Algebraic Groups, Hodge Theory and Transcendence

by

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1. ALGEBRAIC GROUPS AND TRANSCENDENCE

Let G be a connected algebraic group of dimension n defined over a number field K and $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra of invariant derivations which has a natural structure as a K-algebra. Then we denote by $G_{\mathbb{C}}$ the Lie group obtained from G by base extension from K to $\mathbb{C}$. This is a complex Lie group and we have an analytic homomorphism

$$\exp : \mathfrak{g}_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}$$

from $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_K \mathbb{C}$ into $G_{\mathbb{C}}$.

Given a Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$ we obtain by integration a Lie subgroup B of $G_{\mathbb{C}}$ which need not be closed. We say that B is defined over $\overline{\mathbb{Q}}$ if $\mathfrak{b} \subseteq \mathfrak{g} \otimes_K \overline{\mathbb{Q}}$.

It is well-known that the algebraic group G is an extension of an abelian variety A by a linear algebraic group L:

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0 \quad .$$

If G is commutative then we can write L, after a base change, as a product of a power of $\mathbb{G}_a$, the additive group, and of $\mathbb{G}_m$, the multiplicative group,

$$L = \mathbb{G}_a^k \times \mathbb{G}_m^\ell \quad .$$

Let us now consider the set $B(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$ - valued points on B . Suppose that there exists an algebraic subgroup $H \subseteq G$ with

(i) H defined over $\overline{\mathbb{Q}}$,

(ii) $\dim H > 0$

(iii) $H_{\mathbb{C}} \subseteq B$

where $H_{\mathbb{C}}$ is the subgroup of $G_{\mathbb{C}}$ obtained by base change. Then

$$0 \neq H(\overline{\mathbb{Q}}) = H_{\mathbb{C}}(\overline{\mathbb{Q}}) \subseteq B_{\mathbb{C}}(\overline{\mathbb{Q}})$$

so $B_{\mathbb{C}}(\overline{\mathbb{Q}})$ is different from 0 . The following theorem shows that the converse is also true.

THEOREM 1 ([Wü1]). $B_{\mathbb{C}}(\overline{\mathbb{Q}}) \neq 0$ *if and only if there exists an algebraic subgroup* H *of* G *such that* (i), (ii) *and* (iii) *hold*.

REMARK. Actually in [Wü1] the theorem is only proved for G commutative but it is not difficult to extend the proof also to the noncommutative case.

Theorem 1 has a very straight forward corollary which proves an old conjecture of Waldschmidt and is also of some independent interest.

COROLLARY 1 ([Wü2]). *Let* $\mathfrak{b} \subseteq \mathfrak{g}_{\mathbb{C}}$ *be a Lie subalgebra of* $\mathfrak{g}_{\mathbb{C}}$ *generated over* $\overline{\mathbb{Q}}$ *by algebraic tangent vectors* b , i.e.

$b \in \exp^{-1}(G(\overline{\mathbb{Q}}))$, <u>and</u> $\overline{\mathfrak{h}}$ <u>the smallest Lie subalgebra defined over</u> $\overline{\mathbb{Q}}$ <u>containing</u> $\mathfrak{h}$. <u>Then</u> $\overline{\mathfrak{h}}$ <u>is the Lie algebra of an algebraic subgroup defined over</u> $\overline{\mathbb{Q}}$.

Theorem 1 is actually a transcendence statement as one can see very easily. A striking example is Baker's theorem in its non-effective version. For this let $\alpha_1,\ldots,\alpha_n$ be non-zero algebraic numbers.

<u>COROLLARY 2</u> ([Ba]). <u>If</u> $\beta_1 \log\alpha_1 + \ldots + \beta_n \log\alpha_n = 0$ <u>for algebraic numbers</u> $\beta_1,\ldots,\beta_n$ <u>not all zero then there exist rational integers</u> $b_1,\ldots,b_n$ <u>not all zero such that</u>

$$b_1 \log\alpha_1 + \ldots + b_n \log\alpha_n = 0 \quad .$$

<u>PROOF</u>. Let $G = \mathbb{C}_m^n$. Then $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}^n$ and

$$\exp : z = (z_1,\ldots,z_n) \in \mathfrak{g}_{\mathbb{C}} \longmapsto (e^{z_1},\ldots,e^{z_n}) \in G_{\mathbb{C}} \quad .$$

Now let $\mathfrak{h}$ be the Lie subalgebra generated by the tangent vector

$$a = (\log\alpha_1 , \ldots , \log\alpha_n) \quad .$$

Then $\exp(a) = (\alpha_1,\ldots,\alpha_n)$ is algebraic. The dimension of $\overline{\mathfrak{h}}$ is strictly less than n since $\beta_1 z_1 + \ldots + \beta_n z_n = 0$ defines a proper Lie subalgebra containing $\overline{\mathfrak{h}}$.

