, & \tilde{c}(1\sigma) &= 1\tilde{d}(\sigma),\\ \tilde{d}(0\sigma) &= 0\sigma, & \tilde{d}(1\sigma) &= 1\tilde{b}(\sigma). \end{aligned}$$

Then $\tilde{G}$ is the group generated by $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$. Clearly all these generators are of order 2, and $\{\tilde{b}, \tilde{c}, \tilde{d}\}$ is elementary abelian of order 8. It can be defined using the second automaton in Figure 1, or as the dynamical system

$$a(z) = \begin{cases} z + \frac{1}{2} & \text{if } z < \frac{1}{2}\\ z - \frac{1}{2} & \text{if } z \geq \frac{1}{2}, \end{cases}$$

$$\tilde{b}(z) = \begin{array}{|c|c|c|c|} \hline \qquad a \qquad & \quad 1 \quad & 1 & a.. \\ \hline \end{array}$$

$$\tilde{c}(z) = \begin{array}{|c|c|c|c|} \hline \qquad 1 \qquad & \quad 1 \quad & a & 1.. \\ \hline \end{array}$$

$$\tilde{d}(z) = \begin{array}{|c|c|c|c|} \hline \qquad 1 \qquad & \quad a \quad & 1 & 1.. \\ \hline \end{array}$$

Recall the map ϕ defined in (1); it restricts to an embedding $\phi : \tilde{H} \to \tilde{G} \times \tilde{G}$ given by

$$\phi : \begin{cases} \tilde{b} \to (a, \tilde{c}), & \tilde{b}^a \to (\tilde{c}, a) \\ \tilde{c} \to (1, \tilde{d}), & \tilde{c}^a \to (d, 1) \\ \tilde{d} \to (1, \tilde{b}), & \tilde{d}^a \to (\tilde{b}, 1), \end{cases}$$

where $\tilde{H} = \operatorname{st}_{\tilde{G}}(1) = \langle \tilde{b}, \tilde{c}, \tilde{d} \rangle^{\tilde{G}}$ is an index-2 subgroup. Consider also $\tilde{K} = \langle (a\tilde{b})^2, (a\tilde{d})^2 \rangle^{\tilde{G}}$. Let again e be the infinite sequence 0^∞; set $\tilde{P}_n = \operatorname{st}_{\tilde{G}}(0^n)$ of index 2^n, and $\tilde{P} = \operatorname{st}_{\tilde{G}}(e) = \cap_{n \geq 0} \tilde{P}_n$ of infinite index.

We note the following facts about G, proved in [BG98]: it

- is an infinite group containing $G = \langle a, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{d}\tilde{b} \rangle$ as an infinite-index subgroup, and has 2-torsion elements as well as infinite-order elements;
- is of intermediate growth, and therefore is amenable;
- is fractal, and regular branch on its subgroup $\tilde{K}$;
- is just infinite;
- has a recursive presentation with infinitely many relators; these relators are obtained as iterates of a substitution on a finite set of words. All of $\tilde{G}$'s relators have even length with respect to the generating set $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$, so its Cayley graph is bipartite;
- is residually finite, and more precisely has a natural sequence of finite approximating quotients $\tilde{G}_n = \tilde{G}/\operatorname{st}_{\tilde{G}}(n)$, of order $2^{13 \cdot 2^{n-4}+2}$ for $n \geq 4$ (and order $2^{2^n - 1}$ for $n \leq 4$).
- has a faithful action on the set $\tilde{G}/\tilde{P}$, of linear growth by Proposition 2.11;
- has a faithful action an $\partial\mathcal{T}$ whose orbits are confinality classes.

The decomposition of $\rho_{\tilde{G}/\tilde{P}_n}$ in irreducibles is given by the following lemma, combined with Lemma 2.15:

Lemma 2.19. *$\tilde{P}_n$ has $n+1$ orbits in Σ^n; they are 0^n and the $0^i 1 \Sigma^{n-1-i}$ for $0 \leq i < n$. The orbits of $\tilde{P}$ in $\mathcal{T}_\Sigma$ are the $0^i\Sigma^*$ for all $i \in \mathbb{N}$.*

2.6. *GGS* **groups.** We next study three examples of a family of groups called *GGS* groups (the terminology was introduced by Gilbert Baumslag [Bau93] and refers to Rostislav Grigorchuk, Narain Gupta and Said Sidki). Let p be a prime number. Denote a the automorphism of $\mathcal{T}_p$ permuting cyclically the top p branches. Let $\epsilon = (\epsilon_0, \dots, \epsilon_{p-2}) \in (\mathbb{Z}/p)^{p-1}$. Define recursively the automorphism t_ϵ of $\mathcal{T}_p$ by

$$t(xy\sigma) = x(y + \epsilon_x)\sigma \text{ if } 0 \leq x \leq p-2, \qquad t((p-1)\sigma) = (p-1)t(\sigma).$$

Then G_ϵ is the subgroup of $\operatorname{Aut}(\mathcal{T}_p)$ generated by $\{a, t\}$.

The following results have their roots in [Gri80, GS83b] and are known as part of folklore; for a proof see [Gri98]:

Theorem 2.20. *G_ϵ is an infinite group if an only if $\epsilon \neq (0, \dots, 0)$. It is a torsion group if and only if $\sum \epsilon_i = 0$.*

Theorem 2.21 ([Bar99])**.** *For all ϵ, the group G_ϵ is of subexponential growth, and therefore is amenable.*

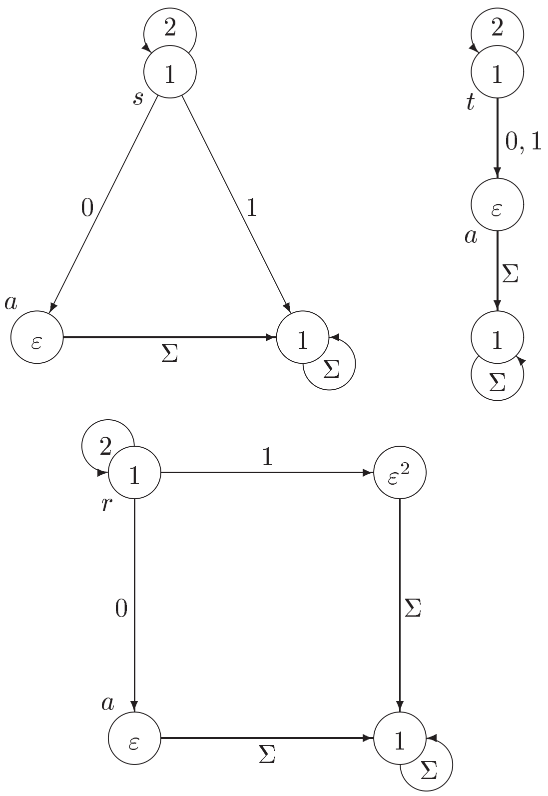


FIGURE 2. The Automata for Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

Let $G = G_\epsilon$ be as above. Let $e = (p-1)^\infty$ be the rightmost path in $\mathcal{T}_p$ and let $P = \mathrm{st}_G(e)$. Proposition 2.11 applies, so G/P has polynomial growth of degree at most $\log_2(p)$.

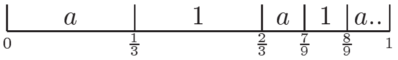
2.7. **The Group Γ.**

As always denote by a the rooted automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let s be the automorphism of $\mathcal{T}_3$ defined recursively by

$$s(0x\sigma) = 0\overline{x}\sigma, \qquad s(1x\sigma) = 1x\sigma, \qquad s(2\sigma) = 2s(\sigma).$$

Then Γ is the subgroup of $\mathrm{Aut}(\mathcal{T}_3)$ generated by $\{a, s\}$; its growth was studied by Jacek Fabrykowski and Narain Gupta [FG91]. It can also be defined using automata, as in Figure 2, or as the dynamical system

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$s(z) =$ 

Set $H = \mathrm{st}_\Gamma(1) = \langle s\rangle^\Gamma$; then $\phi : H \to \Gamma \times \Gamma \times \Gamma$ can be expressed as

$$\phi : \Big\{ s \to (a, 1, s), \quad s^a \to (s, a, 1), \quad s^{a^2} \to (1, s, a). \tag{3}$$

Define the elements $x = as$, $y = sa$ of Γ, and let K be the subgroup of Γ generated by x and y. Then K is normal in Γ, because $y^s = x^{-1}y^{-1}$, $y^{a^{-1}} = y^{-1}x^{-1}$, $y^{s^{-1}} = y^a = a$, and similar relations hold for conjugates of x. Moreover K is of index 3 in Γ, with transversal $\langle a\rangle$. Let L be the subgroup of K generated by $[K, K]$ and cubes in K.

Theorem 2.22 ([BG98]). *Γ is a regular branch, fractal group. The subgroup K of Γ is torsion-free; thus Γ is virtually torsion-free. The finite quotients $\Gamma_n = \Gamma/H_n$ of Γ have order $3^{3^{n-1}+1}$ for $n \geq 2$, and 3 for $n = 1$.*

2.8. **The Group $\overline{\Gamma}$.** Let a be as in the previous subsection, and let t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{x}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\Gamma}$ is the subgroup of $\mathrm{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$. The associated dynamical system is

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$$t(z) = \lfloor \quad a \quad | \quad a \quad | \ a \ | \ a \ | a.. \rfloor.$$

Set $H = \mathrm{st}_{\overline{\Gamma}}(1) = \langle t\rangle^{\overline{\Gamma}}$; then $\phi : H \to \overline{\Gamma} \times \overline{\Gamma} \times \overline{\Gamma}$ can be expressed as

$$\phi : \Big\{ t \to (a, a, t), \quad t^a \to (t, a, a), \quad t^{a^2} \to (a, t, a).$$

Define the elements $x = ta^{-1}$, $y = a^{-1}t$ of $\overline{\Gamma}$, and let $\overline{K}$ be the subgroup of $\overline{\Gamma}$ generated by x and y. Then $\overline{K}$ is normal in $\overline{\Gamma}$, because $x^t = y^{-1}x^{-1}$, $x^a = x^{-1}y^{-1}$, $x^{t^{-1}} = x^{a^{-1}} = y$, and similar relations hold for conjugates of y. Moreover $\overline{K}$ is of index 3 in $\overline{\Gamma}$, with transversal $\langle a\rangle$.

Theorem 2.23 ([BG98]). *$\overline{\Gamma}$ is a fractal group and is weak branch, but not branch. The subgroup $\overline{K}$ of $\overline{\Gamma}$ is torsion-free; thus $\overline{\Gamma}$ is virtually torsion-free. The finite quotients $\overline{\Gamma}_n = \overline{\Gamma}/H_n$ of $\overline{\Gamma}$ have order $3^{\frac{1}{4}(3^n+2n+3)}$ for $n \geq 2$, and $3^{\frac{1}{2}(3^n-1)}$ for $n \leq 2$.*

2.9. **The Group $\overline{\overline{\Gamma}}$.** Let again a denote the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now r be the automorphism of $\mathcal{T}_3$ defined recursively by

$$r(0x\sigma) = 0\overline{x}\sigma, \qquad r(1x\sigma) = 1\overline{\overline{x}}\sigma, \qquad r(2\sigma) = 2r(\sigma).$$

Then $\overline{\overline{\Gamma}}$ is the subgroup of $\mathrm{Aut}(\mathcal{T}_3)$ generated by $\{a, r\}$; it was studied by Gupta and Sidki [GS83a, GS83b, Sid87a, Sid87b]. The associated dynamical system is

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$$r(z) = \lfloor \quad a \quad | \quad a^2 \quad | \; a \; | \; a^2 \; | \; a.. \rfloor .$$

Set $H = \mathrm{st}_{\overline{\overline{\Gamma}}}(1) = \langle r \rangle^{\overline{\overline{\Gamma}}}$; then $\phi : H \to \overline{\overline{\Gamma}} \times \overline{\overline{\Gamma}} \times \overline{\overline{\Gamma}}$ con be expressed as

$$\phi : \Big\{ r \to (a, a^2, r), \quad r^a \to (r, a, a^2), \quad r^{a^2} \to (a^2, r, a).$$

Theorem 2.24 ([BG98]). *$\overline{\overline{\Gamma}}$ is a just infinite torsion 3-group. It is branch and fractal. The finite quotients $\overline{\overline{\Gamma}}_n = \overline{\overline{\Gamma}}/H_n$ of Γ have order $3^{2\cdot 3^{n-2}+1}$ for $n \geq 2$, and 3 for $n = 1$.*

3. Unitary Representations and Hecke type Operators

The five groups introduced in the previous section share the property of acting faithfully on a regular rooted tree; natural representations arise from this fact. In this chapter we suppose G is any group acting level-transitively on a regular tree.

We defined in Subsection 2.1 the boundary $\partial\mathcal{T}$ of the tree on which G acts. Since G preserves the uniform measure on this boundary, we have a unitary representation π of G in $L^2(\partial\mathcal{T}, \nu)$, or equivalently in $L^2([0,1], m)$. Let $\mathcal{H}_n$ be the subspace of $L^2(\partial\mathcal{T}, \nu)$ spanned by the characteristic functions χ_σ of the rays e starting by σ, for all $\sigma \in \Sigma^n$. It is of dimension d^n, and can equivalently be seen as spanned by the characteristic functions in $L^2([0,1], m)$ of intervals of the form $[(i-1)d^{-n}, id^{-n}]$, $1 \leq i \leq d^n$. These $\mathcal{H}_n$ are invariant subspaces, and afford representations $\pi_n = \pi_{|\mathcal{H}_n}$. As clearly π_{n-1} is a subrepresentation of π_n, we set $\pi_n^\perp = \pi_n \ominus \pi_{n-1}$, so that $\pi = \oplus_{n=0}^\infty \pi_n^\perp$.

