ON THE STANDARD CONJECTURE FOR COMPLEX ABELIAN SCHEMES OVER SMOOTH PROJECTIVE CURVES*

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Abstract. We reduce the Hodge conjecture for Abelian varieties to the question about the existence of an algebraic isomorphism $H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \widetilde{\rightarrow} H^0(C, R^i\pi_*\mathbb{Q})$ for all $i \geq 2$ and for all complex principally polarized Abelian schemes $\pi : X \to C$ of relative dimension d over smooth projective curves. If canonically defined Hodge cycles $\alpha_i(X/C) \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ are algebraic for all natural numbers $i \geq 2$, then the Grothendieck standard conjecture $B(X)$ on the algebraicity of the operators Λ and $*$ holds for X. We prove $B(X)$ for an Abelian scheme under the assumption that $\mathrm{End}(X_s) = \mathbb{Z}$ for some geometric fiber X_s of non-exceptional dimension.

Introduction

Let X be a smooth projective d-dimensional variety over the field $\mathbb{C}$ of complex numbers. The choice of $\sqrt{-1} \in \mathbb{C}$ determines the isomorphism $\mathbb{Q}(1) \simeq \mathbb{Q}$. We denote by $\mathrm{cl}_X(Z) \in H^{2i}(X, \mathbb{Q})$ the class of an algebraic cycle Z of codimension i on X.

The Hodge conjecture $\mathrm{Hodge}^i(X)$ [13]. *Given a natural number i, the $\mathbb{Q}$-space $H^{2i}(X, \mathbb{Q}) \cap H^{i,i}(X, \mathbb{C})$ is generated by classes of algebraic cycles of codimension i on X.*

If $\mathrm{Hodge}^i(X)$ is true for any natural number i, then we say that the Hodge conjecture $\mathrm{Hodge}(X)$ holds for X. By the Lefschetz theorem, $\mathrm{Hodge}^1(X)$ always holds.

The surveys of T.Shioda [27] and B.Gordon [10] give a good introduction to the Hodge conjecture for Fermat varieties and for Abelian varieties.

Recall that a linear map $u^* : H^i(X, \mathbb{Q}) \to H^j(X, \mathbb{Q})$ is called algebraic if it is the restriction of a map $H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ induced by an element of the $\mathbb{Q}$-vector space generated by the algebraic classes ([14], 1.3.6). In other words, u^* is algebraic, if there exists an algebraic cycle Z with rational coefficients such that $u = \mathrm{cl}_{X\times X}(Z)$ and

$$u^* : H^*(X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^*(X \times X, \mathbb{Q}) \xrightarrow{\cup u} H^*(X \times X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^*(X, \mathbb{Q}),$$

where pr_1, pr_2 are canonical projections.

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Let H be a smooth hyperplane section of X. We denote by L the Lefschetz operator on $H^*(X,\mathbb{Q})$ defined by the formula $Lx = \mathrm{cl}_X(H) \cup x$. By the strong Lefschetz theorem, the map

$$L^{d-i} : H^i(X,\mathbb{Q}) \to H^{2d-i}(X,\mathbb{Q})$$

is an isomorphism for any $i \leq d$. Let $\Lambda^{d-i} : H^{2d-i}(X,\mathbb{Q}) \xrightarrow{\sim} H^i(X,\mathbb{Q})$ be the isomorphism inverse to L^{d-i}.

The Grothendieck standard conjecture $B^i(X)$ [12]. *Given a natural number $i \leq d$, the isomorphism $\Lambda^{d-i} : H^{2d-i}(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is algebraic.*

If $B^i(X)$ is true for any natural number i, then we say that the standard conjecture $B(X)$ holds for X.

The conjecture $B(X)$ implies the algebraicity of the classical operators ${}^{\mathrm{c}}*$ and ${}^{\mathrm{c}}\Lambda$ of the Hodge theory for X, the coincidence of the numerical and homological equivalences of algebraic cycles on X, the algebraicity of the Künneth components of the class of an algebraic cycle on $X \times X$.

The property $B(X)$ is independent of the polarization L and is stable under product, smooth hyperplane section and specialization. It is satisfied by projective space and the other usual rational varieties having cellular decompositions, by curves, by surfaces and by Abelian varieties. In § 11, we add to this list some complex Abelian schemes over smooth projective curves.

Now let k be a finitely generated extension of a prime field, and let X be a smooth projective variety over k. We denote by k^{sep} the maximal separable algebraic extension of k.

The Tate conjecture $\mathrm{Tate}^i(X)$ [32]. *Given a natural number i, there exists a finite Galois extension k' of k such that the $\mathbb{Q}_l$-space*

$$\left[H^{2i}_{et}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l(i))\right]^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}$$

is generated by classes of k'-rational algebraic cycles of codimension i on $X \otimes_k k'$.

The Tate conjecture and the corresponding conjecture about the poles of zeta functions are discussed in [32], [33].

The significance of the conjecture $B(X)$ is based on the following important remarks:

i) By the Y.André theorem ([1], Theorem 0.6.2), the algebraicity of

$$\Lambda^{d+1-i} : H^{2d+2-i}(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$$

for all complex Abelian schemes $\pi : X \to C$ of relative dimension d over smooth projective curves C and all i implies the Hodge conjecture for all Abelian varieties.

ii) By the Pohlmann theorem ([25], Theorem 4), the Hodge conjecture for a complex Abelian variety X of CM-type is equivalent to the Tate conjectures [32] for algebraic cycles on a model X_0 of X over a number field and for poles of zeta function of X_0 ([25], Theorem 5).

iii) By the Milne theorem ([22], Theorem 7.1), the Hodge conjecture for all Abelian varieties of CM-type over $\mathbb{C}$ implies the Tate conjectures [32] for algebraic cycles and poles of zeta functions for all Abelian varieties over finite fields.

Let $\pi : X \to C$ be a principally polarized complex Abelian scheme over a smooth projective curve, $j : X \times_C X \hookrightarrow X \times X$ the canonical embedding, $\tau : X \times_C X \to C$ the structure morphism, $\Delta : X \hookrightarrow X \times X$ the diagonal embedding.

We denote by $\mathcal{P}$ the Poincaré bundle on $X \times_C \hat{X} \widetilde{\to} X \times_C X$, and let v be the image of $\mathrm{cl}_{X\times_C X}(\mathcal{P}) \in H^2(X \times_C X, \mathbb{Q})$ under the canonical edge morphism $H^2(X \times_C X, \mathbb{Q}) \to H^0(C, R^2\tau_*\mathbb{Q})$ of the Leray spectral sequence $E_2^{p,q} = H^p(C, R^q\tau_*\mathbb{Q}) \Rightarrow H^*(X \times_C X, \mathbb{Q})$. Then $j_*v^i = j_*(v^{\cup i})$ is a canonically defined algebraic class, and it determines an algebraic isomorphism

$$[j_*v^i]^* : H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \widetilde{\to} H^2(C, R^i\pi_*\mathbb{Q}).$$

We develop the Y.André approach and reduce the Hodge conjecture for Abelian varieties to the question about the existence of an algebraic isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \widetilde{\to} H^0(C, R^i\pi_*\mathbb{Q})$$

for all $i \geq 2$ and for all complex principally polarized Abelian schemes $\pi : X \to C$ of relative dimension d over smooth projective curves.

In this paper, we prove the following technical result:

Let $\pi : X \to C$ be a principally polarized complex Abelian scheme of relative dimension d over a smooth projective curve C. Then for any natural number $i \geq 2$, there exists the unique Hodge cycle

$$\alpha_i = \alpha_i(X/C) \in [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cap H^{i,i}(X \times X, \mathbb{C}),$$

characterized by the following condition: $\alpha_i \cup \mathrm{cl}_{X\times X}(X \times X_s)$ determines an isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \widetilde{\to} H^2(C, R^i\pi_*\mathbb{Q}),$$

which coincides with an algebraic isomorphism

$$[j_*v^i]^* : H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \widetilde{\to} H^2(C, R^i\pi_*\mathbb{Q}).$$

If α_i is algebraic, then the algebraic class

$$[\Delta_*\mathrm{cl}_X(X_s)] \circ [j_*v^{i-2}] + j_*v^{i-1} + \alpha_i$$

induces an algebraic isomorphism $H^{2d+2-i}(X, \mathbb{Q}) \widetilde{\to} H^i(X, \mathbb{Q})$, and the conjecture $B^i(X)$ holds (Theorem 10.1).

If canonically defined Hodge cycles $\alpha_i(X/C) \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ are algebraic for all natural numbers $i \geq 2$, then the Grothendieck standard conjecture $B(X)$ on the algebraicity of the operators Λ and $*$ holds for X.

We also prove $B(X)$ for an Abelian scheme under the assumption that the Hodge group of the generic geometric fiber $X_{0\overline{\eta}}$ is a simple algebraic group of type C_d (Theorem 11.2). This assumption holds for an Abelian scheme $\pi : X \to C$, if $\mathrm{End}(X_s) = \mathbb{Z}$ for some geometric fiber X_s and d is a non-exceptional dimension (for example, if d is an odd integer) (Corollaries 11.4, 11.5, 11.3).

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§ 1. On some conjectures concerning algebraic cycles

1.1. Let X be a smooth d-dimensional projective variety over the field $\mathbb{C}$ of complex numbers. For $j \geq 0$, we have the Lefschetz primitive decomposition

$$H^j(X,\mathbb{Q}) = \bigoplus_{k\geq \max(0,j-d)} L^k P^{j-2k}(X),$$

where $P^i(X) = H^i(X,\mathbb{Q}) \cap \operatorname{Ker} L^{d-i+1} \quad (i \leq d)$ is a primitive part of $H^i(X,\mathbb{Q})$ ([4], § 3, (3.4)).

