$U(1)^m$ Modular Invariants, N=2 Minimal Models, and the Quantum Hall Effect

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Abstract

The problem of finding all possible effective field theories for the quantum Hall effect is closely related to the problem of classifying all possible modular invariant partition functions for the algebra $\widehat{u(1)}^{\oplus m}$, as was argued recently by Cappelli and Zemba. This latter problem is also a natural one from the perspective of conformal field theory. In this paper we completely solve this problem, expressing the answer in terms of self-dual lattices, or equivalently, rational points on the dual Grassmannian $G_{m,m}(\mathbb{R})^*$. We also find all modular invariant partition functions for affine $su(2)\oplus u(1)^{\oplus m}$, from which we obtain the classification of all $N=2$ superconformal minimal models. The 'A-D-E classification' of these, though often quoted in the literature, turns out to be a very coarse-grained one: e.g. associated with the names E_6, E_7, E_8, respectively, are precisely 20,18,8 different partition functions. As a by-product of our analysis, we find that the list of modular invariants for $\widehat{su(2)}$ lengthens surprisingly little when commutation with T – i.e. invariance under $\tau \mapsto \tau+1$ – is ignored: the other conditions are far more essential.

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1. Introduction

The quantum Hall effect for 2-dimensional incompressible quantum fluids has received considerable attention in recent years, both from theorists and experimentalists (see e.g. [1]). First observed experimentally in the early 1980s, a major theoretical step was done by Laughlin and his theory of plateaux. Experimentally, one observes universality – i.e. some features of the effect, e.g. the possible values of the filling factor ν, are largely independent of impurities and geometry, for example.

One is thus led to study universality classes of incompressible quantum Hall fluids by effective field theories, in the long-distance/low-temperature limit. There are at least two main approaches to this. One (see e.g. [2] and references therein) starts with an abelian Chern-Simons theory, while the other (see e.g. [3] and references therein) expresses incompressibility algebraically and investigates $W_{1+\infty}$ conformal field theories. The two approaches are related, and in recent work [4] (see also [5]) proposed looking at modular invariant partition functions for these theories. Both these approaches correspond to looking at modular invariant sesquilinear combinations of the characters of the affine algebra $U_m := \widehat{u(1)}^{\oplus m}$ at some (matrix-valued) level k.

There is a family of rational conformal field theories (RCFTs) for each choice of current(=nontwisted affine Kac-Moody) algebra g (see e.g. [6] and references therein for a review of this problem), and the choice $g = U_m$ is a natural one from this perspective as well. In this paper we find all such partition functions. The solution has a simple geometric description in terms of self-dual lattices, or equivalently rational points on the dual of the Grassmannian $G_{m,m}(\mathbb{R})$. The theories in [2] correspond to a small subset of these, namely the diagonal partition functions. [4] have suggested that some of the non-diagonal partition functions provide a natural explanation of some of the plateaux falling out of the Jain sequence, which have been experimentally observed (e.g. $\nu = 4/11$).

Of course the connection between lattices and the quantum Hall effect is well-known (see e.g. [2]). The difference here is that the lattices are all self-dual, and have dimension $2m$ (instead of m).

As is well-known, the quantum Hall theorists are plagued by the difficulty of having too many possible effective field theories to choose from – far more than have been observed experimentally. What still seems to be missing is an understanding of the stability, i.e. width, of the plateaux – it appears only heuristic proposals have so far been made. This short paper cannot contribute to this difficult problem, except indirectly by providing a complete list of the possible effective theories (more precisely, a complete list of the possible partition functions, which provide all possible spectra of these theories).

The second classification we obtain in this paper is that of the $N = 2$ superconformal minimal models. The conformal (i.e. $N = 0$) minimal models are classified in [7], and the $N = 1$ ones in [8]. The $N = 2$ super-Virasoro algebra is of great interest because $N = 1$ space-time supersymmetry in string theory is related to $N = 2$ world-sheet supersymmetry (e.g. [9] uses the $N = 2$ minimal models to compactify the heterotic string), and also because of the possible relation of the $N = 2$ models with Calabi-Yau manifolds and with Landau-Ginzburg theories. The classification of the $N = 2$ minimal models has been addressed many times in the literature (see e.g. [10,9]). An A-D-E classification is often claimed. To our knowledge this paper gives the first rigourous and complete classification of the possible $N = 2$ minimal model partition functions. The previous attempts generally assume that some sort of factorisation holds here at the level of the individual partition functions themselves, an assumption which is simply wrong. Thus we find many more partition functions, and unfortunately there seems no natural relation between our list and the A-D-E pattern.

The relation between the effective field theories for the quantum Hall effect, and the $N = 2$ minimal models, is that their classifications reduce to the modular invariant classifications of U_m and $\hat{A}_1 \oplus U_2$, respectively, and the techniques used to solve U_m help to solve $\hat{A}_1 \oplus U_m$.

The activity at present concerning the classification of modular invariants is following a clearly defined program (see e.g. [6]) aiming at achieving this classification for all *simple* affine algebras. The present paper falls outside this program. Its justification is that it accomplishes the classification for two infinite families of (non-simple) algebras, both of which concern problems of immediate physical interest.

2. The two problems

The notions of *lattice* Λ, its *dual* Λ^*, and its *determinant* $|\Lambda|$, are well-known. An *integral* lattice obeys $\Lambda \subset \Lambda^*$, and a *self-dual* one obeys $\Lambda = \Lambda^*$. Equivalently, Λ is self-dual iff it is integral and has determinant $|\Lambda| = 1$. An integral lattice is *even* if all its norms x^2 are even, otherwise it is called *odd.* The operation $\oplus$ denotes orthogonal direct sum. See e.g. Chapter 2 of [11] for definitions.

An RCFT possesses a finite set P_+ of labels (weights), and a complex-valued function (character) χ_a for each $a \in P_+$. The modular group $SL_2(\mathbb{Z})$ acts on these χ_a:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \chi_a = \sum_{b \in P_+} S_{a,b}\, \chi_b \tag{2.1a}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \chi_a = \sum_{b \in P_+} T_{a,b}\, \chi_b \ . \tag{2.1b}$$

S and T are unitary and symmetric, and T is diagonal. There is a distinguished weight $0 \in P_+$ for which

$$S_{0,a} \geq S_{0,0} > 0 \ . \tag{2.1c}$$

The spectrum of the RCFT is encoded in its (genus 1) partition function

$$\mathcal{Z} = \sum_{a,b \in P_+} M_{a,b}\, \chi_a\, \chi_b^* \ . \tag{2.2a}$$

(Strictly speaking, the a and b in (2.2a) may come from different sets P_+^L, P_+^R, respectively – such M are called *heterotic* and do occur in this paper. For notational simplicity we will usually ignore this technicality. For example, we will never write e.g. S^L or 0_L – no confusion should result.) The coefficient matrix M obeys

$$M_{a,b} \in \mathbb{Z}_{\geq} \qquad \forall a, b \in P_+ \ , \tag{2.2b}$$
$$M_{0,0} = 1 \ . \tag{2.2c}$$

Usually in a RCFT one requires invariance of $\mathcal{Z}$ under the full modular group $SL_2(\mathbb{Z})$:

$$SM = MS \tag{2.3a}$$
$$TM = MT \ . \tag{2.3b}$$

By *physical invariant* is meant any matrix M, or equivalently the corresponding function $\mathcal{Z}$ in (2.2a), obeying (2.2b), (2.2c), (2.3a) and (2.3b). We will use the term *weak invariant* to denote any M (or $\mathcal{Z}$) obeying (2.2b), (2.2c) and (2.3a). In this paper we classify all the physical/weak invariants for certain choices of χ_a, motivated by the quantum Hall effect and the $N = 2$ super-Virasoro algebra. The physical invariant classification for other χ_a has been the subject of much work – see e.g. [7,8,6] and references therein.

Incidently, equations (2.3) require the χ_a to be linearly independent. This is usually accomplished in practice by giving them full variable dependence (i.e. including zero-mode oscillations).

A rich source of RCFT data comes from the representations of affine algebras. The representation theory of the affine algebra $U_m = \widehat{u(1)^{\oplus m}}$ at level k (more concisely, $U_{m,k}$) is well-known – see e.g. Ch. 12 of [12]. k here is an $m \times m$ symmetric integral matrix – it is common to call it 'level' by analogy with the other affine algebras although its nature here is a little different. Let Γ_k be the corresponding integral lattice, i.e. it will have a basis $\{e_1, \ldots, e_m\}$ satisfying $e_i \cdot e_j = k_{ij}$. Let $\{e_1^*, \ldots, e_m^*\}$ be the corresponding dual basis. There is an integrable representation of U_m for each choice of highest weight $\lambda \in \overline{P}^k := \Gamma_k^*/\Gamma_k$; its character $\overline{\chi}_\lambda^k(\tau, z)$ is proportional to the m-dimensional theta function θ_λ, where the proportionality constant is independent of λ, and where

$$\theta_S(z|\tau) = \sum_{x \in S} \exp[\pi \mathrm{i}\, \tau x^2 + 2\pi \mathrm{i}\, z \cdot x]$$

for any set $S \subset \mathbb{Q} \otimes \Gamma_k$. Here, $z \in \mathbb{C} \otimes \Gamma_k$, $\tau \in \mathbb{C}$; when S is the translate of an m-dimensional lattice, θ_S will converge for $\mathrm{Im}(\tau) > 0$ provided the lattice is Euclidean, i.e. provided in our case k is positive definite.

The simplest case is when k is diagonal, in which case $\Gamma_k = (\sqrt{k_{11}}\,\mathbb{Z}) \oplus \cdots \oplus (\sqrt{k_{mm}}\,\mathbb{Z})$, $\Gamma_k^* = (\sqrt{\frac{1}{k_{11}}}\mathbb{Z}) \oplus \cdots \oplus (\sqrt{\frac{1}{k_{mm}}}\mathbb{Z})$, $e_i^* = e_i/k_{ii}$, and θ_λ reduces to a product of 1-dimensional theta functions. We will usually denote a weight $\lambda \in \overline{P}^k$ for such k by its (integer) components with respect to this dual basis e_i^*.

Strictly speaking, highest weight representations of U_m require k to be diagonal (as well as positive definite and integral). However all of our formulas and arguments are independent of this restriction. Moreover, the more important structure for CFT is the chiral algebra(=vertex operator algebra), which is well-defined for any positive definite, integral k (non-integral k would correspond to irrational CFTs).

We will thus assume throughout this paper that the level k is positive definite and integral, though not necessarily diagonal. Positive-definiteness is necessary for convergence of the partition function, and hence for the existence of a RCFT. Physically (i.e. for the quantum Hall effect discussed below), this would correspond to all the excitations on each edge having equal chirality; the more general situation (of mixed chiralities) can be easily accommodated within this picture by using the following recipe: first find an orthogonal sublattice of Γ_k, using Gram-Schmidt – i.e. find independent vectors $v_i \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_i$, $i = 1, \ldots, m$, such that $v_i \cdot v_j = 0$ for $i \neq j$. Provided each $v_i^2 \neq 0$, define k' to be the diagonal matrix $(|v_1^2|) \oplus \cdots \oplus (|v_m^2|)$. What we have effectively done is moved all the excitations with wrong chirality to the opposite edge. The original mixed chirality theory will then be constructable from one of the ones at level k' by returning all the excitations to their proper edge. This is precisely what we do below with e.g. $U_1 \oplus \hat{A}_{m-1,1}$ theories – see (2.6) – as well as in the correspondence between $N = 2$ minimal models and $\hat{A}_1 \oplus U_2$ theories in (6.1d). This recipe breaks down when some of the $v_i^2 = 0$, but for such a case it would be doubtful that the theory could correspond naturally to a RCFT.

The modular transformation properties of $\overline{\chi}_\lambda^k$ are given by the matrices

$$\overline{T}^2_{\lambda,\mu} = \delta_{\lambda,\mu} \exp[2\pi\mathrm{i}\,(\lambda^2 - \frac{m}{12})], \tag{2.4a}$$

$$\overline{S}_{\lambda,\mu} = \frac{1}{\sqrt{|k|}} \exp[-2\pi\mathrm{i}\,\lambda \cdot \mu]. \tag{2.4b}$$

These correspond to the transformations $(\tau, z) \mapsto (\tau + 2, z)$, and $(\tau, z) \mapsto (-1/\tau, z/\tau)$, respectively (in this second transformation we are ignoring a multiplicative factor which is not important for our purposes). Both matrices are unitary and symmetric. We use the notation $\overline{T}^2$ purely formally here – its square-root $\overline{T}$ will exist iff each k_{ii} is even.

The partition function $\overline{\mathcal{Z}}$ built from these $\overline{\chi}_\lambda^k$ enters naturally into the classification problem of effective field theories for incompressible quantum Hall fluids for generic hierarchical plateaux, where it describes the pairings of excitations on the two edges of an annulus – see [4,5] for a discussion. For example, $m = \dim \Gamma_k$ corresponds to the number of independent bosons, i.e. edge currents, and equals the central charge of the RCFT ($m = 1$ for Laughlin fluids). There are two main differences introduced here from the generic RCFT situation. One is that for quantum Hall fluids equation (2.3b) should be weakened to

$$T^2 M = M T^2\ , \tag{2.5a}$$

where of course $T^2 = \overline{T}^2$ here. The other difference is that there is a vector $t \in \Gamma_k^*$ in terms of which the charge of the edge excitation $\lambda \in \overline{P}^k$ is given by $t \cdot \lambda$. For quantum Hall fluids we should have M commuting with the matrices U_t and V_t defined by

$$(U_t)_{\lambda,\mu} = \delta_{\lambda,\mu}\ \exp[2\pi\mathrm{i}\, t \cdot \lambda] \tag{2.5b}$$

$$(V_t)_{\lambda,\mu} = \delta_{\lambda+t,\mu}\ \exp[-\pi\mathrm{i}\,\mathrm{Re}(\tau)\, t^2 - 2\pi\mathrm{i}\,\mathrm{Re}(t \cdot z)]\ . \tag{2.5c}$$

Physically, U_t says that edge excitations should have integer total charge, while V_t is related to spectral flow. There are other properties that $\overline{\mathcal{Z}}$ is expected to obey in order for the theory to have a chance at being physical [4], but these are all which will be considered here.

