THREE-DIMENSIONAL TERMINAL CONTRACTIONS

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ABSTRACT. The analog of Miles Reid's general elephant conjecture is that every three-dimensional terminal extremal contraction is not exceptional and hence the canonical divisor K has 1-, 2-, 3-, 4- or 6-complement. In this paper the approach to the conjecture is suggested. The main results are the following ones. The structure theorem about the inductive log surface $(\overline{S}, \mathrm{Diff}_{\overline{S}}(0))$ is proved. Shokurov's conjecture on the characterization of toric varietis is proved for the three-dimensional small contractions. The basic special log flips of terminal type are classified.

Introduction

The problem of the existence of a "good" divisor in the (multiple) anticanonical linear system is a fundamental one in the classification of singularities, contractions and varieties arising in the log minimal model program. In solving this problem the powerful method is a theory of complements. It allows to reduce the question to some log variety of a smaller dimension, called *an inductive log variety* (for instance, see the papers [23], [18], [15]).

Roughly speaking, an *n-complement* of a canonical divisor K is an effective $\mathbb{Q}$-divisor D^+ such that $n(K + D^+) \sim 0$ and $K + D^+$ is a log canonical divisor. Other advantages of this method are that it works in any dimension, every n-complement of our local contraction, singularity gives an n-complement of some inductive log variety and every n-complement of the inductive log variety is extended to an n-complement of our local contraction.

For the three-dimensional terminal contractions the following conjecture takes place. It is the analog of Miles Reid's general elephant conjecture in the theory of complements.

Conjecture 0.1. *Let $f\colon X \to (Z \ni P)$ be a contraction of three-dimensional variety X with the following properties:*

(1) *the variety X has the $\mathbb{Q}$-factorial terminal singularities;*

This work was done with the partial support of the Russian Foundation for Basic Research (grant no. 02-01-00441), the Leading Scientific Schools (grant no. 00-15-96085) and INTAS-OPEN (grant no. 2000#269.)

(2) *the divisor* $-K_X$ *is* f*-ample;*
(3) $\dim Z > 0$*,* Z *is a germ near* P *and* $\rho(X/Z) = 1$.

Then this contraction is not exceptional, in particular there exists 1-, 2-, 3-, 4- *or* 6-*complement of a divisor* K_X.

See chapter 4 for the discussion of this conjecture.

We suggest the very direct and natural approach to prove it. Assume that f is exceptional. Equivalently, the inductive log surface $(\overline{S}, \mathrm{Diff}_{\overline{S}}(0))$ is exceptional (see chapter 3). The exceptional log surfaces belong to the finite number of algebraic families [23]. Therefore it is possible to obtain their complete and detailed classification. Such classification was done in the papers [8], [23] for $\delta\big(\overline{S}, \mathrm{Diff}_{\overline{S}}(0)\big) = 1, 2$, where

$$\delta\big(\overline{S}, \mathrm{Diff}_{\overline{S}}(0)\big) = \#\Big\{E \mid E \text{ is a divisor (not nessecarily exceptional)} \\ \text{with a discrepancy } a(E, \mathrm{Diff}_{\overline{S}}(0)) \leq -\frac{6}{7}\Big\}.$$

Since $\delta \leq 2$ [23, Theorem 5.1] it remains to classify the log surfaces with $\delta = 0$. In the case $\delta = 0$ the models are already constructed [9].

Let us remember the construction of the inductive log surface $(\overline{S}, \mathrm{Diff}_{\overline{S}}(0))$ (see chapter 3). At first we take an inductive blow-up (purely log terminal blow-up) of a variety X at some terminal point and obtain the pair (Y, S). After it the log minimal model program is applied and we obtain the pair $(\overline{Y}, \overline{S})$ required.

Problem. Describe the class of all log del Pezzo surfaces, generic $\mathbb{P}^1$-fibrations which can be the exceptional divisors of some purely log terminal blow-ups of three-dimensional terminal singularities.

The class mentioned above is restricted. Thus, it is “difficult” to obtain (after some transformation) the exceptional log surface $(\overline{S}, \mathrm{Diff}_{\overline{S}}(0))$ from “very” non-exceptional log surface $(S, \mathrm{Diff}_S(0))$. Unfortunately, this information is not enough to prove conjecture 0.1. Therefore we suggest to study the birational map $Y \dashrightarrow \overline{Y}$. It is the composition of anti-log flips with respect to $-(K_Y + S)$. These log flips are special and have the terminal singularities outside S. They are called *the special log flips of terminal type* (see definition 2.14).

Problem. Describe the three-dimensional special log flips of terminal type.

The advantage of described above appoach is that if it is realized then the classification of terminal extremal contractions (may be not very explicit) will be obtained immediately.

This paper is organized in the following way. In chapter 1 the main definitions and preliminary results are collected. In chapter 2 the method of a study of special flips is given. The idea is to make a reduction to the toric special flips. For this purpose Shokurov's conjecture on the characterization of toric varieties is proved for the three-dimensional small contractions (see theorem 2.4). Also the basic special log flips of terminal type are classified (see theorem 2.20). In chapter 3 we recall the construction of model. In chapter 4 the theorem about the structure of a transformation $(S, \mathrm{Diff}_S(0)) \dashrightarrow (\overline{S}, \mathrm{Diff}_{\overline{S}}(0))$ is proved (see theorem 4.7).

I am grateful to Professors Yu.G. Prokhorov and V.V. Shokurov for useful discussions.

The work has been completed during my stay at Max-Planck-Institut für Mathematik in 2003. I would like to thank MPIM for the hospitality, support and stimulating atmosphere.

1. **Preliminary facts**

All varieties are algebraic and are assumed to be defined over $\mathbb{C}$, the complex number field. The main definitions, notations and notions used in the paper are given in [5], [16].

Definition 1.1. Put $\Phi_{\mathbf{sm}} = \{1 - 1/m \mid m \in \mathbb{Z}_{>0} \cup \{\infty\}\}$. A coefficient d is called *standard* if $d \in \Phi_{\mathbf{sm}}$.

Definition 1.2. For fix $n \in \mathbb{N}$ put

$$\mathcal{P}_n = \{a \mid 0 \leq a \leq 1,\ \llcorner (n+1)a \lrcorner \geq na\}.$$

Definition 1.3. Let (X, D) be a contraction of varieties, where D is a subboundary. Then a $\mathbb{Q}$-*complement* of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geq D$, $K_X + D'$ is log canonical and $n(K_X + D') \sim 0$ for some $n \in \mathbb{N}$.

Definition 1.4. Let X be a normal variety and let $D = S + B$ be a subboundary on X such that B and S have no common components, S is an effective integral divisor and $\llcorner B \lrcorner \leq 0$. Then we say that $K_X + D$ is n-*complementary* if there is a $\mathbb{Q}$-divisor D^+ (called an n-*complement*) such that

(1) $n(K_X + D^+) \sim 0$ (in particular, nD^+ is an integral divisor);
(2) the divisor $K_X + D^+$ is log canonical;
(3) $nD^+ \geq nS + \llcorner (n+1)B \lrcorner$.

Definition 1.5. Let (X, D) be a projective log variety of dimension d such that

(1) the pair (X, D) is log canonical;
(2) every coefficient of a divisor D is standard;
(3) the divisor $-(K_X + D)$ is nef and big;
(4) there exists a $\mathbb{Q}$-complement of a divisor $K_X + D$.

Then the pair (X, D) is called a *log Fano variety.*

Put

$$\mathrm{Compl}(X, D) = \min\{m \mid \text{the divisor } K_X + D \text{ has } m\text{-complement}\}$$

and

$$N_d = \sup\{\mathrm{Compl}(X, D) \mid (X, D) \text{ is a log Fano variety of dimension } d\}.$$

It was proved that $N_1 = 6$ [22, Example 5.2] and $N_2 < \infty$ [23]. Hypothetically, $N_d < \infty$ for every $d \geq 1$ and $N_2 = 66$ (see [8, Corollary 2.4]).

Put

$$\Phi^d = \Phi_{\mathbf{sm}} \cup \Big[\frac{N_{d-1}}{N_{d-1}+1}, 1\Big].$$

Note that $a \in \mathcal{P}_n$ if $a \in \Phi^d$ and $n \leq N_{d-1}$ [18, Lemma 1.4].

The invariance of the complements with respect to the log minimal model program and their inductive properties are shown in the following three theorems.