Denote by $\rho_{G/H}$ the quasi-regular representation of G in $\ell^2(G/H)$ and by ρ_{G/H_n} the finite-dimensional representations of G in $\ell^2(G/H_n)$. Since G is level-transitive, the representations π_n and ρ_{G/H_n} are unitary equivalent.

Definition 3.1. *Let G be a group generated by a set S and H a subgroup of G. The* Schreier graph *$\mathcal{S}(G, H, S)$ of G/H is the directed graph on the edge set G/H, with for every $s \in S$ and every $gH \in G/H$ an edge from gH to sgH. The base point of $\mathcal{S}(G, H, S)$ is the coset H.*

Note that $\mathcal{S}(G, 1, S)$ is the Cayley graph of G relative to S. It may happen that $\mathcal{S}(G, P, S)$ have loops and multiple edges even if S is disjoint from H. Schreier graphs are $|S|$-regular graphs, and any degree-regular graph $\mathcal{G}$ containing a 1-factor (i.e. a regular subgraph of degree 1; there is always one if $\mathcal{G}$ has even degree) is a Schreier graph [Lub95, Theorem 5.4].

Definition 3.2. *Let G be a group generated by a finite symmetric set S. The* spectrum $\mathrm{spec}(\tau)$ *of a representation $\tau : G \to \mathcal{U}(\mathcal{H})$ with respect to the given set of generators is the spectrum of $\Delta_\tau = \sum_{s \in S} \tau(s)$ seen as an bounded operator on $\mathcal{H}$.*

As the vertices of $\mathcal{S}(G,H,S)$ coincide with the set G/H, it is easy to see that $\Delta_\tau/|S|-1$ is unitary equivalent to the Laplacian operator on $\mathcal{S}(G,H,S)$, and therefore has same spectrum.

Definition 3.3. *Let G be a group acting on a set X. This action is* amenable *in the sense of von Neumann* [vN29] *if there exists a finitely additive measure μ on X, invariant under the action of G, with $\mu(X)=1$.*

A group G is amenable *if its action on itself by left-multiplication is amenable.*

Amenability can be tested using the following criterion, due to Følner for the regular action [Føl57] (see also [CGH99] and the literature cited there):

Theorem 3.4. *Assume the group G acts on a discrete set X. Then the action is amenable if and only for every if for every $\lambda>0$ and every $g\in G$ there exists a finite subset $F\subset X$ such that $|F\triangle gF|<\lambda|F|$, where $\triangle$ denotes symmetric difference and $|\cdot|$ cardinality.*

Using this criterion it is easy to see that any G-space X of subexponential growth is amenable. In particular the G-spaces G/P are of polynomial growth when the conditions of Proposition 2.11 are fulfilled, and therefore are amenable. The following result belongs to the common lore, and we don't know of a reference to its proof. Our attention was drawn to it by Marc Burger and Alain Valette:

Proposition 3.5. *Let $H<G$ be any subgroup. Then the quasi-regular representation $\rho_{G/H}$ is weakly contained in ρ_G if and only if H is amenable.*

Proof. If H is amenable, then the trivial one-dimensional representation 1_H of H is weakly contained in ρ_H. Inducing up, $\rho_{G/H}=\operatorname{Ind}_H^G 1_H$ is weakly contained in $\rho_G=\operatorname{Ind}_H^G(\rho_H)$.

Conversely, if $\rho_{G/H}$ is weakly contained in ρ_G, we obtain by restricting to H that $\rho_{G/H}{}_{|H}$ is weakly contained in $\rho_{G|H}$. Now 1_H is a subrepresentation of $\rho_{G/H}{}_{|H}$, because the Dirac mass at H is H-fixed; and $\rho_{G|H}=[G:H]\rho_H$ by Frobenius reciprocity. It follows that 1_H is weakly contained in $[G:H]\rho_H$, and therefore that H is amenable. □

The following statements allows one to compare spectra of diverse representations.

Theorem 3.6. *Let G be a group acting on a regular rooted tree, and let π, π_n and $\pi_n^\perp$ be as above.*

1. *If G is weak branch, then $\rho_{G/P}$ is an irreducible representation of infinite dimension.*
2. *π is a reducible representation of infinite dimension whose irreducible components are precisely those of the $\pi_n^\perp$ (and thus are all finite-dimensional). Moreover*
$$\operatorname{spec}(\pi)=\overline{\bigcup_{n\geq 0}\operatorname{spec}(\pi_n)}=\overline{\bigcup_{n\geq 0}\operatorname{spec}(\pi_n^\perp)}.$$
3. *The spectrum of $\rho_{G/P}$ is contained in $\overline{\cup_{n\geq 0}\operatorname{spec}(\rho_{G/P_n})}=\overline{\cup_{n\geq 0}\operatorname{spec}(\pi_n)}$, and thus is contained in the spectrum of π. If moreover either P or G/P are*

amenable, these spectra coincide, and if P is amenable, they are contained in the spectrum of ρ_G:

$$\operatorname{spec}(\rho_{G/P}) = \operatorname{spec}(\pi) = \overline{\bigcup_{n\geq 0} \operatorname{spec}(\pi_n)} \subseteq \operatorname{spec}(\rho_G).$$

4. *Δ_π has a pure-point spectrum, and its spectral radius $r(\Delta_\pi) = s \in \mathbb{R}$ is an eigenvalue, while the spectral radius $r(\Delta_{\rho_{G/P}})$ is not an eigenvalue of $\Delta_{\rho_{G/P}}$. Thus $\Delta_{\rho_{G/P}}$ and Δ_π are different operators having the same spectrum.*

Proof. The first statement follows from Mackey's theorem 2.14 and Proposition 2.13. The second holds because π splits as the direct sum of the $\pi_n^\perp$'s, and is weakly equivalent to the direct sum of the π_n's.

It was mentioned that ρ_{G/P_n} and π_n are equivalent: they are both finite-dimensional and act on G-equivalent sets, namely G/H_n and Σ^n. The third statement then follows from Proposition 3.5 and Propositions 3.8 and 3.9 below.

It is obvious that s is an eigenvalue of Δ_π with constant eigenfunction. Now the fourth follows from Proposition 5 in [GŻ97]. □

Since all of our example groups are amenable, the spectra computed in the next section are included in $\operatorname{spec}(\rho_G)$. Moreover, as $\tilde{G}$ has a bipartite Cayley graph, $\operatorname{spec}(\rho_{\tilde{G}})$ is symmetrical about 1 and contains $[0, 4]$ (as we shall show in Section 4.2), so is $[-4, 4]$.

We finish this subsection by turning to a question of Mark Kać [Kać96]: "Can one hear the shape of a drum?" This question was answered in the negative in [GWW92], and we here answer by the negative to a related question: "Can one hear a representation?" Indeed $\rho_{G/P}$ and π have same spectrum (i.e. cannot be distinguished by hearing), but are not equivalent. Furthermore, if G is a branched group, there are uncountably many nonequivalent representations within $\{\rho_{G/\mathrm{st}_G(e)} | \, e \in \partial\mathcal{T}\}$, as is shown in [BG98].

The same question may be asked for graphs: "are there two non-isomorphic graphs with same spectrum?" There are finite examples, obtained through the notion of *Sunada pair* [Lub95]. Cédric Béguin, Alain Valette and Andrzej Żuk produced the following example in [BVŻ97]: let Γ be the integer Heisenberg group (free 2-step nilpotent on 2 generators x, y). Then $\Delta = x + x^{-1} + y + y^{-1}$ has spectrum $[-2, 2]$, which is also the spectrum of $\mathbb{Z}^2$ for an independent generating set. As a consequence, their Cayley graphs have same spectrum, but are not quasi-isometric (they do not have the same growth).

Using the result of Nigel Higson and Gennadi Kasparov [HK97] (giving a partial positive answer to the Baum-Connes conjecture), we may infer the following

Proposition 3.7. *Let Γ be a torsion-free amenable group with finite generating set $S = S^{-1}$ such that there is a map $\phi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ with $\phi(S) = \{1\}$. Then*

$$\operatorname{spec}(\sum_{s\in S} \rho(s)) = [-|S|, |S|].$$

In particular, there are countably many non-quasi-isometric graphs with the same spectrum, including the graphs of $\mathbb{Z}^d$, of free nilpotent groups and of suitable torsion-free groups of intermediate growth (for the first examples, see [Gri85]).

Proof. Since Γ is amenable, $|S|$ is in its spectrum. By the existence of ϕ, the Cayley graph of Γ with respect to S is two-colourable (i.e. bipartite), so its spectrum is symmetrical, and therefore contains $-|S|$. By the Baum-Connes conjecture, proved for this case by Higson and Kasparov, the C^*-algebra $C^*_r(\rho(\Gamma))$ contains no idempotents. It follows by functional integration that ρ's spectrum is connected, so is $[-|S|,|S|]$ as claimed. □

3.1. **Approximations of Operators and Spectra.** We now prove the claimed inclusions of spectra, in part relying on C^*-algebraic results.

Proposition 3.8. *Let $\{H_n\}_{n\geq 0}$ be a descending family of finite-index subgroups of G, and set $H = \bigcap_{n\geq 0} H_n$. Let τ and τ_n be the quasi-regular representations of G on G/H and G/H_n respectively. Then*

$$\operatorname{spec}\tau \subseteq \overline{\bigcup_{n\geq 0} \operatorname{spec}\tau_n}.$$

Proof. Let $\mathcal{S}$ and $\mathcal{S}_n$ be the Schreier graphs $\mathcal{S}(G,H,S)$ and $\mathcal{S}(G,H_n,S)$ respectively, and mark in them the vertices H and H_n. Then $\mathcal{S}_n \underset{n\to\infty}{\Longrightarrow} \mathcal{S}$ in the sense of [GŻ97], that is, coincide with $\mathcal{S}$ in ever increasing balls centered at H_n; therefore, for their associated spectral measures, $\sigma_n(\lambda) \underset{n\to\infty}{\Longrightarrow} \sigma(\lambda)$, in the sense of weak convergence. Therefore the support of $d\sigma$ is contained in the closure of the union of the supports of $d\sigma_n$, and the proposition follows. □

Alternate Proof. The quasi-regular representation τ is weakly contained in $\oplus_{n\geq 0}\tau_n$, by "approximation of coefficients" [Dix77, Theorem 3.4.9]. Indeed, as δ_H, the Dirac function at H, is a cyclic vector for τ, it is enough to see that the function

$$g \mapsto \langle \tau(g)\delta_H | \delta_H\rangle = \begin{cases} 1 & \text{if } g \in H \\ 0 & \text{otherwise}\end{cases}$$

can be pointwise approximated on G be coefficients of the τ_n's. But

$$\langle \tau_n(g)\delta_{H_n} | \delta_{H_n}\rangle = \begin{cases} 1 & \text{if } g \in H_n \\ 0 & \text{otherwise}\end{cases},$$

so

$$\lim_{n\to\infty} \langle \tau_n(g)\delta_{H_n} | \delta_{H_n}\rangle = \langle \tau(g)\delta_H | \delta_H\rangle.$$

We then have a surjection $C^*(\oplus_{n\geq 0}\tau_n) \twoheadrightarrow C^*(\tau)$, where $C^*(\rho)$ is the C^*-algebra generated by the image of ρ. The spectrum inclusions follow. □

Proposition 3.9. *Let $\{H_n\}_{n\geq 0}$ be a descending family of finite-index subgroups of G, and set $H = \bigcup_{n\geq 0} H_n$. Let τ and τ_n be the quasi-regular representations of G on G/H and G/H_n respectively. Assume moreover that the action of G on G/H is amenable, i.e. that $\mathcal{S}(G,H,S)$ is amenable. Then*

$$\operatorname{spec}\tau = \overline{\bigcup_{n\geq 0} \operatorname{spec}\tau_n}.$$

Proof. Let $\mathcal{S}$ be the Schreier graph $\mathcal{S}(G, H, S)$. Choose $\mu \in \operatorname{spec} \tau_n$, with a corresponding eigenvector $v : G/H_n \to \mathbb{C}$. As $\mathcal{S}$ is amenable, there is a Følner sequence $\{F_k\}$ in $\mathcal{S}$, i.e. a family of subsets of $V(\mathcal{S})$ with $|F_k|/|\partial F_k| \to 0$ as $k \to \infty$. Define now functions v_k on $\mathcal{S}$,

$$v_k(gH) = \begin{cases} v(gH_n) & \text{if } gH \in F_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\Delta_\tau v_k - \mu v_k\| \leq |\partial F_k| \cdot \|v\|$. If we let Ω be a fundamental domain of G/H_n in $\mathcal{S}$, of diameter δ, and let $\partial_\delta(F_k)$ denote the δ-neighbourhood of F_k, then we have $|\Omega| \cdot \|v_k\| \geq (|F_k| - |\partial_\delta F_k|)\|v\|$, whence

$$\left\| \Delta_\tau \frac{v_k}{\|v_k\|} - \mu \frac{v_k}{\|v_k\|} \right\| \leq \frac{|\Omega| \cdot |\partial F_k|}{|F_k| - |\partial_\delta F_k|} \to 0,$$

so $\mu \in \operatorname{spec} \tau$. □

3.2. **Spectral Measures.** In parallel to the computation of spectra, interesting questions arise in relation to properties of spectral measures associated to Markovian operators. There are two approaches to spectral measures, one via a solution to the moment problem and one via trace states in von Neumann algebras.