Most observed plateaux lie in the Jain series with filling factor $\nu = m/(ms \pm 1)$, for s even. One intriguing explanation of those involves the $W_{1+\infty}$ minimal models [3], but unfortunately these do not possess [4] a modular invariant partition function in the sense given here and so have an unclear RCFT interpretation. Instead, the partition functions considered here correspond to 'generic' $W_{1+\infty}$ RCFTs. The Jain series is obtained, both in the generic $W_{1+\infty}$ theories and in the abelian Chern-Simons theories, when the U_m algebra extends to $U_1 \oplus \hat{A}_{m-1,1}$, where $\hat{A}_{m-1,1}$ is affine A_{m-1} at level 1. Thus we also would like to know the modular behaviour of the characters of $\hat{A}_{m-1,1}$. These turn out (Thm. 13.8 of [12]) to be given by the complex conjugates of the $\overline{S}$ and $\overline{T}^2$ matrices for U_1 at level $k = m$ (up to an irrelevant constant factor in the T^2 matrix). In particular the $\hat{A}_{m-1,1}$ weight $\lambda = \Lambda_i$ corresponds to the U_1 weight $\lambda = i$. Hence the weak invariants M for $U_{m,k} \oplus \hat{A}_{n,1}$ are in natural one-to-one correspondence with the weak invariants $\overline{M}$ for $U_{m+1,k\oplus(n+1)}$, with the correspondence given by

$$M_{\lambda,\Lambda_i;\mu,\Lambda_j} = \overline{M}_{\lambda,j;\mu,i} \tag{2.6}$$

(note the $i \leftrightarrow j$ switch on the right side).

The other algebra we are interested in is $\hat{A}_1$. The level k here is a nonnegative integer, and its level k weights can be taken to be the set $P_+^k = \{0, 1, \ldots, k\}$. Its characters $\chi_a^k(\tau, z, u)$ can also be expressed using theta functions (Ch. 13 of [12]), and its modular matrices are

$$S_{a,b} = \sqrt{\frac{2}{k+2}}\sin[\pi\frac{(a+1)\,(b+1)}{k+2}] \,, \tag{2.7a}$$

$$T_{a,b} = \delta_{a,b}\,\exp[\pi \mathrm{i}\,\{(a+1)^2/2(k+2) - 1/4\}] \,. \tag{2.7b}$$

In particular, the set of highest weights for $\hat{A}_{1,k} \oplus U_{m,\ell}$ is $P_+^k \times \overline{P}^\ell$, and the modular matrices are $S \otimes \overline{S}$, $T \otimes \overline{T}$. The relation of the $N = 2$ super-Virasoro algebra at $c = \frac{3k}{k+2}$, and $\hat{A}_{1,k} \oplus U_{1,4} \oplus U_{1,2k+4}$, is given at the start of Section 6.

3. The classification of U_m modular invariants

Throughout this paper we use the convenient notation $(x; y) := (x, \sqrt{-1}y)$ for any vector lying in the pseudo-Euclidean vector space $\mathbb{R} \otimes (\Gamma_k; \Gamma_k)$, where likewise $(\Lambda_1; \Lambda_2)$ denotes the indefinite lattice $\Lambda_1 \oplus \sqrt{-1}\Lambda_2$.

Theorem 1.

(a) The set of all weak invariants $\overline{\mathcal{Z}}$ (defined after (2.3)) is in a natural one-to-one correspondence with all self-dual $2m$-dimensional lattices Λ containing $(\Gamma_k; \Gamma_k)$; all these

will automatically obey (2.5a). When each k_{ii} is even, $\overline{T}$ exists and the physical invariants $\overline{\mathcal{Z}}$ (defined after (2.3)) correspond to these Λ which are in addition even.

(b) Choose any $t \in \Gamma_k^*$ and weak invariant $\overline{\mathcal{Z}}$, and let Λ be the corresponding lattice. Then $\overline{\mathcal{Z}}$ commutes with U_t iff it commutes with V_t, iff $(t;t) \in \Lambda$.

In particular, the partition function $\overline{\mathcal{Z}}$ of the theory is proportional to the indefinite theta function

$$\theta_\Lambda((z;z)|\tau) = \sum_{(x_L;x_R)\in\Lambda} \exp[\pi\mathrm{i}\,\tau x_L^2 - \pi\mathrm{i}\,\tau^* x_R^2 + 2\pi\mathrm{i}\,(z\cdot x_L - z\cdot x_R)] \tag{3.1a}$$

of the lattice Λ, and the coefficient matrix M in (2.2a) is given by

$$M_{\lambda,\mu} = \begin{cases} 1 & \text{if } (\lambda;\mu)\in\Lambda \\ 0 & \text{otherwise} \end{cases} . \tag{3.1b}$$

In the following section we discuss how to find these lattices Λ.

An alternate, level-independent, formulation of this classification using Grassmannians is also possible and very intriguing. Let Z_o^m be the set of all weak invariants for U_m, for arbitrary level k, which are not physical (i.e. violate (2.3b)), and let Z_e^m be the physical invariants for U_m. These then correspond to odd (resp. even) self-dual lattices Λ, by the correspondence of Theorem 1(a). Up to transformations in the full orthogonal group $SO(m,m)$, these lattices are unique: $I_{m,m} = (\mathbb{Z}^m;\mathbb{Z}^m)$ (resp. $II_{m,m} = II_{1,1}^m$ where $II_{1,1}$ has basis $\{(e/\sqrt{2};\pm e'/\sqrt{2})\}$. Recall [13] that the Grassmannian $G_{m,n}(\mathbb{R}) = SO(m+n)/(SO(m)\times SO(n))$ is an mn-dimensional compact symmetric space consisting of all m-dimensional subspaces of $\mathbb{R}^{m+n}$. Its dual $G_{m,n}(\mathbb{R})^* = SO(m,n)/(SO(m)\times SO(n))$ is noncompact and consists of all m-dimensional Euclidean subspaces in the pseudo-Euclidean space $\mathbb{R}^{m,n}$. By a rational point in $G_{m,n}(\mathbb{R})^*$ we mean an equivalence class containing a rational matrix, or equivalently a subspace V with a basis $\{f_1,\dots,f_m\}$ which can be written over $\mathbb{Q}$ in terms of the preferred orthonormal basis $\{e_1,\dots,e_m,e_1',\dots,e_n'\}$ of $\mathbb{R}^{m,n}$. Now the group $SO(m,m)$ acting on our lattices Λ will mix its left- and right-sectors and hence change the physics. On the other hand, the group $SO(m)$ acting separately on either side should preserve the physics. Hence we get a natural bijection between the physically distinct $\overline{\mathcal{Z}} \in Z_o^m$ or $\overline{\mathcal{Z}} \in Z_e^m$, and the set of rational points on $G_{m,m}(\mathbb{R})^*$ (or between numerically distinct $\overline{\mathcal{Z}}$'s in Z_o^m or Z_e^m, and rational points on $SO(m,m)$). In particular, the important lattices Λ_L, Λ_R defined shortly are given by $\Lambda_L = V\cap I_{m,m}$ and $\Lambda_R = \sqrt{-1}(V^\perp \cap I_{m,m})$ for all odd Λ, and $\Lambda_L = V\cap II_{m,m}$ and $\Lambda_R = \sqrt{-1}(V^\perp\cap II_{m,m})$ for even Λ.

What makes this picture interesting is that it gives many examples of what the 'moduli space' of certain families of RCFTs looks like. In particular we find that (at least as far as their genus 1 partition functions are concerned) the Wess-Zumino-Witten models corresponding to $\widehat{u(1)}^{\oplus m}$ form a dense subset of a noncompact m^2-dimensional symmetric space.

It is intriguing that the norm condition (2.5a) is redundant here. A more striking example of the irrelevance of (2.3b) or (2.5a) is provided in Thm. 2 below.

Part *1(a)* of the theorem gives the classification of all partition functions for RCFTs corresponding to U_m at any level k. The only other such classifications for all levels of an affine algebra are $\widehat{A_1}$, $\widehat{A_1 \oplus A_1}$, and $\widehat{A_2}$ (see [6] for the original references). In Section 5 we generalise Thm. 1 to the algebra $\hat{A}_1 \oplus U_m$ at any level.

Part *1(b)* of the theorem gives the complete classification of the effective field theories for quantum Hall fluids, assuming they possess a partition function $\overline{\mathcal{Z}}$ discussed in the previous section. The reason for believing they should is given in [4]. As mentioned earlier, this includes all generic (as opposed to minimal) $W_{1+\infty}$ theories, and all abelian Chern-Simons theories considered in e.g. [2].

The relation between Λ (or $\overline{\mathcal{Z}}$) and the physical quantities of the quantum Hall fluid are discussed in e.g. [2,4]. For example, the dimensionless Hall conductivity is $\sigma_H = t \cdot t$. In all cases the relevant level is not k, corresponding to the lattice Γ_k, but rather the matrices k_L and k_R corresponding to the largest m-dimensional sublattice $\Lambda_L := \Gamma_{k_L}$ of Λ, and $\Lambda_R := \Gamma_{k_R}$ of $\sqrt{-1}\Lambda$, which contains Γ_k. That is, we are interested in the 'maximally extended chiral algebras' of the theory, rather than the arbitrarily chosen subalgebra at level k. (In general it is a very difficult problem to find the maximally extended chiral algebras for an RCFT, but for U_m theories it is trivial.) As an example, the Wen topological order gives the degeneracy of the quantum Hall ground state on compact genus g surfaces, and will equal $|\Lambda_L|^g$, as can be seen directly from Verlinde's formula (this is discussed in [4]).

Because of this remark about chiral algebras, the $\overline{\mathcal{Z}}$'s in Theorem 1 will include redundancies caused by an inappropriate original choice of level k (incidently, these redundancies are avoided in the Grassmannian picture). To avoid these redundancies, it suffices to restrict attention to those Λ with $\Gamma_k = \Lambda_L$. But, in order to keep all the $\mathcal{Z}$'s obtained in Theorem 1, we are then required to allow 'heterotic' theories, i.e. theories whose 'left level' k_L need not equal its 'right level' k_R. In order to avoid the redundancies spoken of earlier, we would then supplement the conditions of the previous section with one more:

$$M_{\lambda,0} = \delta_{\lambda,0} \quad \text{and} \quad M_{0,\mu} = \delta_{\mu,0} \ , \qquad \forall \lambda \in \overline{P}^{k_L}, \ \mu \in \overline{P}^{k_R} \ . \tag{3.2a}$$

In order for solutions $\overline{\mathcal{Z}}$ to exist, it is necessary and sufficient to require that [14]

$$\Lambda_L^*/\Lambda_L \cong \Lambda_R^*/\Lambda_R \qquad \text{and} \qquad \mathbb{Q} \otimes \Lambda_L = \mathbb{Q} \otimes \Lambda_R \ . \tag{3.2b}$$

The first condition is the isomorphism of groups, and is required by the maximality property of Λ_L and Λ_R [14]. It says among other things that $|\Lambda_L| = |\Lambda_R|$ (see (4.1a)). The second statement states that Λ_L and Λ_R are rationally equivalent, and because Λ_L and Λ_R are integral is equivalent to the existence of $2m$-dimensional self-dual lattices Λ containing $(\Lambda_L; \Lambda_R)$. ([14] gives a practical algorithm for deciding when two lattices are rationally equivalent.) These two conditions are independent: e.g. $\Lambda_L = \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}$ and $\Lambda_R = A_2$ obey the first condition but not the second. It seems heterotic theories may not be directly physically relevant here, because the corresponding partition function will not be real. We will not consider this redundancy issue again in this paper, and will not impose (3.2a) (until Example 2 in the next section).

Heteroticity applies also to U_t, V_t: in general these will be replaced by U_{t_L}, U_{t_R}, etc. The U-commutativity constraint then becomes $U_{t_L} M = M U_{t_R}$, and Thm. 1(b) then becomes that U-commutativity is equivalent to $(t_L;t_R)\in\Lambda$ (heterotic V-commutativity is more complicated to interpret because of its z-dependence, but would require at least that $t_L^2=t_R^2$ in order to be equivalent to U-commutativity). If we insist that $t_{L,R}$ satisfy

$$x_L\cdot t_L-x_R\cdot t_R\equiv x_L^2-x_R^2\qquad(\mathrm{mod}\ 2)\qquad\forall(x_L;x_R)\in\Lambda\ ,\tag{3.3a}$$

then we can say much about $t_{L,R}$. They always exist (it is easy to see that if $\{e_1,\ldots,e_n\}$ is any basis of Λ and $\{e_1^*,\ldots,e_n^*\}$ is the dual basis, then

$$(t_L;t_R)=\sum_{i=1}^{n}e_i^2\,e_i^*\tag{3.3b}$$

satisfies (3.3a)). Although there is no unique solution $t_{L,R}$ to (3.3a) (if $(t_L;t_R)$ works, so will anything in $2\Lambda+(t_L;t_R)$), the physically important quantities $t_{L,R}^2$ are severely constrained. For example, $t_{L,R}^2$ will be an integral multiple of $1/|k_{L,R}|$ (since they must lie in $\Lambda_{L,R}^*$). Moreover, any $t_{L,R}$ satisfying (3.3a) will obey

$$t_L^2\equiv t_R^2\qquad(\mathrm{mod}\ 8)\ .\tag{3.3c}$$

The physical invariants for $\hat{A}_{m-1,1}$ are classified in [15]. [4] used this analysis to find many (but not all) weak invariants for U_2 obeying (2.5) for the choice $t=e_1^*$. [4] also found several (but not all) weak invariants for U_2 (ignoring (2.5)). Some $\overline{\mathcal{Z}}$ for the most physically interesting theories were found first in [5].