Corollary 1 implies now that $\overline{\mathfrak{h}}$ is algebraic and therefore the

Lie algebra of an algebraic subgroup H of G of codimension at least one. But it is well-known that algebraic subgroups H of $\mathbb{G}_m^n$ are defined in $\mathrm{Lie}\,\mathbb{G}_m^n$ by linear forms with integral coefficients. Hence there exists such a linear form

$$b_1 z_1 + \ldots + b_n z_n \quad ,$$

not identically zero, which vanishes on $\bar{\mathfrak{h}}$ hence on $\mathfrak{a}$. This proves the corollary.

2. SKETCH OF THE PROOF OF THEOREM 1

The proof of Theorem 1 is done by transcendence arguments and is an interplay between analytic, arithmetic and algebraic arguments. We divide it in a natural way into a number of steps. In order to be able to give references and to avoid technical complications we restrict ourselves to the case that G is commutative.

STEP 1. We first embed G into some projective space $\mathbb{P}^N$. For this we write G as an extension of an abelian variety A by a product of a torus $\mathbb{G}_m^k$ and a unipotent group which we assume to be of the form $\mathbb{G}_a^\ell$:

$$0 \longrightarrow \mathbb{G}_m^k \times \mathbb{G}_a^\ell \longrightarrow G \longrightarrow A \longrightarrow 0 \quad .$$

Then we compactify G and obtain a homogeneous space $\overline{G}$ on which G acts. This is described in all details in [F-W] , which is based on ideas of Serre. $\overline{G}$ now is embedded into a projective space $\mathbb{P}^N$ for some N :

$$G \overset{i}{\hookrightarrow} \overline{G} \overset{\varphi}{\hookrightarrow} \mathbb{P}^N \quad .$$

This gives us then the analytic functions we need for the analytic part of the proof. They are by

$$f = \varphi \circ i \circ \exp : \mathbb{C}^n \xrightarrow{\sim} \mathrm{Lie}\, G_{\mathbb{C}} \longrightarrow \mathbb{P}^N \quad .$$

These functions have order of growth at most 2. Furthermore if $f = (f_0,\ldots,f_N)$ then the functions

$$g_i = \frac{f_i}{f_0} \qquad (i = 1,\ldots,N)$$

satisfy a system of differential equations,

$$\frac{\partial}{\partial z_i} g_j = h_{ij}(g_1,\ldots,g_N)$$

with polynomials $h_{ij}(T_1,\ldots,T_N)$ for $1 \leq i \leq n$ and $1 \leq j \leq N$. These polynomials have coefficients in $\overline{\mathbb{Q}}$.

<u>STEP 2</u>. Let $\zeta_1,\ldots,\zeta_b$ be coordinates on $\mathfrak{h} \simeq \mathbb{C}^b$ and $\frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_b}$ the corresponding derivations in $\mathfrak{h} \simeq \mathfrak{g}_{\mathbb{C}}$. Let

$g \in G(\mathbb{C})$ be a complex point with $X_0(g) \neq 0$ and $V \subseteq \overline{G}$ a complete algebraic subset; $U \subseteq \mathbb{P}^N$ (not necessarily reduced) the open set defined by $X_0 \neq 0$ and $V' = V \cap U$. Suppose that $g \in V(\mathbb{C})$. Then we say that g is a point of multiplicity at least T on V along B if for all $F \in I(V')$, $F = F\left(\frac{X_1}{X_0}, \ldots, \frac{X_N}{X_0}\right)$, and all $\gamma = (\gamma_1, \ldots, \gamma_n) \in \exp^{-1}(g)$

$$\left(\frac{\partial}{\partial \zeta_1}\right)^{\tau_1} \cdots \left(\frac{\partial}{\partial \zeta_b}\right)^{\tau_b} F \circ \varphi \circ i \circ \exp(\gamma_1, \ldots, \gamma_n) = 0$$

for all $0 \leq \tau_1, \ldots, \tau_b$ such that

$$\tau_1 + \ldots + \tau_b < T .$$

For simplicity we shall further assume that $b = n-1$. Assume now that $g \neq 0$ is in $B_{\mathbb{C}}(\overline{\mathbb{Q}})$ and again for simplicity that g is not a torsion point. We choose the coordinates $X_0, \ldots, X_N$ such that $X_0(sg) \neq 0$ for all $s \in \mathbb{N}$. Then one constructs by the socalled Siegel Lemma for a certain triple of positive real numbers $S, T, D \gg 1$ such that

$$D^n \gg ST^{n-1}$$

a hypersurface X on $\overline{G}$ of degree at most D that vanishes on the set $\{sg, 0 \leq s < S\}$ to order at least T along B .