In particular, for any j, any $x \in H^j(X,\mathbb{Q})$ can be written uniquely in the form

$$x = \sum_{k\geq \max(0,j-d)} L^k x_{j-2k},$$

where $x_{j-2k} \in P^{j-2k}(X)$ ([4], § 3, (3.3)).

Let ${}^{\mathrm{c}}\Lambda$ be the dual operator for L of the classical Hodge theory, and let ${}^{\mathrm{c}}*$ be the classical operator star ([11], Chapter 0, § 6; [4], § 3). For an element $\varphi \in P^p(X)\otimes_{\mathbb{Q}}\mathbb{C}$,

$${}^{\mathrm{c}}\Lambda L^r\varphi = r(d-p-r+1)L^{r-1}\varphi$$

([4], § 3.1). In particular,

$${}^{\mathrm{c}}\Lambda\, x = \sum_{k\geq \max(0,j-d)} {}^{\mathrm{c}}\Lambda L^k x_{j-2k} =$$

$$\sum_{k\geq \max(0,j-d)} k(d-(j-2k)-k+1)L^{k-1}x_{j-2k}.$$

Hence

$${}^{\mathrm{c}}\Lambda\, x = \sum_{k\geq \max(1,j-d)} k(d-j+k+1)L^{k-1}x_{j-2k}. \tag{1.1}$$

On the space of forms of type (a,b), we define the Weil operator C by the following formula: $C\varphi = (\sqrt{-1})^{a-b}\varphi$. If $\varphi \in P^p(X)\otimes_{\mathbb{Q}}\mathbb{C}$, then

$${}^{\mathrm{c}}* L^r\varphi = \begin{cases} (-1)^{p(p+1)/2}\frac{r!}{(d-p-r)!}L^{d-p-r}C\varphi, & \text{for} \quad r \leq d-p, \\ 0, \quad \text{for} \quad r > d-p \end{cases}$$

([4], § 3, (3.6)). In particular,

$${}^{\mathrm{c}}*\, x = \sum_{k\geq \max(0,j-d)} {}^{\mathrm{c}}*\, L^k x_{j-2k} =$$

$$\sum_{k\geq \max(0,j-d)} (-1)^{(j-2k)(j-2k+1)/2}\frac{k!}{(d-(j-2k)-k)!}L^{d-(j-2k)-k}Cx_{j-2k}.$$

Hence

$${}^{\mathrm{c}}*\, x = \sum_{k\geq \max(0,j-d)} (-1)^{(j-2k)(j-2k+1)/2}\frac{k!}{(d-j+k)!}L^{d-j+k}Cx_{j-2k}. \tag{1.2}$$

The decompositions (1.1) – (1.2) permit to define the following abstract operators:

$$\Lambda x = \sum_{k \geq \max(1, j-d)} L^{k-1} x_{j-2k}, \tag{1.3}$$

$$*x = \sum_{k \geq \max(0, j-d)} (-1)^{(j-2k)(j-2k+1)/2} L^{d-j+k} x_{j-2k}, \tag{1.4}$$

$$p^k x = \delta_{jk} x_m \quad \text{where} \quad m = \max(0, j-d) \quad \text{for} \quad k = 0, \ldots, 2d. \tag{1.5}$$

The formula for p^k corrects ([14], 1.4.2.4). All these formulas are discussed in ([14], § 1.4; [15], § 4). It is easy to check that $*^2 = 1$ and $\Lambda = *L*$ ([14], Proposition 1.4.3).

For $i \leq d$, $\Lambda^{d-i} : H^{2d-i}(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ is known to be inverse to L^{d-i} and ${}^{\mathrm{c}}\Lambda^{d-i} : L^{d-i} P^i(X) \to P^i(X)$ is inverse to a multiple of L^{d-i} ([14], Proposition 1.4.3), so one can define the Lefschetz involution on $H^*(X, \mathbb{Q})$ ([1], § 1.1):

$$*_L\, x = \begin{cases} L^{d-j} x, & \text{for } x \in H^j(X, \mathbb{Q}), \quad j \leq d, \\ \Lambda^{j-d} x, & \text{for } x \in H^j(X, \mathbb{Q}), \quad j > d. \end{cases} \tag{1.6}$$

1.2. It is well known that the Grothendieck standard conjecture
$B(X)$: *"the operator Λ is algebraic"*
is equivalent to each of the following conjectures:
${}^{\mathrm{c}}B(X)$: *"the operator ${}^{\mathrm{c}}\Lambda$ is algebraic"*;
$*(X)$: *"the operator $*$ is algebraic"*
([15], Theorem 4-1);
$\nu(X)$: "for any $i \leq \dim X - 1$, there exists an isomorphism

$$\nu^i : H^{2 \cdot \dim X - i}(X, \mathbb{Q}) \simeq H^i(X, \mathbb{Q})$$

which is algebraic" ([14], p. 360; Theorem 2.9) (for the reader convenience, we reproduce the proof of the equivalence

$$B^i(X) \Leftrightarrow \nu^i(X) :$$

consider the algebraic isomorphism

$$u = \nu^i L^{d-i} : H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q});$$

its characteristic polynomial $P(t)$ has rational coefficients; by the Cayley – Hamilton theorem, $P(u) = 0$; it follows that u^{-1} is a linear combination of $1, u, u^2, \ldots$ with rational coefficients; so, u^{-1} is algebraic; consequently, the isomorphism

$$\Lambda^{d-i} : H^{2d-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X, \mathbb{Q})$$

inverse to $L^{d-i} : H^i(X, \mathbb{Q}) \xrightarrow{\sim} H^{2d-i}(X, \mathbb{Q})$, being equal to $u^{-1}\nu^i$, is algebraic).

Let $p \leq d = \dim X$, and let $H^{2p}_{\mathrm{alg}}(X, \mathbb{Q}) \hookrightarrow H^{2p}(X, \mathbb{Q})$ be the $\mathbb{Q}$-vector space of classes of algebraic cycles of codimension p on X. The standard conjecture $B(X)$ has the following weaker form:

1.3. Conjecture $A(X,L)$. *For any natural number $p \le d/2$, the injection*

$$L^{d-2p} : H^{2p}_{\mathrm{alg}}(X,\mathbb{Q}) \to H^{2d-2p}_{\mathrm{alg}}(X,\mathbb{Q})$$

is an isomorphism.

The conjecture $A(X,L)$ is equivalent to each of the following conjectures:
"$H^*_{\mathrm{alg}}(X,\mathbb{Q})$ is stable under the operator $*$";
"$H^*_{\mathrm{alg}}(X,\mathbb{Q})$ is stable under the operator Λ";
"$H^*_{\mathrm{alg}}(X,\mathbb{Q})$ is stable under the operator ${}^{\mathrm{c}}\Lambda$"
([14], Proposition 2.1). By the classical result of D.Lieberman it is also equivalent to the following conjectures:
"$H^*_{\mathrm{alg}}(X,\mathbb{Q})$ is stable under the operator ${}^{\mathrm{c}}*$";
"if $x \in H^{2p}_{\mathrm{alg}}(X,\mathbb{Q})$ and

$$x = \sum_{k \ge \max(0,2p-d)} L^k x_{2p-2k},$$

where $x_{2p-2k} \in P^{2p-2k}(X)$, then $x_{2p-2k} \in H^{2p-2k}_{\mathrm{alg}}(X,\mathbb{Q})$";
D(X): *"numerical equivalence of algebraic cycles on X coincides with homological equivalence"*
([19], Theorem 1).

The latter conjecture is independent of the polarization L, therefore the conjecture $A(X,L)$ is independent of the polarization, and we denote it simply by $A(X)$.

We claim that

$$A(X) \Rightarrow \left(H^*_{\mathrm{alg}}(X,\mathbb{Q}) \text{ is stable under the Lefschetz involution } *_L\right). \tag{1.7}$$

Indeed, assume that $A(X)$ holds. If $x \in H^{2p}_{\mathrm{alg}}(X,\mathbb{Q})$ and

$$x = \sum_{k \ge \max(0,2p-d)} L^k x_{2p-2k},$$

where $x_{2p-2k} \in P^{2p-2k}(X)$, then $x_{2p-2k} \in H^{2p-2k}_{\mathrm{alg}}(X,\mathbb{Q})$. Since $H^*_{\mathrm{alg}}(X,\mathbb{Q})$ is stable under the operators Λ and L, we see that $*_L\, x \in H^*_{\mathrm{alg}}(X,\mathbb{Q})$.

Recall that

$$B(X) \Rightarrow A(X) \tag{1.8}$$

([14], Corollary 2.2), and

$$A(X \times X) \Rightarrow B(X) \tag{1.9}$$

([15], Theorem 4-1),

$$B(X)\,\&\,B(Y) \Rightarrow B(X \times Y) \tag{1.10}$$

([14], Corollary 2.5).

Moreover,

$$B(X) \Leftrightarrow A(X \times X). \tag{1.11}$$

Indeed, (1.10) gives the relation $B(X) \Rightarrow B(X \times X)$; it remains to note that, in virtue of (1.8) – (1.9), $B(X \times X) \Rightarrow A(X \times X)$ and $A(X \times X) \Rightarrow B(X)$.