The theories corresponding to chiral quantum Hall lattices [2] are a small subset of the theories in *1(b)*. In particular, they correspond to the special cases where M in (3.1b) is the identity matrix. [4] gave a reason why the other M also seem interesting and should be considered: it has to do with finding simple theories corresponding to some experimentally observed plateaux not lying in the Jain sequences. We are not claiming however that our theorem trivialises in any way the work in e.g. [2]. They are really addressing the formidable task of finding explicit lists of those $\overline{\mathcal{Z}}$ in *1(b)* lying within the subclass of interest to them. As will be described in the next section, this is so challenging that it is hopeless in general, but is possible if one restricts to sufficiently small k and m, as they do.

The remainder of this section is devoted to a proof of the theorem. The argument closely follows the one given in Lemma 3.1 of [6], and is surprisingly simple.

Note first that (2.3a),(2.4b) and the unitarity of $\overline{S}$ implies

$$M_{\lambda,\mu}=\sum_{\alpha,\beta\in\overline{P}^k}\overline{S}_{\lambda,\alpha}\,M_{\alpha,\beta}\,\overline{S}_{\beta,\mu}^{\,*}=\frac{1}{|k|}\sum_{\alpha,\beta\in\overline{P}^k}\exp[2\pi\mathrm{i}\,(\mu\cdot\beta-\lambda\cdot\alpha)]\,M_{\alpha,\beta}\ .\tag{3.4a}$$

Taking absolute values and using the triangle inequality, (3.4a) becomes $|M_{\lambda,\mu}|\le|M_{0,0}|$ with equality, thanks to (2.2b) and (2.2c), iff the following holds:

$$M_{\alpha,\beta}\neq0\qquad\Longrightarrow\qquad\lambda\cdot\alpha\equiv\beta\cdot\mu\quad(\mathrm{mod}\ 1)\ ,\tag{3.4b}$$

for all $\alpha, \beta \in \overline{P}^k$. We know then that each $M_{\lambda,\mu} \in \{0,1\}$. Define a set Λ by

$$\Lambda = \bigcup_{\substack{\lambda,\mu \in \overline{P}^k \\ M_{\lambda,\mu}=1}} (\lambda;\mu) + (\Gamma_k;\Gamma_k) \ . \tag{3.4c}$$

Then (3.4b) implies that Λ is closed under addition and under multiplication by -1, and therefore is a lattice. Also, (3.4b) says that whenever $x, y \in \Lambda$, then $x \cdot y \in \mathbb{Z}$ – i.e. Λ is integral. Putting $\lambda = \mu = 0$ in (3.4a) says $\|\Lambda/(\Gamma_k;\Gamma_k)\| = |k| = |\Gamma_k|$ and hence that $|\Lambda| = 1$ (see (4.1a) below). Thus Λ is self-dual.

The rest of the theorem now follows quickly. Commutation with U_t says that $(t;t) \in \Lambda^* = \Lambda$, and while commutation with V_t says $(t;t) + \Lambda = \Lambda$. Hence both are equivalent to $(t;t) \in \Lambda$.

4. Finding the self-dual lattices Λ

In this section we address the question of finding all the self-dual Λ occurring in Theorem 1, i.e. making the classification of the partition functions $\overline{\mathcal{Z}}$ somewhat more explicit. It would seem however that *this problem is completely intractible* for large m, simply because the number of such Λ becomes so great. For example it includes, as a small subset, the classification of all Euclidean self-dual lattices of dimension m, and though there are only 28 of these for $m = 20$, there are over 8×10^{16} for $m = 32$ (see e.g. Tables 2.2 and 16.3 of [11]). Also, we learned in the last section that our lattices Λ for fixed m (and varying level) form a dense subset of an m^2-dimensional manifold! These considerations give some indication of the numbers of $\overline{\mathcal{Z}}$'s involved. But apparently this is not a serious issue, because stability considerations [2] seem to require small m and k.

A point worth repeating is that, up to transformations in the orthogonal group $SO(m,m)$, each Λ is equivalent either to the lattice $I_{m,m}$ (if odd) or the lattice $II_{m,m}$ (if even). However $SO(m,m)$ mixes up quite thoroughly the excitations on the two edges, and so those transformations will not respect the physics in any way. On the other hand, transformations from the smaller group $SO(m) \times SO(m)$ should preserve the physics, and we will usually identify lattices related by such transformations.

At least for small m, lattices are easy to work with and are conducive to explicit computations. We begin this section with some general statements [14] about how to find these Λ, given Γ_k, and then we specialise to $m \leq 2$. A basic geometrical fact, easily provable by considering volumes of fundamental regions, is the following: if $\Lambda_1 \subset \Lambda_2$ are two integral lattices, then

$$\|\Lambda_2/\Lambda_1\| = \sqrt{|\Lambda_1|/|\Lambda_2|} \ ; \tag{4.1a}$$

also, Λ_2^*/Λ_2 must be a subgroup of Λ_1^*/Λ_1, and Λ_2 a sublattice of Λ_1^*.

The first step to solving our problem consists of finding all possible m-dimensional integral lattices Λ_L which contain Γ_k. Any m-dimensional integral Λ_L containing Γ_k can be written as

$$\Lambda_L = \Gamma_k + \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_m \ , \tag{4.1b}$$

where $g_i \in \Gamma_k^*/\Gamma_k$ obey $g_i \cdot g_j \in \mathbb{Z}$ (any or all g_i may be 0). Thus the task of finding all possible Λ_L reduces to a finite search ($\|\Gamma_k^*/\Gamma_k\| = |k|$). (For lattice calculations such as required here, it is sometimes convenient to begin by using the Gram-Schmidt orthogonalisation process to find an orthogonal lattice Γ_D contained in Γ_k, since its dual and inner products are easy to compute.)

Now choose any two such Λ_L, and call the second one Λ_R. We may or may not have $\Lambda_L = \Lambda_R$, but we must have (3.2b). Let $h_1, \ldots, h_n$ be linearly independent generators of the group Λ_L^*/Λ_L. Find some $h_i' \in \Lambda_R^*/\Lambda_R$ such that $h_i \cdot h_j \equiv h_i' \cdot h_j' \pmod 1$. Again this is a finite search. From this we obtain

$$\Lambda = (\Lambda_L;\Lambda_R) + \mathbb{Z}(h_1;h_1') + \cdots + \mathbb{Z}(h_n;h_n') \ . \tag{4.1c}$$

Such a Λ will be self-dual and contain $(\Gamma_k;\Gamma_k)$, and all such Λ can be obtained in this way.

This manner of constructing lattices is called 'gluing' (see e.g. Chapter 4 of [11]). There is another standard method, called 'shifting' [14], which is more elegant in some ways. We will only state a special case of it here. Let Λ be a self-dual lattice, $V = \{v_1, \ldots, v_n\} \subset \mathbb{Q} \otimes \Lambda$, with each $v_i \cdot v_j \in \mathbb{Z}$. Define

$$\Lambda(V) := \{x + \sum_{i=1}^{n} \ell_i v_i \mid \ell_i \in \mathbb{Z},\ x \in \Lambda,\ \text{and}\ \forall j,\ x \cdot v_j \in \mathbb{Z}\}. \tag{4.2}$$

Then $\Lambda(V)$ will also be self-dual.

Example 1. $g = \widehat{u(1)}$ *at level* $k \in \mathbb{Z}_{\geq}$

Here $\Gamma_k = \sqrt{k}\,\mathbb{Z}$. The possible Λ_L are given by $d/\sqrt{k}\,\mathbb{Z}$ where $d \in \mathbb{Z}_{\geq}$ obeys

$$d|k\ , \quad k|d^2\ . \tag{4.3a}$$

Here $\Lambda_L = \Lambda_R$ is forced, by $|\Lambda_L| = |\Lambda_R|$. Now choose any $\ell \in \{1, \ldots, d^2/k\}$ obeying

$$\ell^2 \equiv 1 \qquad (\mathrm{mod}\ d^2/k)\ . \tag{4.3b}$$

To any such (ℓ, d) there corresponds a distinct self-dual lattice $\Lambda_{d,\ell}$ given by

$$\Lambda_{d,\ell} = (d/\sqrt{k}\,\mathbb{Z}\,; d/\sqrt{k}\,\mathbb{Z}) + \mathbb{Z}(\sqrt{k}/d\,; \ell\sqrt{k}/d)\ , \tag{4.3c}$$

and hence a weak invariant $\overline{\mathcal{Z}}_{d,\ell}$. Conversely, any weak invariant for U_1 at level k is of this form.

A simple counting argument shows that there is exactly one such partition function $\overline{\mathcal{Z}}$ for each divisor of k if k is odd, or for each divisor of $2^{a-2}k$ when k is even, where 2^a is the exact power of 2 dividing k. For example, for $k = 1, 2, \ldots, 10$ there are precisely 1, 1, 2, 3, 2, 2, 2, 5, 3, 2 different Λ's, respectively. When k is odd, we can make this correspondence explicit using shifting (4.2): to any divisor d of k, it is given by $d \mapsto \Lambda_1(\{(\sqrt{k}/d; -\sqrt{k}/d)\})$, where $\Lambda_1 = (\Gamma_k;\Gamma_k) + \mathbb{Z}(1/\sqrt{k}; 1/\sqrt{k})$.

The relationship between the notation here and that of equation (4.26) of [4], we find that $k \mapsto p$, $\ell \mapsto \omega^{-1}$ (or ω_i^{-1} if k is even), and $d \mapsto p/\delta$ (or p/δ' if k is even). However their list appears to miss some $\overline{\mathcal{Z}}$. For example, for $k = 8$, they get six $\overline{\mathcal{Z}}$'s, but two of them are redundant. There are in fact five distinct solutions – they miss the one with $d = 4$. In general they will miss some $\overline{\mathcal{Z}}$ when k is even.

If we consider even k and impose the stronger condition (2.3b), we find the resulting lattices are in a one-to-one relationship with divisors of $k/2$. This result was first obtained in [16].

If we impose commutation with U_t for $t = 1/\sqrt{k}$, then of course only one solution survives: $\ell = 1$, $d = k$.

Example 2. $g = \widehat{u(1) \oplus u(1)}$

It is difficult and unenlightening to state the solution for general k, although the list of Λ is easy to find for fixed k. Instead we will give all the self-dual lattices Λ with $|k_L| \leq 10$. For convenience we will mod out by $SO(2) \times SO(2)$. Table 15.1 of [11] is a list of all 2-dimensional integral lattices Γ_k of small determinant. These lattices give the possible values of levels k, through the correspondence $\Gamma_k \mapsto k$. To avoid unnecessary redundancy, by level here we will mean the minimal possible, namely $k_{L,R}$ – i.e. we impose (3.2a). A priori, the two levels k_L, k_R need not be equal, but for the small determinants considered here, (3.2b) usually forces them to be. For convenience here, we will give components of weights in terms of the e_i, not e_i^* as before. Recall that $e_i \cdot e_j = k_{i,j}$. We will also give for each of these Λ the smallest values of $t_{L,R}^2$ for $t_{L,R}$ satisfying (3.3a).

$|k| = 1$. The only choice of level is $k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The only Λ here is $I_{2,2}$. $t_{L,R}^2 = 2$ is the smallest.

$|k| = 2$. Here $k = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The weights $\lambda \in \overline{P}^k = \Gamma_k^*/\Gamma_k$ are generated linearly by $h = (0, \frac{1}{2})$. There is only one Λ here, corresponding to the 'diagonal glue' $(h; h)$ (see (4.1c)). $t_{L,R}^2 = 1$ is minimal.

$|k| = 3$. There are two possibilities here: $k' = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Each of these produce exactly one Λ, in both cases corresponding again to the diagonal glue, for the generators $h' = (0, \frac{1}{3})$ and $h'' = (\frac{1}{3}, -\frac{2}{3})$, respectively. Minimal ${t'_{L,R}}^2$, ${t''_{L,R}}^2$ are $\frac{4}{3}$ and 0, respectively.

$|k| = 4$. Again there are two possibilities: $k' = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The generators here are $h' = (0, \frac{1}{4})$, and $h_1'' = (\frac{1}{2}, 0)$ and $h_2'' = (0, \frac{1}{2})$ respectively. There are two Λ for k', but one of them involves a level reduction to $|k| = 1$ and so will be discarded. The other corresponds to the diagonal glue $(h'; h')$. There is only one Λ for k'', corresponding to the diagonal glues $(h_1''; h_1'')$ and $(h_2''; h_2'')$. Minimal ${t'_{L,R}}^2, {t''_{L,R}}^2$ are 1 and 0.

$|k| = 5$. We have $k' = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. $h' = (0, \frac{1}{5})$, and $h'' = (\frac{1}{5}, -\frac{2}{5})$. Again, there is only one Λ each, corresponding in each case to the diagonal glue, with minimal $t_{L,R}^2$ being $\frac{6}{5}$ and $\frac{2}{5}$, resp.

$|k| = 6$. We have $k' = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, with $h' = (0, \frac{1}{6})$, and $h''_1 = (\frac{1}{2}, 0)$ and $h''_2 = (0, \frac{1}{3})$. There is a unique Λ for each choice, again given by the diagonal glues, with minimal $t^2_{L,R}$ being 1 and $\frac{1}{3}$.