<u>STEP 3</u>. By the socalled Schwarz Lemma one shows next that for certain real numbers S', T' with

$$S'T'^{n-1} >> D^n$$

the hypersurface constructed in Step 2 vanishes on the set $\{sg, 0 \leq s < S'\}$ to order at least T' .

STEP 4. The vanishing statement in Step 3 can be transformed into a statement in linear algebra. It means that the coefficients of the defining equation of X have to satisfy a system of

$$S' \binom{T' + n-1}{n-1} >> S'T'^{n-1}$$

linear equations and the condition $S'T'^{n-1} >> D^n$ just means that the number of equations is much larger than the number of coefficients which is about D^n . In order to get a contradiction one hopes that the system has maximal rank which would imply that all coefficients are zero; in other words $X = G$ which contradicts the definition of a hypersurface. Hence the system can not have maximal rank. By means of some complicated machinery, the socalled "multiplicity estimates on group varieties" developed in [Ma-Wü1], [Ma-Wü2] and [Wü3], this statement is translated into statements on the group generated by g and the analytic subgroup B . The results in [Wü3] imply that the analytic subgroup B is degenerate in a certain sense; more precisely they say that B contains an algebraic subgroup H such that (i) - (iii) hold. This completes the sketch of the proof of Theorem 1.

3. HODGE THEORY

Let X be a smooth proper variety of dimension n over an algebraic number field K. Then there are various ways for associating with X cohomology theories.

The first one is the singular cohomology. For this let $\sigma : K \longrightarrow \mathbb{C}$ be an embedding and define

$$X_{\mathbb{C}} := X \otimes_K \mathbb{C}$$

via this embedding. Then $X_{\mathbb{C}}$ is a complex manifold and its singular cohomology $H^i(X_{\mathbb{C}},\mathbb{Z})$ is defined and one puts

$$H^i_B(X) := H^i(X_{\mathbb{C}},\mathbb{Q}) := H^i(X_{\mathbb{C}},\mathbb{Z}) \otimes \mathbb{Q} \qquad (i=0,1,\ldots) ,$$

the socalled Betti-cohomology.

The de Rham cohomology is defined as follows. Let $\Omega^i_{X/K}(i=0,1,\ldots)$ be the sheaf of algebraic differential i-forms. One obtains the complex

$$\Omega^*_{X/K} : \mathcal{O}_{X/K} \xrightarrow{d} \Omega^1_{X/K} \xrightarrow{d} \Omega^2_{X/K} \xrightarrow{d} \cdots \quad .$$

Then the algebraic de Rham cohomology is defined to be

$$H^*_{DR}(X) := \mathbb{H}^*(X,\Omega^*_{X/K})$$

where $\mathbb{H}^*(X,\Omega^*_{X/K})$ is the hypercohomology defined in turn as follows. Let

$$\Omega^*_{X/K} \longrightarrow J^{*,*}$$

be a resolution of $\Omega^*_{X/K}$ by sheaves of injecitve $\mathcal{O}_X$-modules. Take its associated simple complex K^* defined as

$$K^i = \bigoplus_{a+b=i} J^{a,b} \quad .$$

Then

$$\mathbb{H}^*(X,\Omega^*_{X/K}) := H^*(X,\Gamma(X,K^*)) \quad .$$

Then there is the Hodge-cohomology $H^*_{Hodge}(X)$ defined as

$$H^i_{Hodge}(X) := \bigoplus_{a+b=i} H^a(X,\Omega^b_{X/K}) \qquad (i=0,1,\ldots) \quad .$$

The two cohomologies $H^*_{DR}(X)$ and $H^*_{Hodge}(X)$ are related by the usual spectral sequence

$$E_1^{a,b} = H^b(X,\Omega^a_{X/K}) \Longrightarrow H^{a+b}_{DR}(X)$$

and defines a filtration on $H^*_{DR}(X)$, the Hodge filtration. Now a well-known result says that

$$H^*(X,\mathbb{C}) \xrightarrow{\sim} H^*_{DR}(X) \otimes_K \mathbb{C} \quad .$$