1.4. Recall that
$$B(X) \Rightarrow C(X),$$
where

$C(X)$: "the Künneth components $\pi^i \otimes H^{2\cdot\dim X - i}(X,\mathbb{Q}) \in H^i(X,\mathbb{Q})$ of the diagonal class Δ, which correspond to the projection operators
$$\pi^i : H^*(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}),$$
are algebraic"
([14], Lemma 2.4). On the other hand, $C(X)$ is equivalent to the following assertion:
"$H^*_{\mathrm{alg}}(X \times X,\mathbb{Q})$ is stable under Künneth decomposition"
([14], Proposition 2.6).

1.5. Theorem. *Let X, Y be smooth projective varieties over the field of complex numbers. If there exists a surjective morphism $f : X \to Y$, then $A(X) \Rightarrow A(Y)$.*

Proof. Recall that the conjecture $A(X)$ is equivalent to the following statement: numerical and homological equivalences coincide on the set of algebraic cycles on X ([19], Theorem 1).

Assume that $\dim X > \dim Y$. Let X' be a smooth hyperplane section of X. Then $A(X) \Rightarrow A(X')$. Indeed, let $i : X' \hookrightarrow X$ be the inclusion morphism of a smooth hyperplane section; then $\Lambda' = i^*\Lambda^2 i_*$ ([14], Proposition 1.4.7(vii)), and therefore
$$\Lambda'\left(H^*_{\mathrm{alg}}(X',\mathbb{Q})\right) = i^*\Lambda^2 i_*\left(H^*_{\mathrm{alg}}(X',\mathbb{Q})\right) \hookrightarrow$$
$$i^*\Lambda^2\left(H^*_{\mathrm{alg}}(X,\mathbb{Q})\right) \hookrightarrow i^*\left(H^*_{\mathrm{alg}}(X,\mathbb{Q})\right) \hookrightarrow H^*_{\mathrm{alg}}(X',\mathbb{Q})$$
because i_*, Λ, i^* transform algebraic classes into algebraic classes. Since Λ' transforms algebraic classes into algebraic classes, the conjecture $A(X')$ holds.

On the other hand, $f_{|X'} : X' \to Y$ is a surjective morphism. Therefore one can use an induction and reduce the problem to the case $\dim X = \dim Y$. In the latter case, let Z_Y be an algebraic cycle of codimension r on Y, which is numerically equivalent to zero on Y. Then for any algebraic cycle Z_X of codimension $d - r$ on X, the projection formula gives the following relation:
$$\deg(f_*(f^*(\mathrm{cl}_Y(Z_Y)) \cup \mathrm{cl}_X(Z_X))) = \deg(\mathrm{cl}_Y(Z_Y) \cup f_*(\mathrm{cl}_X(Z_X))) = 0; \qquad (1.12)$$
on the other hand, $f^*(\mathrm{cl}_Y(Z_Y)) \cup \mathrm{cl}_X(Z_X) \in H^{2d}(X,\mathbb{Q}) \simeq \mathbb{Q}$, and so $f_* : H^{2d}(X,\mathbb{Q}) \to H^{2d}(Y,\mathbb{Q})$ is an isomorphism. Therefore (1.12) gives an equality $f^*(\mathrm{cl}_Y(Z_Y)) \cup \mathrm{cl}_X(Z_X) = 0$ for all algebraic cycles Z_X of codimension $d - r$ on X. In particular, $f^*(\mathrm{cl}_Y(Z_Y))$ is numerically equivalent to zero. Since $A(X)$ holds, we get an equality $f^*(\mathrm{cl}_Y(Z_Y)) = 0$. On the other hand, the projection formula shows that
$$f_*f^*(\mathrm{cl}_Y(Z_Y)) = \deg f \cdot \mathrm{cl}_Y(Z_Y)$$
([9], § 2.3). Consequently, $\mathrm{cl}_Y(Z_Y) = 0$ and Z_Y is homologically equivalent to zero. Another proof: since f^* is injective ([14], Proposition 1.2.4), we have the relation
$$f^*(\mathrm{cl}_Y(Z_Y)) = 0 \;\Rightarrow\; \mathrm{cl}_Y(Z_Y) = 0.$$
Therefore the numerical and homological equivalences of algebraic cycles coincide on Y, and so $A(Y)$ holds. Theorem 1.5 is proved.

1.6. Theorem. *Let X, Y be smooth projective varieties over the field of complex numbers. If there exists a surjective morphism $f : X \to Y$, then $B(X) \Rightarrow B(Y)$.*

Proof. Since $f \times f : X \times X \to Y \times Y$ is a surjective morphism, Theorem 1.5 gives the relation $A(X \times X) \Rightarrow A(Y \times Y)$, and the statement follows from (1.11). Theorem 1.6 is proved.

1.7. Corollary. *Let $\mathrm{Sym}^n(C)$ be a symmetric power of a smooth projective curve C over the field of complex numbers. Then the conjecture $B(\mathrm{Sym}^n(C))$ holds.*

Indeed, since $B(X)$ trivially holds for curves, it remains to apply (1.10) and Theorem 1.6 to the natural surjective morphism $C^n = C \times \cdots \times C \to \mathrm{Sym}^n(C)$.

1.8. Corollary (D.Lieberman's theorem) ([19], Theorem 3; [14], Theorem 2A11). *The conjecture $B(X)$ holds for all Abelian varieties.*

Proof. For any smooth projective curve C of genus g, there exists a morphism $\mathrm{Sym}^g(C) \to \mathrm{Pic}^0(C)$ of finite degree, which is defined by the following formula:

$$f(c_1 + c_2 + \cdots + c_g) = \mathcal{O}_C(c_1 + c_2 + \cdots + c_g - g \cdot e)) \in \mathrm{Pic}^0(C),$$

where e is the fixed point of C ([11], Chapter II, § 7; [26], Chapter V, § 1, Proposition 2, § 3.11). By Corollary 1.7 and Theorem 1.6, $B(X)$ holds for all Jacobian varieties. On the other hand, any Abelian variety is isomorphic to a factor of a product of Jacobian varieties ([26], Chapter VII, § 2.13, Corollary). Since $B(X)$ is stable under product (property (1.10)), it remains to apply Theorem 1.6.

1.9. Let X be a smooth projective d-dimensional variety over the field $\mathbb{C}$ of complex numbers. Recall that there exists a canonical orientation isomorphism

$$\mathrm{tr} : H^{2d}(X, \mathbb{Q}) \widetilde{\to} \mathbb{Q},$$

and, by the Poincaré duality theorem ([14], 1.2.A), the bilinear form

$$\mathrm{tr}(x \cup y) : H^i(X, \mathbb{Q}) \times H^{2d-i}(X, \mathbb{Q}) \to \mathbb{Q}$$

is a perfect duality (which identifies $H^i(X, \mathbb{Q})$ with the dual of $H^{2d-i}(X, \mathbb{Q})$).

§ 2. On the Y.André theorem

2.1. Y.André's theorem ([1], Theorem 0.6.2). *Each Hodge cycle ξ on an Abelian variety J is motivated (motivé). More precisely, ξ is a sum of cycles of the form $\mathrm{pr}_{J*}(\alpha \cup *_L \beta)$, where α and β are algebraic cycles (with coefficients in $\mathbb{Q}$) on $J \times I \times Y_1 \times \cdots \times Y_n$, I is an Abelian variety, Y_i is a total space of a compact pencil of Abelian varieties, and pr_J is the canonical projection onto J.*

Recall that the Lefschetz involution $*_L$ is defined by the formula (1.6). If the conjecture $A(J \times I \times Y_1 \times \cdots \times Y_n)$ holds, then in virtue of (1.7), $*_L$ transforms algebraic classes into algebraic classes, and (in virtue of the Y.André theorem), the Hodge conjecture holds for an abelian variety J. On the other hand, by the relation (1.8),

$$B(J \times I \times Y_1 \times \cdots \times Y_n) \Rightarrow A(J \times I \times Y_1 \times \cdots \times Y_n).$$

2.2. By the property (1.10), $B(X)$ is stable under product; in other words, $B(X)$, $B(Y)$ imply $B(X \times Y)$. Consequently, the Hodge conjecture holds for an Abelian variety J if the standard conjecture $B(X)$ holds for all Abelian varieties and for all total spaces of compact pencils of Abelian varieties. By the classical

D.Lieberman theorem (Corollary 1.8), $B(X)$ holds for all Abelian varieties. It remains to prove $B(X)$ for a total space of a compact pencil $\pi : X \to C$ of Abelian varieties.

Note that for some ramified covering $C' \to C$, the morphism $X \times_C C' \to C'$ has a section (in other words, $X \times_C C' \to C'$ is an Abelian scheme over C'). The canonical morphism $X \times_C C' \to X$ is finite, and in virtue of Theorem 1.6, one may assume that $\pi : X \to C$ is an Abelian scheme.