$|k| = 7$. We have $k' = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, where $h' = (0, \frac{1}{7})$ and $h'' = (\frac{1}{7}, -\frac{2}{7})$. The diagonal glues have minimal $t^2_{L,R}$ equal to $\frac{8}{7}$ and 0. Here for the first time we have a heterotic possibility: $k_L = k'$, $k_R = k''$, with $(h'; 2h'')$ as the glue and minimal $t^2_{L,R} = \frac{8}{7}$. The remaining lattice has $k_L = k''$, $k_R = k'$, with glue $(h''; 3h')$ and $t^2_{L,R} = \frac{8}{7}$.

$|k| = 8$. There are three possible levels here: $k' = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$, $k'' = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, and $k''' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. The generators are: $h' = (0, \frac{1}{8})$; $h''_1 = (\frac{1}{2}, 0)$ and $h''_2 = (0, \frac{1}{4})$; $h''' = (\frac{1}{8}, -\frac{3}{8})$. The choice k' for the level yields two Λ's, one with the diagonal glue $(h'; h')$ (with $t^2_{L,R} = 1$) and the other with $(h'; 3h')$ (with $t^2_{L,R} = \frac{3}{2}$). The choice k'' also has two, but one reduces to $|k| = 2$ so can be ignored. The other is given by the diagonal glues (with $t^2_{L,R} = 0$). The final choice k''' also has two Λ's, corresponding to $(h'''; h''')$ ($t^2_{L,R} = \frac{1}{2}$) and $(h'''; 3h''')$ ($t^2_{L,R} = 1$).

$|k| = 9$. There are three levels here as well: $k' = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$; $k'' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$; and $k''' = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$. The generators are: $h' = (0, \frac{1}{9})$; $h''_1 = (\frac{1}{3}, 0)$ and $h''_2 = (0, \frac{1}{3})$; $h''' = (\frac{1}{9}, -\frac{2}{9})$. There is one Λ for each choice of level, and each is given by the diagonal glues (a second Λ for k' reduces to $|k| = 1$ and so is ignored) (with $t^2_{L,R} = \frac{10}{9}$, $\frac{2}{3}$, and $\frac{2}{9}$ resp.).

$|k| = 10$. We have $k' = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$ and $k'' = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, with $h' = (0, \frac{1}{10})$, $h''_1 = (\frac{1}{2}, 0)$, and $h''_2 = (0, \frac{1}{5})$. As usual, there is exactly one Λ for each level, and it corresponds to the diagonal glues, and $t^2_{L,R} = 1, \frac{1}{5}$, respectively.

In summary, there are exactly 1, 1, 2, 2(+1), 2, 2, 4, 5(+1), 3(+1), 2, respectively, distinct self-dual lattices Λ (hence partition functions $\overline{\mathcal{Z}}$) for each $|k| \leq 10$. Obviously this example can be pushed considerably further.

An additional two-dimensional example is $U_1 \oplus \hat{A}_{m-1,1}$, with $t = e_1^*$. As mentioned in [4], these will be given by tensor products of the diagonal $\overline{\mathcal{Z}}$ for U_1, with the various weak invariants for $\hat{A}_{m-1,1}$. The latter were completely classified in Example 1.

5. Extensions

A considerable amount of attention in the literature has been paid to the classification of modular invariants for affine algebras g – see e.g. [7,17,6] and references therein. Two of the more useful and general (i.e. valid for any RCFT) concepts that have come from this are *simple currents* and a certain *Galois action.*

A simple current – see e.g. [17] – can be defined as any label $a \in P_+$ for which $S_{0,a} = S_{0,0}$ (compare (2.1c)). From Verlinde's formula one then finds that to any such

a corresponds a distinct permutation J_a of P_+ and a function $Q_{J_a} : P_+ \to \mathbb{Q}$, such that $J_a 0 = a$ and [17]

$$S_{J_a b,c} = \exp[2\pi\mathrm{i}\, Q_{J_a}(c)]\, S_{b,c} \ . \tag{5.1a}$$

From this fundamental equation can be derived (see Lemma 3.1 in [6], though the arguments are similar to that of Thm. 1 given above) the following important facts, valid for any weak invariant M and any simple currents J, J':

$$M_{J0,J'0} \in \{0,1\} \ ; \tag{5.1b}$$

$$M_{J0,J'0} = 1 \Longrightarrow M_{Ja,J'b} = M_{a,b} \qquad \forall a,b \in P_+ \ ; \tag{5.1c}$$

$$M_{J0,J'0} = 1 \Longleftrightarrow \ Q_J(a) \equiv Q_{J'}(b) \ (\mathrm{mod}\ 1)\ \forall\, a,b \in P_+ \text{ with } M_{a,b} \neq 0 \ . \tag{5.1d}$$

Let $\mathcal{I}$ denote the set (in fact, an abelian group) of all simple currents of P_+. An important subset of P_+ are the *fixed points* of $\mathcal{I}$, defined by $a \in Ja$ for some nonzero $J \in \mathcal{I}$.

The Galois action also concerns the matrix S. Verlinde's formula implies [18] that each entry $S_{a,b}$ will lie in some cyclotomic extension K of $\mathbb{Q}$. Choosing any Galois automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, one finds [18]

$$\sigma S_{a,b} = \epsilon_\sigma(a)\, S_{a^\sigma,b} \tag{5.2a}$$

for some map $\epsilon_\sigma : P_+ \to \{\pm 1\}$ and permutation $\lambda \mapsto \lambda^\sigma$ of P_+. This together with (2.3a) and (2.2b) implies

$$M_{a,b} = \epsilon_\sigma(a)\, \epsilon_\sigma(b)\, M_{a^\sigma,b^\sigma} \ . \tag{5.2b}$$

The most important consequence of (5.2b) is that, because of (2.2b), we get the selection rule

$$M_{a,b} \neq 0 \quad \Longrightarrow \quad \epsilon_\sigma(a) = \epsilon_\sigma(b) \quad \forall \sigma \in \mathrm{Gal}(K/\mathbb{Q}) \ . \tag{5.2c}$$

Example 3. $g = U_m$

Here every $\lambda \in \overline{P}^k$ is a simple current, with $J_\lambda \mu = \lambda + \mu$ and $Q_\lambda(\mu) = -\lambda \cdot \mu$. That $\overline{P}^k$ consists only of simple currents is precisely the reason the classification of U_m weak invariants is so easy. For this reason their classification would also follow from the work in [19]. By way of comparison, [20] concerns the next simplest such class of algebras, $\widehat{A_1^{\oplus m}}$, which turns out to be far more complicated.

The cyclotomic field K here can be taken to be $\mathbb{Q}(\zeta_{|k|})$, where $\zeta_n := \exp[2\pi\mathrm{i}\,/n]$, and all $\epsilon_\sigma(\lambda) = +1$. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ can be identified with the multiplicative group $\mathbb{Z}_n^*$ consisting of the integers ℓ mod n coprime to n. We will write this correspondence as $\sigma \leftrightarrow \sigma_\ell$. For ℓ coprime to $|k|$, $\sigma_\ell \lambda = \ell\lambda$ is the Galois action for U_m.

Example 4. $g = \hat{A}_1$

Here there is exactly one simple current J, and it maps a to $k-a$ and has $Q_J(a) = a/2$. The only fixed point is $k/2$. Write $\overline{k} = k+2$. The field K here can be taken to be $\mathbb{Q}(\zeta_{8\overline{k}})$. For any ℓ coprime to $8\overline{k}$,

$$\epsilon_\ell(a) = \left(\frac{2\overline{k}}{\ell}\right) \cdot \begin{cases} +1 & \text{if } \langle \ell \frac{a+1}{2\overline{k}} \rangle < \frac{1}{2} \\ -1 & \text{otherwise} \end{cases} , \tag{5.3a}$$

where $\langle x\rangle$ denotes the unique number between 0 and 1 congruent to x (mod 1). The factor $\left(\frac{2\overline{k}}{\ell}\right)$ is a Jacobi symbol and, since it is independent of a, is irrelevant here. Also,

$$\sigma_\ell a = \begin{cases} 2\overline{k}\,\langle\frac{\ell(a+1)}{2\overline{k}}\rangle & \text{if } \epsilon_\ell(a)=+1 \\ 2\overline{k}-2\overline{k}\,\langle\frac{\ell(a+1)}{2\overline{k}}\rangle & \text{if } \epsilon_\ell(a)=-1 \end{cases} . \tag{5.3b}$$

It is possible to completely solve the constraint (5.2c) for $\hat{A}_1$ at any level k. We find, provided $\gcd(a+1,b+1,\overline{k})=1$, that the right-side of (5.2c) is equivalent to $b\in\{a,Ja\}$, with the following exceptions:

$$k=4: \qquad a,b\in\{0,\,2,\,4\}\ ; \tag{5.4a}$$

$$k=8: \qquad a,b\in\{0,\,2,\,6,\,8\}\ ; \tag{5.4b}$$

$$k=10: \qquad a,b\in\{0,\,4,\,6,\,10\}\ ; \tag{5.4c}$$

$$k=28: \qquad a,b\in\{0,\,10,\,18,\,28\} \quad \text{or} \quad a,b\in\{6,\,12,\,16,\,22\}\ . \tag{5.4d}$$

When the gcd condition is not satisfied, simply divide through by the common divisor (so $a'+1=\frac{a+1}{d}$, $b'+1=\frac{b+1}{d}$, $k'+2=\frac{\overline{k}}{d}$), in order to apply this result. This result is actually far stronger than we need – in general it is necessary only to look at (5.2c) for $a=0$, which for $\hat{A}_1$ was solved in Lemma 3 of [20], but our general solution (5.4) is an easy consequence (sketch: we can assume $\overline{k}$ is even, a odd, b odd; if some prime p divides $\gcd(a+1,\overline{k})$ but not $b+1$, then it can be shown that p must equal 3; thus $\gcd(b+1,\overline{k})=1$ and hence b can get mapped by (5.3b) to 0).

The physical invariants for $\hat{A}_1$ were first classified by [7] using methods considerably different from the ones we use in this paper. The newer techniques permit for example the following interesting generalisation of their important work:

Theorem 2. The list of all weak invariants of $\hat{A}_1$ at level k is:

$$\mathcal{A}_k = \sum_{a=0}^{k} |\chi_a|^2\ , \qquad \forall k; \tag{5.5a}$$

$$\mathcal{D}_k = \sum_{a=0}^{k} \chi_a\chi^*_{J^a a}\ , \qquad \forall\,\text{even } k; \tag{5.5b}$$

$$\mathcal{D}'_k = \sum_{a=0}^{[(k-2)/4]} |\chi_{2a}+\chi_{k-2a}|^2 + \begin{cases} 2|\chi_{k/2}|^2 & \text{if } 4|k \\ 0 & \text{otherwise} \end{cases}\ , \qquad \forall\,\text{even } k; \tag{5.5c}$$

$$\mathcal{E}_4 = |\chi_0+\chi_4|^2 + \chi_2(\chi_0+\chi_4)^* + (\chi_0+\chi_4)\chi_2^*\ , \qquad k=4; \tag{5.5d}$$

$$\mathcal{E}_8 = |\chi_0+\chi_2+\chi_6+\chi_8|^2\ , \qquad k=8; \tag{5.5e}$$

$$\mathcal{E}_{10} = |\chi_0+\chi_6|^2 + |\chi_4+\chi_{10}|^2 + |\chi_3+\chi_7|^2\ , \qquad k=10; \tag{5.5f}$$

$$\mathcal{E}'_{10} = |\chi_0+\chi_4+\chi_6+\chi_{10}|^2, \qquad k=10; \tag{5.5g}$$

$$\mathcal{E}_{16} = |\chi_0+\chi_{16}|^2 + |\chi_4+\chi_{12}|^2 + |\chi_6+\chi_{10}|^2 + (\chi_2+\chi_{14})\chi_8^* + \chi_8(\chi_2+\chi_{14})^* + |\chi_8|^2\ , \qquad k=16; \tag{5.5h}$$

$$\mathcal{E}_{28} = |\chi_0+\chi_{10}+\chi_{18}+\chi_{28}|^2 + |\chi_6+\chi_{12}+\chi_{16}+\chi_{22}|^2\ , \qquad k=28. \tag{5.5i}$$

This theorem is proved in the Appendix. The weak invariants in equations (5.5) differ from the physical invariant list of [7], in which (2.3b) was also imposed, only in that there are a few extra exceptionals $\mathcal{E}_4$, $\mathcal{E}_8$ and $\mathcal{E}'_{10}$, and that $\mathcal{D}_k$ is defined now for $4|k$ and $\mathcal{D}'_k$ for $4|(k-2)$. It is surprising how irrelevant T-invariance is for the $\hat{A}_1$ classification. Of these, only $\mathcal{E}_4$ and $\mathcal{E}_8$ violate T^2-invariance. The names $\mathcal{A}_k$, $\mathcal{D}_k$, etc here are introduced purely by analogy with the A-D-E classification in [7] – it would be very interesting however if some similar interpretation of this list can be found.

A similar result to Theorem 2 can be expected at least for $\hat{A}_2$ – (5.2c) has also been solved for it. More generally, it is easy to show that there are only finitely many weak invariants at each level k, for each affine algebra $\hat{g}$.