Let $\mathcal{G}$ be a graph, and let M be its Markovian operator. For any $x, y \in V(\mathcal{G})$ and $n \in \mathbb{N}$ let $p_{x,y}^n$ be the probability that a simple random walk starting at x be at y after n steps. Recall that if M_v be the characteristic vector of the vertex v, then

$$p_{x,y}^n = \langle M^n \delta_x | \delta_y \rangle.$$

Define the spectral measures $\sigma_{x,y}$ by

$$p_{x,y}^n = \int_{-1}^{1} \lambda^n d\sigma_{x,y}(\lambda) \qquad \forall n \in \mathbb{N},$$

or equivalently as

$$\sigma_{x,y}(\lambda) = \langle M(\lambda)\delta_x | \delta_y \rangle,$$

where $M(\lambda)$ is the spectral decomposition of M (the operator coinciding with M on eigenfunctions whose eigenvalue is at most λ). Set $\sigma_x = \sigma_{x,x}$. These measures are called the *Kesten spectral measures*, as they were introduced in [Kes59].

Proposition 3.10. *If $\mathcal{G}$ is connected, then all the measures $\sigma_{x,y}$ are equivalent.*

Therefore there is only one type of Kesten spectral measure, up to equivalence.

Now let N be a von Neumann algebra with a finite state τ. For any self-adjoint element $a \in N$ one can define a spectral measure τ_a by $\tau_a(\lambda) = \tau(a(\lambda))$, where $a(\lambda)$ is the spectral decomposition of a. We shall call τ_a the *von Neumann spectral measure* associated to a.

For instance, such a situation appears if N is finite-dimensional, or more generally of finite type. Another important example is when N is the von Neumann algebra generated by the left-regular representation of a group. In this case the von Neumann spectral measure is given by $\tau_a(\lambda) = \langle a(\lambda)\delta_1, \delta_1 \rangle$, where δ_1 is the Dirac function in the identity of the group. Therefore in this case the von Neumann and Kesten spectral measures of a Markovian operator coincide. A similar situation occurs if N is a hyperfinite algebra of type II_1, but there δ_1 should be replaced by any cyclic vector. If N is any algebra of type II_1 there is a canonical trace of the form $\tau(a) = \sum_i \langle ax_i, x_i \rangle$, for some (in general infinite) sequence of vectors (x_i).

Let P be a subgroup of a group G and suppose that the von Neumann algebra generated by $\rho_{G/P}$ has finite trace. Then for any symmetric system S of generators of G the von Neumann spectral measure $\tau_a = \tau_{\sum_{s\in S} s}$ can be defined. We also call τ_a the von Neumann spectral measure of the Schreier graph $\mathcal{G} = \mathcal{S}(G, P, S)$.

In case $\mathcal{G}$ is finite, the von Neumann spectral measure is just a "histogram", counting in any given interval the "average number of eigenvalues" that belong to it.

Suppose now that $P = \cap_{n=1}^{\infty} P_n$, where the P_n are subgroups of finite index of G. Then the sequence of finite graphs $\mathcal{G}_n = \mathcal{S}(G, P_n, S)$ converges to the graph $\mathcal{G}$ in the sense of [GŻ97], and the following statement holds:

Proposition 3.11 ([GŻ97]). *Let σ_n and σ be the Kesten spectral measures of the Schreier graphs $\mathcal{G}_n$ and $\mathcal{G}$ based at P_n and P respectively.*

Then we have $\sigma_n(\lambda) \Longrightarrow \sigma(\lambda)$ in the sense of weak convergence.

In case the subgroups P_n are normal and $P = 1$, we deduce from this proposition the convergence of the corresponding spectral measures τ_n to τ, since in this case $\sigma_n = \tau_n$ and $\sigma = \tau$. These results were obtained by Wolfgang Lück [Lüc94] and Michael Farber [Far98] using different methods.

It is interesting to study the conditions under which the spectral measures τ_n of finite graphs $\mathcal{S}(G, P_n, S)$ converge to some limit τ_* (which we call the *empiric* spectral measure), and, in case the von Neumann spectral measure τ is well defined for a graph $\mathcal{S}(G, P, S)$ (for instance, for a quasi-regular representation $\rho_{G/P}$ generating a von Neumann algebra of finite type), under which conditions we have $\tau = \tau_*$. Also, in the last case, when do we have $\sigma = \tau$?

Unfortunately, in our situation, the von Neumann algebra generated by $\rho_{G/P}$ is the algebra of all bounded operators so it has no a good state.

More investigations should be done in order to clarify the meaning of our computations of empiric spectral measures in the examples that follow.

3.3. **Operator Recursions.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and suppose $\Phi : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ is an isomorphism, where the domain of Φ is a sum of $d \geq 2$ copies of $\mathcal{H}$. Let S be a finite subset of $\mathcal{U}(\mathcal{H})$, and suppose that for all $s \in S$, if we write $\Phi^{-1} s \Phi$ as an operator matrix $(s_{i,j})_{i,j\in\{1,\dots,d\}}$ where the $s_{i,j}$ are operators in $\mathcal{H}$, then $s_{i,j} \in S \cup \{0, 1\}$.

This is precisely the setting in which we will compute the spectra of our five example groups: for G, we have $d = 2$ and $S = \{a, b, c, d\}$ with

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},$$
$$c = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

For $\tilde{G}$, we also have $d = 2$, and $S = \{a, \tilde{b}, \tilde{c}, \tilde{d}\}$ given by

$$b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

For $\Gamma = \langle a, s\rangle$, $\overline{\Gamma} = \langle a, t\rangle$ and $\overline{\overline{\Gamma}} = \langle a, r\rangle$, we have $d = 3$ and

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix},$$
$$t = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & t \end{pmatrix}, \qquad r = \begin{pmatrix} a & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & r \end{pmatrix}.$$

Each of these operators is unitary. The families $S = \{a, b, c, d\}, \dots$ generate subgroups $G(S)$ of $\mathcal{U}(\mathcal{H})$. The choice of the isomorphism Φ defines a unitary representation of $\langle S\rangle$.

We note, however, that the expression of each operator as a matrix of operators does not uniquely determine the operator, in the sense that different isomorphisms Φ can yield non-isomorphic operators satisfying the same recursions. Even if Φ is fixed, it may happen that different operators satisfy the same recursions. We consider two types of isomorphisms in this paper: $\mathcal{H} = \ell^2(G/P)$, where Φ is derived from the ϕ defined in (1); the second case considered is $\mathcal{H} = L^2(\partial\mathcal{T})$, where $\Phi : \mathcal{H} \to \mathcal{H}^\Sigma$ is defined by $\Phi(f)(\sigma) = (f(0\sigma), \dots, f((d-1)\sigma))$, for $f \in L^2(\partial\mathcal{T})$ and $\sigma \in \partial\mathcal{T}$. There are actually uncountably many non-equivalent isomorphisms, giving uncountably many non-equivalent representations of the same group, as we indicated just before Subsection 3.1.

4. Computations of Finite Spectra

Here we compute explicitly the spectra of the representations π_n described in Section 3. For our five examples, the general principle will be the same: obtain recurrence relations on the matrices of the representation and compute eigenvalues by recurrence. We end each subsection with a computation of the spectral measure τ_Δ.

4.1. **The group G.**

Recall the finite quotient G_n acts faithfully on $\{0, 1\}^n$. If we denote by a_n, b_n, c_n, d_n the permutation matrices of this representation, we have

$$a_0 = b_0 = c_0 = d_0 = (1),$$
$$a_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix},$$
$$c_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, \qquad d_n = \begin{pmatrix} 1 & 0 \\ 0 & b_{n-1} \end{pmatrix}.$$

The Hecke type operator of π_n is

$$\Delta_n = a_n + b_n + c_n + d_n = \begin{pmatrix} 2a_{n-1} + 1 & 1 \\ 1 & \Delta_{n-1} - a_{n-1} \end{pmatrix},$$

and we wish to compute its spectrum. We start by proving a slightly stronger result: define

$$Q_n(\lambda,\mu) = \Delta_n - (\lambda+1)a_n - (\mu+1)$$

and

$$\begin{aligned}
\Phi_0 &= 2-\mu-\lambda,\\
\Phi_1 &= 2-\mu+\lambda,\\
\Phi_2 &= \mu^2-4-\lambda^2,\\
\Phi_n &= \Phi_{n-1}^2 - 2(2\lambda)^{2^{n-2}} \qquad (n\geq 3).
\end{aligned}$$

Lemma 4.1. *For $n \geq 2$ we have*

$$|Q_n(\lambda,\mu)| = (4-\mu^2)^{2^{n-2}} \left| Q_{n-1}\left(\frac{2\lambda^2}{4-\mu^2}, \mu + \frac{\mu\lambda^2}{4-\mu^2}\right)\right| \qquad (n \geq 2).$$

Proof.

$$\begin{aligned}
|Q_n(\lambda,\mu)| &= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ -\lambda & \Delta_{n-1}-a_{n-1}-(1+\mu)\end{vmatrix}\\
&= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ (2a_{n-1}-\mu)\frac{-\lambda(2a_{n-1}+\mu)}{4-\mu^2} & \Delta_{n-1}-a_{n-1}-(1+\mu)\end{vmatrix}\\
&= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ 0 & \Delta_{n-1}-a_{n-1}-(1+\mu)-\frac{\lambda^2}{4-\mu^2}(2a_{n-1}+\mu)\end{vmatrix}\\
&= |2a_{n-1}-\mu| \cdot \left|\Delta_{n-1} - \left(1+\frac{2\lambda^2}{4-\mu^2}\right)a_{n-1} - \left(1+\mu+\frac{\mu\lambda^2}{4-\mu^2}\right)\right|;
\end{aligned}$$

now

$$|2a_{n-1}-\mu| = \begin{vmatrix} -\mu & 2 \\ 2 & -\mu \end{vmatrix}_{2^{n-1}} = (\mu^2-4)^{2^{n-2}}$$

so the lemma follows. ☐

Lemma 4.2. *For all n we have*

$$|Q_n| = \Phi_0\Phi_1\cdots\Phi_n.$$

Proof. By direct computation,

$$Q_0(\lambda,\mu) = (2-\mu-\lambda), \qquad Q_1(\lambda,\mu) = \begin{pmatrix} 2-\mu & -\lambda \\ -\lambda & 2-\mu \end{pmatrix},$$

so $|Q_0| = \Phi_0$ and $|Q_1| = \Phi_0\Phi_1$.

Let us temporarily agree to write

$$\lambda' = \frac{2\lambda^2}{4-\mu^2} \qquad\qquad \Phi_i = \Phi_i(\lambda,\mu)$$

$$\mu' = \mu + \frac{\mu\lambda^2}{4-\mu^2} \qquad\qquad \Phi_i' = \Phi_i(\lambda',\mu').$$

We will prove by recurrence that

$$\begin{aligned}(2-\mu)\Phi_0' &= \Phi_0\Phi_1,\\ (2+\mu)\Phi_1' &= -\Phi_2,\\ (4-\mu^2)\Phi_2' &= -\Phi_3,\\ (4-\mu^2)^{2^{n-2}}\Phi_n' &= \Phi_{n+1} \qquad (n\geq 3).\end{aligned}$$

Indeed

$$\begin{aligned}(2-\mu)\Phi_0' &= (2-\mu)(2-\mu) - \frac{\mu\lambda^2}{2+\mu} - \frac{2\lambda^2}{2+\mu}\\ &= (2-\mu)^2 - \lambda^2 = \Phi_0\Phi_1;\\ (2+\mu)\Phi_1' &= (2+\mu)(2-\mu) - \frac{\mu\lambda^2}{2-\mu} + \frac{2\lambda^2}{2-\mu}\\ &= 4-\mu^2+\lambda^2 = -\Phi_2;\\ (4-\mu^2)\Phi_2' &= (4-\mu^2)(\mu^2-4) + 2\mu^2\lambda^2 + \frac{(\mu\lambda^2)^2}{4-\mu^2} - \frac{(2\lambda^2)^2}{4-\mu^2}\\ &= -\left[(\mu^2-4)^2 - 2\mu^2\lambda^2 + \lambda^4\right]\\ &= -\left[(\mu^2-4-\lambda^2)^2 - 8\lambda^2\right] = -\Phi_3;\end{aligned}$$

and for $n \geq 3$,

$$\begin{aligned}(4-\mu^2)^{2^{n-2}}\Phi_n' &= (4-\mu^2)^{2^{n-2}}(\Phi_{n-1}')^2 - (4-\mu^2)^{2^{n-2}}2(2\lambda')^{2^{n-2}}\\ &= \left((4-\mu^2)^{2^{n-3}}\Phi_{n-1}'\right)^2 - 2(4\lambda^2)^{2^{n-2}}\\ &= (\pm\Phi_n)^2 - 2(2\lambda)^{2^{n-1}} = \Phi_{n+1}.\end{aligned}$$