Consequently, we arrive to the following result ([1], p. 33, Remark 2):

2.3. Corollary. *If the Grothendieck standard conjecture $B(X)$ holds for all Abelian schemes $\pi : X \to C$ over smooth projective curves C, then the Hodge conjecture holds for all Abelian varieties.*

2.4. Let K be a function field with the field of constants k, and let A be an Abelian variety over K. By definition, the K/k-trace of A is the pair (B, τ) consisting of an Abelian variety B over k and a K-homomorphism $\tau : B \to A$ which has the following universal property: for any Abelian variety J over k and for any K-homomorphism $\varphi : J \to A$, there exists the unique k-homomorphism $\varphi_* : J \to B$ such that the following diagram is commutative:

$$\begin{tikzcd} J \arrow[r, "\varphi"] \arrow[dr, "\varphi_*"'] & A \\ & B. \arrow[u, "\tau"'] \end{tikzcd}$$

It is well known that K/k-trace exists ([16], Chapter VIII, § 3, Theorem 8), and τ is injective. We recall the Lang – Néron theorem:

2.5. Theorem ([17], Chapter 6, Theorem 2). *Let K be a function field with the field of constants k, A an Abelian variety over K and the pair (B, τ) is a K/k-trace of A. Then the group $A(K)/\tau B(k)$ is finitely generated.*

2.6. Lemma. *Let $\pi : X \to C$ be an Abelian scheme over a smooth projective curve C, $J \times C \xrightarrow{\mathrm{pr}_2} C$ be a maximal constant Abelian subscheme of $\pi : X \to C$, and let $\pi' : Y \to C$ be the corresponding Abelian scheme factor. Then*

$$B(X) \Leftrightarrow B(Y)$$

and

$$H^0(C, R^1\pi'_*\mathbb{Q}) = 0.$$

Proof. By Theorem 1.6, $B(X) \Rightarrow B(Y)$ because there exists a canonical surjective morphism $X \to Y$. On the other hand, the Poincaré reducibility theorem shows that there exists an isogeny of Abelian C-schemes

$$Y \times J = Y \times_C (J \times C) \to X.$$

By Theorem 1.6, $B(Y \times J) \Rightarrow B(X)$. By the property (1.10), $B(Y)\&B(J) \Rightarrow B(Y \times J)$. Since $B(J)$ holds for all Abelian varieties (by the D.Lieberman theorem), we get a relation $B(Y) \Rightarrow B(X)$.

Let $K = \mathbb{C}(C)$ be the field of rational functions on a curve C. In the situation under consideration, the generic fiber Y_η of the morphism $\pi' : Y \to C$ has a trivial $K/\mathbb{C}$-trace. By the Lang – Néron theorem 2.5, $Y_\eta(K)$ is a finitely generated group.

The canonical exponential map defines an exact sequence of sheaves of sections

$$0 \to T_{\mathbb{Z}}(Y) \to [e^*\Omega^1_{Y/C}]^\wedge \to Y \to 0,$$

where $e : C \to Y$ is a zero section of $\pi' : Y \to C$ and

$$T_{\mathbb{Z}}(Y) = [R^1\pi'_*\mathbb{Z}]^\wedge$$

is a local system dual to $R^1\pi'_*\mathbb{Z}$. We arrive to an exact cohomology sequence

$$0 \to H^0(C, T_{\mathbb{Z}}(Y)) \to H^0(C, [e^*\Omega^1_{Y/C}]^\wedge) \to H^0(C, Y) \tag{2.1}$$

In virtue of Lemma 1 in Supplement to § 2 of [2], the set of all extensions of $e : C \to Y$ to C-morphisms $C[\varepsilon]/(\varepsilon^2) \to Y$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_C}(e^*\Omega^1_{Y/C}, \mathcal{O}_C)$. Therefore

$$\mathrm{Hom}_{\mathcal{O}_C}(e^*\Omega^1_{Y/C}, \mathcal{O}_C) = H^0(C, [e^*\Omega^1_{Y/C}]^\wedge)$$

is the tangent space at the origin of the $\mathbb{C}$-scheme $\mathrm{Sect}_{Y/C}$ of sections of Y over C, and $\mathrm{Sect}_{Y/C}(\mathbb{C}) = Y_\eta(K)$. In virtue of the Lang – Néron theorem 2.5, $\dim H^0(C, [e^*\Omega^1_{Y/C}]^\wedge)$ is equal to the dimension of $K/\mathbb{C}$-trace of Y_η. Therefore

$$H^0(C, [e^*\Omega^1_{Y/C}]^\wedge) = 0,$$

and an exact sequence (2.1) shows that

$$H^0(C, T_{\mathbb{Z}}(Y)) = 0. \tag{2.2}$$

Since $T_{\mathbb{Z}}(Y) \otimes \mathbb{Q}$ is a polarizable system of Hodge $\mathbb{Q}$-structures of weight -1, there exists an isomorphism of systems of Hodge $\mathbb{Q}$-structures

$$R^1\pi'_*\mathbb{Q} = [T_{\mathbb{Z}}(Y) \otimes \mathbb{Q}]^\wedge \simeq T_{\mathbb{Z}}(Y) \otimes \mathbb{Q}(-1)$$

([6], § 4.2.3). Therefore (2.2) gives the relations

$$H^0(C, R^1\pi'_*\mathbb{Q}) = H^0(C, T_{\mathbb{Z}}(Y) \otimes \mathbb{Q}(-1)) = H^0(C, T_{\mathbb{Z}}(Y)) \otimes \mathbb{Q}(-1) = 0.$$

Lemma 2.6 is proved.

2.7. Remark. We may assume that an Abelian scheme $\pi : X \to C$ satisfies the following additional assumption: for any (ramified) covering $C' \to C$ an Abelian scheme $X \times_C C' \to C'$ contains no nontrivial constant Abelian subschemes; in particular,

$$H^0(C, R^1\pi_*\mathbb{Q}) = 0. \tag{2.3}$$

It is well known that for an Abelian scheme $\pi : X \to C$, there exists an isogeny $X' \to X$ over C such that an Abelian scheme $X' \to C$ has a principal polarization. By Theorem 1.6, $B(X') \Rightarrow B(X)$. Therefore we may also assume that $\pi : X \to C$ has a principal polarization.

§ 3. On cohomology of complex Abelian schemes over smooth projective curves

3.1. Let C be a smooth projective curve over the field of complex numbers and let $\pi : X \to C$ be an Abelian scheme of relative dimension d. Since π is a smooth morphism of compact Kähler varieties, the Leray spectral sequence $E_2^{p,q} = H^p(C, R^q\pi_*\mathbb{Q}) \Rightarrow H^*(X, \mathbb{Q})$ degenerates at the E_2-term, so

$$H^i(X, \mathbb{Q}) = \bigoplus_{a+b=i} H^a(C, R^b\pi_*\mathbb{Q}) \tag{3.1}$$

([5]; [11], § 3.5).

Let Θ_n be the endomorphism of X induced by multiplication by n in the fibers of $\pi : X \to C$, for any integer $n > 1$. Since $[\Theta_n]^*$ commute with the d_2 differentials of the Leray spectral sequence $E_2^{p,q} = H^p(C, R^q\pi_*\mathbb{Q}) \Rightarrow H^*(X, \mathbb{Q})$, we see that $H^a(C, R^b\pi_*\mathbb{Q})$ coincides with the n^b-eigenspace $H^{<a,b>}(X, \mathbb{Q})$ of $[\Theta_n]^*$ in $H^r(X, \mathbb{Q})$ ([14], Lemma 2A3).

So we get the well known decomposition of Hodge $\mathbb{Q}$-structures

$$H^i(X, \mathbb{Q}) = \bigoplus_{a+b=i} H^{<a,b>}(X, \mathbb{Q}) = \bigoplus_{a+b=i} H^a(C, R^b\pi_*\mathbb{Q}); \tag{3.2}$$

moreover, by the same reason, the $\cup$-product in $H^*(X, \mathbb{Q})$ has the following property:

$$H^a(C, R^b\pi_*\mathbb{Q}) \cup H^p(C, R^q\pi_*\mathbb{Q}) \hookrightarrow H^{a+p}(C, R^{b+q}\pi_*\mathbb{Q}). \tag{3.3}$$

It is easy to see that $H^a(C, R^b\pi_*\mathbb{Q}) = (0)$ for all $a \geq 3$. Indeed, the l-cohomological dimension $\mathrm{cd}_l(C_{\mathrm{et}}) \leq 2 \cdot \dim C = 2$ ([21], Chapter VI, § 1, Theorem 1.1); therefore $H^a_{\mathrm{et}}(C, R^b\pi_*\mathbb{Z}/l^n\mathbb{Z}) = (0)$ for $a \geq 3$, and it remains to use the comparison theorem for the étale cohomology and classical cohomology ([21], Chapter III, § 3, Theorem 3.12). Consequently, we get the following variant of the decomposition (3.2):

$$H^i(X, \mathbb{Q}) = H^0(C, R^i\pi_*\mathbb{Q}) \bigoplus H^1(C, R^{i-1}\pi_*\mathbb{Q}) \bigoplus H^2(C, R^{i-2}\pi_*\mathbb{Q}). \tag{3.4}$$

3.2. Recall that $R^p\pi_*\mathbb{Q}$ is completely determined by the monodromy representation $\pi_1(C, s) \to \mathrm{GL}(H^p(X_s, \mathbb{Q}))$ ([11], Chapter 3, § 5).