Using this isomorphism one defines the periods as follows: Let

$$e^1,\ldots,e^{N_i} \in H^i(X,\mathbb{Z}) \qquad (i=0,1,\ldots)$$

be a basis of $H^i(X,\mathbb{C})$ and $e_1,\ldots,e_{N_i}$ the dual basis in $H_i(X,\mathbb{Z})$. Furthermore let

$$\omega_1,\ldots,\omega_{N_i} \in H^i_{DR}(X) \qquad (i=0,1,\ldots)$$

be a basis for $H^i_{DR}(X)$. Then

$$\omega_k = \sum_{\ell=1}^{N_i} \omega_{k,\ell} e^{\ell} \qquad (k=1,\ldots,N_i) \quad .$$

The coefficients $\omega_{k,\ell}$ are called the periods (of weight i) and can be expressed in the usual way; namely by integrating ω_k along $e_r \in H_i(X,\mathbb{Z})$ one finds

$$\int_{e_r} \omega_k = \sum_{\ell=1}^{N_i} \omega_{k,\ell} \cdot \int_{e_r} e^{\ell} = \sum_{\ell=1}^{N_i} \omega_{k,\ell}\, \delta_{r,\ell} = \omega_{k,r} \quad .$$

So the complex numbers $\omega_{k,\ell}$ are indeed what are classically known to be periods.

The vector spaces $H^i(X,\mathbb{C})$ carry a socalled Hodge structure, i.e. they admit a canonical decomposition

$$H^i(X,\mathbb{C}) = \bigoplus_{a+b=i} H^{a,b}$$

such that $\overline{H^{a,b}} = H^{b,a}$. They are said to be of weight i .

<u>EXAMPLE 1</u>: We consider an elliptic curve

$$E : y^2 = 4x^3 - g_2x - g_3 \ , \ g_2, g_3 \in K \quad .$$

Then $E_{\mathbb{C}}$ is the complex torus $\mathbb{C}/L$ for a lattice L of rank 2 that generates $\mathbb{C}$ over the reals. Hence

$$H_1(X,\mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 \quad .$$

The de Rham cohomology is obtained as follows. Let

$$\omega = \frac{dx}{y} \ , \quad \eta = x\frac{dx}{y} \quad .$$

Then ω is a differential of the first and η one of the second kind. They generate $H^1_{DR}(E)$ over K . The first one is everywhere holomorphic and therefore in $H^0(E,\Omega^1_{E/K})$. The second one has two poles without residues. The periods are

$$\omega_i = \int_{\gamma_i} \omega \quad , \quad \eta_i = \int_{\gamma_i} \eta \qquad (i = 1,2) \quad ,$$

and L can be taken to be the lattice generated by ω_1 and ω_2 .

REMARK 1. It is not necessary to restrict ourselves to proper schemes. If X is arbitrary the theory of "logarithmic differentials" leads also to an analogous theory which includes differentials of the third kind.

REMARK 2. Everything extends to socalled variations of Hodge structures which arise as follows. Let S be a smooth scheme over $\mathbb{C}$ and $f : X \longrightarrow S$ a proper and smooth morphism. Then one has to replace the functors H^i by the higher direct image functors $R^i f_*$ and obtains socalled variations of Hodge structures.

REMARK 3. One often finds the situation that a Hodge structure of weight i is isomorphic to a Hodge structure coming from a Hodge structure of weight j for some j. This is where one has to introduce the socalled Tate-twists $\mathbb{Q}_B(1)$, $\mathbb{Q}_{DR}(1)$ defined as

$$\mathbb{Q}_B(1) = 2\pi i\,\mathbb{Q} \quad , \quad \mathbb{Q}_{DR}(1) = K$$

and for integers m

$$H^*_B(X)(m) = H^*_B(X) \otimes \mathbb{Q}_B(1)^{\otimes m} \ ,$$
$$H^*_{DR}(X)(m) = H^*_{DR}(X) \otimes \mathbb{Q}_B(1)^{\otimes m}$$

and writes $\mathbb{Q}_B(m) = \mathbb{Q}_B(1)^{\otimes m}$; and the same for $\mathbb{Q}_{DR}(m)$.