Theorem 1.6. [22, Lemma 5.4] *Let $f\colon X \to Y$ be a birational contraction and let D be a subboundary on X. Assume that $K_X + D$ is n-complementary for some $n \in \mathbb{N}$. Then $K_Y + f(D)$ is also n-complementary.*

Theorem 1.7. [23, Lemma 4.4] *Let $f\colon X \to Z$ be a birational contraction of varieties and let D be a subboundary on X. Assume that*

(1) *the divisor $K_X + D$ is f-nef;*
(2) *the coefficient of every non-exceptional component of D meeting $\mathrm{Exc}\, f$ belongs to $\mathcal{P}_n$;*
(3) *the divisor $K_Z + f(D)$ is n-complementary.*

Then the divisor $K_X + D$ is also n-complementary.

Theorem 1.8. [16, Proposition 4.4.1] *Let $f\colon X \to (Z \ni P)$ be a contraction and D be a boundary on X. Put $S = \llcorner D \lrcorner$ and $B = \{D\}$. Assume that*

(1) *the divisor $K_X + D$ is purely log terminal;*
(2) *the divisor $-(K_X + D)$ is f-nef and f-big;*
(3) *$S \neq 0$ near $f^{-1}(P)$;*
(4) *every coefficient of D belongs to $\mathcal{P}_n$.*

Further, assume that near $f^{-1}(P) \cap S$ there exists an n-complement $K_S + \mathrm{Diff}_S(B)^+$ of $K_S + \mathrm{Diff}_S(B)$. Then near $f^{-1}(P)$ there exists an n-complement $K_X + S + B^+$ of $K_X + S + B$ such that $\mathrm{Diff}_S(B)^+ = \mathrm{Diff}_S(B^+)$.

Definition 1.9. Let $(X/Z \ni P, D)$ be a contraction of varieties, where D is a boundary.

(1) Assume that Z is not a point (local case). Then $(X/Z \ni P, D)$ is said to be *exceptional* over P if for any $\mathbb{Q}$-complement $K_X + D'$ of $K_X + D$ near the fiber over P there exists at most one divisor E such that $a(E, D') = -1$.

(2) Assume that Z is a point (global case). Then (X, D) is said to be *exceptional* if every $\mathbb{Q}$-complement $K_X + D'$ of $K_X + D$ is kawamata log terminal.

Example 1.10. Let $(Z \ni P)$ be a normal singularity. Suppose that there is an effective divisor H such that (Z, H) is a log canonical pair and $\llcorner H \lrcorner \neq 0$. Then the singularity $(Z \ni P)$ is not exceptional. Therefore the three dimensional terminal singularity is not exceptional because there is a divisor having only Du Val singularities in the anticanonical linear system $|-K_Z|$ [21, §6.4]. Hence every three-dimensional divisorial contraction $f\colon X \to (Z \ni P)$ is not exceptional if the singularity $(Z \ni P)$ is terminal.

2. **Three-dimensional toric flips**

2.1. We refer the reader to [12] for the basics of a toric geometry. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. For a fan Δ in N the corresponding toric variety is denoted by $T_N(\Delta)$. For a k-dimensional cone $\sigma \in \Delta$ the closure of corresponding orbit is denoted by $V(\sigma)$. This is a closed subvariety of codimension k in $T_N(\Delta)$.

Example 2.2. 1) Let

$$\sigma = \langle e_1, e_2, e_3, e_4 \rangle = \langle (1,0,0), (0,1,0), (0,0,1), (1,1,-1) \rangle$$

be a cone in $N \cong \mathbb{Z}^3$ and $\Delta' = \{\sigma, \text{ all faces of } \sigma\}$ be a fan. Then $T_N(\Delta')$ is a three-dimensional terminal hypersurface singularity $x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0)$. Let

$$\Delta = \{\langle e_1, e_2, e_3 \rangle, \langle e_1, e_2, e_4 \rangle, \text{ their faces}\}$$

and

$$\Delta^+ = \{\langle e_1, e_3, e_4 \rangle, \langle e_2, e_3, e_4 \rangle, \text{ their faces}\}$$

be the fans. Then $T_N(\Delta) \to T_N(\Delta')$ and $T_N(\Delta^+) \to T_N(\Delta')$ are the small resolutions. The birational map $T_N(\Delta) \dashrightarrow T_N(\Delta^+)$ is the famous Atiyah's flop.
2) Let $f\colon X \to Z$ be a toric flopping contraction (with respect to K_X) of a smooth 3-fold X. Then f is the previous Atiyah's contraction, in particular $\rho(X/Z) = 1$ [24, Example 6]. The generalization of this fact is in proposition 2.3.

Proposition 2.3. *Let $(Z \ni P)$ be a three-dimensional toric terminal singularity. Then either $(Z \ni P) \cong_{\mathrm{an}} (\mathbb{C}^3 \ni 0)/\mathbb{Z}_r(1, -1, q)$, $(r, q) = 1$ or $(Z \ni P) \cong_{\mathrm{an}} (x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0))$.*

Proof. We give an non-combinatorical proof. If the singularity $(Z \ni P)$ is $\mathbb{Q}$-factorial then we have the first possibility by terminal lemma (for example, see [12, §1.6]). Assume that the singularity $(Z \ni P)$ is not $\mathbb{Q}$-factorial. Let $(Z' \ni P') \to (Z \ni P)$ be a canonical cover. Let $\widetilde{Z} \to Z'$ be a $\mathbb{Q}$-factorialization. Since the variety $\widetilde{Z}$ has the $\mathbb{Q}$-factorial toric terminal singularities of index one only then $\widetilde{Z}$ is a smooth variety. By [24, Example 6] we obtain that

$$(Z' \ni P') \cong_{\mathrm{an}} (x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0)).$$

Hence $(Z' \ni P') = (Z \ni P)$ by the classification of terminal singularities [11] (of course, it is clear without classification). $\square$

In the next theorem Shokurov's conjecture on the characterization of toric varieties is proved for the three-dimensional small contractions.

Theorem 2.4. *Let $f\colon (X, D = \sum_{i=1}^r D_i) \to (Z \ni P)$ be a small contraction of three-dimensional normal $\mathbb{Q}$-gorenstein variety X. Assume that the divisor $K_X + D$ is log canonical and the divisor $-(K_X + D)$ is f-nef. Then*

$$r \le \sigma(X/Z) + 3,$$

where $\sigma(X/Z)$ is the rank of group Weil divisors on X modulo the numerical equivalence over Z. Moreover, the equality holds if and only if the log variety $(X/Z \ni P, D)$ is analytically isomorphic to a toric pair, in particular $K_X + D \sim 0$ over Z.

Proof. The strategy of proof is the same one as in the paper [19]. The question is local, so $(Z \ni P)$ is a sufficiently small analytical neighborhood. The sufficiency is well-known. Let $r \ge \sigma(X/Z) + 3$ and a divisor $K_X + D'$ be a $\mathbb{Q}$-complement of $K_X + D$. It exists since we can add to the divisor D the necessary number of general hyperplane sections of a variety X. So, by abundance theorem [5, Theorem 8.4] the $\mathbb{Q}$-complement required is constructed. Let $D' = \sum d_i D'_i$. Then $\sum d_i \geq \sigma(X/Z)+3$. Actually, in lemma 2.9 we will prove that $D' = D$.

Proposition 2.5. *Let Y be a log canonical variety. Let B be an effective $\mathbb{Q}$-Weil divisor not containing any minimal log canonical center of $LCS(Y, 0)$. Then there exists a small contraction ($\mathbb{Q}$-factorialization) $\widetilde{Y} \to Y$ such that $K_{\widetilde{Y}}$ and $\widetilde{B}$ are $\mathbb{Q}$-Cartier divisors, where $\widetilde{B}$ is the proper transform of a divisor B.*

Proof. In the proof of this proposition we use the log minimal model program and abundance theorem [5, Theorem 8.4], which are proved in dimension ≤ 3. Let $\varphi\colon Y' \to Y$ be a minimal log terminal modification [16, Corollary 3.1.3]. If Y is a kawamata log terminal variety then φ is a required small contraction. Assume that Y is not a kawamata log terminal variety. Write $K_{Y'} + \sum E_i = \varphi^* K_Y$. By hypothesis of proposition the center of every E_i on Y does not lie in $\operatorname{Supp} B$. Run $(K_{Y'} + \sum E_i + \varepsilon B')$–MMP over Y for $0 < \varepsilon \ll 1$, where B' is a proper transform of B. After some log flips we obtain a log variety $(\overline{Y}, \sum \overline{E}_i + \varepsilon \overline{B})$. By abundance theorem the linear system $|m(K_{\overline{Y}} + \sum \overline{E}_i + \varepsilon \overline{B})|$ is free over Y for some $m \gg 0$ and gives a birational contraction $q\colon \overline{Y} \to \widetilde{Y}$. It is obvious that q contracts every $\overline{E}_i$. Moreover, if $(K_{\overline{Y}} + \sum \overline{E}_i + \varepsilon \overline{B}) \cdot C = 0$ for a curve C then $\overline{B} \cdot C = (K_{\overline{Y}} + \sum \overline{E}_i) \cdot C = 0$. Hence $K_{\widetilde{Y}}$ is a $\mathbb{Q}$-Cartier divisor and $\widetilde{Y}$ is a required variety. □

Proposition 2.6. [5] *Let $(Y \ni P, B = \sum b_i B_i)$ be a log canonical pair such that every component of a boundary B is $\mathbb{Q}$-Cartier. Then $\sum b_i \leq \dim Y$. Moreover,*

a) if the equality holds then $(Y \ni P, \llcorner B \lrcorner)$ is a $\mathbb{Q}$-factorial toric pair;

b) if $(Y \ni P)$ is not a kawamata log terminal singularity then $\sum b_i \leq \dim Y - 2$.