Let $\tau : X \times_C X \to C$ be the structure morphism. Then $R^p\tau_*\mathbb{Q}$ is completely determined by the corresponding monodromy representation

$$\pi_1(C, s) \to \mathrm{GL}(H^p(X_s \times X_s, \mathbb{Q})) = \mathrm{GL}\left(\bigoplus_{a+b=p} H^a(X_s, \mathbb{Q}) \otimes H^b(X_s\mathbb{Q})\right),$$

where $\pi_1(C, s)$ acts on $H^a(X_s, \mathbb{Q}) \otimes H^b(X_s\mathbb{Q})$ via the diagonal embedding $\pi_1(C, s) \hookrightarrow \pi_1(C, s) \times \pi_1(C, s) = \pi_1(C \times C, s \times s)$ ([11], Chapter 3, § 5), and the Künneth decomposition

$$H^p(X_s \times X_s, \mathbb{Q}) = \bigoplus_{a+b=p} H^a(X_s, \mathbb{Q}) \otimes H^b(X_s, \mathbb{Q})$$

is compatible with monodromy representations. Therefore we have the Künneth decomposition

$$R^p\tau_*\mathbb{Q} = \bigoplus_{a+b=p} R^a\pi_*\mathbb{Q} \otimes R^b\pi_*\mathbb{Q}.$$

Since $\tau : X \times_C X \to C$ is an Abelian scheme over a curve C, we have the decompositions

$$H^p(X \times_C X, \mathbb{Q}) =$$
$$H^0(C, R^p\tau_*\mathbb{Q}) \bigoplus H^1(C, R^{p-1}\tau_*\mathbb{Q}) \bigoplus H^2(C, R^{p-2}\tau_*\mathbb{Q}) =$$
$$\left[\bigoplus_{a+b=p} H^0(C, R^a\pi_*\mathbb{Q} \otimes R^b\pi_*\mathbb{Q})\right] \bigoplus \left[\bigoplus_{a+b=p-1} H^1(C, R^a\pi_*\mathbb{Q} \otimes R^b\pi_*\mathbb{Q})\right] \bigoplus$$
$$\left[\bigoplus_{a+b=p-2} H^2(C, R^a\pi_*\mathbb{Q} \otimes R^b\pi_*\mathbb{Q})\right]. \tag{3.5}$$

Similarly,

$$H^p(X \times X, \mathbb{Q}) =$$
$$\bigoplus_{a+b=p} \left[H^a(X, \mathbb{Q}) \bigotimes H^b(X, \mathbb{Q})\right] =$$
$$\bigoplus_{a+b=p} \left[H^0(C, R^a\pi_*\mathbb{Q}) \bigoplus H^1(C, R^{a-1}\pi_*\mathbb{Q}) \bigoplus H^2(C, R^{a-2}\pi_*\mathbb{Q})\right] \bigotimes$$
$$\left[H^0(C, R^b\pi_*\mathbb{Q}) \bigoplus H^1(C, R^{b-1}\pi_*\mathbb{Q}) \bigoplus H^2(C, R^{b-2}\pi_*\mathbb{Q})\right]. \tag{3.6}$$

We denote by $u_{pqrt} \in H^p(C, R^q\pi_*\mathbb{Q}) \bigotimes H^r(C, R^t\pi_*\mathbb{Q})$ the corresponding component of $u \in H^{2i}(X \times X, \mathbb{Q})$ in the decomposition (3.6). In particular,

$$u_{0i0i} \in H^0(C, R^i\pi_*\mathbb{Q}) \bigotimes H^0(C, R^i\pi_*\mathbb{Q}).$$

3.3. Lemma. *If* $u \in H^{2i}_{\text{alg}}(X \times X, \mathbb{Q})$, *then* $u_{0i0i} \in H^{2i}_{\text{alg}}(X \times X, \mathbb{Q})$.

Proof. The following is a modification of the D.Lieberman method ([14], the proof of Theorem 2A11). In virtue of (3.2), we have

$$[\Theta_n \times \mathrm{id}_X]^* u_{pqrt} = n^q \cdot u_{pqrt},$$
$$[\mathrm{id}_X \times \Theta_n]^* u_{pqrt} = n^t \cdot u_{pqrt}. \tag{3.7}$$

For the sake of simplicity assume that $i = 2$. Then

$$u = u_{0004} + u_{0013} + u_{0022} + u_{0103} + u_{0112} + u_{0121}+$$
$$u_{0202} + u_{0211} + u_{0220} + u_{1102} + u_{1111} + u_{1120} + u_{2002} + u_{2011} + u_{2020}+$$
$$u_{0301} + u_{0310} + u_{1201} + u_{1210} + u_{2101} + u_{2110} + u_{0400} + u_{1300} + u_{2200}.$$

Applying $[\Theta_n \times \mathrm{id}_X]^* - [\mathrm{id}_X \times \mathrm{id}_X]^*$ to this algebraic class, we get the relation

$$([\Theta_n \times \mathrm{id}_X]^* - [\mathrm{id}_X \times \mathrm{id}_X]^*)(u) \in H^{2i}_{\text{alg}}(X \times X, \mathbb{Q}).$$

In virtue of (3.7), the latter relation has the form

$$\begin{gathered}(n-1)\cdot u_{0103}+(n-1)\cdot u_{0112}+(n-1)\cdot u_{0121}+(n^2-1)\cdot u_{0202}+\\ (n^2-1)\cdot u_{0211}+(n^2-1)\cdot u_{0220}+(n-1)\cdot u_{1102}+(n-1)\cdot u_{1111}+\\ (n-1)\cdot u_{1120}+(n^3-1)\cdot u_{0301}+(n^3-1)\cdot u_{0310}+(n^2-1)\cdot u_{1201}+\\ (n^2-1)\cdot u_{1210}+(n-1)\cdot u_{2101}+(n-1)\cdot u_{2110}+(n^4-1)\cdot u_{0400}+\\ (n^3-1)\cdot u_{1300}+(n^2-1)\cdot u_{2200}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}$$

Consequently,

$$\begin{gathered}u_{0103}+u_{0112}+u_{0121}+(n+1)\cdot u_{0202}+(n+1)\cdot u_{0211}+(n+1)\cdot u_{0220}+u_{1102}+\\ u_{1111}+u_{1120}+(n^2+n+1)\cdot u_{0301}+(n^2+n+1)\cdot u_{0310}+(n+1)\cdot u_{1201}+\\ (n+1)\cdot u_{1210}+u_{2101}+u_{2110}+(n^3+n^2+n+1)\cdot u_{0400}+\\ (n^2+n+1)\cdot u_{1300}+(n+1)\cdot u_{2200}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}$$

Applying $[\mathrm{id}_X\times\Theta_n]^*-[\mathrm{id}_X\times\mathrm{id}_X]^*$ to the latter algebraic class, we get the relation

$$\begin{gathered}(n^3-1)\cdot u_{0103}+(n^2-1)\cdot u_{0112}+(n-1)\cdot u_{0121}+(n+1)(n^2-1)\cdot u_{0202}+\\ (n+1)(n-1)\cdot u_{0211}+(n^2-1)\cdot u_{1102}+(n-1)\cdot u_{1111}+\\ (n^2+n+1)(n-1)\cdot u_{0301}+(n+1)(n-1)\cdot u_{1201}+\\ (n-1)\cdot u_{2101}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}$$

Hence,

$$\begin{gathered}(n^2+n+1)\cdot u_{0103}+(n+1)\cdot u_{0112}+u_{0121}+\\ (n+1)^2\cdot u_{0202}+(n+1)\cdot u_{0211}+(n+1)\cdot u_{1102}+u_{1111}+\\ (n^2+n+1)\cdot u_{0301}+(n+1)\cdot u_{1201}+u_{2101}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}\tag{3.8}$$

Applying $[\Theta_n\times\mathrm{id}_X]^*-[\mathrm{id}_X\times\mathrm{id}_X]^*$ to this algebraic class, we get the relation

$$\begin{gathered}(n^2+n+1)(n-1)\cdot u_{0103}+(n+1)(n-1)\cdot u_{0112}+(n-1)\cdot u_{0121}+\\ (n+1)^2(n^2-1)\cdot u_{0202}+(n+1)(n^2-1)\cdot u_{0211}+\\ (n+1)(n-1)\cdot u_{1102}+(n-1)\cdot u_{1111}+(n^2+n+1)(n^3-1)\cdot u_{0301}\\ +(n+1)(n^2-1)\cdot u_{1201}+(n-1)\cdot u_{2101}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}$$

Therefore

$$\begin{gathered}(n^2+n+1)\cdot u_{0103}+(n+1)\cdot u_{0112}+u_{0121}+\\ (n+1)^3\cdot u_{0202}+(n+1)^2\cdot u_{0211}+(n+1)\cdot u_{1102}+u_{1111}+\\ (n^2+n+1)^2\cdot u_{0301}+(n+1)^2\cdot u_{1201}+u_{2101}\in H^{2i}_{\mathrm{alg}}(X\times X,\mathbb{Q}).\end{gathered}\tag{3.9}$$

Subtracting (3.8) from (3.9), we get the following relation:

$$(n+1)^2 n \cdot u_{0202} + (n+1)n \cdot u_{0211} + \\ (n^2+n+1)n(n+1) \cdot u_{0301} + (n+1)n \cdot u_{1201} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q}).$$

Consequently,

$$(n+1) \cdot u_{0202} + u_{0211} + (n^2+n+1) \cdot u_{0301} + u_{1201} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q}). \quad (3.10)$$

Applying $[\mathrm{id}_X \times \Theta_n]^* - [\mathrm{id}_X \times \mathrm{id}_X]^*$ to the latter algebraic class, we get the relation

$$(n+1)(n^2-1) \cdot u_{0202} + (n-1) \cdot u_{0211} + \\ (n^2+n+1)(n-1) \cdot u_{0301} + (n-1) \cdot u_{1201} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q}).$$

Therefore

$$(n+1)^2 \cdot u_{0202} + u_{0211} + (n^2+n+1) \cdot u_{0301} + u_{1201} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q}). \quad (3.11)$$

Subtracting (3.10) from (3.11), we arrive to the relation $(n+1)n \cdot u_{0202} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q})$. In particular, $u_{0i0i} \in H^{2i}_{\mathrm{alg}}(X \times X, \mathbb{Q})$. Lemma 3.3 is proved.