$\mathbb{Q}_B(1)$ defines a Hodge structure of weight -2 and its Hodge decomposition is

$$\mathbb{Q}_B(1) \otimes \mathbb{C} = H^{-2,0} \oplus H^{-1,-1} \oplus H^{0,-2}$$

with $H^{-2,0} = H^{0,-2} = 0$, hence

$$\mathbb{Q}_B(1) \otimes \mathbb{C} = H^{-1,-1} \quad .$$

The comparison isomorphism relates the algebraic de Rham cohomology with the analytic Betti cohomology. This interplay between analytic and algebraic leads to a transcendence problem.

CONJECTURE. Let X be a smooth proper scheme of dimension n over K . Then the periods of X of weight i are either zero or transcendental.

For $i = 1$ this conjecutre is due to Grothendieck ([Gr]). The following theorem solves this conjecture of Grothendieck.

THEOREM 2. *Let X be a smooth proper scheme over K of dimension n . Then the periods of weight 1 are either zero or transcendental.*

The proof of this theorem uses Theorem 1 and will be published in the Mathematische Annalen ([Wü4]).

In many situations one can reduce the study of the periods of higher weight to those of weight one.

<u>EXAMPLE 2</u>. Let $n = 2m+1 \geq 3, n, m$ be integral and for $\underline{a} = (a_1, \ldots, a_r)$ with integers $a_i > 1$ let $V_n(\underline{a}) \subseteq \mathbb{P}^{n+r}$ be a complete intersection of dimension n of hypersurfaces of degree $a_1, \ldots, a_r$ of "niveau de Hodge 1" . This means that

(i) $H^q\left(X, \Omega_X^p\right) = 0$ for $p+q=n$, $|q-p| > 1$,

(ii) $H^m\left(X, \Omega_X^{m+1}\right) \neq 0$.

The list of possible $V_n(\underline{a})$ can be found in [Ra] . Then there exists an isomorphism of polarized Hodge structures

$$\alpha : H_B^n(X)(m) \xrightarrow{\sim} H_B^1(J(X))$$

where $J(X)$ is the socalled intermediate Jacobian of X and defined as follows. It is the complex torus

$$J(X) = H^n(X, \mathbb{Z}) \setminus H^n(X, \mathbb{C}) / H^{m+1,m}$$

where

$$H^n(X, \mathbb{C}) = H^{m+1,m} \oplus H^{m,m+1}$$

is the Hodge decomposition. Hence

$$J(X) = H^{m,m+1} / H^n(X, \mathbb{Z}) \quad .$$

One obtains then the following corollary.

COROLLARY. *The periods of the Hodge structure* $H_B^n(X)(m)$ *are either zero or transcendental.*

EXAMPLE 3. Let X be an algebraic K3-surface over K. Then again one associates with $H_B^2(X)(1)$ via the Clifford algebra constructed from the primitive cohomology and the intersection pairing on X a certain abelian variety and again one can apply Theorem 2.

4. VECTOR SPACES ASSOCIATED WITH RATIONAL INTEGRALS

We consider a smooth quasiprojective variety X defined over a number field K and having a K-rational point, and $\omega \in H^0(X,\Omega^1_{X/K})$ a holomorphic differential form on X. We are interested in the periods

$$\int_\gamma \omega \quad , \quad \gamma \in H_1(X,\mathbb{Z}) \quad .$$

The link between these periods and Theorem 1 is given by the following result which generalizes a result of Serre [Se].

THEOREM 3. ([F-W]). *Let* $\omega \in \Gamma(X,\Omega^1_X)$ *be a holomorphic differential form with* $d\omega = 0$. *Then there exists a commutative algebraic group* G *over* K, *a morphism* $\varphi : X \longrightarrow G$ *and an* $\alpha \in \mathfrak{g}^* = \Gamma(G,\Omega^1_G)^{\text{const.}}$, $\mathfrak{g} := \text{Lie}\, G$, *such that*

$$\omega = \varphi^{*}\alpha \quad .$$

The proof of this theorem uses the construction of the socalled generalized Albanese variety. If one combines this result and Theorem 1 one obtains the following theorem.

THEOREM 4. ([Wü4]). *The numbers*

$$\int_{\gamma} \omega \qquad (\gamma \in H_1(X,\mathbb{Z}))$$

are either zero or transcendental.

It is a natural question to look at vector spaces generated by numbers

$$\int_{\gamma} \omega \qquad (\gamma \in H_1(X,\mathbb{Z}) \ , \omega \in H^0(X,\Omega^1_{X/K})) \quad .$$

Since in general $H^0(X,\Omega^1_{X/K})$ is not finite dimensional one has to look at canonical finite dimensional subspaces of this space.