The complete list of weak invariants for $\hat{A}_1 \oplus U_m$ which obey (2.5a), at level (k,ℓ) for any positive integer k and any positive definite integer matrix ℓ, is:

AU.sc 'simple current invariants': these are given in (A.8b), (A.9c), or are of the form $\mathcal{D}'_k\,\overline{\mathcal{Z}}$ (equivalently the tensor product of the corresponding matrices) for any $\overline{\mathcal{Z}}$ in Thm. 1;

AU.4 for $k=4$, there are the '$\mathcal{E}_7$-type exceptionals' given in (A.14);

AU.10 for $k=10$, there are the exceptionals given by the product $\mathcal{Z}=\mathcal{E}'_{10}\,\overline{\mathcal{Z}}$, as well as the exceptionals given by matrix product $M=M'\,(M_{10}\otimes\overline{I})$, where M' is any simple current invariant in AU.sc, and $\overline{I}$ is the identity matrix for $\overline{P}^{\ell}$;

AU.16 for $k=16$, there are the exceptionals $\mathcal{Z}=\mathcal{E}_{16}\,\overline{\mathcal{Z}}$;

AU.28 for $k=28$, there are the exceptionals $\mathcal{Z}=\mathcal{E}_{28}\,\overline{\mathcal{Z}}$.

The completeness of this list is also proved in the Appendix. The only 'new' invariants here (i.e. ones which cannot be generated by standard simple current tricks from those of $\hat{A}_1$) are the $k=4$ exceptionals, the simplest of which occur for $A_{1,4}\oplus U_{1,6}$ and $A_{1,4}\oplus U_{1,9}$. The constraint (2.5a) is imposed here to shorten the proof; if instead we drop (2.5a) then we get new exceptionals only at $k=4,8,10$, and these can be easily found using the methods of the Appendix. Note that few of our $\mathcal{Z}$ factorise completely into a product of $m+1$ $\mathcal{Z}_i$'s – in fact about half fail to factorise into an $\hat{A}_1$ part and a U_m part. This is characteristic of modular invariant classifications for semi-simple algebras and unfortunately means that the semi-simple classifications do not reduce to the simple ones. In general, there are many more physical (or weak) invariants, including exceptionals, for semi-simple algebras than would be expected from the list for simple ones.

6. The $N=2$ superconformal minimal models

The reason the classification AU.sc–AU.28 of the previous section will permit us to read off the $N=2$ rational minimal model classification is because [21] gives a description of the $N=2$ super-Virasoro algebra at $c=3(1-\frac{2}{k+2})$ for $k\in\mathbb{Z}_>$, in terms of the coset $(SU(2)_k\times U(1)_4)/U(1)_{2k+4}$, and [22] explains how to reduce physical invariant classifications for cosets to those for semi-simple algebras ($\hat{A}_1\oplus U_2$ in our case).

The partition function of a superconformal field theory will not be built directly from the super-Virasoro characters, since the super-Virasoro algebra contains fields of half-integer conformal dimension. We are required here to use certain projections to split the characters into two parts ($\tilde{\chi}_c^{a(b)}$ and $\tilde{\chi}_c^{a(b+2)}$ in the notation below), in other words to consider the possible spin-structures. The modular transformations mix these spin-structures (apart from the periodic-periodic one, which contributes an additive constant – the Witten index for the Ramond sector – to the partition function and will be ignored).

The partition functions for $N=2$ minimal models will be of the form [9]

$$\widetilde{\mathcal{Z}} = \sum_{a,a'\in P_+^k} \sum_{c,c'=0}^{2k+3} \widetilde{M}_{a,c;a',c'}\, \tilde{\chi}_c^{a(b)}\, \tilde{\chi}_{c'}^{a'(b')*} , \tag{6.1a}$$

where the $\widetilde{M}$'s are non-negative integers, $\widetilde{M}_{0,0;0,0}=1$, the $\tilde{\chi}$'s are the 'half-characters' alluded to above, and $b=0$ or 1 depending on whether $a+c$ is even or odd (similarly for b'). This must be invariant under the full $SL_2(\mathbb{Z})$. As with the quantum Hall effect (see the comments after (2.3)), we should either regard $\tilde{\mathcal{Z}}$ as a function of additional variables z (other than just τ), or equivalently, formally assume that all $\tilde{\chi}$'s are distinct.

Many $\tilde{\mathcal{Z}}$ can be found in the literature (see e.g. [10,9]), but the complete list appears here for the first time. The structure of $N=2$ minimal models have been studied in e.g. [23]. It is shown in [24] that any rational model of the $N=2$ super-Virasoro algebra is unitary and hence is one of the minimal models given below (this surprising result is in sharp contrast to the $N=0$ and $N=1$ cases).

[22] tells us how to interpret this classification in terms of the $A_{1,k}\oplus U_{1,4}\oplus U_{1,2k+4}$ one: in particular, $\tilde{\chi}_c^{a(b)}$ has identical modular behaviour as $\chi_a^k\overline{\chi}_b^4\overline{\chi}_c^{2k+4\,*}+\chi_{Ja}^k\overline{\chi}_{b+2}^4\overline{\chi}_{c+k+2}^{2k+2\,*}$ and hence the classification of $\widetilde{\mathcal{Z}}$ in (6.1a) is identical to that of the physical invariants

$$\mathcal{Z} = \sum_{a,a'\in P_+^k} \sum_{b,b'=0}^{3} \sum_{c,c'=0}^{2k+3} M_{a,b,c;a',b',c'}\, \chi_a^k\, \overline{\chi}_b^4\, \overline{\chi}_c^{2k+4}\, \chi_{a'}^{k*}\, \overline{\chi}_{b'}^{4*}\, \overline{\chi}_{c'}^{2k+4*} , \tag{6.1b}$$

subject to the additional conditions that

$$M_{J0,2,0;0,0,k+2} = M_{0,0,k+2;J0,2,0} = 1 . \tag{6.1c}$$

The precise relation between M and $\widetilde{M}$ is given by [22]

$$\widetilde{M}_{a,c;a',c'} = M_{a,b,c';a',b',c} \tag{6.1d}$$

(note the $c\leftrightarrow c'$ switch), where $b,b'\in\{0,1\}$ are as defined after (6.1a). The classification of these $\mathcal{Z}$ is an elementary application of the list AU.sc–AU.28 – all we need to do is impose (2.3b) and (6.1c).

For convenience write $\overline{k}$ for $k+2$.

Theorem 3. The complete list of distinct physical invariants $\widetilde{M}$ for $N=2$ minimal models at level k are given by (6.1d), for each of the following choices of M:

k *odd* : There is only one kind of M here: its only nonzero entries are

$$M^o_{a,b,c;J^{a+b+c}a,2a+bw+2c,av+bv+2zc+2\ell v} = 1 \qquad \forall \ell\in\mathbb{Z} , \tag{6.2a}$$

for any a,b,c provided $vc/\overline{k} \in \mathbb{Z}$, where $w \in \{1,3\}$ is arbitrary, and $z \in \{1,2,\ldots,\frac{v^2}{\overline{k}}\}$ and the divisor v of $\overline{k}$ are any solutions to

$$(4z^2-1)\overline{k}/v^2 \equiv v^2/\overline{k} \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.2b}$$

$k/2$ *odd* : There are three kinds of M's here: we will call them $M^{2,0}$, $M^{2,1}$ and $M^{2,2}$. The nonzero entries of $M^{2,0}$ are

$$M^{2,0}_{a,b,c;J^\ell a,ax+2dy+2e+2\ell,zc+ayv+2mv} = 1 \qquad \forall \ell,m \in \mathbb{Z}\ , \tag{6.3a}$$

for all a,b,c provided $d := vc/\overline{k} \in \mathbb{Z}$ and $e := (b-a)/2 \in \mathbb{Z}$, where $x \in \{1,3\}$, $y \in \{0,1\}$, $z \in \{1,2,\ldots,2v^2/\overline{k}\}$, and v is a divisor of $\overline{k}/2$, such that

$$v^2/\overline{k} \equiv \overline{k}\,(z^2-1)/4v^2 + y/2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.3b}$$

The nonzero entries of $M^{2,1}$ are

$$M^{2,1}_{a,b,c;J^\ell a,ax+2d\,(z+1)+2e+2m+2\ell,va+zd\overline{k}/v+2mv} = 1 \qquad \forall \ell,m \in \mathbb{Z}\ , \tag{6.3c}$$

for all a,b,c provided $d := (c-av)v/\overline{k} \in \mathbb{Z}$ and $e := (b-a)/2$, where $x \in \{1,3\}$, $z \in \{1,2,\ldots,2v^2/\overline{k}\}$, and v is a divisor of $\overline{k}/2$, such that

$$1/2 + v^2/\overline{k} \equiv \overline{k}\,(z^2-1)/2v^2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.3d}$$

The nonzero entries of $M^{2,2}$ are

$$M^{2,2}_{a,b,c;J^\ell a,ax+2e+wc+2\ell,av(x-w)/2+cz+\overline{k}\ell+2vm} = 1 \qquad \forall \ell,m \in \mathbb{Z}\ , \tag{6.3e}$$

for all a,b,c with $cv/\overline{k} \in \mathbb{Z}$ and $e := (b-a-d)/2 \in \mathbb{Z}$, where $x,w \in \{1,3\}$, $z \in \{1,\ldots,2v^2/\overline{k}\}$, $\overline{k}/v$ is odd, and

$$v^2/\overline{k} \equiv \overline{k}\,(z^2-1)/4v^2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.3f}$$

$k/2$ *even* : There are four kinds of M's here: $M^{4,0}$, $M^{0,1}$, $M^{04,2}$, and $M^{04,3}$. $M^{4,0}$ exists only for $k \equiv 4$ (mod 8). Its nonzero entries are

$$M^{4,0}_{a,b,c;J^{ax+bx'+d}a,ay+by'+2d(z+1)+2\ell,av+bv+dz\overline{k}/v+2\ell v} = 1 \qquad \forall \ell \in \mathbb{Z}\ , \tag{6.4a}$$

for all a,b,c provided $d := (c-vb)v/\overline{k} \in \mathbb{Z}$, where $y,y' \in \{1,3\}$, $x \in \{0,1\}$, $z \in \{1,2,\ldots,2v^2/\overline{k}\}$, $x' = (y-y')/2$, and v is a divisor of $\overline{k}/2$, such that

$$x/2 + 1/8 + v^2/4\overline{k} \equiv \overline{k}\,(z^2-1)/v^2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.4b}$$

$M^{0,1}$ exists only when 8 divides k. Its nonzero entries are

$$M^{0,1}_{a,b,c;J^\ell a,bk/4+dy,bxv/2+2cz+2mv} = 1 + \delta_{a,k/2} \qquad \forall \ell,m \in \mathbb{Z}\ , \tag{6.4c}$$

for all a,b,c provided a is even and $d := vc/\overline{k} \in \mathbb{Z}$, where $x,y \in \{1,3\}$, $z \in \{1,2,\ldots,v^2/\overline{k}\}$, and v is a divisor of $\overline{k}$, such that

$$v^2/\overline{k} \equiv \overline{k}\,(4z^2-1)/4v^2 + 1/8 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.4d}$$

$M^{04,2}$ exists whenever $k/2$ is even. Its nonzero entries are

$$M^{04,2}_{a,b,c;J^\ell a,bx+2d(z+1)+2m,bv+dz\overline{k}/v+2mv} = 1+\delta_{a,k/2} \qquad \forall \ell,m \in \mathbb{Z}\ , \tag{6.4e}$$

for all a,b,c provided a is even and $d := (c-bv)v/\overline{k} \in \mathbb{Z}$, where $x \in \{1,3\}$, $z \in \{1,2,\ldots,2v^2/\overline{k}\}$, and v is a divisor of $\overline{k}/2$ such that

$$1/2 + v^2/\overline{k} \equiv \overline{k}\,(z^2-1)/2v^2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.4f}$$

$M^{04,3}$ also exists whenever $k/2$ is even. Its nonzero entries are

$$M^{04,3}_{a,b,c;J^\ell a,ax+2d+cx+2\ell,cz+2\ell v} = 1 \qquad \forall \ell\ , \tag{6.4g}$$

for all a,b,c for which $d := (b-a-2cv/\overline{k})/2 \in \mathbb{Z}$, where $x \in \{1,3\}$, $z \in \{1,2,\ldots,2v^2/\overline{k}\}$ and v is a divisor of $\overline{k}/2$ such that

$$1/2 + v^2/\overline{k} \equiv \overline{k}\,(z^2-1)/4v^2 \equiv 0 \qquad (\mathrm{mod}\ 1)\ . \tag{6.4h}$$

$k=10$. In addition to the ones mentioned in (6.3), there are precisely 20 exceptionals, given by the matrix product $M'\,(\mathcal{E}_{10}\otimes\overline{I})$, where M' here is any of the 20 matrices (4 $M^{2,0}$'s and 16 $M^{2,2}$'s) in (6.3) for $k=10$.

$k=16$. In addition to the ones mentioned in (6.4), there are precisely 18 exceptionals, given by the tensor product $\mathcal{E}_{16}\otimes\overline{M}$, where $\overline{M}$ here is the projection to the last two components of any of the 12 matrices $M^{0,1}$ or 6 matrices $M^{04,2}$ for $k=16$ (the matrices of type $M^{0,1}$ and $M^{04,2}$ in (6.4) are always of the form $\mathcal{D}'_k\otimes\overline{M}$).

$k=28$. In addition to the ones mentioned in (6.4), there are precisely 8 exceptionals, given by the tensor product $\mathcal{E}_{28}\otimes\overline{M}$, where $\overline{M}$ here comes from the 8 matrices $M^{04,2}$ for $k=28$.

As usual for the classifications considered in this paper, there is an unavoidable problem with being explicit, at least for general k. The number of level k $N=2$ minimal models for $k<30$, is: 4, 10, 4, 14, 4, 14, 6, 14, 4, 40, 4, 14, 8, 18, 4, 40, 4, 20, 8, 14, 4, 28, 6, 14, 8, 20, 4, 36, 4.