Now using Lemma 4.1 we have for $n \geq 3$

$$\begin{aligned}|Q_n(\lambda,\mu)| &= (4-\mu^2)^{2^{n-2}}|Q_{n-1}(\lambda',\mu')|\\ &= (4-\mu^2)^{2^{n-2}}\Phi_0'\Phi_1'\cdots\Phi_{n-1}'\\ &= (2-\mu)\Phi_0'(2+\mu)\Phi_1'(4-\mu^2)\Phi_2'\cdots(4-\mu)^{2^{n-3}}\Phi_{n-1}'\\ &= \Phi_0\Phi_1\Phi_2\cdots\Phi_n,\end{aligned}$$

proving the claim. □

Proposition 4.3. *For all n we have*

$$\begin{aligned}\{(\lambda,\mu) : Q_n(\lambda,\mu) \text{ non invertible}\} &= \{(\lambda,\mu) : \Phi_0(\lambda,\mu)=0\}\cup\{(\lambda,\mu) : \Phi_1(\lambda,\mu)=0\}\\ &\cup\{(\lambda,\mu) : 4-\mu^2+\lambda^2+4\lambda\cos\left(\frac{2\pi j}{2^n}\right) = 0 \text{ for some } j=1,\dots,2^{n-1}-1\}.\end{aligned}$$

Proof. We prove by recurrence that for all n,k with $0 \leq k \leq n-2$ we have

$$\Phi_n = \prod_{t=0}^{2^k-1}\left(\Phi_{n-k} - 2(2\lambda)^{2^{n-2-k}}\cos\left(\frac{2\pi(2t+1)}{2^{k+2}}\right)\right). \tag{4}$$

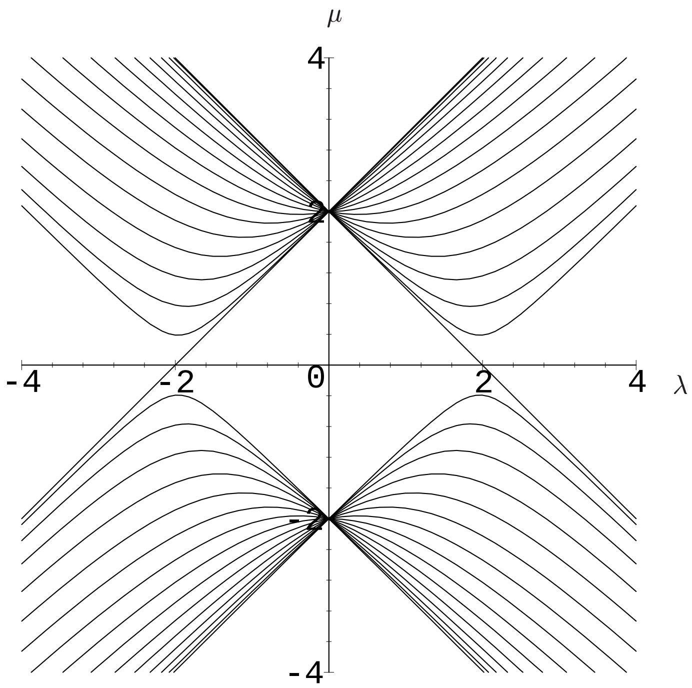


FIGURE 3. The spectrum of $Q_n(\lambda, \mu)$ for G and $\tilde{G}$

Indeed for $k = 0$ equality holds trivially, and if $k > 0$ we combine the terms for t and $2^k - 1 - t$, with $t < 2^{k-1}$: letting $A_{k,t}$ designate the t-th term,

$$\begin{aligned} A_{k,t}A_{k,2^k-1-t} &= \Phi_{n-k}^2 - 4(2\lambda)^{2^{n-k-1}} \cos^2\left(\frac{2\pi(2t+1)}{2^{k+2}}\right) \\ &= \Phi_{n-k}^2 - 2\left(1 + (2\lambda)^{2^{n-k-1}} \cos\left(\frac{2\pi(2t+1)}{2^{k+1}}\right)\right) \\ &= \Phi_{n-k+1} - 2(2\lambda)^{2^{n-k-1}} \cos\left(\frac{2\pi(2t+1)}{2^{k+1}}\right) \\ &= A_{k-1,t}, \end{aligned}$$

so $\prod_{t=0}^{2^k-1} A_{k,t} = \prod_{t=0}^{2^{k-1}-1} A_{k-1,t}$. Letting $k = n-2$ in (4) proves the proposition, in light of Lemma 4.2. □

In the (λ, μ) system, the spectrum of Q_n is thus a collection of 2 lines and $2^{n-1}-1$ hyperbolæ. The spectrum of Δ_n is precisely the set of θ such that $|Q_n(-1, \theta-1)| = 0$. From Proposition 4.3 we obtain

$$\operatorname{spec}(\Delta_n) = \{1 \pm \sqrt{5 - 4\cos\phi} : \phi \in 2\pi\mathbb{Z}/2^n\} \setminus \{0, -2\}.$$

The first eigenvalues are 4; 2; $1 \pm \sqrt{5}$; $1 \pm \sqrt{5 \pm 2\sqrt{2}}$; $1 \pm \sqrt{5 \pm 2\sqrt{2 \pm \sqrt{2}}}$; etc.

The numbers of the form $\pm\sqrt{\lambda\pm\sqrt{\lambda\pm\sqrt{\dots}}}$ appear as preimages of the quadratic map $z^2-\lambda$, and after closure produce a Julia set for this map (see [Bar88]). In the given example this Julia set is just an interval. In the examples that follow in Subsection 4.3 the spectra are simple transformations of Julia sets which are totally disconnected—this behaviour is explained by Lemma 4.12 and the remark after its proof.

Corollary 4.4. *The spectrum of* π*, for the group* G*, is*

$$\operatorname{spec}(\Delta)=[-2,0]\cup[2,4].$$

We now investigate the empiric spectral measure τ_Δ, as defined in Subsection 3.2. We constructed in the previous paragraph a one-to-one map $\chi:[0,\pi]\times\{\pm1\}\to\mathbb{R}$ defined by

$$\chi(\theta,\epsilon)=1+\epsilon\sqrt{5+4\cos\theta}.$$

The spectrum is uniformly distributed in $[0,\pi]\times\{\pm1\}$ by Proposition 4.3, and χ is by assumption a measure-preserving map, so the measure $\tau(A)$ of any subset A of $\mathbb{R}$ can be evaluated as

$$\tau(A)=\operatorname{vol}(\chi^{-1}(A)),$$

where vol is the uniform measure on $[0,\pi]\times\{\pm1\}$. The measure $d\tau_\Delta(x)=g(x)dx$ we are seeking is thus given by

$$g(x)=\frac{1}{2\pi}\frac{\partial}{\partial x}\chi^{-1}(x)=\frac{1-x}{4\pi\sqrt{1-\left(\frac{(x-1)^2-5}{4}\right)^2}}.$$

The eigenvectors of Δ_n can be expressed as follows. Let λ be an eigenvalue and define by induction the sequence

$$v_1=1;\quad v_2=\lambda-3;\quad v_i=\begin{cases}\frac{(\lambda-1)v_{i-1}-v_{i-2}}{2} & \text{if } i\geq 3, i\equiv 1[2];\\ (\lambda-1)v_{i-1}-2v_{i-2} & \text{if } i\geq 3, i\equiv 0[2].\end{cases}$$

Define also by induction for all $i\geq 0$ the following ordering of Σ^i: if the ordering of Σ^{i-1} is $(\sigma_1,\dots,\sigma_{2^{i-1}})$, the ordering of Σ^i is

$$(1\sigma_1,0\sigma_1,0\sigma_2,1\sigma_2,1\sigma_3,\dots,0\sigma_{2^{i-1}},1\sigma_{2^{i-1}}).$$

Lemma 4.5. *If* λ *is an eigenvalue of* Δ_n*, and* $(\sigma_1,\dots,\sigma_{2^n})$ *is the ordering described above, then the eigenvector corresponding to* λ *has value* v_i *on the vertex* σ_i*.*

Proof. Consider the Schreier graph $\mathcal{G}_n=\mathcal{S}(G,P_n,S)$ described in Subsection 5.1. It has vertex set $\{1,\dots,2^n\}$, a simple edge between $2i-1$ and $2i$ for all i, a double edge between $2i$ and $2i+1$ for all i, a loop at each i, and a triple loop at 1 and 2^n. The homogeneous space G_n/P_n is isomorphic to X graph through the correspondence $\sigma_i\mapsto i$. The choices of v_i clearly define an eigenvector for λ, because $v_{2^n}=0$ if λ is an eigenvalue for Δ_n. □

It is then possible to express the characteristic function of any vertex as a linear combination of the eigenvectors described above, and to explicitly compute the

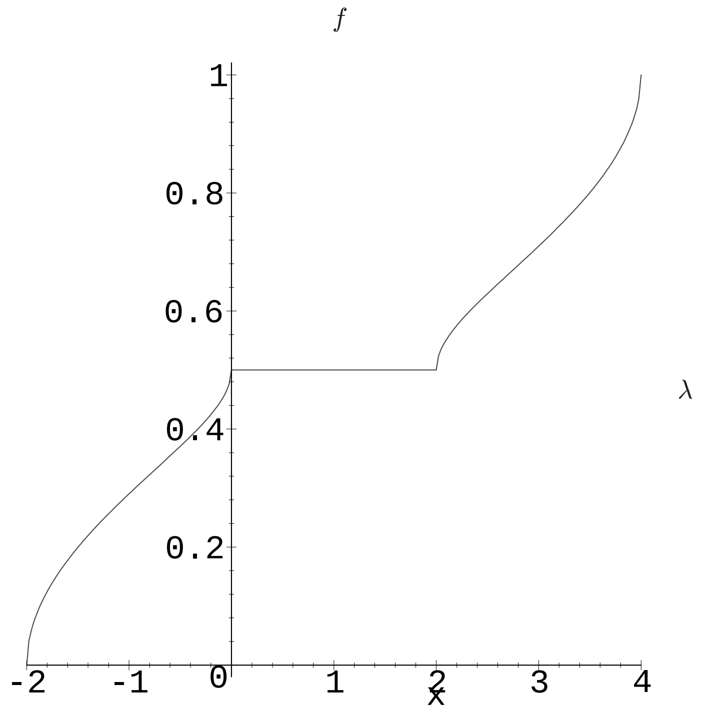


FIGURE 4. The empiric spectral measure for G

corresponding Kesten spectral measure. We shall not develop here this topic; but we note that the coordinates of the eigenvectors (of norm 1) vary greatly in absolute value, so the spectral measure may be very different to the empiric spectral measure.

4.2. **The Group $\tilde{G}$.** The computation of the spectrum of π_n for $\tilde{G}$ amounts to the following proposition. As before, we define $\tilde{\Delta}_n = a_n + \tilde{b}_n + \tilde{c}_n + \tilde{d}_n$, viewed as a $2^n \times 2^n$ matrix, and let

$$\tilde{Q}_n(\lambda,\mu) = \tilde{\Delta}_n - (\lambda+1)a_n - (\mu+2).$$

Proposition 4.6. *Then for all n we have*

$$\tilde{Q}_n(\lambda,\mu) = \frac{1}{2}Q_n(2\lambda, 2\mu).$$

Proof. The proof follows by recurrence on n: first it is readily checked that $\tilde{Q}_0(\lambda,\mu) = (1-\mu-\lambda)$; then for $n \geq 1$ we compute

$$\begin{aligned}\frac{1}{2}Q_n(2\lambda,2\mu) &= \frac{1}{2}\begin{pmatrix}2a_{n-1}-2\mu & -2\lambda\\ -2\lambda & \Delta_{n-1}-a_{n-1}-(2\mu+1)\end{pmatrix}\\ &= \begin{pmatrix}a_{n-1}-\mu & -\lambda\\ -\lambda & \frac{1}{2}Q_{n-1}(0,2\mu)\end{pmatrix} = \begin{pmatrix}a_{n-1}-\mu & -\lambda\\ -\lambda & \tilde{Q}_{n-1}(0,\mu)\end{pmatrix}\\ &= \begin{pmatrix}a_{n-1}-\mu & -\lambda\\ -\lambda & \tilde{\Delta}_{n-1}-a_{n-1}-(\mu+2)\end{pmatrix} = \tilde{Q}_n(\lambda,\mu).\end{aligned}$$

□

We can now obtain the spectrum of π_n by setting $\lambda = -1$ in $\tilde{Q}_n(\lambda,\mu)$ and solving for μ; in view of the previous proposition and the computations performed for G_n, we obtain:

Proposition 4.7. *The spectrum of $\tilde{G}_n$ is*

$$\left\{2+2\cos\left(\frac{2\pi j}{2^{n+1}}\right)\middle| j=0,\dots,2^n-1\right\}.$$

Proof. The spectrum consists precisely of the $\mu+2$ such that $\tilde{Q}_n(-1,\mu)=0$; this amounts to $Q_n(-2,2\mu)=0$. Now this holds when $\Phi_0(-2,2\mu)=0$, $\Phi_1(-2,2\mu)=0$ or $4-(2\mu)^2+4-4\cos(2\pi j/2^n)(-2)=0$ for some $j=1,\dots,2^{n-1}-1$. These give respectively $\mu=2$, $\mu=0$ and $\mu=\pm\sqrt{2-2\cos(2\pi j/2^n)}$, which after simplification yield the proposition. □

The first eigenvalues are 4; 2; $2\pm\sqrt{2}$; $2\pm\sqrt{2\pm\sqrt{2}}$; $2\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2}}}$; etc.

Corollary 4.8. *The spectrum of π, for the group $\tilde{G}$, is*

$$\operatorname{spec}(\Delta) = [0,4].$$

Note that the spectrum of Δ is positive! This can never happen to regular representations [HRV93a], and indeed the spectrum of the regular representation of $\tilde{G}$ is $[-4,4]$, since it contains $[0,4]$ and is symmetrical about 0 (as the Cayley graph $\mathcal{C}(\tilde{G},S)$ is bipartite).