Proof. The first and second statements are followed from [5, Theorems 18.22, 18.23]. Let us prove the last one. The proposition is trivial if $\dim Y = 2$. Let us prove it by induction. We have $\dim Y \geq 3$, $\sum b_i > \dim Y - 2$ and hence P is not a minimal log canonical center of $LCS(Y, 0)$. By taking cyclic covers we may assume that $B_i = \{f_i = 0\}$ are Cartier divisors. Let $B = \{\sum c_i f_i = 0\}$ for general $c_i \in \mathbb{C}$. By the choice of a divisor B we obtain that $a(E, B) \geq a(E, B_i)$ for all i and every E. Therefore the pair $(Y, B + \sum b'_i B_i)$ is log canonical, where $b'_i \geq 0$ and $\sum b'_i = \sum b_i - 1$. By inversion of adjunction [5, Theorem 17.6] the variety B is log canonical and not kawamata log terminal. By the inductive hypothesis the proposition is proved for $(B, \sum b'_i B_i|_B)$. So, $\sum b'_i \leq \dim Y - 3$, a contradiction. □

Continue the proof of our theorem.

Lemma 2.7. *The variety X has the kawamata log terminal singularities.*

Proof. Replace the variety X with its $\mathbb{Q}$-factorialization outside minimal log canonical centers of $LCS(X,0)$ by proposition 2.5. In particular, D'_i is a $\mathbb{Q}$-Cartier divisor for every i. By applying some log minimal model program over Z we can contract every curve from $\operatorname{Exc} f$, which does not contain any minimal log canonical center of $LCS(X,0)$. We obtain $X \dashrightarrow W$ and $W \neq Z$ by proposition 2.6. Therefore it can be assumed that every curve from $\operatorname{Exc} f$ contains a minimal log canonical center of $LCS(X,0)$. By connectedness lemma [5, Theorem 17.4] we obtain that $LCS(X,0) \cap \operatorname{Exc} f$ is a curve. Also, if there is an exceptional curve $C \not\subset LCS(X,0)$ then $C \cap D' = \emptyset$. Thus there exists an irreducible exceptional curve lying in $LCS(X,0)$ such that at least three components of D' intersect and do not contain it. Let $X \dashrightarrow X^+$ be a log flip in this curve and D'^+ be a proper transform of D'. Then the divisor $K_{X^+} + D'^+$ is not log canonical along a flipped locus, a contradiction. $\square$

Replace the variety X with its $\mathbb{Q}$-factorialization. By the log minimal model program we can assume without loss of generality that $\operatorname{Exc} f = C$ is an irreducible curve ($\rho(X/Z) = 1$).

Definition 2.8. Let $B = \sum b_i B_i$ be a $\mathbb{Q}$-Weil divisor. Then $||B|| = \sum b_i$.

Put

$$D^{\text{hor}} = \sum_{i\colon D'_i \cdot C > 0} d_i D'_i \quad \text{and} \quad D^{\text{vert}} = \sum_{i\colon D'_i \cdot C \leq 0} d_i D'_i.$$

Actually, corollary 2.10 implies that $D'_i \cdot C \neq 0$ for all i. Let $f^+\colon X^+ \to Z$ be a log flip of f and $\operatorname{Exc} f^+ = C^+$.

Lemma 2.9. [19, Lemma 2.10] *We have $||D^{\text{hor}}|| = ||D^{\text{vert}}|| = 2$. Hence $D = D'$.*

Proof. Since the divisor $K_X + D$ is log canonical then $||D^{\text{vert}}|| \leq 2$. Since the divisor $K_{X^+} + D^+$ is log canonical then $||D^{\text{hor}}|| \leq 2$. $\square$

Corollary 2.10. *We have $C \not\subset \operatorname{Supp} D^{\text{hor}}$ and $C^+ \not\subset \operatorname{Supp}(D^{\text{vert}})^+$.*

Let S be an irreducible component of a divisor D^{vert} and $F = D - S$. The divisorial log contraction $(S, \operatorname{Diff}_S(F)) \to (f(S) \ni P)$ is toric by [23, Theorem 6.4]. In particular, it is analitically isomorphic to the toric blow-up of a cyclic singularity. This implies that the singularities of X are toric by proposition 2.6. Also we may assume that the divisor $-(K_X+S)$ is f-ample by replacing X with X^+ and $S\cdot C<0$ by corollary 2.10. Moreover, by taking the toric blow-ups of X it can be assumed that the surface S is smooth and the variety X is smooth outside C. Consider two possibilities A) and B).

2.11. A) Let X be a smooth variety.

Lemma 2.12. *Our theorem is enough to prove for a flopping contraction (with respect to K_X) of a smooth variety X.*

Proof. Since the divisor $-K_S$ is f-ample then $f\colon S\to f(S)$ is the contraction of a (–1) curve C and $(K_X+S)\cdot C=-1$. Let $S\cdot C=-m$, $K_X\cdot C=m-1$, where $m\geq 2$. Using the natural section of $\mathcal{O}_X(S)$ we can construct a degree m-cyclic cover $\varphi\colon \widetilde{X}\to X$ ramified along S (cf. [5, Theorem 5.4]). Let $\widetilde{C}=\varphi^{-1}(C)$, $\widetilde{Z}$ be the normalization of Z in the function field of the variety X. Let $\widetilde{f}\colon\widetilde{X}\to\widetilde{Z}$ be the induced contraction of a curve $\widetilde{C}$. By the ramification formula

$$K_{\widetilde{X}}\cdot\widetilde{C}=\varphi^*\Big(K_X+\frac{m-1}{m}S\Big)\cdot\widetilde{C}=K_X\cdot C+\frac{m-1}{m}S\cdot C=0.$$

□

We obtain that $(Z\ni P)\cong_{\mathrm{an}}(g=0\subset(\mathbb{C}^4,0))$ is an isolated cDV singularity and

$$\big(K_Z+f(S)+f(F)\big)|_{f(S)}=K_{f(S)}+L_1+L_2,$$

where $L_k=D_{i_k}|_{f(S)}$. Also

$$(f(S),L_1+L_2)\cong_{\mathrm{an}}(\mathbb{C}^2\ni 0,\{x=0\}+\{y=0\}).$$

Hence the pair $(Z\ni P,H+f(S)+D_{i_1})$ is log canonical, where H is a general hyperplane section (induced from the general hyperplane section of $\mathbb{C}^4$) passing through P. Thus $(H,f(S)|_H+D_{i_1}|_H)$ is a log canonical pair, that is, H has a cyclic (Du Val) singularity at the point P. By [1], [3] the singularity of H is $\mathbb{A}_1$. So,

$$(Z\ni P)\cong_{\mathrm{an}}(xy+z^2+t^{2l}=0\subset(\mathbb{C}^4,0))$$

and $f(D)=\{x=0\}+\{y=0\}$. Since the pair $(Z\ni P,f(D))$ is log canonical then $l=1$. Hence $f\colon X\to(Z\ni P)$ is the Atiyah's toric contraction (see example 2.2).

2.13. B) Let X be a singular variety along the curve C, that is, X is isomorphic to $\frac{1}{m}(1,q)\times C$ in some small analytical neighborhood of C by [14, Lemma 3.9]. We can take the toric blow-ups along C and reduce our problem to case A) (all conditions remain valid and varieties do not have singularities along flipping curves). This completes the proof of our theorem. □

Definition 2.14. Let X be a normal $\mathbb{Q}$-factorial n-dimensional variety and $S = \sum S_i \subset X$ be a reduced Weil divisor. Let $f\colon (X, S) \to (Z \ni P)$ be a small contraction with $\rho(X/Z) = 1$. Then f is called *a special flipping contraction* if the following conditions are satisfied:

(1) the divisor $K_X + S$ is log terminal;
(2) the divisor $-(K_X + S)$ is f-ample;
(3) the divisor $-S_i$ is f-ample for every i.