§ 4. Poincaré bundles

4.1. Let $\pi : X \to C$ be an Abelian scheme of relative dimension d over a smooth projective curve C, and let $s \in C$. Recall that there exists a Poincaré bundle $\mathcal{P}_s$ on $X_s \times \hat{X}_s$, uniquely determined up to isomorphisms by the following properties:

i) $\mathcal{P}_s|_{X_s \times \{L_s\}} \xrightarrow{\sim} L_s$ for every $L_s \in \hat{X}_s$,

ii) $\mathcal{P}_s|_{\{0\} \times \hat{X}_s}$ is trivial ([18], Chapter 2, § 5).

We claim that $\mathrm{cl}_{X_s \times \hat{X}_s}(\mathcal{P}_s)$ has Künneth type (1,1). Indeed, by the Künneth formula we have

$$\mathrm{cl}_{X_s \times X_s}(\mathcal{P}_s) \in H^2(X_s \times X_s, \mathbb{Q}) = \\ [H^2(X_s, \mathbb{Q}) \otimes H^0(X_s, \mathbb{Q})] \bigoplus [H^0(X_s, \mathbb{Q}) \otimes H^2(X_s, \mathbb{Q})] \bigoplus \\ [H^1(X_s, \mathbb{Q}) \otimes H^1(X_s, \mathbb{Q})].$$

On the other hand, by the properties i), ii) above,

$$\mathcal{P}_s | X_s \times \{0\} \xrightarrow{\sim} \mathcal{O}_{X_s}, \qquad \mathcal{P}_s | \{0\} \times X_s \xrightarrow{\sim} \mathcal{O}_{X_s}.$$

Therefore the (2,0) and (0,2) components of the Künneth decomposition of $\mathrm{cl}_{X_s \times X_s}(\mathcal{P}_s)$ are trivial.

4.2. Let $e : C \to X$ be the zero section, and let $\hat{\pi} : \hat{X} = \mathrm{Pic}^0(X/C) \to C$ be the dual Abelian scheme with a zero section $\hat{e}$. For any invertible sheaf $\mathcal{L}$ on X, a *rigidification* of $\mathcal{L}$ is an isomorphism $\mathcal{O}_C \xrightarrow{\sim} e^*\mathcal{L}$. By definition of $\mathrm{Pic}^0(X/C)$, for any C-scheme T we have:

$$\begin{gathered}\mathrm{Hom}_C(T, \mathrm{Pic}^0(X/C)) = \\ \{ \text{ isomorphism classes of invertible sheaves } \mathcal{L} \text{ on } X \times_C T \text{ with rigidification} \\ \text{along } e_T = e \times_C T \text{ such that } \forall t \in T \quad \mathcal{L}|_{X \times_C t} \text{ is algebraically equivalent} \\ \text{to zero on } X \times_C t \}\end{gathered}$$

([8], Chapter I, § 1, Definition 1.4).

The tautological invertible sheaf $\mathcal{P}$ on $X \times_C \hat{X}$ corresponding to the identity morphism $\mathrm{id}_{\hat{X}} : \hat{X} \to \hat{X} = \mathrm{Pic}^0(X/C)$ is called the Poincaré bundle. By definition, $\mathcal{P}$ is rigidified along $e \times_C \hat{X}$, and $\forall t \in \hat{X} \quad \mathcal{P}|_{X \times_C t}$ represents the element of $\mathrm{Pic}^0(X \times_C t)$ given by t under the identification $\mathrm{Pic}^0(X \times_C t) = \hat{X}_{\overset{\wedge}{\pi}(t)}$.

It is easy to see that $\mathcal{P}$ induces the Poincaré bundle $\mathcal{P}_s$ on $X_s \times \hat{X}_s$ via the canonical embedding $X_s \times \hat{X}_s \hookrightarrow X \times_C \hat{X}$. Indeed, we have the canonical embedding of C-schemes

$$\hat{X}_s \hookrightarrow \hat{X}.$$

It is determined by an invertible sheaf $\mathcal{L} = \mathcal{P}|_{X \times_C \hat{X}_s}$ on $X \times_C \hat{X}_s = X_s \times \hat{X}_s$, and for any $t \in \hat{X}_s$,

$$\mathcal{L}|_{X_s \times_C t} = \mathcal{L}|_{X_s \times t} = \mathcal{P}|_{X \times_C t}$$

represents the element of $\mathrm{Pic}^0(X_s \times t) = \mathrm{Pic}^0(X_s)$ given by t under the identification $\mathrm{Pic}^0(X_s) = \hat{X}_s$. Therefore $\mathcal{L} = \mathcal{P}_s$.

4.3. By Remark 2.7, we may assume that there exists a principal polarization

$$\lambda : X \xrightarrow{\sim} \hat{X},$$

so we may consider λ as an identification $X = \hat{X}$.

§ 5. Correspondences and algebraic isomorphisms of rational Hodge structures

5.1. Recall some basic properties of correspondences ([14], § 1.3; [20]).

Let X, Y be smooth projective varieties over the field of complex numbers. There are canonical $\mathbb{Q}$-linear isomorphisms

$$\begin{aligned} H^*(X \times Y, \mathbb{Q}) &\xrightarrow{\sim} H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Y, \mathbb{Q}) \qquad \text{(Künneth formula)} \\ &\xrightarrow{\sim} H_*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Y, \mathbb{Q}) \qquad \text{(Poincaré duality)} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(H^*(X, \mathbb{Q}), H^*(Y, \mathbb{Q})). \end{aligned}$$

Specifically, an element $u = a \otimes b \in H^*(X \times Y, \mathbb{Q})$ corresponds to the $\mathbb{Q}$-linear map $u^* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ defined by $u^*(c) = \deg(c \cup a) \cdot b$, where $\deg : H^*(X) \to \mathbb{Q}$ is defined by the following formula:

$$\deg = \begin{cases} \text{the orientation isomorphism } \mathrm{tr} : H^{2 \cdot \dim X}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}, \\ 0, \text{ on } H^i(X, \mathbb{Q}) \quad \text{for } i < 2 \cdot \dim X. \end{cases}$$

Again, an arbitrary element $u \in H^*(X \times Y, \mathbb{Q})$ corresponds to the composition

$$u^* : H^*(X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^*(X \times Y, \mathbb{Q}) \xrightarrow{w \to w \cup u} H^*(X \times Y, \mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^*(Y, \mathbb{Q}).$$

5.2. A linear map $u^* : H^i(X, \mathbb{Q}) \to H^j(Y, \mathbb{Q})$ is called *algebraic* if it is the restriction of a map $u^* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ induced by an algebraic class $u \in H^*_{\mathrm{alg}}(X \times Y, \mathbb{Q})$.

Let $V \hookrightarrow H^i(X, \mathbb{Q})$ and $W \hookrightarrow H^j(Y, \mathbb{Q})$ be Hodge $\mathbb{Q}$-substructures. We call a linear map $V \to W$ *algebraic* if it is the restriction of a map $u^* : H^i(X, \mathbb{Q}) \to H^j(Y, \mathbb{Q})$ induced by an algebraic class $u \in H^*_{\mathrm{alg}}(X \times Y, \mathbb{Q})$.

5.3. Composition of linear maps defines a composition of correspondences: if $u \in H^*(X \times Y, \mathbb{Q})$ and $v \in H^*(Y \times Z, \mathbb{Q})$, then

$$v \circ u = \mathrm{pr}_{13*}(u \otimes 1_Z \cup 1_X \otimes v) \in H^*(X \times Z, \mathbb{Q}),$$

where $\mathrm{pr}_{13} : X \times Y \times Z \to X \times Z$ is the projection. If $X = Y = Z$, then the composition $v \circ u$ of algebraic correspondences is again an algebraic correspondence ([14], Proposition 1.3.7). It is defined by an algebraic class

$$v \circ u = \mathrm{pr}_{13*}(u \otimes 1_X \cup 1_X \otimes v) \in H^*(X \times X, \mathbb{Q}).$$

§ 6. Basic construction

6.1. Let $\mu_s : X_s \times X_s \to X_s$ be the group law for X_s (the sum morphism). Recall some results of S.L.Kleiman and D.Lieberman.

6.2. Lemma (S.L.Kleiman) ([14], Lemma 2A12). *Let X_s be an Abelian variety, Y_s an arbitrary smooth projective variety and $u \in H^2(X_s \times Y_s, \mathbb{Q})$ an element of Künneth type $(1,1)$. Then the correspondence*

$$\exp(u) = \sum u^i/i! : H^*(X_s, \mathbb{Q}) \to H^*(Y_s, \mathbb{Q})$$

takes Pontrjagin product

$$\alpha \bigvee \beta = \mu_{s*}\left(\alpha \bigotimes \beta\right)$$

into cup product.

6.3. Lemma (D.Lieberman) ([14], Remark 2A13). *Let X_s be an Abelian variety, $\hat{X}_s$ the dual Abelian variety and $u \in H^2(X_s \times \hat{X}_s, \mathbb{Q})$ the $(1,1)$-Künneth component of a Poincaré divisor. Then the correspondence*

$$\exp(u) = \sum u^i/i! : H^*(X_s, \mathbb{Q}) \to H^*(\hat{X}_s, \mathbb{Q})$$

is an algebraic isomorphism of the Pontrjagin algebra $H^*(X_s, \mathbb{Q})$ with the cup-product algebra $H^*(\hat{X}_s, \mathbb{Q})$.