Again we may compute the empiric spectral measure τ_Δ. Recall that there is a measure-preserving map $\chi:[0,\pi]\to\mathbb{R}$ given by $\theta\mapsto 2+2\cos\theta$, where $[0,\pi]$ has the Lebesgue measure. The measure $d\tau_\Delta = g(x)dx$ we are seeking is then associated to

$$g(x) = \frac{1}{\pi}\frac{\partial}{\partial x}\chi^{-1}(x) = \frac{1}{\pi\sqrt{4x-x^2}}.$$

4.3. **The Group Γ.** Recall that the finite quotient Γ_n acts faithfully on $\{0,1,2\}^n$. Denote by a_n and s_n the matrices of the action. We have

$$a_0 = s_0 = (1),$$
$$a_n = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \qquad s_n = \begin{pmatrix}a_{n-1}&0&0\\0&1&0\\0&0&s_{n-1}\end{pmatrix}.$$

The combinatorial Laplacian of Γ_n is

$$\Delta_n = a_n + a_n^{-1} + s_n + s_n^{-1} = \begin{pmatrix} a_{n-1} + a_{n-1}^{-1} & 1 & 1 \\ 1 & a_{n-1} + a_{n-1}^{-1} & 1 \\ 1 & 1 & s_{n-1} + s_{n-1}^{-1} \end{pmatrix}.$$

For ease of notation, let us define operators

$$A_n = a_n + a_n^{-1}, \qquad S_n = s_n + s_n^{-1},$$
$$Q_n(\lambda, \mu) = S_n + \lambda A_n - \mu,$$

and polynomials

$$\alpha = 2 - \mu + \lambda, \qquad \beta = 2 - \mu - \lambda,$$
$$\gamma = \mu^2 - \lambda^2 - \mu - 2, \qquad \delta = \mu^2 - \lambda^2 - 2\mu - \lambda.$$

Lemma 4.9. *We have*

$$|Q_0(\lambda, \mu)| = 2 + 2\lambda - \mu = \alpha + \lambda,$$
$$|Q_1(\lambda, \mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^2 = (\alpha + \lambda)\beta^2,$$
$$|Q_n(\lambda, \mu)| = (\alpha\beta\gamma^2)^{3^{n-2}} \left| Q_{n-1}\left(\frac{\lambda^2\beta}{\alpha\gamma}, \mu + \frac{2\lambda^2\delta}{\alpha\gamma}\right)\right| \quad (n \geq 2).$$

Proof. We compute the determinant:

$$|Q_n(\lambda, \mu)| = |S_n + \lambda A_n - \mu| = \begin{vmatrix} A_{n-1} - \mu & \lambda & \lambda \\ \lambda & 2 - \mu & \lambda \\ \lambda & \lambda & S_{n-1} - \mu \end{vmatrix}$$
$$= \begin{vmatrix} 2 - \mu - \lambda & \lambda + \mu - A_{n-1} & 0 \\ \lambda & A_{n-1} - \mu & \lambda \\ \lambda & \lambda & S_{n-1} - \mu \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ * & (2 - \mu)A_{n-1} + (\mu^2 - \lambda^2 - 2\mu) & \lambda \\ * & \lambda A_{n-1} + \lambda(2 - 2\lambda - 2\mu) & S_{n-1} - \mu \end{vmatrix}$$
$$= \left| S_{n-1} - \mu - \frac{\lambda^2(1 - 2\lambda - \mu + A_{n-1})}{(1 - \mu)A_{n-1} + \mu^2 - \lambda^2} \right| \cdot |(1 - \mu)A_{n-1} + \mu^2 - \lambda^2|$$
$$= \left| S_{n-1} - \mu + \lambda^2 \frac{\beta A_{n-1} + 2\delta}{\alpha\gamma} \right| (\alpha\beta\gamma^2)^{3^{n-2}},$$

which completes the proof of the lemma, using the easily verified equations

$$\frac{1}{\pi A_n + \rho} = \frac{\pi A_n - \pi - \rho}{(2\pi + \rho)(\pi - \rho)},$$
$$|\pi A_n + \rho| = \begin{vmatrix} \rho & \pi & \pi \\ \pi & \rho & \pi \\ \pi & \pi & \rho \end{vmatrix}_{3^n} = \big((\pi - \rho)^2(2\pi + \rho)\big)^{3^{n-1}}$$

valid for all scalar π and ρ. □

Consider now the quadratic forms

$$H_\theta = \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda,$$

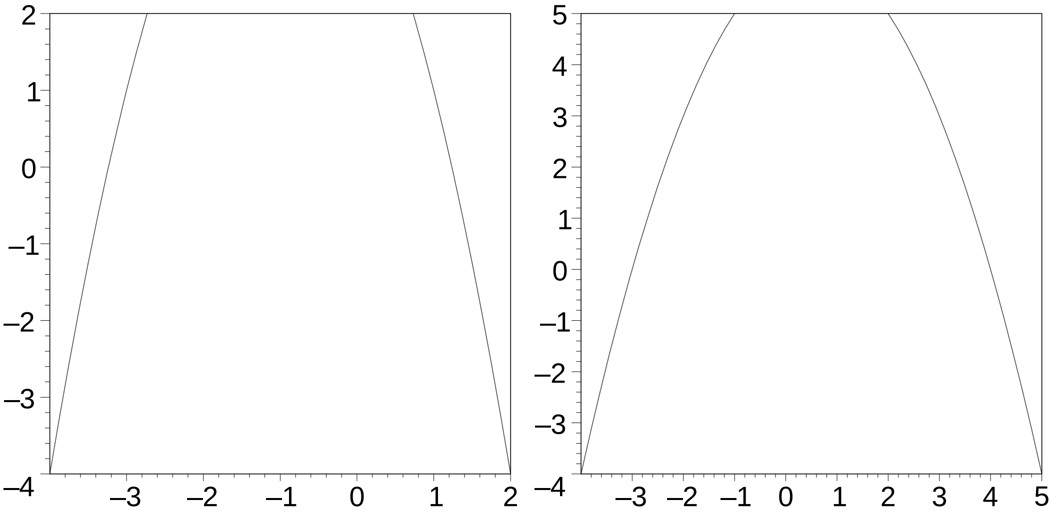


FIGURE 5. The Functions F for Γ and $\overline{\Gamma}$

and the function $F : [-4, 5] \to [-4, 5]$ (see Figure 5 left) given by

$$F(\theta) = 4 - 2\theta - \theta^2.$$

Let $X_2 = \{-1\}$, and iteratively define $X_n = F^{-1}(X_{n-1})$ for all $n \geq 3$. Note that $|X_n| = 2^{n-2}$.

Lemma 4.10. *We have for all $n \geq 2$ the factorization*

$$|Q_n(\lambda, \mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^{3^{n-1}+1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1}. \tag{5}$$

Proof. Let us write temporarily

$$\lambda' = \frac{\lambda^2 \beta}{\alpha\gamma}, \qquad \mu' = \mu + \frac{2\lambda^2\delta}{\alpha\gamma},$$

and $P' = P(\lambda', \mu')$ for P in $\{\alpha, \beta, \gamma, \delta, H_\theta\}$. Then we have

$$\alpha' + \lambda' = \frac{\beta(\alpha + \lambda)}{\alpha}, \qquad \beta' = \frac{\beta H_{-1}}{\gamma},$$
$$H_\theta(\lambda', \mu') = \frac{\beta \prod_{\nu \in F^{-1}(\theta)} H_\nu(\lambda, \mu)}{\alpha\gamma}.$$

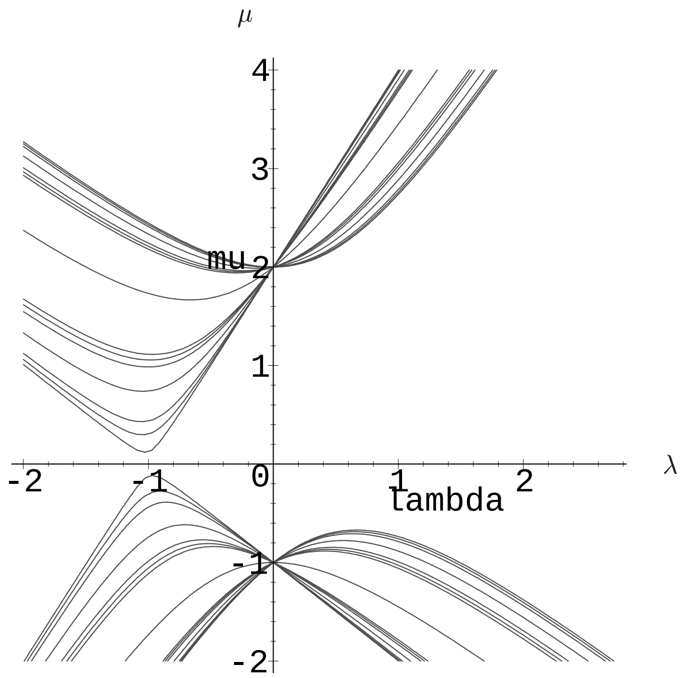


FIGURE 6. The spectrum of $Q_n(\lambda,\mu)$ for Γ

Now, first $Q_2(\lambda,\mu)=(\alpha+\lambda)\beta^4H_{-1}^2$ as claimed; then by induction, using Lemma 4.9, we have for $n\geq 3$

$$\begin{aligned}|Q_n(\lambda,\mu)| &= (\alpha\beta\gamma 2)^{3^{n-2}}\,|Q_{n-1}(\lambda',\mu')|\\ &= (\alpha\beta\gamma^2)^{3^{n-2}}(\alpha'+\lambda')\beta'^{3^{n-2}+1}\prod_{\substack{2\leq m\leq n-1\\ \theta\in X_m}} H_\theta'^{3^{n-1-m}+1}\\ &= (\alpha\beta\gamma^2)^{3^{n-2}}\frac{\beta(\alpha+\lambda)}{\alpha}\left(\frac{\beta H_{-1}}{\gamma}\right)^{3^{n-2}+1}\left(\frac{\beta}{\alpha\gamma}\right)^{3^{n-2}-1}\prod_{\substack{2\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1}\\ &= (\alpha+\lambda)\beta^{3^{n-1}+1}\prod_{\substack{2\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1}\end{aligned}$$

as claimed. ◻

Thus, according to the previous proposition, the spectrum of Q_n is a collection of two lines and $2^{n-1}-1$ hyperbolæ H_θ with $\theta\in X_2\sqcup X_3\sqcup\cdots\sqcup X_n$. The spectrum of Δ_n is obtained by solving $|Q_n(1,\mu)|=0$, as given in the following proposition.

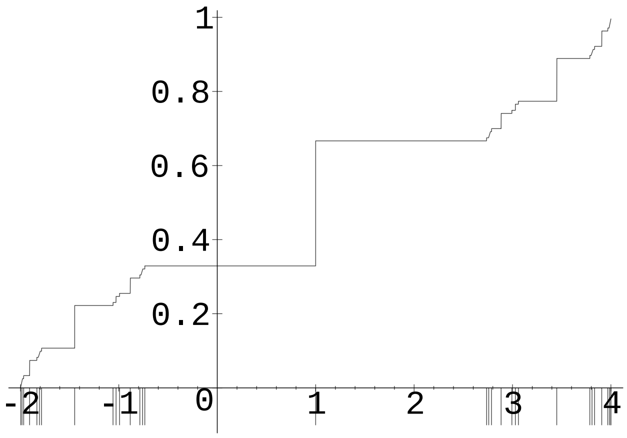


FIGURE 7. The Cantor-Set spectrum of Δ_n for Γ

Proposition 4.11. *Let $\pi_\pm : [-4,5] \to [-2,4]$ be defined by $\pi_\pm(\theta) = 1 \pm \sqrt{5-\theta}$. Then*

$$\begin{aligned}\operatorname{spec}\Delta_0 &= \{4\};\\ \operatorname{spec}\Delta_1 &= \{1,4\};\\ \operatorname{spec}\Delta_n &= \{1,4\} \cup \bigcup_{2\le m\le n} \pi_\pm(X_m) \qquad (n \ge 2).\end{aligned}$$

Proof. Solving $H_\theta(1,\mu) = 0$ gives $\mu = \pi_\pm(\theta)$. The result then follows from the factorization of $|Q_n|$ as a product of H_θ's given by Lemma 4.15. □

Lemma 4.12. *Let $F : [a,b] \to \mathbb{R}$ be a smooth unimodal map with negative Schwartzian derivative (for instance, a quadratic map), and $F(a) = F(b) = a$. Choose $\xi \in [a,b]$, and form*

$$K = \bigcap_{n=0}^{\infty} F^{-n}([a,b]), \qquad L = \overline{\bigcup_{n=0}^{\infty} F^{-n}(\{\xi\})}.$$

Then the following cases may occur:

$\mathbf{F}([\mathbf{a},\mathbf{b}]) \subset [\mathbf{a},\mathbf{b}]$: *Then $K = [a,b]$.*
$\mathbf{F}([\mathbf{a},\mathbf{b}]) = [\mathbf{a},\mathbf{b}]$: *Then $K = L = [a,b]$.*

$\mathbf{F([a,b]) \supset [a,b]}$: *Then K is a Cantor set of null Lebesgue measure. If $\xi \in K$, then $L = K$, otherwise $L = K \sqcup C$, where C is a countable set of isolated points accumulating on K.*

Proof. In the first case: K is clearly $[a,b]$, because $F^{-1}([a,b]) = [a,b]$.