In chapter 3 we will consider some elementary transformations. In the terminal category they have a special structure. Therefore the next definition is introduced.

The special flipping contraction is called *a special flipping contraction of terminal type* if the following conditions are satisfied:

(1) the divisor S is irreducible, that is, (X, S) is a purely log terminal pair;
(2) the variety X^+ is terminal outside S^+, where $f^+\colon X^+ \to Z$ is a flip of f and S^+ is a proper transform of S.

Remark 2.15. In example 2.2 the toric singularity is not $\mathbb{Q}$-factorial. The well-known criterion asserts that the variety $T_N(\Delta)$ is $\mathbb{Q}$-factorial if and only if σ is a simplicial cone for all $\sigma \in \Delta$ (for example, see [10, Lemma 14-1-1]).

2.16. Now let us remind the local description of three-dimensional $\mathbb{Q}$-factorial toric flips [20] (see also [10, §14]). Let X be a three-dimensional $\mathbb{Q}$-factorial toric variety and $f\colon X \to Z$ be a toric small contraction of extremal ray (in some log minimal model program). By definition $\rho(X/Z) = 1$ and hence $\operatorname{Exc} f = C$ is an irreducible curve. Let $f^+\colon X^+ \to Z$ be a flip of f. Also $\operatorname{Exc} f^+ = C^+$ is an irreducible curve. We have the following local description: $X = T_N(\Delta)$, $X^+ = T_N(\Delta^+)$, $Z = T_N(\Delta')$, where

$$\Delta = \{\langle e_1, e_2, e_3\rangle, \langle e_1, e_2, e_4\rangle, \text{ their faces}\},$$

$$\Delta' = \{\langle e_1, e_2, e_3, e_4\rangle, \text{ their faces}\}$$

and

$$\Delta^+ = \{\langle e_1, e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \text{ their faces}\}.$$

There is a relation

$$a_1e_1 + a_2e_2 + a_3e_3 + e_4 = 0,$$

where $a_1 < 0$, $a_2 < 0$, $a_3 > 0$. Note that $C = V(\langle e_1, e_2\rangle)$ and $C^+ = V(\langle e_3, e_4\rangle)$.

Definition 2.17. In examples 2.18, 2.19 we will use the notation of section 2.16 and the next agreements. Let $e_1 = (1,0,0)$, $e_2 = (q,r,0)$, $e_3 = (0,0,1)$, where $0 \leq q < r$. Put $S = V(\langle e_2\rangle)$ on X and $S^+ = V(\langle e_2\rangle)$ on X^+.

Example 2.18. Let $q = 0$, $r = 1$, that is, X is an non-singular variety. Put $e_4 = (1, a, -1)$, where $a \geq 1$. Then f is a special flipping contraction of terminal type and we have the following description.

(1) The variety X^+ has a singularity of type $\frac{1}{a}(-1,1,1)$ outside S^+ only.
(2) The surface S^+ is non-singular.
(3) The relation is $-e_1 - ae_2 + e_3 + e_4 = 0$ and $K_X \cdot C = a - 1$, $S \cdot C = -a$.

Example 2.19. Let $0 < q < r$, $(q,r) = 1$. In all cases the variety X has a singularity along the curve C only. Its type is $\frac{1}{r}(-q,1) \times C$. Let us consider four cases.

I) Let $e_4 = (a, 1, -1)$, where $a = 1, 2$. Then f is a special flipping contraction of terminal type and we have the following description.

(1) The variety X^+ has a singularity of type $\frac{1}{ar-q}(-r,-r,1)$ on S^+ only.
(2) The surface S^+ has a singularity of type $\frac{1}{ar-q}(1,1)$. It is non-singular if and only if $a = 1$, $r = q + 1$.
(3) The relation is $-(a - \frac{q}{r})e_1 - \frac{1}{r}e_2 + e_3 + e_4 = 0$ and $K_X \cdot C = a - 2 - \frac{q-1}{r}$, $S \cdot C = -\frac{1}{r}$.

II) Let $e_4 = (2a + 1, 2, -1)$, where either $a = 0$ and $\frac{q}{r} > \frac{1}{2}$, or $a = 1$. Then f is a special flipping contraction of terminal type and we have the following description.

(1) The variety X^+ has the singularities of types $\frac{1}{2}(1,1,1)$ outside S^+ and $\frac{1}{(2a+1)r-2q}(-q + ra, -q + ra, 1)$ on S^+.
(2) The surface S^+ has a singularity of type $\frac{1}{(2a+1)r-2q}(1,1)$. It is non-singular if and only if $a = 0$, $r = 2q + 1$.
(3) The relation is $-(2a+1-\frac{2q}{r})e_1 - \frac{2}{r}e_2 + e_3 + e_4 = 0$ and $K_X \cdot C = 2a - 1 - \frac{2q-2}{r}$, $S \cdot C = -\frac{2}{r}$.

III) Let $e_4 = (a + 1, a, -1)$, where $3 \leq a < \frac{r}{r-q}$. Then f is a special flipping contraction of terminal type and we have the following description.

(1) The variety X^+ has the singularities of types $\frac{1}{a}(-1,1,1)$ outside S^+ and $\frac{1}{a(r-q)+r}(1,r-q,r-q)$ on S^+.
(2) The surface S^+ has a singularity of type $\frac{1}{a(r-q)+r}(1,1)$. It is singular always.
(3) The relation is $-(a+1-\frac{qa}{r})e_1-\frac{a}{r}e_2+e_3+e_4=0$ and $K_X\cdot C = a-1-\frac{qa-a}{r}$, $S\cdot C=-\frac{a}{r}$.

IV) Let $e_4=(1,a,-1)$, where $3\le a<\frac{r}{q}$. Then f is a special flipping contraction of terminal type and we have the following description.

(1) The variety X^+ has the singularities of types $\frac{1}{a}(-1,1,1)$ outside S^+ and $\frac{1}{(r-aq}(-q,-q,1)$ on S^+.
(2) The surface S^+ has a singularity of type $\frac{1}{r-aq}(1,1)$. It is non-singular if and only if $a=\frac{r-1}{q}$.
(3) The relation is $-(1-\frac{qa}{r})e_1-\frac{a}{r}e_2+e_3+e_4=0$ and $K_X\cdot C = -1-\frac{qa-a}{r}$, $S\cdot C=-\frac{a}{r}$.

Theorem 2.20. *Let $f\colon (X,S)\to(Z\ni P)$ be a three-dimensional special flipping contraction of terminal type. Assume that S is an non-singular surface and X is an non-singular variety outside* $\operatorname{Exc} f$. *If X is a singular variety then we also assume that* $\operatorname{Exc} f$ *is an irreducible curve. Then f is a toric contraction from examples* 2.18 *and* 2.19.

Proof. If X is an non-singular variety then every irreducible exceptional curve is (-1) curve on a surface S. Therefore $\operatorname{Exc} f$ is an irreducible curve always. By the same argument as in lemma 2.12 and section 2.13 it is enough to prove that f is toric for a flopping contraction (with respect to K_X) of a smooth variety X. Similarly as in proof of theorem 2.4 (after lemma 2.12) we can show that

$$(Z\ni P)\cong_{\mathrm{an}}(xy+z^2-t^{2l}\subset(\mathbb{C}^4,0)) \quad\text{and}\quad f(S)=\{x=z-t^l=0\}.$$

Hence the surface S has a Du Val singularity $\mathbb{A}_{l-1}$. Therefore $l=1$ and $f\colon X\to(Z\ni P)$ is the Atiyah's toric contraction (see example 2.2).

Put $S=V(\langle e_2\rangle)$. Consider two possibilities.

Let X be an non-singular variety. Then we may assume that $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ and $e_4=(a,b,c)$, where $a>0$, $b>0$, $c<0$. Since $T_N(\langle e_1,e_2,e_4\rangle)$ is an non-singular variety then $c=-1$. After the log flip $T_N(\langle e_2,e_3,e_4\rangle)$ is an non-singular variety by [22, Corollary 3.7]. Hence $a=1$. We obtain that these log flips are from example 2.18.