The often-claimed A-D-E pattern to the $N=2$ minimal model classification is rather obscure from the standpoint of our theorem, and is at best 'one-to-many' (e.g. the exceptional E_6 corresponds to 20 different partition functions). An example of its apparent inappropriateness is that the so-called A and D partition functions for $k\equiv 2$ (mod 4) correspond to the value 0 and 1, respectively, of the seemingly insignificant parameter x in (6.3)!

The remainder of this section is devoted to the proof of Theorem 3. Note the definitions of $\mathcal{I}_{L,R}(M)$, $\mathcal{P}_{L,R}(M)$ given in (A.2b),(A.2c). Note also that (6.1c), (5.1d), and (2.3b) force

$$M_{a,b,c;a',b',c'} \neq 0 \Longrightarrow a+b+c' \equiv a'+b'+c \equiv 0 \pmod 2 \tag{6.5a}$$

$$M_{a,b,c;a',b',c'} \neq 0 \Longrightarrow \frac{(a+1)^2}{4\overline{k}} + \frac{b^2}{8} + \frac{c^2}{4\overline{k}} \equiv \frac{(a'+1)^2}{4\overline{k}} + \frac{b'^2}{8} + \frac{c'^2}{4\overline{k}} \pmod 1 \tag{6.5b}$$

6.1. The automorphism invariants

These are the physical invariants obeying (A.1a), i.e. $M_{0,0,0;a,b,c} = \delta_{a,0}\delta_{b,0}\delta_{c,0}$. We know $\phi(0,0,1) = (J^x 0, y, z)$ for some x, y, z, hence by linearity (A.8b) we have $\phi(0,0,\overline{k}) = (J^{x\overline{k}}0, \overline{k}y, \overline{k}z) = (J0, 2, 0)$, where the last equality holds by (6.1c). For even k, this is impossible, and hence for these k there are no such automorphism invariants. The automorphism invariants for odd k turn out to be a special case ($v = \overline{k}$) of the treatment in the next paragraph.

6.2. The ADE_7 invariants

These are the physical invariants obeying (A.2a). Consider first k odd. We want $M_{J^x 0,y,z;0,0,0} = 1$ for some x, y, z. (6.5) says $x = y = 0$ and z is even, hence $v = z/2$ satisfies (6.2b). We can choose v so that it divides $\overline{k}$. We find $\mathcal{I}_L(M) = \mathcal{I}_R(M)$ is forced here by the constraint (A.2d). There are no fixed points here and so M is given by (A.2f) for some ϕ. The way to show M satisfies (6.2a) is simply to look at the possibilities for $\phi(1,0,0)$, $\phi(0,1,0)$, and $\phi(0,0,\overline{k}/v)$, and to solve the various constraints coming from (A.2g) and (6.5).

Next consider $k/2$ odd. Again, there are no fixed points, so (A.2f) applies. Note that if $(1,e,e') \in \mathcal{P}_L(M)$ for some even e, e', then $(1,e,e') \mapsto (*,*,o)$ where o is odd, in which case (6.5b) cannot be satisfied. This forces either $(J,2,0) \in \mathcal{I}_L(M)$ or $(J,2,\overline{k}) \in \mathcal{I}_L(M)$. Consider first the former possibility. Then $(0,0,\overline{k}) \in \mathcal{I}_R(M)$ by (6.1c), hence we find

$$\mathcal{I}_L(M) = \mathcal{I}_R(M) = \langle (J,2,0),\, (0,0,\overline{k}),\, (0,2u,2v) \rangle \ ,$$

where $u \in \{0,1\}$. These two possible values of u should be treated separately, and produce the invariants $M^{2,u}$ of (6.3). The second possibility for $\mathcal{I}_L(M)$ either reduces to the former, or we have

$$\mathcal{I}_L(M) = \mathcal{I}_R(M) = \langle (J,2,\overline{k}),\, (0,2u,2v) \rangle \ ,$$

where either $\overline{k}/v$ is odd (if $u = 0$) or $\overline{k}/2v$ is odd (if $u = 1$). We then find $u = 1$ violates (6.5b). $u = 0$ here produces $M^{2,2}$.

Finally, consider k a multiple of 4. Suppose first there is nothing in $\mathcal{I}_L(M)$ of the form $(J,*,*)$, i.e. $\mathcal{I}_L(M) = \langle (0,2u,2v) \rangle$. Then $u = 1$ because otherwise $x_0 = (1,0,0)$ and $x_1 = (0,1,0)$ will have $\phi(x_0) \cdot \phi(x_1) \equiv \frac{1}{4} \pmod{\frac{1}{2}}$ by (6.5), violating (A.8a). Now $\phi(1,0,0) = (J^x 1, y, w)$ where y, w are odd. This means $(0,2,2w) \in \mathcal{I}_R(M)$ (by (A.8b), $\phi(2,0,0) = (2,2,2w)$ must lie in $\mathcal{I}_R(M)(2,0,0)$) and hence (counting powers of 2) (A.2d) forces $\mathcal{I}_L(M) = \mathcal{I}_R(M)$ and $w = v$. It is now straightforward to verify M is given by (6.4a). The constraint $k \equiv 4 \pmod 8$ is a consequence of (6.4b).

Next, suppose $(J,0,0) \in \mathcal{I}_L(M)$. Then $(J,0,0) \in \mathcal{I}_R(M)$, because of the argument in the Appendix after (A.9a), so $\mathcal{I}_L(M) = \mathcal{I}_R(M) = \langle (J,0,0),\, (0,2u,2v) \rangle$. M factors in this case into $\mathcal{D}'_k \otimes \overline{M}$ (or $\mathcal{E}_{16} \otimes \overline{M}$ for $k = 16$), so it suffices to find $\overline{M}$. As usual, the analysis depends on whether $u = 0$ or 1: the former case yields (6.4c) and the latter case yields (6.4e).

The remaining possibility is that neither $\mathcal{I}_{L,R}(M)$ contain $(J, 0, 0)$, but both contain something of the form $(J, u_{L,R}, v_{L,R})$. Then $v_{L,R} \neq 0$ by (6.5). Once again we find by the usual arguments that $u_{L,R}, v_{L,R}$ can be chosen so that $\mathcal{I}_L(M) = \mathcal{I}_R(M) = \langle (J, 2u, 2v) \rangle$. We must have $u = 1$, since otherwise v would be even and hence $\frac{k}{2}(J, 0, 2v) = (J, 0, 0)$ would have to lie in $\mathcal{I}_L(M)$, contrary to hypothesis. We find M is given by (6.4g).

6.3. Exceptional levels

The only exceptional levels obeying (2.3b) are $k = 10$ and 28 ($\Delta^2_{L,R} \equiv \frac{4}{3}$ (mod 2) for $k = 4$). AU.10 and AU.28 of the previous section allow us to read off the answer from the M of (6.3) and (6.4c), (6.4e).

7. Conclusion

In this paper we accomplish two main modular invariant partition function classifications: that of the possible effective field theories for the quantum Hall fluids; and that of the $N = 2$ superconformal minimal models. The answer to the former is given in Section 3 in terms of self-dual lattices, where we also provide a prettier but less practical formulation of the classification in terms of rational points on Grassmannians. The answer to the second problem is given in Section 6. This latter classification is often claimed to fall into an A-D-E pattern, but from the complete list of partition functions obtained here this claim looks rather artificial – e.g. arbitrarily large numbers of partition functions are assigned to the same name A_ρ (or D_ρ) for large ρ. Certainly it is far less convincing a match as the A-D-E of the A_1 classification [7].

The connection between these two problems lies in their symmetry algebras: $u(1)^{\oplus m}$ versus $(A_{1,k} \oplus u(1)_4)/u(1)_{2k+4}$. Solving the first takes us a long way towards solving the second. In fact, in Section 5 we find the partition functions for the algebra $A_1 \oplus u(1)^{\oplus m}$ – the choice $m = 2$ then yields the $N = 2$ classification.

The moduli space picture of rational points on the dual Grassmannian $G_{m,m}(\mathbb{R})^*$ is a very intriguing one, reminiscent of the compactification of heterotic strings on tori studied in [25]. It should be possible to find a natural lattice interpretation for the $A_1 \oplus u(1)^{\oplus m}$ classification given here, and from this perhaps an analogous description of its moduli space.

An interesting consequence of the work here is the list of all A_1 modular invariants, when invariance under $\tau \mapsto \tau + 1$ is dropped. This is given in Section 5. What is found is the list is surprisingly little changed from the A-D-E list of Cappelli-Itzykson-Zuber. One is not always interested in invariance under the full $SL_2(\mathbb{Z})$ (witness the quantum Hall effect; see also e.g. [26]), and at least in this case the classification is little different and is achieved by similar methods.

I appreciate the hospitality of the MPIM, and have benefitted from useful communications with Andrea Cappelli, Wolfgang Eholzer, Christoph Schweigert and Mark Walton.

The possibility of classifying the $N = 2$ minimal models was first suggested to me by Jean-Bernard Zuber.

Appendix. Proofs for Section 5

We begin the Appendix with a sketch of the arguments which we will use for obtaining the classifications given in Section 5. First consider any weak invariant M satisfying

$$M_{a,0} = \delta_{a,0}, \quad M_{0,b} = \delta_{b,0}, \qquad \forall a \in P_+^L,\ b \in P_+^R \ . \tag{A.1a}$$

Such M are called *automorphism invariants.* It is possible to show (e.g. this is a special case of Lemma 3.1(b)(iii) and Lemma 3.2(b) in [6]) that for any such M, there exists a bijection $\phi : P_+^L \to P_+^R$ such that

$$M_{a,b} = \delta_{b,\phi a} \ . \tag{A.1b}$$

For example, $\phi 0 = 0$ by (2.2c). Then (2.3a) reduces to

$$S_{a,b} = S_{\phi a,\phi b} \qquad \forall a, b \in P_+^L \ . \tag{A.1c}$$

To find all such M, i.e. all such ϕ, we follow the technique first developed in [27]. In particular, let G be any subset of P_+^L with the property that for any $a, b \in P_+^L$,

$$S_{c,a}/S_{0,a} = S_{c,b}/S_{0,b} \qquad \forall c \in G \quad \Longrightarrow \quad a = b \ . \tag{A.1d}$$

Any such set G is called a *fusion-generator* for P_+^L – e.g. for $U_{m,k}$ we can take G to be set of any linear generators of $\overline{P}^k$, while for $g = A_{r,k}^{(1)}$ we can take G to be the set of fundamental weights $\{w^1, \ldots, w^{(r+1)/2}\}$. M is uniquely determined by how ϕ acts on G. See [27] for details.

The next step consists of weakening the constraint (A.1a) to

$$M_{a,0} \neq 0 \quad \Rightarrow \quad a \in \mathcal{I}_L 0, \quad \text{and } M_{0,b} \neq 0 \quad \Longrightarrow \quad b \in \mathcal{I}_R 0 \ , \tag{A.2a}$$

where $\mathcal{I}_{L,R}$ are the sets of simple currents in $P_+^{L,R}$, respectively. Any such M is called an ADE$_7$-invariant [6], since these are precisely the physical invariants of $\hat{A}_1$ satisfying (A.2a). Useful definitions are

$$\mathcal{I}_L(M) := \{J \in \mathcal{I}_L \mid M_{J0,0} \neq 0\} \ , \tag{A.2b}$$
$$\mathcal{P}_L(M) := \{a \in P_+^L \mid \exists b \in P_+^R \text{ such that } M_{a,b} \neq 0\} \ , \tag{A.2c}$$

and define $\mathcal{I}_R(M) = \mathcal{I}_L(M^t)$, $\mathcal{P}_R(M) = \mathcal{P}_L(M^t)$. In the special case of an ADE$_7$-invariant, Lemma 3.1(b) of [6] says that $I_{L,R}(M)$ are subgroups of $I_{L,R}$ obeying

$$\|\mathcal{I}_L(M)\| = \|\mathcal{I}_R(M)\| \ , \tag{A.2d}$$
$$\mathcal{P}_{L,R}(M) = \{a \in P_+^{L,R} \mid Q_J(a) \in \mathbb{Z} \ , \ \forall J \in \mathcal{I}_{L,R}(M)\} \ . \tag{A.2e}$$

The notion of fusion-generator $G(\mathcal{I}')$ for a group $\mathcal{I}'$ of simple currents can be defined analogously to (A.1d), but some extra care is required (see Def. 3.3 of [6]). For $g = A_{1,k}$, $G(\{0, J\}) = \{2\}$. Generically, no $a \in G(\mathcal{I}_{L,R}(M)) =: G_{L,R}$ will be a fixed point of $\mathcal{I}_{L,R}(M)$, and for each $a \in G_L$ we will have $M_{a,f} = 0$ for all fixed points f of $\mathcal{I}_R(M)$ (and similarly for each $b \in G_R$). When this happens, the situation turns out to resemble the automorphism invariant one: there will exist a bijection $\phi : \mathcal{P}_L(M)/\mathcal{I}_L(M) \to \mathcal{P}_R(M)/\mathcal{I}_R(M)$ such that

$$M_{a,b} = \frac{\|\mathcal{I}_L(M)\|}{\sqrt{\|\mathcal{I}_L a\| \, \|\mathcal{I}_R b\|}} \, \delta_{\mathcal{I}_R(M)b, \phi(\mathcal{I}_L(M)a)} \ , \tag{A.2f}$$

and again this ϕ is uniquely determined by its value on G_L. Moreover, if neither a nor b are fixed points of G_L,

$$S_{a,b} = S_{\phi a, \phi b} \ . \tag{A.2g}$$

(We will often write $\phi(a)$ for any element of $\phi(\mathcal{I}_L(M)a)$.) This is Lemma 3.3(b) of [6]. For example, $\phi(\mathcal{I}_L(M)) = \mathcal{I}_R(M)$. In order to prove that this generic case holds for a given choice of $P_+^{L,R}$, one must look at the constraints on $M_{a,f}$ when a is not a fixed point of $\mathcal{I}_L(M)$ but f is one of $\mathcal{I}_R(M)$. In this paper we are only interested in the case $\|\mathcal{I}_L(M)\| = 2$, in which case

$$M_{a,f} \neq 0 \quad \Longrightarrow \quad S_{0,f}/S_{0,a} \in \{1, 2\} \ . \tag{A.2h}$$

This is proved by evaluating (2.3a) at $(a, 0)$ and $(0, f)$.