Now assume that $F([a,b]) \supseteq [a,b]$, and define two continuous maps $g_1, g_2 : [a,b] \to [a,b]$ such that $F^{-1}(x) = \{g_1(x), g_2(x)\}$. Then $\Pi = \langle g_1, g_2 \rangle$ is a free semigroup, because g_1 and g_2 have disjoint image-interiors.

In the second case: K is obviously all $[a,b]$, because $F^{-1}([a,b]) = [a,b]$. Clearly Π-orbit of ξ, is dense in $[a,b] = g_1([a,b]) \cup g_2([a,b])$, so its closure L is $[a,b]$.

In the last case: we first show that $K \subseteq L$. Since $K = g_1(K) \sqcup g_2(K)$, every point in K is specified by an infinite sequence of g_i's applied to some point. Since the g_i's are contracting, the choice of that point is unimportant — hence we may choose ξ, and this proves the claim. If $\xi \in K$, then clearly $L \subseteq K$ and we are done. Otherwise, set $C = \cup F^{-n}(\{\xi\})$, which is a countable set of isolated points. Clearly K and C are disjoint (else ξ would be in K). Finally C accumulates on the infinite orbits under $\langle g_1, g_2 \rangle$, hence on K.

Let $O = [a,b] \setminus F^{-1}([a,b])$. Then O disconnects $[a,b]$, and $F^{-1}(O)$ disconnects each of the connected components of $[a,b] \setminus O$, etc., so K is disconnected. Take any point $x \in K$; then for all $n \in \mathbb{N}$ we may write $x = w(x_n)$ for some $x_n \in K$ and some word $w \in \Pi$ of length n. Taking some $x'_n \in K$ close to x_n and letting n tend to infinity gives a sequence in K converging to x. Since K is clearly closed, it is perfect, so is a Cantor set. Finally $m(K) = \lim_{n\to\infty}(1 - m(O))^n = 0$, where m denotes the Lebesgue measure. □

Let us make a few remarks concerning this lemma:

- The behaviour of L in the first case seems unpredictable. It is fortunately not needed for our purposes.
- The second case arises in connection with G and $\tilde{G}$, the third (with $\xi \notin K$) with Γ and (with $\xi \in K$) with $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$.
- Much more is known on the forward and inverse orbits of points under a unimodal map. See for instance [dMvS93, pages 10 and 327–351] for more information.
- K is the Julia set of the dynamical system F, i.e. the closure of the set of points whose (forward) orbit does not diverge to infinity. Indeed K consists precisely in those points whose orbit remains in $[a,b]$. Examples of Julia sets of a similar nature (occurring in the study of the quadratic map), with nested square-root expressions, appear in [Bar88, page 277] — see also the bibliography there.

The previous lemma gives the structure of the spectrum of Γ:

Corollary 4.13. *The spectrum of π for the group Γ is the closure of the set*

$$\left\{ \begin{array}{rl} 4 & (\mu = 0) \\ 1 & (\mu = \frac{1}{3}) \\ 1 \pm \sqrt{6} & (\mu = \frac{1}{9}) \\ 1 \pm \sqrt{6 \pm \sqrt{6}} & (\mu = \frac{1}{3^3}) \\ 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}} & (\mu = \frac{1}{3^4}) \\ \dots & \end{array} \right\}.$$

It is the union of a Cantor set of null Lebesgue measure that is symmetrical about 1*, and a countable collection of isolated points supporting the empiric spectral measure, which has the values indicated as μ.*

4.4. **The Group $\overline{\Gamma}$.** Recall the finite quotient Γ_n acts faithfully on $\{0,1,2\}^n$. Denote by a_n and t_n the matrices of this action. We have

$$a_0 = t_0 = (1),$$
$$a_n = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad t_n = \begin{pmatrix} a_{n-1} & 0 & 0 \\ 0 & a_{n-1} & 0 \\ 0 & 0 & t_{n-1} \end{pmatrix}.$$

The combinatorial Laplacian of Γ_n is

$$\Delta_n = a_n + a_n^{-1} + t_n + t_n^{-1} = \begin{pmatrix} a_{n-1} + a_{n-1}^{-1} & 1 & 1 \\ 1 & a_{n-1} + a_{n-1}^{-1} & 1 \\ 1 & 1 & t_{n-1} + t_{n-1}^{-1} \end{pmatrix}.$$

For ease of notation, let us define operators

$$A_n = a_n + a_n^{-1}, \qquad T_n = t_n + t_n^{-1},$$
$$Q_n(\lambda, \mu) = T_n + \lambda A_n - \mu,$$

and polynomials

$$\alpha = 2 - \mu + \lambda, \qquad \beta = 2 - \mu - \lambda,$$
$$\gamma = 1 + \mu + \lambda, \qquad \delta = 1 + \mu - \lambda.$$

Lemma 4.14. *We have*

$$|Q_0(\lambda, \mu)| = \alpha + \lambda,$$
$$|Q_1(\lambda, \mu)| = (\alpha + \lambda)\beta^2,$$
$$|Q_n(\lambda, \mu)| = (\gamma\delta)^{2 \cdot 3^{n-2}} (\alpha\beta)^{3^{n-2}} \left| Q_{n-1}\left(\frac{-2\lambda^2}{\alpha\delta}, \mu + \frac{2\lambda^2(\mu - \lambda - 1)}{\alpha\delta} \right) \right| \quad (n \geq 2).$$

Proof. We compute the determinant:

$$
\begin{aligned}
|Q_n(\lambda,\mu)| = |T_n + \lambda A_n - \mu| &= \begin{vmatrix} A_{n-1}-\mu & \lambda & \lambda \\ \lambda & A_{n-1}-\mu & \lambda \\ \lambda & \lambda & T_{n-1}-\mu \end{vmatrix} \\
&= \begin{vmatrix} A_{n-1}-\mu-\lambda & 0 & 0 \\ \lambda & A_{n-1}-\mu+\lambda & \lambda \\ \lambda & 2\lambda & T_{n-1}-\mu \end{vmatrix} \\
&= \left| T_{n-1}-\mu-\frac{2\lambda^2}{A_{n-1}-\mu+\lambda} \right| \cdot |A_{n-1}-\mu+\lambda| \cdot |A_{n-1}-\mu-\lambda| \\
&= \left| T_{n-1}-\mu-2\lambda^2\frac{A_{n-1}+\mu-\lambda-1}{(2-\mu+\lambda)(1+\mu-\lambda)} \right| \cdot |A_{n-1}-\mu+\lambda| \cdot |A_{n-1}-\mu-\lambda|
\end{aligned}
$$

which completes the proof of the lemma, using the easily verified equations

$$
\frac{1}{A_n-\theta} = \frac{A_n+\theta-1}{(2-\theta)(1+\theta)},
$$

$$
|A_n-\theta| = \begin{vmatrix} -\theta & 1 & 1 \\ 1 & -\theta & 1 \\ 1 & 1 & -\theta \end{vmatrix}_{3^{n-1}} = \left((\theta+1)^2(2-\theta)\right)^{3^{n-1}}
$$

valid for all scalar θ. ∎

Consider again the quadratic forms

$$
H_\theta = \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda,
$$

and consider the real function (see Figure 5 right)

$$
F(\theta) = \frac{12+\theta-\theta^2}{2}.
$$

Let $X_3 = \{2\}$, $Y_3 = \{-1\}$ and iteratively define $X_n = F^{-1}(X_{n-1})$ and $Y_n = F^{-1}(Y_{n-1})$ for all $n \geq 4$. Note that $|X_n| = |Y_n| = 2^{n-3}$.

Lemma 4.15. *We have for all $n \geq 2$ the factorization*

(6)

$$
|Q_n(\lambda,\mu)| = (\alpha+\lambda)\beta^{3^{n-2}+1}\gamma^{3^{n-1}-1}(\delta-\lambda)^{3^{n-2}-1} \prod_{\substack{3\leq m\leq n \\ \theta\in X_m}} H_\theta^{3^{n-m}+1} \prod_{\substack{3\leq m<n \\ \theta\in Y_m}} H_\theta^{3^{n-m}-1} \prod_{\theta\in X_{n+1}} H_\theta^2.
$$

Proof. Let us write temporarily

$$
\lambda' = \frac{-2\lambda^2}{\alpha\delta}, \qquad \mu' = \mu + \frac{2\lambda^2(\mu-\lambda-1)}{\alpha\delta},
$$

and $P' = P(\lambda',\mu')$ for P in $\{\alpha,\beta,\gamma,\delta,H_\theta\}$. Then we have

$$
\alpha'+\lambda' = \frac{\beta(\alpha+\lambda)}{\alpha}, \qquad \beta' = \frac{-H_2}{\delta}, \qquad \gamma' = \frac{\gamma(\delta-\lambda)}{\delta}, \qquad \delta'-\lambda' = \frac{-H_{-1}}{\alpha},
$$

$$
H_\theta(\lambda',\mu') = \frac{\prod_{\nu\in F^{-1}(\theta)} H_\nu(\lambda,\mu)}{\alpha\delta}.
$$

Now, first $Q_2(\lambda,\mu)=(\alpha+\lambda)\beta^2\gamma^2H_2^2$ as claimed; then by induction, using Lemma 4.14, we have for $n\geq 3$

$$\begin{aligned}
|Q_n(\lambda,\mu)| &= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}\,|Q_{n-1}(\lambda',\mu')|\\
&= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}(\alpha'+\lambda'){\beta'}^{3^{n-3}+1}{\gamma'}^{3^{n-2}-1}(\delta'-\lambda')^{3^{n-3}-1}\\
&\qquad \prod_{\substack{3\leq m\leq n-1\\ \theta\in X_m}} {H'_\theta}^{3^{n-1-m}+1} \prod_{\substack{3\leq m<n-1\\ \theta\in Y_m}} {H'_\theta}^{3^{n-1-m}-1} \prod_{\theta\in X_n} {H'_\theta}^2\\
&= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}\frac{\beta(\alpha+\lambda)}{\alpha}\left(\frac{-H_2}{\delta}\right)^{3^{n-3}+1}\left(\frac{\gamma(\delta-\lambda)}{\delta}\right)^{3^{n-2}-1}\left(\frac{-H_{-1}}{\alpha}\right)^{3^{n-3}-1}\\
&\qquad (\alpha\delta)^{-2\cdot 3^{n-3}} \prod_{\substack{4\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1} \prod_{\substack{4\leq m<n\\ \theta\in Y_m}} H_\theta^{3^{n-m}-1} \prod_{\theta\in X_{n+1}} H_\theta^2.\\
&= (\alpha+\lambda)\beta^{3^{n-2}+1}\gamma^{3^{n-1}-1}(\delta-\lambda)^{3^{n-2}-1} \prod_{\substack{3\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1} \prod_{\substack{3\leq m<n\\ \theta\in Y_m}} H_\theta^{3^{n-m}-1} \prod_{\theta\in X_{n+1}} H_\theta^2
\end{aligned}$$

as claimed. □

Thus, according to the previous proposition, the spectrum of Q_n is a collection of 4 lines and $(2^{n-1}-1)+(2^{n-3}-1)$ hyperbolæ H_θ with $\theta\in X_3\sqcup\cdots\sqcup X_{n+1}\sqcup Y_3\sqcup\cdots\sqcup Y_{n-1}$. The spectrum of Δ_n is obtained by solving $|Q_n(1,\mu)|=0$, as given in the following proposition.

Proposition 4.16. *Let $\pi_\pm:[-4,5]\to[-2,4]$ be defined by $\pi_\pm(\theta)=1\pm\sqrt{5-\theta}$. Then*

$$\begin{aligned}
\operatorname{spec}\Delta_0 &= \{4\};\\
\operatorname{spec}\Delta_1 &= \{1,4\};\\
\operatorname{spec}\Delta_n &= \{-2,1,4\}\cup\bigcup_{3\leq m\leq n+1}\pi_\pm(X_m)\cup\bigcup_{3\leq m\leq n-1}\pi_\pm(Y_m)\qquad (n\geq 2).
\end{aligned}$$

Proof. Solving $H_\theta(1,\mu)=0$ gives $\mu=\pi_\pm(\theta)$. The result then follows from the factorization of $|Q_n|$ as a product of H_θ's given by Lemma 4.15. □

Corollary 4.17. *The spectrum of π for the group $\overline{\Gamma}$ is the closure of the set*

$$\left\{\begin{array}{rl}
4 & (\mu=0)\\
-2 & (\mu=\frac13)\\
1 & (\mu=\frac29)\\
1\pm\sqrt{\frac{9\pm3}{2}} & (\mu=\frac{2}{27})\\
1\pm\sqrt{\frac{9\pm\sqrt{45\pm4\cdot3}}{2}} & (\mu=\frac{2}{3^4})\\
1\pm\sqrt{\frac{9\pm\sqrt{45\pm4\sqrt{45\pm4\cdot3}}}{2}} & (\mu=\frac{2}{3^5})\\
\dots &
\end{array}\right\}.$$

It is a Cantor set of null Lebesgue measure that is symmetrical about 1. The empiric spectral measure is concentrated on the above algebraic numbers and has the values indicated as μ.