Let X be a singular variety along curve $C=\operatorname{Exc} f$ and X is isomorphic to $\frac{1}{r}(-q,1)\times C$ in some small analytical neighborhood of C, where $q > 0$. Then we may assume that $e_1 = (1, 0, 0)$, $e_2 = (q, r, 0)$, $e_3 = (0, 0, 1)$ and $e_4 = (a, b, c)$, where $c < 0$, $b > 0$ and $a > \frac{qb}{r} > 0$. Easy calculations show that $c = -1$ always. By terminal lemma (for example, see [12, §1.6]) the vectors e_1, e_3, e_4 coincide in some $\mathbb{Z}$-basis of N with $(1, 0, 0)$, $(0, 1, 0)$, $(1, p, t)$ up to permutation, where either $p = 0$, $t = 1$, or $0 < p < t$, $(p, t) = 1$. Easy calculations show that $e_4 = (\alpha, 1, -1)$, $e_4 = (2\alpha + 1, 2, -1)$ or $e_4 = (1 + k\alpha, \alpha, -1)$. Taking into account that $(K_X + S) \cdot C < 0$ we obtain that these log flips are from example 2.19. $\square$

Example 2.21. In an non-$\mathbb{Q}$-factorial case the toric flips have more complicated structure. Let

$$\Delta = \{\langle e_1, e_2, e_3, e_4\rangle, \langle e_3, e_4, e_5\rangle, \text{ their faces}\},$$

$$\Delta' = \{\langle e_1, e_2, e_3, e_4, e_5\rangle, \text{ their faces}\}$$

and

$$\Delta^+ = \{\langle e_1, e_2, e_5\rangle, \langle e_2, e_3, e_5\rangle, \langle e_1, e_4, e_5\rangle, \text{ their faces}\},$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, -1, 1)$, $e_4 = (-1, 0, 1)$, $e_5 = (a, b, c)$, $a > 0$, $b > 0$, $c < 0$, $a+b+c < 0$, $(a, b, c) = 1$. Let $X = T_N(\Delta)$, $X^+ = T_N(\Delta^+)$, $Z = T_N(\Delta')$, $f\colon X \to Z$ and $f^+\colon X^+ \to Z$. The reader can check the following statements:

(1) $\operatorname{Exc} f = C = V(\langle e_3, e_4\rangle)$ is an irreducible curve;
(2) $\operatorname{Exc} f^+ = C_1^+ \cup C_2^+ = V(\langle e_1, e_5\rangle) \cup V(\langle e_2, e_5\rangle)$ is an union of two irreducible curves;
(3) the variety X has (at most) two singularities on C: $\frac{1}{-(a+b+c)}(b, a, 1)$, $x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0)$;
(4) the variety X^+ has (at most) three singularities: $\frac{1}{-c}(-a, -b, 1)$ at the point $C_1^+ \cap C_2^+$, $\frac{1}{a}(-c, -b - c, 1)$ on C_1^+, $\frac{1}{b}(-a - c, -c, 1)$ on C_2^+;
(5) the transformation $X \dashrightarrow X^+$ is an anti-log flip with respect to $K_X + D$, where $D = V(\langle e_5\rangle)$.

Note also that the singularities on X and X^+ can be non-isolated. If $(a, b, c) = (1, 1, -3)$ then this log flip takes place in example 4.1.

3. **Construction of model**

In this section we consider the construction of model for a local contraction [18], [15].

3.1. Let $f\colon (X, D) \to (Z \ni P)$ be a contraction of d-dimensional variety X with the following properties:

(1) every coefficient of a boundary D belongs to Φ^d;
(2) the pair (X, D) is kawamata log terminal;
(3) the divisor $-(K_X + D)$ is f-nef and f-big;
(4) $\dim Z > 0$, Z is a germ near P;
(5) $\rho(X/Z) = 1$ and the variety X is $\mathbb{Q}$-factorial.

Let us remark that same construction of model takes place without condition (5) (see [18]). In our case we obtain some information about $S \dashrightarrow \overline{S}$ (see proposition 3.8 and section 4).

3.2. By Kodaira's lemma there exists an effective $\mathbb{Q}$-divisor D' such that the pair $(X, D + D')$ is kawamata log terminal and the divisor $-(K_X + D + D')$ is ample over Z.

Let $H = \sum h_i H_i$ be an effective divisor on Z containing the point P, where H_i is a $\mathbb{Q}$-Cartier divisor for all i. Let $c = c(X, D + D', f^*H)$ be the log canonical threshold of a divisor f^*H for the pair $(X, D + D')$. Put $D'' = D' + cf^*H$. Then the divisor $K_X + D + D''$ is anti-ample over Z, log canonical and not kawamata log terminal. Let us consider two cases A) and B).

3.3. A) Let the divisor K_X+D+D'' be not purely log terminal. Take an inductive blow-up $g\colon Y \to X$ of the pair $(X, D + D'')$ [16, Proposition 3.1.4]. Write

$$K_Y + S + D_Y + D''_Y = g^*(K_X + D + D''),$$

where D_Y, D''_Y are the proper transforms of D and D''. By the properties of an inductive blow-up the variety X is $\mathbb{Q}$-factorial, $\operatorname{Exc} g = S$ is an irreducible divisor, $\rho(Y/X) = 1$, the divisor $K_Y+S+D_Y+(1-\varepsilon)D''_Y$ is purely log terminal and anti-ample over X for any $\varepsilon > 0$.

If the divisor $-(K_Y + S + D_Y)$ is nef over Z then (Y, S) is a required model. Put $\overline{Y} = Y$, $\overline{S} = S$, $\overline{D}_Y = D_Y$ and $q = f \circ g$. It is clear that theorems 3.5 and 3.6 hold in this case. Therefore we assume that the divisor $-(K_Y + S + D_Y)$ is not nef over Z.

The Mori cone $\overline{\mathrm{NE}}(Y/Z)$ is polyhedral and generated by two extremal rays since the divisor $K_Y + S + D_Y + (1-\delta)D''_Y$ is purely log terminal and anti-ample over Z for $0 < \delta \ll 1$ (see [18, Lemma 3.9]). Let the prime divisors F_1, F_2 generate $N^1(Y/Z)$. For $n \gg 0$ take a general element

$$F \in |-n(K_Y + S + D_Y + (1-\delta)D''_Y) - F_1 - F_2|$$

and put

$$B = (1-\delta)D''_Y + \frac{1}{n}(F + F_1 + F_2).$$

We obtain $K_Y + S + D_Y + B \equiv 0$ over Z. For some $0 < \varepsilon \ll 1$ the divisor $K_Y + S + D_Y + (1+\varepsilon)B$ is purely log terminal. Apply $(K_Y + S + D_Y + (1+\varepsilon)B)$–MMP over Z. In the result one of two possibiliites holds [18], [15].

I) After a sequence of $(K_Y + S + D_Y + (1+\varepsilon)B)$–flips we obtain the pair $(\overline{Y}, \overline{S} + \overline{D}_Y)$ such that the divisor $-(K_{\overline{Y}} + \overline{S} + \overline{D}_Y)$ is nef over Z, $\rho(\overline{Y}/Z) = 2$.

$$\begin{tikzcd}
Y = Y_1 \arrow[rr, dashed] \arrow[d, "g = g_1"'] \arrow[dr, "r_1"] & & Y_2 \arrow[dl, "g_2"'] & \cdots \arrow[dr, "r_{n-1}"] & & Y_n = \overline{Y} \arrow[dl, "g_n"'] \arrow[dd, "q"] \\
X \arrow[d, "f"'] & X_1 \arrow[d] & & \cdots & X_{n-1} \arrow[d] & \\
Z \arrow[r, equal] & Z \arrow[r, equal] & & \cdots & Z \arrow[r, equal] & Z
\end{tikzcd}$$

Proposition 3.4. [18, Lemma 3.11]. *The divisor $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\overline{D}_Y))$ is nef and big over $q(\overline{S})$.*

Proof. It is easily shown that the Mori cone $\overline{\mathrm{NE}}(Y_i/Z)$ is polyhedral and generared by two extremal rays. Hence, by base point free theorem the linear system $|-m(K_{\overline{Y}}+\overline{S}+\overline{D}_Y)|$ is free over Z and gives a contraction $h \colon \overline{Y} \to W$, where $m \gg 0$. For every curve C in the fiber of h we have $C \cdot \overline{B} = 0$. Therefore $C \cdot \overline{B}_i = 0$ for some component $\overline{B}_i$ of a divisor $\overline{B}$ by the construction of B. Hence $C \subset \mathrm{Supp}(\overline{B})$. $\square$

II) After a sequence of $(K_Y + S + D_Y + (1+\varepsilon)B)$–flips we obtain the pair $(Y_n, S_n + (D_Y)_n)$ and a divisorial contraction $r_n \colon Y_n \to \overline{Y}$ such that the divisor $K_{Y_n} + S_n + (D_Y)_n$ is r_n-ample. After it the divisor $-(K_{\overline{Y}} + \overline{S} + \overline{D}_Y)$ is ample over Z, $\rho(\overline{Y}/Z) = 1$.

$$\begin{tikzcd}
Y = Y_1 \arrow[rr, dashed] \arrow[d, "g = g_1"'] \arrow[dr, "r_1"] & & Y_2 \arrow[dl, "g_2"'] & \cdots \arrow[dr, "r_{n-1}"] & & Y_n \arrow[dl, "g_n"'] \arrow[dr, "r_n"] & \\
X \arrow[d, "f"'] & X_1 \arrow[d] & & \cdots & X_{n-1} \arrow[d] & & Y_{n+1} = \overline{Y} \arrow[d, "q"] \\
Z \arrow[r, equal] & Z \arrow[r, equal] & & \cdots & Z \arrow[rr, equal] & & Z
\end{tikzcd}$$

Note that the divisor $-(K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\overline{D}_Y))$ is ample over $q(\overline{S})$.