The final step in these classifications is to consider arbitrary weak invariants M and solve the constraints for those $b \in P_+^R$ satisfying $M_{0,b} \neq 0$. One constraint is given by (5.2c) with $a = 0$. Another useful constraint is [20]

$$\sum_{a \in P_+^R} M_{0,a} \, S_{a,b} \geq 0 \qquad \forall b \in P_+^R \ , \tag{A.3a}$$

$$\sum_{a \in P_+^R} M_{0,a} \, S_{a,b} = 0 \qquad \Longleftrightarrow \qquad b \notin \mathcal{P}_R(M) \ . \tag{A.3b}$$

Of course similar equations hold for P_+^L and $\mathcal{P}_L(M)$. These are proved by evaluating (2.3a) at $(0, a)$ and using (2.1c) and (2.2b). These are severe constraints and we find that for almost all M, (A.2a) will be satisfied.

Now let us turn to the proof of Thm. 2. Let M be any weak invariant for $A_{1,k}$. Recall the discussions about $A_{1,k}$ at the end of Section 2 and in Example 2. Write $\overline{k} := k + 2$. The automorphism invariants are easy to find: (A.1c) and (2.7a) say

$$\sin(2\pi/\overline{k}) = \sin(\pi\,(\phi 1 + 1)/\overline{k}) \ ,$$

and hence $\phi 1 \in \{1, J1\}$. $S_{1,1} = S_{\phi 1, \phi 1}$ says $\phi 1 = J1$ is only possible when k is even. Since $G = \{1\}$, we are now done: we find $M = \mathcal{A}_k$ if $\phi 1 = 1$, and $M = \mathcal{D}_k$ if $\phi 1 = J1$.

Next, consider the ADE$_7$-invariants which are not automorphism invariants. Then (A.2d) says that $\mathcal{I}_L(M) = \mathcal{I}_R(M) = \{0, J\}$. By (5.1d), k must be even. We choose the fusion-generator $G_{L,R} = \{2\}$. The only fixed point f is $f = k/2$ (for odd $k/2$, $f \notin \mathcal{P}_{L,R}(M)$ and can be ignored). We want to show that $M_{2,f} = 0$ (except for the trivial case where $k = 4$, when $2 = f$). The only solution to (A.2h) is $k = 16$, and $M_{2,f} \neq 0$ or $M_{f,2} \neq 0$ for $k = 16$ is easily seen to yield $M = \mathcal{E}_{16}$ (see e.g. Section 7.2 of [6]). Otherwise, M will be 'generic'; in this case $\phi(2) = 2$ is forced by (A.2g) at (2,0), so uniqueness forces $M = \mathcal{D}'_k$.

Finally, consider an arbitrary weak invariant M for A_1. We learned in (5.4) that (A.2a) is forced, except possibly when $k = 4, 8, 10, 28$. These succumb to a case-by-case analysis.

Consider first M for $k = 4$ violating (A.2a). Then by (5.4a), we may assume without loss of generality that $M_{0,2} \neq 0$. Put $b = 2$ in (A.3a) and use (5.4a) to obtain

$$\sin(3\pi/6) - M_{0,2}\sin(3\pi/6) + M_{0,4}\sin(3\pi/6) > 0 \ . \tag{A.4}$$

Therefore $M_{0,2} \leq M_{0,4}$ and hence $M_{0,2} = M_{0,4} = 1 = M_{4,0} = M_{4,4}$ by (5.1b), (5.2b) and (5.1c). Computing (2.3a) at (0,0) now forces $M_{2,0} = 1$. From (5.1d) we know that $M_{a,b} = 0$ if either a or b is odd. That $M_{2,2} = 0$ follows from (2.3b) at (0,2). Hence $M = \mathcal{E}_4$, given in (5.5d).

The argument for $k = 8$ is similar. Suppose $M_{0,2}+M_{0,6} \geq 1$. Using (5.4b) and putting $b = 4$ into (A.3a) gives

$$(1 + M_{0,8} - M_{0,2} - M_{0,6})\ \sin(5\pi/10) \geq 0 \tag{A.5a}$$

while (2.3b) at (0,1) gives

$$(M_{1,1} + M_{7,1})\ \sin(2\pi/10) = (1 - M_{0,8})\ \sin(2\pi/10) + (M_{0,2} - M_{0,6})\ \sin(4\pi/10) \ . \tag{A.5b}$$

(A.5b) forces $M_{0,2} = M_{0,6}$, since $\sin(4\pi/10)/\sin(2\pi/10)$ is irrational. Then (A.5a) forces $M_{0,8} = 1$. (5.2b) with $\sigma = \sigma_3$ then gives $M_{2,0} = M_{2,2} = 1$ (see (5.3b)). By (5.1d) and (A.3b), $\mathcal{P}_L(M) = \mathcal{P}_R(M) = \{0, 2, 6, 8\}$, so we are done by (5.1c).

For $k = 10$, (5.4c) and $b = 1, 2, 3$ in (A.3a) tells us

$$1 - M_{0,10} \geq |M_{0,4} - M_{0,6}| \tag{A.6a}$$

$$1 + M_{0,10} \geq M_{0,4} + M_{0,6} \ . \tag{A.6b}$$

The only difficult task here is eliminating the possibility $M_{0,10} = 0$, $M_{0,4} \neq 0$. In this case, $M_{0,4} = 1$ and $M_{0,6} = 0$. (5.2b) then implies $M_{a,0} = M_{0,a}$ for all a, so by (A.3b) $2 \notin \mathcal{P}_L(M)$. (2.3a) at (0,1) and (2,1) give $2M_{5,1}+M_{1,1}+M_{9,1} = 2$ and $M_{1,1}+M_{9,1} = M_{5,1}$ using (5.4c), i.e. $M_{5,1} = 2/3$, which is impossible.

For $k = 28$ use (5.4d) and $b = 1, 2, 3$ in (A.3a), and then $\sigma = \sigma_7, \sigma_{11}$ in (5.2b). The rest of the argument is as before. This concludes the proof of Thm. 2.

Now we turn to the classification of weak invariants M for $U_{m,\ell} \oplus A_{1,k}$. Though much more complicated notationally than for $\hat{A}_1$, and involving many more cases, the arguments are very similar to those used in Thm. 2. For later convenience we will replace the level ℓ of U_m with $\ell_{L,R}$, where $|\ell_L| = |\ell_R|$. The sets of highest weights here are $P_{L,R} = \overline{P}^{\ell_{L,R}} \times P_+^k$. The possible simple currents are $\mathcal{I}_{L,R} = \overline{P}^{\ell_{L,R}} \times \{0, J\}$. Let $\mathcal{P}'_{L,R}(M)$ denote the projections of $\mathcal{P}_{L,R}(M)$ onto $\overline{P}^{\ell_{L,R}}$.

Any weak invariants M' for $U_{m,\ell}$ and M'' for $A_{1,k}$, give us a weak invariant $M = M' \otimes M''$ of $U_{m,\ell} \oplus A_{1,k}$. The converse unfortunately is not true. We begin with the following very useful fact, true for any g (not just $g = A_{1,k}$), which tells us when M actually does factorise.

Claim Let M be a weak invariant for $U_{m,\ell} \oplus g$. Suppose that for each $x \in \mathcal{P}'_L(M)$, there exists $x' \in \overline{P}^{\ell_R}$ for which $M_{x,0;x',0} \neq 0$, and conversely that for each $y \in \mathcal{P}'_R(M)$ there is a $y'' \in \overline{P}^{\ell_L}$ such that $M_{y'',0;y,0} \neq 0$. Then $M = M' \otimes M''$ for some weak invariants M' and M'' of $U_{m,\ell}$ and g, resp.

Proof Define $M'_{x,y} := M_{x,0;y,0}$, $M''_{a,b} := M_{0,a;0,b}$. We want to show

$$M_{x,a;y,b} = M'_{x,y}\, M''_{a,b}\ . \tag{A.7}$$

Suppose $M_{x,a;y,b} = 0$. Then either $M'_{x,y} = 0$, or $M_{x,0;y,0} \neq 0$ and (by (5.1c)) $M_{0,a;0,b} = 0$ – in either case (A.7) holds.

If instead $M_{x,a;y,b} \neq 0$, then again by (5.1c) applied to $(v'', 0; v, 0)$ and $(0, a; y - x', b)$, $M_{0,a;y-x',b} \neq 0$. Now consider any $M_{a,c;v,d} \neq 0$; by hypothesis, there exists a v'' such that $M_{v'',0;v,0} \neq 0$, and hence by (5.1d) $v \cdot (y - x') \in \mathbb{Z}$. Thus again by (5.1d) applied to $(0, 0; y - x', 0)$ and $(a, c; v, d)$, we must have $M_{0,0;y-x',0} = 1$, i.e. $M_{x,0;y,0} = 1$. Then (5.1c) again forces (A.7). *QED*

As before, consider first the automorphism invariants M obeying (A.1a). (A.1c) with $a = (0,0)$ forces $\phi(x,c) \in \mathcal{I}_R(0,c)$. For a fusion-generator choose $G = \{(x_1, 0), \ldots, (x_n, 0), (0,1)\}$, where the x_i span $\overline{P}^{\ell_L}$. For each i, write $(y_i, J^{a_i}0) := \phi(x_i, 0)$, and also $(y_0, J^{a_0}1) := \phi(0,1)$. Then by (A.1c) these must obey

$$0 \equiv x_i \cdot x_j - y_i \cdot y_j + k a_i a_j/2 \equiv y_i \cdot y_0 - (a_i + k a_i a_0)/2 \equiv y_0^2 - k a_0/2 \pmod 1\ , \tag{A.8a}$$

for all $i, j \in \{1, \ldots, n\}$. By (5.1c) and the usual fusion arguments [27], we find

$$\phi(\sum_{i=1}^n c_i x_i,\, c_0) = (\sum_{i=0}^n c_i y_i,\, \prod_{i=0}^n J^{c_i a_i}\, c_0)\ . \tag{A.8b}$$

It is straightforward to verify from (A.8a) that (A.8b) satisfies (A.1c), and that ϕ is one-to-one. Hence it defines an automorphism invariant, and all automorphism invariants are of this form.

Next we consider the more general condition (A.2a). If both $\mathcal{I}_{L,R}(M) \subset \overline{P}^{\ell_{L,R}} \times \{0\}$, then this just reduces to the automorphism invariant case considered in the previous paragraph: 'renormalise' the levels $\ell_{L,R}$ by replacing the lattices $\Gamma_{L,R}$ with the denser lattices $\Gamma_{L,R} + \mathcal{I}_{L,R}(M)$.

So assume $\mathcal{I}_R(M) \not\subset \overline{P}^{\ell_R} \times \{0\}$. Again, by renormalising the levels, we can require $\mathcal{I}_L(M) = \{0, (x_L, J^{a_L})\}$ and $\mathcal{I}_R(M) = \{0, (x_R, J)\}$, where $2x_{L,R} = 0$, $(x_L, J^{a_L}) \neq 0$, and

$$x_L^2 + ka_L/2 \equiv x_R^2 + k/2 \equiv 0 \pmod 1 . \tag{A.9a}$$

Suppose $x_R = 0$ (so k is even) but $x_L \neq 0$. Then we have a problem with (5.4), since $1 \in \mathcal{P}'_L(M)$ and $o \notin \mathcal{P}'_R(M)$ for any odd o. Therefore $x_L = 0$ iff $x_R = 0$.

So assume next that both $x_{L,R} \neq 0$. The fusion-generator G_L for the left side can be chosen to be of the form $\{(x_1, 0), \ldots, (x_n, 0), (x_0, 1)\}$, where the $(x_i, 0)$ generate all $(x, 0) \in \mathcal{P}_L(M)$, and $(x_0, 1) \in \mathcal{P}_L(M)$. For each $i \geq 0$, choose some $y_i \in \overline{P}^{\ell_R}$ such that

$$y_i \cdot x_R + \delta_{i,0}/2 \equiv x_i \cdot x_j - y_i \cdot y_j \equiv 0 \pmod 1 , \tag{A.9b}$$

for all $i, j \geq 0$. Define M by (A.2f), with

$$\phi(\sum_{i=0}^{n} c_i x_i,\, c_0) = (\sum_{j=0}^{n} c_i y_i,\, c_0) . \tag{A.9c}$$

Our M must be of this form, and the reader can readily verify that any such M is a well-defined weak invariant.

The final possibility for a weak ADE$_7$-invariant here is that (after renormalising the levels) $\mathcal{I}_L(M) = \mathcal{I}_R(M) = \{0, (0, J)\}$. Assume first that all $M_{x_i,0;y,k/2} = 0$ for all x, y. Then for each $(x, 0) \in \mathcal{P}_L(M)$ there exists a x' such that $M_{x,0;x',0} \neq 0$. Using (5.1c), $x \mapsto x'$ must be a bijection here, and so by the Claim M factorises.