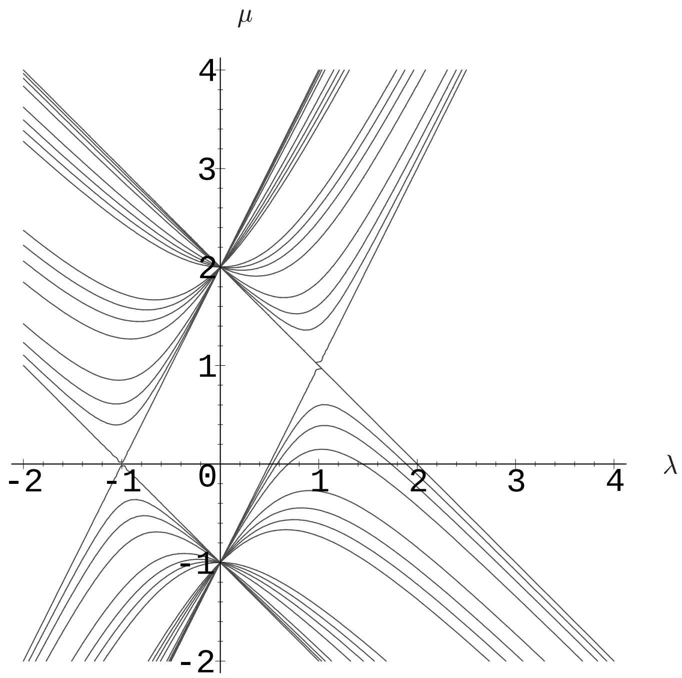


FIGURE 8. The spectrum of $Q_n(\lambda,\mu)$ for $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

4.5. **The Group $\overline{\overline{\Gamma}}$.** Although $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ greatly differ in structure—the former is virtually torsion-free while the latter is torsion—their representations π_n have the same spectrum:

Proposition 4.18. *The spectrum of $\overline{\overline{\Gamma}}$ is the same as that of $\overline{\Gamma}$; that is, a Cantor set symmetrical about* 1 *and spanning from* -2 *to* 4*.*

Proof. It suffices to note that for all n we have $r_n + r_n^{-1} = t_n + t_n^{-1}$; this follows by induction on n. □

5. Schreier Graphs

Recall from Section 3 the definition of a Schreier graph. In the general setting of a group G acting level-transitively on a tree Σ^*, and a subgroup P defined as the stabilizer of an infinite path, the quotient space G/P is naturally identified with an orbit, i.e. a countable subset, of $\Sigma^{\mathbb{N}}$. For the finite quotients G_n, the space G_n/P_n is identified with Σ^n. We set $\mathcal{G}_n$ denote the Schreier graph associated to this homogeneous space. Due to the fractal (or recursive) nature of the five examples we study, there are simple local rules producing $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$, the limit of this

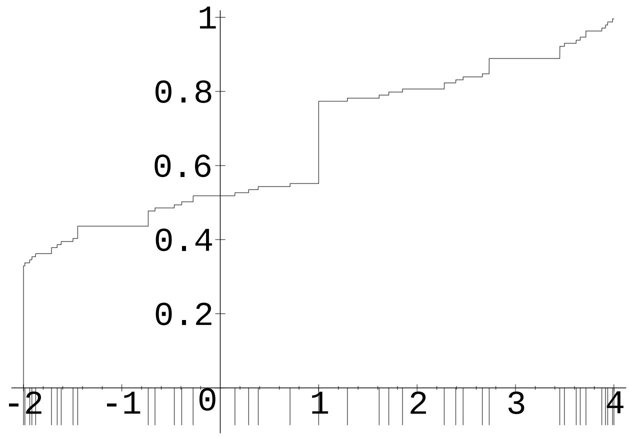


FIGURE 9. The Cantor-Set spectrum of Δ_n for $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

process being the Schreier graph of G/P. We describe these rules for our examples: $G, \tilde{G}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$.

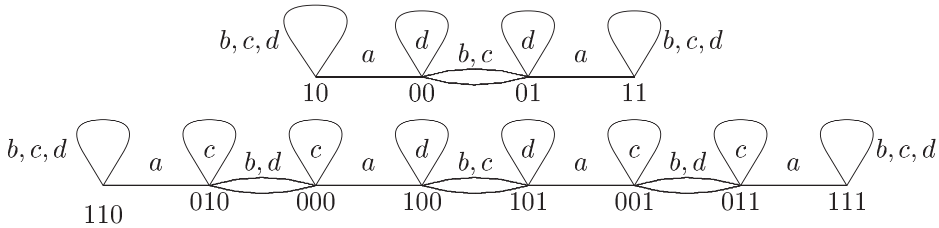
5.1. $\mathcal{S}(G, P, S)$. We describe the graphs $\mathcal{G}_n = \mathcal{S}(G_n, P_n, S)$. They will be with edges labeled by S (and not oriented, because all $s \in S$ are involutions) and vertices labeled by Σ^n, where $\Sigma = \{0, 1\}$.

First, it is clear that $\mathcal{G}_0$ is a graph on one vertex, labeled by the empty sequence $\emptyset$, and four loops at this vertex, labeled by a, b, c, d. Next, $\mathcal{G}_1$ has two vertices, labeled by 0 and 1; an edge labeled a between them; and three loops at 0 and 1 labeled by b, c, d.

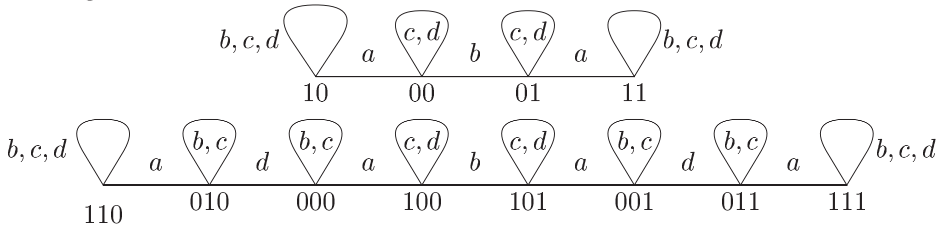
Now given $\mathcal{G}_n$, for some $n \geq 1$, perform on it the following transformation: replace the edge-labels b by d, d by c, c by b; replace the vertex-labels σ by 1σ; and replace all edges labeled by a connecting σ and τ by: edges from 1σ to 0σ and from 1τ to 0τ, labeled a; two edges from 0σ to 0τ labeled b and c; and loops at 0σ and 0τ labeled d. We claim the resulting graph is $\mathcal{G}_{n+1}$.

To prove the claim, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If $b(\sigma) = \tau$, then $d(1\sigma) = 1\tau$, and similarly for c and d; this explains the cyclic permutation of the labels b, c, d. The other substitutions are verified similarly.

As an illustration, here are $\mathcal{G}_2$ and $\mathcal{G}_3$ for G:



The graphs $\mathcal{G}_2$ and $\mathcal{G}_3$ for $\tilde{G}$ are similar:



5.2. $\mathcal{S}(\Gamma, P, S)$. Recall that for Γ we take $\Sigma = \{0, 1, 2\}$. Let us write $\mathcal{G}_n = \mathcal{S}(\Gamma, P_n, S)$. First, $\mathcal{G}_0$ has one vertex, labeled by the empty sequence $\emptyset$, and four loops, labeled a, a^{-1}, t, t^{-1}. Next, $\mathcal{G}_1$ has three vertices, labeled $0, 1, 2$, cyclically connected by a triangle labeled a, a^{-1}, and with two loops at each vertex, labeled t, t^{-1}. In the pictures only geometrical edges, in pairs $\{a, a^{-1}\}$ and $\{t, t^{-1}\}$, are represented.

Now given $\mathcal{G}_n$, for some $n \geq 1$, perform on it the following transformation: replace the vertex-labels σ by 2σ; replace all triangles labeled by a, a^{-1} connecting ρ, σ, τ by: three triangles labeled by a, a^{-1} connecting respectively $0\rho, 1\rho, 2\rho$ and $0\sigma, 1\sigma, 2\sigma$ and $0\tau, 1\tau, 2\tau$; a triangle labeled by t, t^{-1} connecting $0\rho, 0\sigma, 0\tau$; and loops labeled by t, t^{-1} at $1\rho, 1\sigma, 1\tau$. We claim the resulting graph is $\mathcal{G}_{n+1}$.

As above, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If $a(\rho) = \sigma$ and $t(\rho) = \tau$, then $t(0\rho) = 0\sigma$, $t(1\rho) = 1\rho$ and $t(2\sigma) = 2\tau$. The verification for a edges is even simpler.

5.3. **Substitutional graphs.** The three Schreier graphs presented in the previous subsection are special cases of *substitutional graphs*, which we define below.

Substitutional graphs were introduced in the late 70's to describe growth of multicellular organisms. They bear a strong similarity to L-systems [RS80], as was noted by Mikhael Gromov [Gro84]. Another notion of graph substitution has been studied by [Pre98], where he had the same convergence preoccupations as us.

Let us make a conventions in this section: all graphs $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ shall be *labeled*, i.e. endowed with a map $E(\mathcal{G}) \to C$ for a fixed set C of colours, and *pointed*, i.e. shall have a distinguished vertex $* \in V(\mathcal{G})$. A *graph embedding* $\mathcal{G}' \hookrightarrow \mathcal{G}$ is just an injective map $E(\mathcal{G}') \to E(\mathcal{G})$ preserving the adjacency operations.

Definition 5.1. *A* substitutional rule *is a tuple* $(U, R_1, \ldots, R_n)$*, where* U *is a finite* d*-regular edge-labeled graph, called the* axiom*, and each* R_i *is a rule of the form* $X_i \to Y_i$*, where* X_i *and* Y_i *are finite edge-labeled graphs. The graphs* X_i *are required to have no common edge. Furthermore, there is a inclusion, written* ι_i*, of the*

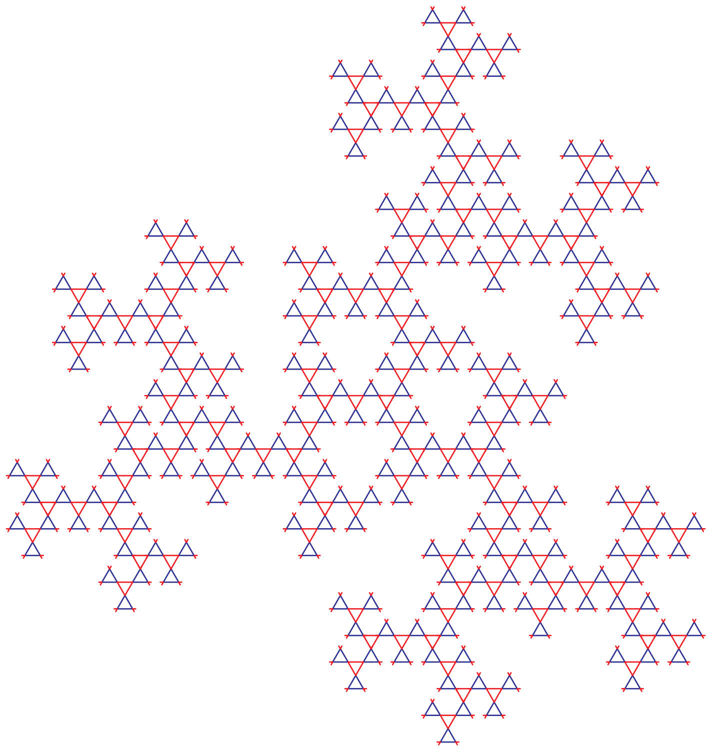


FIGURE 10. The Schreier Graph of Γ_6. The edges represent the generators s and a.

vertices of X_i in the vertices of Y_i; the degree of $\iota_i(x)$ is the same as the degree of x for all $x \in V(X_i)$, and all vertices of Y_i not in the image of ι_i have degree d.

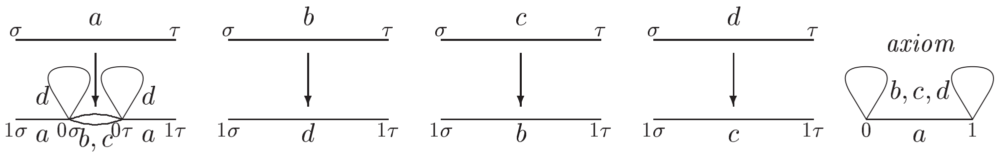
Given a substitutional rule, one sets $\mathcal{G}_0 = U$ and constructs iteratively $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$ by listing all embeddings of all X_i in $\mathcal{G}_n$ (they are disjoint), and replacing them by the corresponding Y_i. If the base point $*$ of $\mathcal{G}_n$ is in a graph X_i, the base point of $\mathcal{G}_{n+1}$ will be $\iota_i(*)$.

Note that this expansion operation preserves the degree, so $\mathcal{G}_n$ is a d-regular finite graph for all n. We are interested in fixed points of this iterative process.

For any $R \in \mathbb{N}$, consider the balls $B_n(*, R)$ of radius R at the base point $*$ in $\mathcal{G}_n$. Since there is only a finite number of rules, there is an infinite sequence $n_0 < n_1 < \dots$ such that the balls $B_{n_i}(*, R) \subseteq \mathcal{G}_{n_i}$ are all isomorphic. We consider $\mathcal{G}$, a limit graph in the sense of [GŻ97] (the limit exists), and call it a *substitutional graph.*

Note that in case some rule $X_i \to Y_i$ does not satisfy $X_i \subset Y_i$, there may be more than one limit graph — this is the case in the following example (where there are in fact three limit graphs, according to whether the next-to-rightmost edge is b, c or b, d or c, d).

Theorem 5.2. *The following four substitutional rules describe the Schreier graph of G:*



where the vertex inclusions are given by $\sigma \mapsto 1\sigma$ and $\tau \mapsto 1\tau$. The base point is the vertex $111\dots$.

Proof. Consider the Schreier graph $\mathcal{G}_n$ associated to the action of G on Σ^n, the n-th level of the tree $\mathcal{T}_\Sigma$. The vertex set of $\mathcal{G}_n$ is Σ^n, and its edges are described by the action of G. Note first that the axiom is $\mathcal{G}_0$.