In both cases I) and II) we have the following properties of the constructed model.

Theorem 3.5. [18, Lemma 3.13] *Let there exists an n-complement of $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\overline{D}_Y)$ near $q^{-1}(P)$. Then there exists an n-complement of $K_X + D$ near $f^{-1}(P)$.*

Proof. The n-complement of a divisor $K_{\overline{S}}+\mathrm{Diff}_{\overline{S}}(\overline{D}_Y)$ is extended to an n-complement $K_{\overline{Y}}+\overline{S}+\overline{D}_Y^+$ of a divisor $K_{\overline{Y}}+\overline{S}+\overline{D}_Y$ by theorem 1.8. In the case I) the image of $K_{\overline{Y}}+\overline{S}+\overline{D}_Y^+$ is an n-complement $K_Y+S+D_Y^+$ of a divisor K_Y+S+D_Y. Therefore $K_X+g(D_Y^+)$ is an n-complement of a divisor K_X+D by theorem 1.6. In the case II) a pull-back of $K_{\overline{Y}}+\overline{S}+\overline{D}_Y^+$ is an n-complement of a divisor $K_{Y_n}+S_n+(D_Y)_n$ by theorem 1.7. As above we obtain an n-complement of a divisor K_X+D. □

Theorem 3.6. [18, Corollary 3.18] *The local contraction $f\colon (X,D)\to(Z\ni P)$ is exceptional if and only if $f(g(S)) = q(\overline{S}) = P$ and the log variety $(\overline{S}, \mathrm{Diff}_{\overline{S}}(\overline{D}_Y))$ is exceptional.*

Definition 3.7. The images of divisors, curves of Y on Y_i are denoted with index i.

Proposition 3.8. 1) *The divisor $-S_i$ is g_i-ample. In particular,* $\mathrm{Exc}\, g_i \subset S_i$ [17, Lemma 4.3].

2) *The sets* $\mathrm{Exc}(S \dashrightarrow \overline{S})$ *and* $\mathrm{Exc}(\overline{S} \dashrightarrow S)$ *are connected.*

Proof. Let us prove the first statement at first (it was proved in [17, Lemma 4.3] for the non-small contractions). Let the Mori cone $\overline{\mathrm{NE}}(Y_i/Z)$ be generated by the rays G_i and R_i. They gives the contractions g_i and r_i respectively.

We claim that the divisor $-S_i$ is not nef over Z for $i = 1,\dots,n-1$. Assume the converse. Consider three cases.

1) Let $\mathrm{Exc}(f) \neq f^{-1}(P)$. Let $l \subset g^{-1}(f^{-1}(P'))$ be a curve, where $P' \neq P$. It is obvious that l_i does not generate an extremal ray. Since $0 = S_i\cdot l_i = S_i\cdot(\alpha R_i + \beta Q_i)$, where $\alpha,\beta \geq 0$ then $S_i\cdot R_i = S_i\cdot Q_i = 0$. Hence $S_i \equiv 0$ over Z, a contradiction.

2) Let $\mathrm{Exc}(f) = f^{-1}(P)$ and f be a divisorial contraction to a point P. Then $S_i = \mathrm{Exc}(Y_i \to Z)$, a contradiction with the construction.

3) Let $\mathrm{Exc}(f) = f^{-1}(P)$ and f be a small contraction. Then $S_i = \mathrm{Exc}(Y_i \to Z)$. It is enough to prove that $-(K_{Y_i} + S_i + (D_Y)_i)$ is nef over Z. Let $c' = c(X, D, f^*H)$ be a log canonical threshold. Without loss of generality it can be assumed that g is the inductive blow-up of a divisor $K_X + D + cf^*H$. Let B' be a proper transform of cf^*H on Y. We have $B' + \alpha S \equiv 0$ over Z for some $\alpha > 0$. Hence $B'_i + \alpha S_i \equiv 0$ over Z and B'_i is nef over Z. Since $\rho(X/Z) = 1$ there are two possibilities.

Let $K_Y + S + D_Y + B' \equiv 0$ over Z. Then the divisor $-(K_{Y_i} + S_i + (D_Y)_i) = B'_i$ is nef over Z.

Let $K_Y + S + D_Y + B' \not\equiv 0$ over Z. Then $-(K_Y + S + D_Y + (1-\varepsilon)B')$ is ample over Z for some $0 < \varepsilon \ll 1$. It is clear that $(K_Y + S + D_Y + (1-\varepsilon_1)B') \cdot R_1 = 0$ and $(K_Y + S + D_Y + (1-\varepsilon_1)B') \cdot Q_1 < 0$ for some $\varepsilon < \varepsilon_1 < 1$. After the log flip $Y \dashrightarrow Y_2$ we have $B'_2 \cdot Q_2 > 0$, $(K_{Y_2} + S_2 + (D_Y)_2 + (1-\varepsilon_1)B'_2) \cdot Q_2 = 0$ and $(K_{Y_2} + S_2 + (D_Y)_2 + (1-\varepsilon_1)B'_2) \cdot R_2 < 0$. In fact, if $(K_{Y_2} + S_2 + (D_Y)_2 + (1-\varepsilon_1)B'_2) \cdot R_2 > 0$ then the divisor $K_{Y_2} + S_2 + (D_Y)_2 + (1-\varepsilon_1+\delta)B'_2$ is nef over Z for some $0 < \delta \ll \varepsilon_1 - \varepsilon$, a contradiction with [5, Lemma 6.3]. By the same argument on every step there is $\varepsilon_{j-2} < \varepsilon_{j-1} < 1$ such that $(K_{Y_j} + S_j + (D_Y)_j + (1-\varepsilon_{j-1})B'_j) \cdot Q_j = 0$ and $(K_{Y_j} + S_j + (D_Y)_j + (1-\varepsilon_{j-1})B'_j) \cdot R_j < 0$. Hence the divisor

$$-(K_{Y_i} + S_i + (D_Y)_i) = -(K_{Y_i} + S_i + (D_Y)_i + (1-\varepsilon_{i-1})B'_i) + (1-\varepsilon_{i-1})B'_i$$

is nef over Z. This completes the proof of claim.

It is clear that $S \cdot Q_1 < 0$ and $S \cdot R_1 > 0$. Hence $S_2 \cdot Q_2 < 0$. Assume our statement is proved on Y_{i-1}. Then $S_{i-1} \cdot Q_{i-1} < 0$ and hence $S_{i-1} \cdot R_{i-1} > 0$. After the log flip $Y_{i-1} \dashrightarrow Y_i$ we have $S_i \cdot Q_i < 0$. This completes the proof of first statement.

Let us prove the second one. Let it is true for $S \dashrightarrow S_{i-1}$. Let us prove it for $S \dashrightarrow S_i$. Assume the converse. We have three non-trivial factorizations of $Y_{i-1} \to Z$. These are $Y_{i-1} \to X_{i-2} \to Z$, $Y_{i-1} \to X_{i-1} \to Z$ and $Y_{i-1} \to W \to Z$, where $Y_{i-1} \to W$ is induced by r_i. We obtain a contradiction with $\rho(Y_{i-1}/Z) = 2$ by [5, Lemma 6.1]. □

3.9. B) Let the divisor $K_X + D + D''$ be purely log terminal. Put $S = \llcorner D + D'' \lrcorner$. By connectedness lemma [5, Theorem 17.4] the divisor S is irreducible and hence normal. Let $D = \sum d_i D_i$. Put

$$D_X = \sum_{i\colon D_i \neq S} d_i D_i \text{ and } D''_X = D + D'' - D_X - S.$$

Let the divisor $-(K_X + S + D_X)$ be nef over Z. Then theorems 3.5 and 3.6 hold, where $\overline{Y} = Y$, $\overline{S} = S$, $\overline{D}_Y = D_X$ and $q = f$. Let us remark that the local contraction $f\colon (X, D) \to (Z \ni P)$ can be exceptional only in the cases when f is a divisorial contraction to a point P or $\dim Z = 1$.