6.4. In the situation under consideration, $X_s = \hat{X}_s$ and $\mathrm{cl}_{X_s \times X_s}(\mathcal{P}_s)$ has Künneth type $(1,1)$, so (by Lemma 6.3) for any natural number i, the map

$$H^{2d-i}(X_s, \mathbb{Q}) \to H^i(X_s, \mathbb{Q})$$

defined by

$$[\mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)]^{\cup i}=\mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)\cup\cdots\cup\mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)\in H^{2i}(X_s\times X_s,\mathbb{Q}),$$

is an algebraic isomorphism. More precisely, this isomorphism is a composition of the following isomorphisms:

$$H^{2d-i}(X_s,\mathbb{Q})\overset{\mathrm{pr}_{1s}^*}{\widetilde{\longrightarrow}}H^{2d-i}(X_s,\mathbb{Q})\otimes H^0(X_s,\mathbb{Q})\xrightarrow[\sim]{\cup[\mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)]^{\cup i}}$$
$$H^{2d}(X_s,\mathbb{Q})\otimes H^i(X_s,\mathbb{Q})\overset{\mathrm{pr}_{2s*}}{\widetilde{\longrightarrow}}H^i(X_s,\mathbb{Q}),\tag{6.1}$$

where $\mathrm{pr}_{ks}:X_s\times X_s\to X_s$ is the canonical projection.

6.5. Let $i_s:X_s\hookrightarrow X$ be the canonical embedding. Then we have canonical embeddings

$$X_s\times X_s\overset{i_s\times i_s}{\hookrightarrow}X\times_C X\overset{j}{\hookrightarrow}X\times X.$$

By the Deligne theorem ([6], 4.1.2), the canonical restriction map

$$H^2(X\times_C X,\mathbb{Q})\to H^2(X_s\times X_s,\mathbb{Q})$$

is a composition of the canonical surjective edge morphism (of rational Hodge structures)

$$H^2(X\times_C X,\mathbb{Q})\to H^0(C,R^2\tau_*\mathbb{Q})$$

associated to the Leray spectral sequence $E_2^{p,q}=H^p(C,R^q\tau_*\mathbb{Q})\Rightarrow H^*(X\times_C X,\mathbb{Q})$ and the canonical maps

$$H^0(C,R^2\tau_*\mathbb{Q})\widetilde{\to}H^2(X_s\times X_s,\mathbb{Q})^{\pi_1(C,s)}\hookrightarrow H^2(X_s\times X_s,\mathbb{Q}).$$

It is evident that $\mathrm{cl}_{X\times_C X}(\mathcal{P})\in H^2(X\times_C X,\mathbb{Q})$ maps to an element $v_s=\mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)\in H^2(X_s\times X_s,\mathbb{Q})^{\pi_1(C,s)}$ of Künneth type (1,1) because

$$[i_s\times i_s]^*\mathrm{cl}_{X\times_C X}(\mathcal{P})=\mathrm{cl}_{X_s\times X_s}([i_s\times i_s]^*(\mathcal{P}))$$

([14], 1.2.C) and $[i_s\times i_s]^*(\mathcal{P})=\mathcal{P}_s$. Therefore

$$\mathrm{cl}_{X\times_C X}(\mathcal{P})\in$$
$$H^0(C,R^1\pi_*\mathbb{Q}\otimes R^1\pi_*\mathbb{Q})\bigoplus\left[\bigoplus_{a+b=1}H^1(C,R^a\pi_*\mathbb{Q}\otimes R^b\pi_*\mathbb{Q})\right]\bigoplus H^2(C,\mathbb{Q}).$$

In virtue of the Lefschetz theorem, the $H^0(C,R^2\tau_*\mathbb{Q})$-component v of an algebraic class $\mathrm{cl}_{X\times_C X}(\mathcal{P})$ is again algebraic. On the other hand, $v\in H^0(C,R^1\pi_*\mathbb{Q}\otimes R^1\pi_*\mathbb{Q})$.

6.6. We claim that the algebraic class

$$v^i=v\cup v\cup\cdots\cup v\in H^0(C,R^i\pi_*\mathbb{Q}\otimes R^i\pi_*\mathbb{Q})\hookrightarrow H^{2i}(X\times_C X,\mathbb{Q})$$

determines a canonical isomorphism of local systems

$$\nu^i:R^{2d-i}\pi_*\mathbb{Q}\widetilde{\to}R^i\pi_*\mathbb{Q}$$

over C. Indeed, let ν^i be the composition

$$R^{2d-i}\pi_*\mathbb{Q} \overset{p_1^*}{\widetilde{\longrightarrow}} R^{2d-i}\pi_*\mathbb{Q}\otimes R^0\pi_*\mathbb{Q} \overset{\cup v^i}{\widetilde{\longrightarrow}} R^{2d}\pi_*\mathbb{Q}\otimes R^i\pi_*\mathbb{Q} \overset{p_{2*}}{\widetilde{\longrightarrow}} R^i\pi_*\mathbb{Q},$$

where p_1, p_2 are canonical projections $X\times_C X\to X$; in terms of monodromy representations of $\pi_1(C,s)$ we have:

$$v^i\in H^0(C,R^i\pi_*\mathbb{Q}\otimes R^i\pi_*\mathbb{Q}) \widetilde{\to} \left[H^i(X_s,\mathbb{Q})\otimes H^i(X_s,\mathbb{Q})\right]^{\pi_1(C,s)},$$

and ν^i is the composition of isomorphisms of $\pi_1(C,s)$-modules

$$H^{2d-i}(X_s,\mathbb{Q}) \overset{\mathrm{pr}_{1s}^*}{\widetilde{\longrightarrow}} H^{2d-i}(X_s,\mathbb{Q})\otimes H^0(X_s,\mathbb{Q}) \overset{\cup v_s^i}{\widetilde{\longrightarrow}}$$
$$H^{2d}(X_s,\mathbb{Q})\otimes H^i(X_s,\mathbb{Q}) \overset{\mathrm{pr}_{2s*}}{\widetilde{\longrightarrow}} H^i(X_s,\mathbb{Q}).$$

These sequences induce the following sequence of isomorphisms:

$$H^k(C,R^{2d-i}\pi_*\mathbb{Q}) \overset{p_1^*}{\widetilde{\longrightarrow}} H^k(C,R^{2d-i}\pi_*\mathbb{Q}\otimes R^0\pi_*\mathbb{Q}) \overset{\cup v^i}{\widetilde{\longrightarrow}}$$
$$H^k(C,R^{2d}\pi_*\mathbb{Q}\otimes R^i\pi_*\mathbb{Q}) \overset{p_{2*}}{\widetilde{\longrightarrow}} H^k(C,R^i\pi_*\mathbb{Q}).$$

It is evident that $p_k=\mathrm{pr}_k\circ j$, therefore the latter sequence may be rewritten as

$$H^k(C,R^{2d-i}\pi_*\mathbb{Q}) \overset{j^*\mathrm{pr}_1^*}{\widetilde{\longrightarrow}} H^k(C,R^{2d-i}\pi_*\mathbb{Q}\otimes R^0\pi_*\mathbb{Q}) \overset{\cup v^i}{\widetilde{\longrightarrow}}$$
$$H^k(C,R^{2d}\pi_*\mathbb{Q}\otimes R^i\pi_*\mathbb{Q}) \overset{\mathrm{pr}_{2*}j_*}{\widetilde{\longrightarrow}} H^k(C,R^i\pi_*\mathbb{Q}).$$

Let $x\in H^k(C,R^{2d-i}\pi_*\mathbb{Q})$. Then the projection formula gives the relation

$$\mathrm{pr}_{2*}j_*(j^*\mathrm{pr}_1^*(x)\cup v^i)=\mathrm{pr}_{2*}(\mathrm{pr}_1^*(x)\cup j_*v^i).$$

In particular, an algebraic class $j_*v^i\in H^{2i+2}(X\times X,\mathbb{Q})$ determines an algebraic isomorphism

$$H^k(C,R^{2d-i}\pi_*\mathbb{Q}) \overset{[j_*v^i]^*}{- - -\widetilde{\to}} H^k(C,R^i\pi_*\mathbb{Q}). \tag{6.2}$$

6.7. Consider the canonical maps

$$\alpha: H^1(\mathrm{Alb}(X),\mathbb{Q})\to H^1(X,\mathbb{Q}),$$
$$\beta: H^{2d+1}(X,\mathbb{Q})\to H^1(\mathrm{Pic}^0(X),\mathbb{Q})$$

from ([14], 2A1), where β is induced by a Poincare divisor D on $X\times\mathrm{Pic}^0(X)$. It is known that α is an injection and β is a surjection ([14], Theorem 2A9(1)). Since we are in characteristic zero, both maps are isomorphisms. Moreover, there exists a divisor E on $X\times X$, which induces the multiple of the map $\Lambda^d: H^{2d+1}(X,\mathbb{Q})\to$

$H^1(X,\mathbb{Q})$ inverse to L^d ([14], Theorem 2A9(5)). In particular, Λ^d is algebraic, so the conjecture $B^1(X)$ holds.

Another proof of $B^1(X)$: in virtue of Lemma 2.6, we may assume that $H^0(C,R^1\pi_*\mathbb{Q})=0$; in this case,

$$H^1(X,\mathbb{Q}) = H^1(C,\mathbb{Q}),$$
$$H^{2d+1}(X,\mathbb{Q}) = H^1(C,R^{2d}\pi_*\mathbb{Q}) \bigoplus H^2(C,R^{2d-1}\pi_*\mathbb{Q}) = H^1(C,R^{2d}\pi_*\mathbb{Q})$$

because $R^{2d}\pi_*\mathbb{Q} \widetilde{\rightarrow} \mathbb{Q}$; in virtue of (6.2), an algebraic class $j_*1 \in H^2(X\times X,\mathbb{Q})$ determines an algebraic isomorphism $H^1(C,R^{2d}\pi_*\mathbb{Q}) \widetilde{\rightarrow} H^1(C,\mathbb{Q})$, and the conjecture $B^1(X)$ holds.