If instead $M_{x,0;y,k/2} \neq 0$ for some x, then (A.2h) forces $k = 4$. For a given $x \in \mathcal{P}'_L$, it is easy to show (by evaluating (2.3a) at $(x, 0; 0, 0)$) that there must exist some $y \in \mathcal{P}'_R$ such that either

$$M_{x,0;z,a} = \delta_{z,y}\,(\delta_{a,0} + \delta_{a,4}) \tag{A.10a}$$

$$M_{x,0;z,a} = \delta_{z,y}\,\delta_{a,2} . \tag{A.10b}$$

Let $\mathcal{I}_L(a)$ be the x satisfying (A.10a), and $\mathcal{I}_L(b)$ be those satisfying (A.10b) – we know both $\mathcal{I}_L(a)$ and $\mathcal{I}_L(b)$ are nonempty ($0 \in \mathcal{I}_L(a)$, and $\mathcal{I}_L(b) = \emptyset$ would mean M factorises).

Similarly, given any $y \in \mathcal{P}'_R$, there exists $x, x' \in \mathcal{P}'_L$ such that either

$$M_{z,a;y,2} = (\delta_{z,x} + \delta_{z,x'})\,(\delta_{a,0} + \delta_{a,4}) \tag{A.11a}$$

$$M_{z,a;y,2} = \delta_{z,x}\,(\delta_{a,0} + \delta_{a,4}) + \delta_{z,x'}\delta_{a,2} \tag{A.11b}$$

$$M_{z,a;y,2} = (\delta_{z,x} + \delta_{z,x'})\,\delta_{a,2} . \tag{A.11c}$$

Similarly put each y in $\mathcal{I}_R(A), \mathcal{I}_R(B), \mathcal{I}_R(C)$, respectively. Of course similar remarks hold for M^t.

Choose any $x \in \mathcal{I}_L(a)$, and x_1, x'_1, y_1 satisfying any of (A.11). Let $\Delta_L := x_1 - x'_1$. Then $\Delta_L \cdot x \in \mathbb{Z}$, by (5.1d). Moreover, for any $x \in \mathcal{I}_L(b)$, (2.3a) evaluated at $(x, 0; y_1, 2)$ implies $x \cdot \Delta_L \in \pm\frac{1}{3} + \mathbb{Z}$. Since $\mathcal{I}_L(a) \cup \mathcal{I}_L(b) = \mathcal{P}'_L$, what we have shown is that

$$\mathcal{I}_L(a) = \{x \in \mathcal{P}'_L \,|\, x \cdot \Delta_L \in \mathbb{Z}\} \tag{A.12a}$$

$$\mathcal{I}_L(b) = \pm\, a_L + \mathcal{I}_L(a) \qquad \text{for some } a_L . \tag{A.12b}$$

Of course there is a Δ_R playing the identical role for $\mathcal{P}_R(M)$. Moreover, by (5.1d) and (A.11) we have

$$M_{x,a;z,c}\neq 0 \text{ and } M_{y,b;z+i\Delta_R,d}\neq 0 \quad\Longrightarrow\quad x-y\in\{0,\pm\Delta_L\}\ , \tag{A.12c}$$

for any x,y,z,a,b,c,d,i. From (A.12) we get that either $\mathcal{I}_L(A)=\mathcal{I}_R(A)=\emptyset$ and $\mathcal{I}_{L,R}(B)=\mathcal{I}_{L,R}(b)$, or $\mathcal{I}_L(B)=\mathcal{I}_R(B)=\emptyset$ and $\mathcal{I}_{L,R}(A)=\mathcal{I}_{L,R}(a)$.

Let x,x',y satisfy one of (A.11a) or (A.11c). Then $x'=x\pm\Delta_L$ for some choice of sign, and hence by (2.5a)

$$\pm\Delta_L\cdot x\equiv\Delta_L^2 \qquad (\mathrm{mod}\ 1)\ . \tag{A.13a}$$

If x,x',y satisfy (A.11b), then similarly (2.5a) and (A.12c) say $x'=x\pm\Delta_L$, and

$$\pm\Delta_L\cdot x\equiv\Delta_L^2-1/3 \qquad (\mathrm{mod}\ 1)\ . \tag{A.13b}$$

Suppose first that $\Delta_L^2\in\mathbb{Z}$. Then by (A.12a), $M_{\Delta_L,0;x_R,0}=1$ for some x_R – in fact we can choose the sign of Δ_R so that $x_R=\Delta_R$. By (A.13a), we must have $\mathcal{I}_{L,R}(A)=\emptyset$, $\mathcal{I}_{L,R}(B)=\mathcal{I}_{L,R}(c)$. Let $x_1,\ldots,x_{n-1}$ be generators for $\mathcal{I}_L(a)$. Then for each i there is a y_i such that $M_{x_i,0;y_i,0}=1$. Since $a_L\in\mathcal{I}_L(B)$, there is a $b\in\mathcal{I}_R(B)$ such that $M_{a_L,2;b,2}=1$. Note that $x_1,\ldots,x_n:=a_L,y_1,\ldots,y_n:=b$ satisfy (A.8a) with all $a_i=0$ and $y_0=0$. Hence they define an automorphism invariant M'. Write $M''=M'^{-1}M$. It is easily shown that its only nonzero entries are

$$M''_{x,J^i0;x,J^j0}=M''_{y,2;y,2}=M''_{y,J^i0;y+3(\Delta_R\cdot y)\Delta_R,2}=M''_{y,2;y-3(\Delta_R\cdot y)\Delta_R,J^i0}=M''_{x,2;x\pm\Delta_R,2}=1 \tag{A.14a}$$

for all $x\in\mathcal{I}_R(a)$, $y\in\mathcal{I}_R(b)$, $i,j\in\{0,1\}$, and choices of signs. Conversely, any choice of $\Delta_L\in\overline{P}^{\ell_L}$ with $3\Delta_L=0$ and $\Delta_L^2\in\mathbb{Z}$, defines a distinct weak invariant in this way, which obeys (2.5a).

Otherwise, $\Delta_L^2\not\in\mathbb{Z}$. Note that from (A.12c) and (2.5a), $(0,2)\not\in\mathcal{I}_{L,R}(C)$, and hence $\mathcal{I}_{L,R}(B)=\emptyset$, $\mathcal{I}_{L,R}(A)=\mathcal{I}_{L,R}(a)$. $M_{0,2;\Delta_R,0}=M_{\Delta_L,0;0,2}=1$ then forces $\Delta_L^2\equiv\Delta_R^2\equiv\frac{2}{3}$ (mod 1), so again we get an automorphism invariant M' from (A.8), such that $M'':=M'^{-1}M$ has the following nonzero entries:

$$M''_{x,J^i0;x,J^j0}=M''_{x\pm\Delta_R,J^i0;x,2}=M''_{x,2;x\pm\Delta_R,J^j0}=M''_{x\pm\Delta_R,2;x\pm\Delta_R,2}=1\ . \tag{A.14b}$$

Conversely, for any choice of $\Delta_L\in\overline{P}^{\ell_L}$ with $3\Delta_L=0$ and $\Delta_L^2\equiv\frac{2}{3}$ (mod 1), we get a distinct weak invariant of this form which obeys (2.5a).

That exhausts all the weak ADE$_7$-invariants. Again the remaining exceptionals will occur only at $k=4,8,10,28$. We will work out the case $k=10$ in detail – the remaining exceptional levels are easier, and succumb to similar arguments. Assume, by renormalising levels if necessary, that either $\mathcal{I}_L(M)=\mathcal{I}_R(M)=\{(0,0)\}$, or $\|\mathcal{I}_{L,R}(M)\|=2$ and $(x,J)\in\mathcal{I}_R(M)$ for some x. Define $s_a:=\sum_x M_{0,0;x,a}$. Then $s_{10}\in\{0,1\}$ by (5.1b), (5.1c). Putting $b=(0,1)$, (0,2), (0,3) in (A.3a) tells us

$$1-s_{10}\geq|s_4-s_6| \tag{A.15a}$$

$$1+s_{10}\geq s_4+s_6 \tag{A.15b}$$

(compare (A.6)). Hence also $s_4, s_6 \in \{0,1\}$. The argument against the possibility that $s_{10} = 0$ but $s_4 = 1$ is identical to the analogous argument for $A_{1,10}$ (see below (A.6)): evaluate (2.3a) at (0,0;0,1) and (0,2;0,1).

Next suppose $s_{10} = 0$ but $s_6 = 1$. Then $\exists z_6, z'_6$ such that $M_{0,0;z_6,6} = M_{z'_6,6;0,0} = 1$, and for any x, $(x,4)$, $(x,10) \notin \mathcal{P}_{L,R}(M)$, by the usual arguments. Looking at each $b = (x,0)$ in (A.3a) forces $z'_6 = 0 = z_6$. Thus also $(x,a) \notin \mathcal{P}_{L,R}(M)$ $\forall x$, for each $a = 1,2,5$. We find $M_{0,3;x,3} = M_{0,3;x,7} = \delta_{x,z_3}$ for some z_3. We also find $M_{0,4;x,4} = M_{0,4;x,10} = M_{0,10;x,4} = M_{0,10;x,10} = \delta_{x,0}$. Let x_i generate $\mathcal{P}'_L$. Then for each i there is a y_i and a_i such that $M_{x_i,0;y_i,J^{a_i}0} = 1$. The conditions on y_i, a_i, and $y_0 := z_3$ coming from (5.1d) and (2.5a) are precisely the congruences of (A.8a). Hence there is an automorphism invariant M' given by (A.8b), for which $M = M'\,(\overline{I} \otimes \mathcal{E}_{10})$.

The remaining possibility is

$$M_{0,0;0,0} = M_{0,0;z_4,4} = M_{0,0;z_6,6} = M_{0,0;z_{10},10} = 1 \tag{A.16}$$

and all other $M_{0,0;x,a} = 0$. Of course $z_4 = z_6 + z_{10}$ and $2z_{10} = 0$. M^t will obey a similar equation, for parameters z'_4, z'_6, z'_{10} (the argument uses (5.4d): see the argument after (A.9a)). If $z_{10} = 0$ then looking at (2.3a) at $(x,0;0,0)$ we see that the Claim applies and M factorises. For $z_{10} \neq 0$, $\exists z_0$ such that $z_0 \cdot z_{10} \equiv \frac{1}{2}$ (mod 1). Looking at (2.3a) at $(0,0;x,0)$ shows that either $z_6 = 0$, or $z_6 = z_{10}$. But $z_6 = z_{10}$ is ruled out by using an argument similar to that after (A.6b): we would have $\sum_x M_{x,5;x_0,1} = \frac{4}{3}$.

Thus $z_4 = z_{10} \neq 0$, $z_6 = 0$. This succumbs to a similar argument to the $s_{10} = 0$, $s_6 \neq 0$ one: we find the parameters (like $x_R := z_{10}$) obey constraints identical to (A.9b) (provided we impose (2.5a)), and hence defines a weak invariant M'' such that $M = M''\,(\overline{I} \otimes \mathcal{E}_{10})$.

References

[1] R. E. Prange and S. M. Girvin, The Quantum Hall Effect, 2nd edition (Springer-Verlag, New York, 1990)

[2] J. Fröhlich, U. M. Studer, and E. Thiran, A classification of quantum Hall fluids, cond-mat/9503113

[3] A. Cappelli, C. A. Trugenberger, and G. R. Zemba, Nucl. Phys. (Proc. Suppl.) B 45A (1996) 112

[4] A. Cappelli and G. R. Zemba, Modular invariant partition functions in the quantum Hall effect, hep-th/9605127

[5] M. Milovanović and N. Read, Edge excitations of paired fractional quantum Hall states, cond-mat/9602113

[6] T. Gannon, Kac-Peterson, Perron-Frobenius, and the classification of conformal field theories, q-alg/9510026

[7] A. Cappelli, C. Itzykson, and J.-B. Zuber, Commun. Math. Phys. 113 (1987) 1

[8] A. Cappelli, Phys. Lett. B185 (1987) 82

[9] D. Gepner, Nucl. Phys. B296 (1988) 757

[10] F. Ravanini and S.-K. Yang, Phys. Lett. B195 (1987) 202;
Z. Qui, Phys. Lett. B198 (1987) 497;

S.-K. Yang, Nucl. Phys. B285 (1987) 183
[11] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, 2nd edition (Springer-Verlag, Berlin, 1993)
[12] V. G. Kac, Infinite-dimensional Lie algebras, 3rd edition (Cambridge University Press, Cambridge, 1990)
[13] R. V. Gamkrelidze (Ed.), Geometry I, Encyclopedia of Mathematical Sciences Vol. 28 (Springer-Verlag, Berlin, 1991)
[14] T. Gannon and C. S. Lam, Rev. Math. Phys. 3 (1991) 331
[15] C. Itzykson, Nucl. Phys. (Proc. Suppl.) 5B (1988) 150;
P. Degiovanni, Commun. Math. Phys. 127 (1990) 71
[16] D. Gepner and Z. Qui, Nucl. Phys. B285 (1987) 423
[17] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B327 (1993) 673
[18] A. Coste and T. Gannon, Phys. Lett. B323 (1994) 316
[19] M. Kreuzer and A. N. Schellekens, Nucl. Phys. B411 (1994) 97
[20] T. Gannon, J. Math. Phys. 36 (1995) 675
[21] P. Di Vecchia, J. L. Peterson, M. Yu and H. B. Zheng, Phys. Lett. B174 (1986) 280
[22] T. Gannon and M. A. Walton, Commun. Math. Phys. 173 (1995) 175
[23] E. Kiritsis, Phys. Rev. D36 (1987) 3048;
G. Mussardo, G. Sotkov and M. Stanishtov, Phys. Lett. B218 (1989) 191
[24] W. Eholzer and M. R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, hep-th/9601163
[25] K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369
[26] I. G. Koh and P. Sorba, Phys. Lett. B215 (1988) 723
[27] T. Gannon, Invent. Math. 122 (1995) 341