Let the divisor $-(K_X + S + D_X)$ be not nef over Z. By the choice of a divisor H the contraction f is not small. Therefore f is a divisorial contraction and $f(S) \neq 0$. Then an n-complement of $K_{f(S)} + \mathrm{Diff}_{f(S)}(f(D_X))$ near P is extended to an n-complement of $K_Z + f(S) + f(D_X)$ by theorem 1.8. By theorem 1.7 we obtain an

n-complement of $K_X + S + D_X$ and $K_X + D$. In particular, the local contraction $f\colon (X, D) \to (Z \ni P)$ is non-exceptional in this case.

4. Three-dimensional contractions

Now we give two examples illustrating the construction of model. In this examples the variety X is not $\mathbb{Q}$-factorial. We can also apply our construction of model (without a $\mathbb{Q}$-factorialization of X) since the inductive blow-up exists [6, Theorem 1.5]. See the paper [7] for the calculation details. By abuse of notation the coordinates are denoted t, z, x, y for every variety.

Example 4.1. Consider the three-dimensional terminal hypersurface singularity $(Z \ni P) = (t^2 + z^2 + x^2y + y^4 = 0 \subset (\mathbb{C}^4, 0))$. We have a following diagram.

$$
\begin{tikzcd}
 & & (Y,S) \arrow[r, dashed, "\psi"] \arrow[dl, "g"'] & (\widetilde{Y},\widetilde{S}) \arrow[dr, "h"] & \\
 & (X,E) \arrow[dl, "f"'] & & & (\overline{Y},\overline{S}) \arrow[dllll, "\varphi"] \\
 (Z \ni P) & & & &
\end{tikzcd}
$$

The birational morphism $f\colon X \to (Z \ni P)$ is induced by the weighted blow-up of $\mathbb{C}^4$ with weights $(1, 1, 1, 1)$. Therefore

$$\operatorname{Exc} f = E = t^2 + z^2 \subset \mathbb{P}(1, 1, 1, 1) = E^1 + E^2,$$

where E^1 and E^2 are the planes intersecting along the line $l = E^1 \cap E^2$. It is obvious that $\operatorname{Sing} X = P_1 \cup P_2 \subset l$ and

$$\big(X \ni P_1\big) \cong_{\text{an}} \big(t^2 + z^2 + x^2y + y^2 = 0 \subset (\mathbb{C}^4, 0)\big),$$

$$\big(X \ni P_2\big) \cong_{\text{an}} \big(t^2 + z^2 + xy = 0 \subset (\mathbb{C}^4, 0)\big).$$

Thus X has the terminal singularities. Let $D' = \frac{1}{2}\{x = 0\} + \frac{7}{8}\{t = 0\}$ be a divisor on Z. Then the divisor $K_X + D + \frac{3}{8}E = f^*(K_Z + D')$ is log canonical, where D is a proper transform of D'. Moreover, the point P_1 is the unique log canonical center of $K_X + D + \frac{3}{8}E$. Therefore the inductive blow-up $g\colon Y \to X$ of a pair $(X, D + \frac{3}{8}E)$ at the point P_1 is given by the weighted blow-up of $\mathbb{C}^4$ with weights $(2, 2, 1, 2)$. Then

$$\operatorname{Exc} g = S = t^2 + z^2 + xy + y^2 \subset \mathbb{P}(1, 1, 1, 1) = \mathbb{F}_0$$

and $\mathrm{Diff}_S(0) = \frac{1}{2}C = \frac{1}{2}\{x = 0\}$.

The birational morphism $\varphi\colon \overline{Y} \to (Z \ni P)$ is induced by the weighted blow-up of $\mathbb{C}^4$ with weights $(4, 4, 3, 2)$. Therefore

$$\operatorname{Exc}\varphi = \overline{S} = t^2 + z^2 + xy + y^4 \subset \mathbb{P}(2, 2, 3, 1)$$

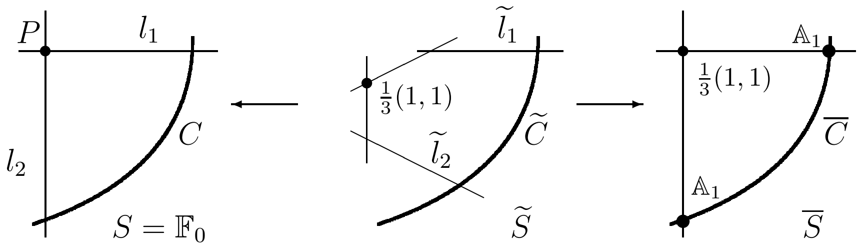
and $\mathrm{Diff}_{\overline{S}}(0) = \frac{1}{2}\overline{C} = \frac{1}{2}\{x = 0\}$. By Noether's formula $\rho(\overline{S}) = 2$.

It is easily shown that the divisors S and $\overline{S}$ have the same valuation of a function field $\mathcal{K}(Z)$, since they correspond to same toric blow-up of $\mathbb{C}^4$.

We can see that the divisor $-(K_Y+S)$ is not nef over Z (for example, $(K_Y + S) \cdot l = 1$). The birational map $\psi\colon Y \dashrightarrow \widetilde{Y}$ is the anti-log flip with respect to $K_Y + S$ in the curve l. This log flip $Y \to X_1 \leftarrow \widetilde{Y}$ is toric by criterion 2.4 since $K_Y + (E_1)_Y + (E_2)_Y + S + D_1 + D_2 \sim 0$ over X_1 and $\sigma(Y/X_1) = 2$, where D_i are the proper transforms of the divisors $\{y = t \pm \sqrt{-1} \cdot z = 0\}$ of Z. We can see that this log flip is of example 2.21. After this log flip the divisor $-(K_{\widetilde{Y}} + \widetilde{S})$ is not nef over Z again (for example, $(K_{\widetilde{Y}} + \widetilde{S}) \cdot \widetilde{l}_i = \frac{1}{2}$).

The birational morphism $h\colon \widetilde{Y} \to \overline{Y}$ induces the contraction of the images of E^1 and E^2.

Let us consider the transformation $S \leftarrow \widetilde{S} \to \overline{S}$.



In this diagram $l_i = S \cap E^i_Y$, where E^i_Y are the proper transforms of E^i on Y. The curves l_i are the rulings of $\mathbb{F}_0$. The log flip ψ induces a divisorial extraction at the point $P = l_1 \cap l_2$. After it $\widetilde{l}_i^2 = -2$ and two exceptional curves appear. They intersect at the singularity $\frac{1}{3}(1, 1)$. The birational morphism h induces the contraction of $\widetilde{l}_1$, $\widetilde{l}_2$.

The divisor $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(0)$ has 1-complement $K_{\overline{S}} + \{x = 0\} + \{y = 0\} \sim 0$. Thus K_X has 1-complement. It is

$$K_X + E + \{x = 0\}_X + \{y = 0\}_X = f^*(K_Z + \{x = 0\} + \{y = 0\}) \sim 0.$$

Example 4.2. Consider the three-dimensional exceptional canonical singularity $(Z \ni P) = (t^3 + z^3 + x^4 + y^n = 0 \subset (\mathbb{C}^4, 0))$, where $n = 5, 7, 11$ [7]. In the notations of example 4.1 let the birational morphisms

f, φ be induced by the weighted blow-ups of $\mathbb{C}^4$ with weights $(1,1,1,1)$ and $(4n,4n,3n,12)$ respectively. We have

$$\Big(\overline{S}, \operatorname{Diff}_{\overline{S}}(0)\Big) = \Big(t^3+z^3+x+y \subset \mathbb{P}(1,1,3,3), \frac{3}{4}\{x=0\}+ \\ +\frac{n-1}{n}\{y=0\}\Big) = \Big(\mathbb{P}(1,1,3), \frac{3}{4}\overline{C}_1 + \frac{n-1}{n}\overline{C}_2\Big),$$

where $\overline{C}_i$ are the general hyperplane sections of a cone.

Also

$$\operatorname{Exc} f = E = t^3+z^3 \subset \mathbb{P}(1,1,1,1) = E^1+E^2+E^3,$$

where E^i are the planes intersecting along the line $l = E^1 \cap E^2 \cap E^3$. It is obvious that $\operatorname{Sing} X = P_1 \subset l$ and

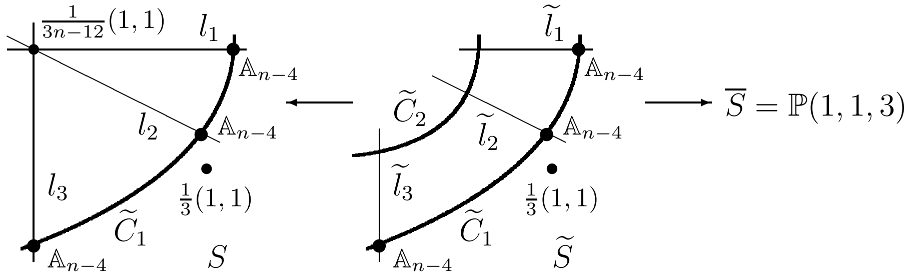
$$\big(X \ni P_1\big) {\cong}_{\mathrm{an}} \big(t^3+z^3+x^4y+y^{n-3}=0 \subset (\mathbb{C}^4,0)\big).$$

Note that $K_X \equiv 0$ over Z (in particular $-K_X$ is f-nef and f-big). Thus the divisors K_Y+S and $K_{\widetilde{Y}}+\widetilde{S}$ are not nef over Z. If $n=5$ then X is a terminal variety.