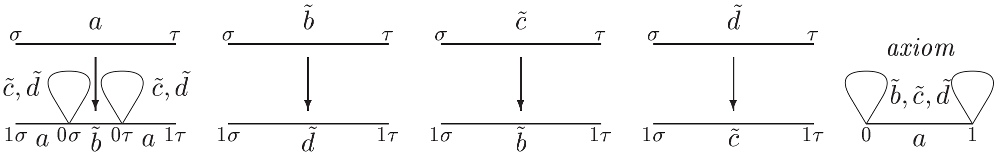
We construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$. Split Σ^{n+1} as $0\Sigma^n \sqcup 1\Sigma^n$. By virtue of the definition of ϕ given in (2), the b, c, d-edges within $1\Sigma^n$ are in bijection to the c, d, b-edges in $\mathcal{G}_n$, while the b, c-edges within $0\Sigma^n$ are in bijection with the a-edges in $\mathcal{G}_n$, and there are d-loops at all $\sigma \in 0\Sigma^n$. Moreover there are "parallel edges" labeled a between 0σ and 1σ for all $\sigma \in \Sigma^n$.

Now consider any b, c, d-edge in $\mathcal{G}_n$, say between σ and τ. In $\mathcal{G}_{n+1}$, it gives rise to a d, b, c-edge between 1σ and 1τ.

Consider then an a-edge in $\mathcal{G}_n$ between σ and τ. In $\mathcal{G}_{n+1}$, it gives rise to the following subgraph: an a-edge from 1σ to 0σ; two edges, labeled b and c, from 0σ to 0τ; an a-edge from 0τ to 1τ; and two loops, labeled d at 0σ and 0τ. This is precisely the substitutional rule for a, completing the proof. □

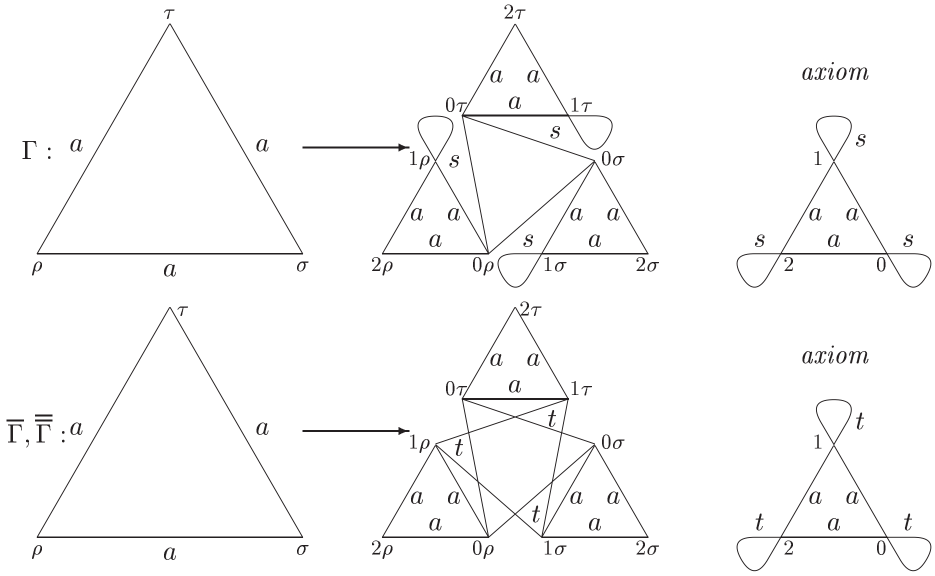
We omit the similar proof of the following result:

Theorem 5.3. *The following four substitutional rules describe the Schreier graph of $\overline{G}$:*



where the vertex inclusions are given by $\sigma \mapsto 1\sigma$ and $\tau \mapsto 1\tau$. The base point is the vertex $111\dots$.

Theorem 5.4. *The substitutional rules producing the Schreier graphs of Γ and $\overline{\Gamma}$ are given below. Remember that the Schreier graphs of $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are isomorphic:*



where the vertex inclusions are given by $\rho \mapsto 2\rho$, $\sigma \mapsto 2\sigma$ and $\tau \mapsto 2\tau$. The base point is the vertex $222\dots$.

Proof. We prove the claim for Γ only, as the same reasoning applies for $\overline{\Gamma}$. Consider the Schreier graph $\mathcal{G}_n$ associated to the action of G on Σ^n, the n-th level of the tree $\mathcal{T}_\Sigma$. The vertex set of $\mathcal{G}_n$ is Σ^n, and its edges are described by the action of G. Note first that the axiom is $\mathcal{G}_0$.

We construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$. Split Σ^{n+1} as $0\Sigma^n \sqcup 1\Sigma^n \sqcup 2\Sigma^n$. By virtue of the definition of ϕ given in (3), the s-edges within $2\Sigma^n$ are in bijection to the s-edges in $\mathcal{G}_n$, while the s-edges within $0\Sigma^n$ are in bijection with the a-edges in $\mathcal{G}_n$, and there are s-loops at every $\sigma \in 1\Sigma^n$. Moreover there are "parallel triangles" labeled a between 0σ, 1σ and 2σ for all $\sigma \in \Sigma^n$.

Now consider any s-edge in $\mathcal{G}_n$, say between σ and τ. In $\mathcal{G}_{n+1}$, it remains an s-edge, but now between 2σ and 2τ.

Consider then an a-edge in $\mathcal{G}_n$ between σ and τ. In $\mathcal{G}_{n+1}$, it gives rise to the following subgraph: an a-triangle between 0σ, 1σ and 2σ; an s-edge between 0σ and 0τ; s-loops at 1σ and 1τ; and an a-triangle between 0τ, 1τ and 2τ. Actually the a-edges form triangles so these subgraphs overlap at the a-triangles and s-loops. This justifies the substitutional rule for a-triangles, completing the proof. $\square$

By Proposition 2.11, the limit graphs have asymptotically polynomial growth of degree $\log_2(3)$.

Note that there are maps $\pi_n : V(\mathcal{G}_{n+1}) \to V(\mathcal{G}_n)$ that locally (i.e. in each copy of some right-hand rule Y_i) are the inverse of the embedding ι_i. In case these π_n

are graph morphisms, and one can consider the projective system $\{\mathcal{G}_n, \pi_n\}$, and its inverse limit $\widehat{\mathcal{G}} = \varprojlim \mathcal{G}_n$, which is a profinite graph [RZ99]. We devote our attention to the discrete graph $\mathcal{G} = \varinjlim \mathcal{G}_n$.

The growth series of $\mathcal{G}$ can often be described as an infinite product. We give such an expression for the graph in Figure 10, making use of the fact that $\mathcal{G}$ "looks like a tree" (even though it is amenable).

Consider the finite graphs $\mathcal{G}_n$; recall that $\mathcal{G}_n$ has 3^n vertices. Let D_n be the diameter of $\mathcal{G}_n$ (maximal distance between two vertices), and let $\gamma_n = \sum_{i\in\mathbb{N}} \gamma_n(i) X^i$ be the growth series of $\mathcal{G}_n$ (here $\gamma_n(i)$ denotes the number of vertices in $\mathcal{G}_n$ at distance i from the base point $*$).

The construction rule for $\mathcal{G}$ implies that $\mathcal{G}_{n+1}$ can be constructed as follows: take three copies of $\mathcal{G}_n$, and in each of them mark a vertex V at distance D_n from $*$. At each V delete the loop labeled s, and connect the three copies by a triangle labeled s at the three V's. It then follows that $D_{n+1} = 2D_n + 1$, and $\gamma_{n+1} = (1+2X^{D_n+1})\gamma_n$. Using the initial values $\gamma_0 = 1$ and $D_0 = 0$, we obtain by induction

$$D_n = 2^n - 1, \qquad \gamma_n = \prod_{i=0}^{n-1} (1 + 2X^{2^i}).$$

We also have shown that the ball of radius 2^n around $*$ contains 3^n points, so the growth of $\mathcal{G}$ is at least $n^{log_2(3)}$. But Proposition 2.11 shows that it is also an upper bound, and we conclude:

Proposition 5.5. *Γ is an amenable 4-regular graph whose growth function is transcendental, and admits the product decomposition*

$$\gamma(X) = \prod_{i\in\mathbb{N}} (1 + 2X^{2^i}).$$

It is planar, and has polynomial growth of degree $\log_2(3)$.

Any graph is a metric space when one identifies each edge with a disjoint copy of an interval $[0, L]$ for some $L > 0$. We turn $\mathcal{G}_n$ in a diameter-1 metric space by giving to each edge in $\mathcal{G}_n$ the length $L = \operatorname{diam}(\mathcal{G}_n)^{-1}$. The family $\{\mathcal{G}_n\}$ then converges, in the following sense:

Let A, B be closed subsets of the metric space (X, d). For any ϵ, let $A_\epsilon = \{x \in X \mid d(x, A) \le \epsilon\}$, and define the *Hausdorff distance*

$$d_X(A, B) = \inf\{\epsilon \mid A \subseteq B_\epsilon, B \subseteq A_\epsilon\}.$$

This defines a metric on closed subsets of X. For general metric spaces (A, d) and (B, d), define their *Gromov-Hausdorff distance*

$$d^{GH}(A, B) = \inf_{X,i,j} d_X(i(A), j(B)),$$

where i and j are isometric embeddings of A and B in a metric space X.

We may now rephrase the considerations above as follows: the sequence $\{\mathcal{G}_n\}$ is convergent in the Gromov-Hausdorff metric. The limit set $\mathcal{G}_\infty$ is a compact metric space.

The limit spaces are then: for G and $\tilde{G}$, the limit $\mathcal{G}_\infty$ is the interval $[0, 1]$ (in accordance with its linear growth, see Proposition 2.11). The limit spaces for Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are fractal sets of dimension $\log_2(3)$.

When the present work was completed Stanislav Smirnov informed us about the article [Mal95] which has some connection to our paper. A substitutional tree $\mathcal{T}$ is constructed in [Mal95], and the spectrum of the Markov operator on $\mathcal{T}$ is of the form $K \sqcup P$, where K is the Julia set of some quadratic map and P is countable. The approximation arguments used in [Mal95] are different from ours and does not use any amenability assumption (indeed, the graphs constructed there have eigenvalues with compactly supported eigenvectors).

An essential difference is that the graph in [Mal95] is not regular and therefore is not a Schreier graph of a group.

6. Concluding Remarks and Problems

Let us draw here some conclusions and formulate a few questions. We hope that our methods will prove useful in the investigation of spectral properties of the Laplace operator Δ for other groups acting on rooted trees. By computing the spectrum of the Hecke type operator of a quasi-regular representation $\rho_{G/P}$ we obtained a subset of the spectrum of Δ; namely, $\operatorname{spec}(\rho_{G/P}) \subseteq \operatorname{spec}(\rho_G)$ as soon as P is amenable. Moreover, in one case (that of $\tilde{G}$) we obtained $\operatorname{spec}(\Delta)$ using the bipartitivity of $\tilde{G}$'s Cayley graph. More research must be done in the nonamenable case, in particular

Question 1. *Under which conditions (besides amenability) does one have* $\operatorname{spec}(\rho_{G/P}) \subseteq \operatorname{spec}(\rho_G)$*? When does one have* $\operatorname{spec}(\rho_{G/P}) = \operatorname{spec}(\rho_G)$*?*

We also hope that the methods in this article can be useful to find residually finite examples of nonamenable groups without free subgroups (all known examples are non-residually-finite). The non-amenability of the group considered could be proven by showing that 1 is an isolated point in the spectrum of the associated dynamical system.

Question 2. *Is there a finitely generated subgroup G of* $\operatorname{Aut}(\mathcal{T})$ *with no non-abelian free subgroup and such that* $1 \notin \operatorname{spec}(\rho_{G/P})$*?*

Examples of groups G with $1 \notin \operatorname{spec}(\rho_{G/P})$ are also interesting because they provide sequences of expanding graphs [Lub94], namely the Schreier graphs $\mathcal{S}(G, P_n, S)$. In some cases these graphs can be Ramanujan graphs. We formulate the following question with Andrzej Żuk:

Question 3. *Are there groups* $G < \operatorname{Aut}(\mathcal{T})$*, generated by a finite set S of finite automata, such that the sequence of graphs* $\mathcal{S}(G, P_n, S)$ *is* (i) *a sequence of expanders;* (ii) *a sequence of Ramanujan graphs?*

The problem of factoring the resolvent of operators of Hecke type is related to the decomposition of ρ_{G/P_n} in irreducible representations. It can be shown that any finite-dimensional irreducible representation of G is a subrepresentation in a tensor product of sufficiently many copies of ρ_{G/P_n} (in other words, the representation ring $R(G)$ is generated by the irreducible subrepresentations of ρ_{G/P_n}). We therefore hope that our methods will extend the knowledge on the irreducible representations of G.

We constructed a virtually torsion-free group, Γ, with totally disconnected spectrum $\mathrm{spec}(\rho_{\Gamma/P})$. To answer in the negative to the Baum-Connes conjecture, as well as to the Kadison-Kaplansky conjecture [Val89], it would suffice to construct a torsion-free group with a gap in the spectrum of its Laplace operator. Therefore one should delete the "virtually" and replace $\rho_{\Gamma/P}$ by ρ_Γ in order to improve our results.

Question 4. *Is there a torsion-free group $G < \mathrm{Aut}(\mathcal{T})$ with totally disconnected spectrum $\mathrm{spec}(\rho_G)$? Or with a gap in its spectrum?*

7. Acknowledgments

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