Special cases of questions of this type have been treated by various authors (Siegel, Schneider, Baker, Coates, Masser). But they all dealed with low dimensional algebraic groups. Till now we have not been able to work out a general result on this question depending intrinsically on the cohomology of X . But in important special cases we can give explicit results.

For this let Y be a smooth projecitve variety and $D \subseteq Y$ an ample divisor both defined over some number field K. Then put $X = Y \setminus D$. In this case we have a surjection

$$\pi : H^0(X, \Omega^1_{X/K}) \longrightarrow H^1_{DR}(Y)$$

by a result of Hodge and Atiyah [H-A]. Therefore we can find a basis for $H^1_{DR}(Y)$ (up to exact forms) from forms in $H^0(X, \Omega^1_{X/K})$. We take therefore a section of π, i.e. a subspace V of $H^0(X, \Omega^1_{X/K})$. Then we denote by $V(X)$ the vector space generated by

$$1, 2\pi i, \int_\gamma \omega \quad (\gamma \in H_1(X, \mathbb{Z}), \omega \in V) \quad .$$

Let $Alb(Y)$ be the Albanese variety of Y. Then up to isogeny we have

$$Alb(Y) = A_1^{n_1} \times \ldots \times A_k^{n_k}$$

for pairwise nonisogenous simple abelian varieties $A_1, \ldots, A_k$ defined over $\overline{K}$. Then we have the following result.

THEOREM 5. ([Wü4]). *We have*

$$\dim V = 2 + 4 \sum_{i=1}^{k} \frac{(\dim A_i)^2}{\dim(\mathrm{End}\, A_i)} \quad .$$

One can for example apply this result in the case that Y is a Fermat curve.

EXAMPLE 4. Let X be the Fermat curve of degree N given by

$$X_0^N + N_1^N + N_2^N = 0$$

where $N \geq 3$. Then a basis for $H^1_{DR}(X)$ is given by the differentials (x,y affine coordinates)

$$\omega_{a,b} = x^{Na-1}\ y^{Nb-N}\ dx \qquad \text{(first kind)}$$

and

$$\eta_{a,b} = x^{N-Na-1}\ y^{-Nb}\ dy \qquad \text{(second kind)}$$

where a and b are rational numbers satisfying $0 < a,b < 1$, $a + b < 1$, $Na, Nb \in \mathbb{Z}$. The periods are given by numbers of the form

$$\int_\gamma \omega_{a,b} = c(a,b)\ B(a,b)$$

$$\int_\gamma \eta_{a,b} = c'(a,b)\ B(1-a,1-b)$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ when defined and $c(a,b)$, $c'(a,b) \in \overline{\mathbb{Q}}$. It is possible to decompose explicitely the Jacobian $J(N)$ of this Fermat curve and to obtain the dimension of the vector space generated by the numbers

$$1, 2\pi i\ ,\ B(a,b)\ ,\ B(1-a,1-b)$$

for the given range of a,b . In particular one obtains the following corollary.

<u>COROLLARY</u>. <u>For rational numbers</u> a,b <u>with</u> $0 < a,b < 1$, $a+b<1$ <u>the numbers</u>

$$1\ ,\ \pi\ ,\ B(a,b) = \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}\ ,\ \frac{\pi}{B(a,b)}$$

<u>are</u> $\overline{\mathbb{Q}}$ - <u>linearly independent</u>.

As an application of this Corollary to a problem of Lang let X be a smooth projecitve curve over $\overline{\mathbb{Q}}$ of genus $g > 1$. Then the universal covering space is the open unit disc B_1 . Let $x \in X(\overline{\mathbb{Q}})$ be a $\overline{\mathbb{Q}}$ - rational point and

$$\varphi_1 : B_1 \longrightarrow X_{\mathbb{C}}$$

a covering map with $\varphi_1(0) = x$. After a homothety of B_1 into B_r for some $r \in \mathbb{C}^{\times}$ one can get a new covering map

$$\varphi_r : B_r \longrightarrow X_{\mathbb{C}}$$

such that $\varphi_r(0) = x$ and $\varphi_r'(0) \in \overline{\mathbb{Q}}$.

<u>THEOREM 6</u>. ([W-W]). <u>Let</u> Γ_1 <u>be a torsion free Fuchsian group of finite index in a triangular group</u> $\Delta_1 \subset \operatorname{Aut} B_1$ <u>with signature</u> (p,q,s) <u>and</u> 0 <u>an elliptic fix point of order</u> $p > 1$ <u>of</u> Δ_1 . <u>Then</u> r <u>is transcendental</u>.

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