Put $D' = \big(1-\frac{n}{12}\big)\{y=0\}$ and take an inductive blow-up g at the point P_1. It is induced by the weighted blow-up of $\mathbb{C}^4$ with weights $(4n-12, 4n-12, 3n-12, 12)$. We obtain

$$\Big(S, \operatorname{Diff}_S(0)\Big) = \Big(t^3+z^3+xy+y^{n-3} \subset \mathbb{P}(n-3,n-3,3n-12,3), \\ \frac{3}{4}C_1 = \frac{3}{4}\{x=0\}\Big).$$

Let us consider the transformation $S \leftarrow \widetilde{S} \to \overline{S}$.



The log flip ψ induces the minimal resolution of a singularity $\frac{1}{3n-12}(1,1)$. After it a new component $\widetilde{C}_2$ of Diff appears, that is, $\widetilde{Y}$ has a singularity along the flipped curve $\widetilde{C}_2$. This log flip $Y \to X_1 \leftarrow \widetilde{Y}$ is toric by criterion 2.4 since $K_Y + (E_1)_Y + (E_2)_Y + S + D \sim 0$ over X_1 and $\sigma(Y/X_1) = 1$, where D is the proper transforms of a divisor

$\{y=0\}$ of Z. This log flip is from example 2.19 I). If $n=5,7,11$ then $(q,r,a)=(2,5,1)$, $(5,7,2)$, $(1,11,2)$ respectively.

The birational morphism h contracts the curves $\widetilde{l}_i=\widetilde{E}^i\cap\widetilde{S}$ to the smooth points of $\overline{S}$, where $\widetilde{E}^i$ are the images of E^i on $\widetilde{Y}$.

The divisor $K_{\overline{S}}+\mathrm{Diff}_{\overline{S}}(0)$ has 5-, 8-, 12-complement for $n=5,7,11$ respectively. Hence K_X has 5-, 8-, 12-complement for $n=5,7,11$ respectively.

The next conjecture is the analog of Reid's general elephant conjecture in the theory of complements.

Conjecture 4.3. *Let $f\colon X\to(Z\ni P)$ be a contraction of three-dimensional variety X with the following properties:*

(1) *the variety X has the $\mathbb{Q}$-factorial terminal singularities;*
(2) *the divisor $-K_X$ is f-ample;*
(3) $\dim Z>0$, *Z is a germ near P and $\rho(X/Z)=1$.*

Then this contraction is not exceptional, in particular there exists 1-, 2-, 3-, 4- *or* 6-*complement of a divisor K_X.*

Remark 4.4. If the contraction $f\colon X\to(Z\ni P)$ is not exceptional then the log surface $(\overline{S},\mathrm{Diff}_{\overline{S}}(0))$ is not exceptional by theorem 3.6. Hence there exists 1-, 2-, 3-, 4- or 6-complement of a divisor $K_{\overline{S}}+\mathrm{Diff}_{\overline{S}}(0)$ by [23, Theorem 2.3] and this n-complement is extended to an n-complement of K_X by theorem 3.5.

Remark 4.5. It is worth to note that previous conjecture 4.3 can be generelized by many ways. For instance, we can reject from the assumptions that the variety X is $\mathbb{Q}$-factorial and $\rho(X/Z)=1$. If the variety X is not $\mathbb{Q}$-factorial then the contraction $X\to(Z\ni P)$ is exceptional if and only if the contraction $\widetilde{X}\to(Z\ni P)$ is exceptional, where $\widetilde{X}$ is a $\mathbb{Q}$-factorialization of X. Let us remark that the divisor $-K_{\widetilde{X}}$ is nef and big over Z and not ample over Z.

Finally note that we cannot reject from the condition (2) and assume only that the divisor $-K_X$ is f-nef and f-big by example 4.2 ($n=5$).

Remark 4.6. 1) Conjecture 4.3 was proved in some cases.

A) If f is a divisorial contraction then it is non-exceptional by example 1.10. Besides, for a divisorial contraction it is enough to require that $(Z\ni P)$ is a terminal singularity only.

B) If f is a divisorial contraction to the point P then the divisor K_X has a strong 1-complement D^+, that is, $K_X+D^+\sim 0$ is a purely log terminal divisor [2].

C) Conjecture 4.3 was proved if any singularity of X has an index one or two and the relative dimension of f is equal to one [4], [13].

D) Some special case was considered in the paper [17].

2) Reid's general elephant conjecture under the assumption that the central fiber (over the point P) is irreducible was proved for the birational contractions [4], [2]. If $X \to (Z \ni P)$ is a Mori conic bundle then see the paper [13] for some results about the general elephant conjecture in this case.

Theorem 4.7. *Conjecture* 4.3 *must be proved in the case* A) I) *for the non-divisorial contractions* (*see chapter* 3) *only. Moreover,*

$$\overline{S} = S_n \to S_{n-1} \to \ldots \to S_1 = S$$

is a birational contraction with the following properties.

(1) *The set* $\operatorname{Exc}(\overline{S} \to S)$ *is connected.*
(2) $\theta_i \colon S_i \to S_{i-1}$ *is an non-trivial divisorial contraction for every* $i = n, \ldots, 2$. *The divisor* $-(K_{S_i} + \operatorname{Diff}_{S_i}(0))$ *is* θ_i*-ample.*
(3) *If* C *is an irreducible curve from* $\operatorname{Exc}\theta_i$ *then there are two possibilities.*
    (a) *Let* $C \not\subset \operatorname{Sing} Y_i$. *Then* $\operatorname{Diff}_{S_i}(0) = (\theta_i)_*^{-1}(\operatorname{Diff}_{S_{i-1}}(0))$ *near a curve* C.
    (b) *Let* $C \subset \operatorname{Sing} Y_i$. *Then* $\operatorname{Diff}_{S_i}(0) = (\theta_i)_*^{-1}(\operatorname{Diff}_{S_{i-1}}(0)) + \frac{n-1}{n}C$ *near a curve* C *for some* $n \geq 2$.

Proof. By example 1.10 and remark 4.6 we can assume that f is an non-divisorial contraction. The log surface $(S, \operatorname{Diff}_S(0))$ is not exceptional since it is exceptional divisor of an inductive blow-up (purely log terminal blow-up) of some terminal singularity [14]. Therefore, if the divisor $-(K_Y + S)$ is nef over Z then f is not exceptional. Hence we have the cases A) and B) in the construction of model. It is obvious that $\dim Z = 1$ in the case B) and in the case A) II). Thus, f is not exceptional by [17, Proposition 3.1]. Therefore, the case A) I) must be considered only.

It remains to show that $S_j \to S_{j-1}$ is a birational morphism by proposition 3.8. The key observation is that if we change the divisor H then the construction of model for an new H will coincide with the previous construction. Let H_i denotes the image of a divisor H on Y_i. If f has a relative dimension one then $H_i \cap S_i$ is a general curve of S_i and H_i does not contain any irreducible curve from $\operatorname{Exc} g_i$. Since $-H_j$ is r_j-nef then any flipping curve of $r_j \colon Y_j \to X_j$ does not lie in S_j.

Consider the last case $\dim Z = 1$. Let F be a general fiber of f (it is a smooth del Pezzo surface). Consider the exact sequence

$$0 \to \mathcal{O}_X(-K_X - F) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_F(-K_F) \to 0.$$

By Kawamata-Viehweg vanishing theorem $R^1 f_* \mathcal{O}_X(-K_X - F) = 0$. Hence there is a surjection

$$H^0(X, \mathcal{O}_X(-K_X)) \to H^0(F, \mathcal{O}_F(-K_F)) \to 0.$$

Therefore $h^0(X, \mathcal{O}_X(-K_X)) \geq 2$. Let $G \in |-K_X|$ be a general element. Consider the G instead of f^*H and apply above-mentioned arguments. $\square$

Remark 4.8. The transformations from items (a) and (b) are illustrated in examples 4.1 and 4.2 respectively. Also such examples can be constructed in the $\mathbb{Q}$-factorial case too.

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