It is evident that for a generator γ of the 1-dimensional space $H^0(X,\mathbb{Q}) = H^0_{\mathrm{alg}}(X,\mathbb{Q})$, the canonical sequence

$$H^{2d+2}(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^{2d+2}(X,\mathbb{Q})\otimes H^0(X,\mathbb{Q}) \xrightarrow{\cup\gamma}$$
$$H^{2d+2}(X,\mathbb{Q})\otimes H^0(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^0(X,\mathbb{Q})$$

determines an algebraic isomorphism

$$H^{2d+2}(X,\mathbb{Q}) \widetilde{\rightarrow} H^0(X,\mathbb{Q}).$$

In particular, $B^0(X)$ holds.

So we may assume that $i\ge 2$.

In this case, algebraic isomorphisms (6.2) determine the following diagram:

$$\begin{array}{ccccc} H^0(C,R^{2d+2-i}\pi_*\mathbb{Q}) & \bigoplus & H^1(C,R^{2d+1-i}\pi_*\mathbb{Q}) & \bigoplus & H^2(C,R^{2d-i}\pi_*\mathbb{Q}) \\ \simeq\downarrow{\scriptstyle [j_*v^{i-2}]^*} & & \simeq\downarrow{\scriptstyle [j_*v^{i-1}]^*} & & \simeq\downarrow{\scriptstyle [j_*v^{i}]^*} \\ H^0(C,R^{i-2}\pi_*\mathbb{Q}) & \bigoplus & H^1(C,R^{i-1}\pi_*\mathbb{Q}) & \bigoplus & H^2(C,R^{i}\pi_*\mathbb{Q}), \end{array} \tag{6.3}$$

where all vertical arrows are algebraic isomorphisms.

Comparing the rows of the diagram with the decompositions

$$H^{2d+2-i}(X,\mathbb{Q}) =$$
$$H^0(C,R^{2d+2-i}\pi_*\mathbb{Q}) \bigoplus H^1(C,R^{2d+1-i}\pi_*\mathbb{Q}) \bigoplus H^2(C,R^{2d-i}\pi_*\mathbb{Q}),$$
$$H^i(X,\mathbb{Q}) = H^2(C,R^{i-2}\pi_*\mathbb{Q}) \bigoplus H^1(C,R^{i-1}\pi_*\mathbb{Q}) \bigoplus H^0(C,R^{i}\pi_*\mathbb{Q}),$$

we see that the middle vertical arrow is good, but the left and the right vertical arrows of the diagram have to be corrected in order to obtain an algebraic isomorphism $H^{2d+2-i}(X,\mathbb{Q}) \widetilde{\rightarrow} H^i(X,\mathbb{Q})$.

§ 7. Correction of the left vertical arrow

7.1. Let H be a hyperplane section of X, and let $h\in H^0(C,R^2\pi_*\mathbb{Q})$ be the component of

$$\mathrm{cl}_X(H) \in H^2(X,\mathbb{Q}) = H^0(C,R^2\pi_*\mathbb{Q}) \bigoplus H^1(C,R^1\pi_*\mathbb{Q}) \bigoplus H^2(C,\mathbb{Q}).$$

The Lefschetz theorem shows that h is an algebraic class, inducing the polarization on each fiber X_s. By the strong Lefschetz theorem, the canonical map $H^i(X_s,\mathbb{Q}) \xrightarrow{\cup h^{d-i}} H^{2d-i}(X_s,\mathbb{Q})$ is an isomorphism of $\pi_1(C,s)$-modules ($i \le d$). Therefore h^{d-i} determines an isomorphism

$$H^0(C,R^i\pi_*\mathbb{Q}) = H^i(X_s,\mathbb{Q})^{\pi_1(C,s)} \xrightarrow[\sim]{\cup h^{d-i}}$$
$$H^{2d-i}(X_s,\mathbb{Q})^{\pi_1(C,s)} = H^0(C,R^{2d-i}\pi_*\mathbb{Q}),$$

and this isomorphism is induced by an algebraic class $\Delta_* h^{d-i}$, where $\Delta : X \hookrightarrow X\times X$ is the diagonal embedding ([14], Proposition 1.3.5). We also have an equality

$$\dim H^0(C,R^i\pi_*\mathbb{Q}) = \dim H^0(C,R^{2d-i}\pi_*\mathbb{Q}). \tag{7.1}$$

7.2. We have a canonical nondegenerate pairing of $\pi_1(C,s)$-modules

$$H^{2d-i}(X_s,\mathbb{Q}) \times H^i(X_s,\mathbb{Q}) \to H^{2d}(X_s,\mathbb{Q}) \xrightarrow[\sim]{\mathrm{tr}_{X_s}} \mathbb{Q}.$$

It gives an identification of $\pi_1(C,s)$-modules

$$H^i(X_s,\mathbb{Q})^\wedge = H^{2d-i}(X_s,\mathbb{Q}),$$

so we have an identification

$$[H^i(X_s,\mathbb{Q})^\wedge]^{\pi_1(C,s)} = H^{2d-i}(X_s,\mathbb{Q})^{\pi_1(C,s)}.$$

In particular,

$$H^0(C,R^i\pi_*\mathbb{Q})^\wedge = [H^i(X_s,\mathbb{Q})^{\pi_1(C,s)}]^\wedge =$$
$$[H^i(X_s,\mathbb{Q})^\wedge]^{\pi_1(C,s)} = H^{2d-i}(X_s,\mathbb{Q})^{\pi_1(C,s)} = H^0(C,R^{2d-i}\pi_*\mathbb{Q}),$$

and the canonical pairing

$$H^0(C,R^{2d-i}\pi_*\mathbb{Q}) \times H^0(C,R^i\pi_*\mathbb{Q}) \to H^0(C,R^{2d}\pi_*\mathbb{Q})$$

is nondegenerate: for any element $0 \neq y \in H^0(C,R^i\pi_*\mathbb{Q})$ there exists an element $x \in H^0(C,R^{2d-i}\pi_*\mathbb{Q})$ such that

$$0 \neq x \cup y \in H^0(C,R^{2d}\pi_*\mathbb{Q});$$

let c be the image of $\mathrm{cl}_X(e(C)) \in H^{2d}(X,\mathbb{Q})$ under the canonical morphism

$$H^{2d}(X,\mathbb{Q}) \to H^0(C,R^{2d}\pi_*\mathbb{Q});$$

we have

$$\mathrm{cl}_X(e(C)) = c + c_1 + c_2,$$

where $c_j \in H^j(C, R^{2d-j}\pi_*\mathbb{Q})$ $(j = 1, 2)$; since $\mathrm{cl}_X(X_s) \in H^2(C, \mathbb{Q})$ is a generator of 1-dimensional space $H^2(C, \mathbb{Q})$, we see that

$$\mathrm{cl}_X(X_s) \cup c_j = 0 \ \ (j = 1, 2),$$

and

$$\mathrm{cl}_X(X_s) \cup c = \mathrm{cl}_X(X_s) \cup \mathrm{cl}_X(e(C)) = \mathrm{cl}_X(X_s \cap e(C)) = \mathrm{cl}_X(e(s)) \neq 0;$$

in particular,

$$0 \neq \mathrm{cl}_X(X_s) \cup H^0(C, R^{2d}\pi_*\mathbb{Q});$$

since the latter space is contained in 1-dimensional space $H^2(C, R^{2d}\pi_*\mathbb{Q})$, we get the equalities

$$H^2(C, R^{2d}\pi_*\mathbb{Q}) = \mathrm{cl}_X(X_s) \cup H^0(C, R^{2d}\pi_*\mathbb{Q}) = H^2(C, \mathbb{Q}) \cup H^0(C, R^{2d}\pi_*\mathbb{Q});$$

consequently, the canonical morphism

$$H^0(C, R^{2d}\pi_*\mathbb{Q}) \ \xrightarrow{\cup \mathrm{cl}_X(X_s)} \ H^2(C, R^{2d}\pi_*\mathbb{Q})$$

is bijective, and

$$0 \neq \mathrm{cl}_X(X_s) \cup x \cup y \in H^2(C, R^{2d}\pi_*\mathbb{Q});$$

in particular,

$$0 \neq \mathrm{cl}_X(X_s) \cup y \in H^2(C, R^i\pi_*\mathbb{Q})$$

for any $0 \neq y \in H^0(C, R^i\pi_*\mathbb{Q})$; so the canonical morphism

$$H^0(C, R^i\pi_*\mathbb{Q}) \ \xrightarrow{\cup \mathrm{cl}_X(X_s)} \ H^2(C, R^i\pi_*\mathbb{Q}) \tag{7.2}$$

is injective.

Note that elements of $H^2(C, R^i\pi_*\mathbb{Q})$ annihilate (by the $\cup$-product) the components $H^1(C, R^{2d-i-1}\pi_*\mathbb{Q})$ and $H^2(C, R^{2d-i-2}\pi_*\mathbb{Q})$ of the space

$$H^{2d-i}(X, \mathbb{Q}) = \\ H^0(C, R^{2d-i}\pi_*\mathbb{Q}) \bigoplus H^1(C, R^{2d-i-1}\pi_*\mathbb{Q}) \bigoplus H^2(C, R^{2d-i-2}\pi_*\mathbb{Q}),$$

so the nondegeneracy of the canonical pairing

$$H^{i+2}(X, \mathbb{Q}) \times H^{2d-i}(X, \mathbb{Q}) \to H^{2d+2}(X, \mathbb{Q}) \overset{\mathrm{tr}}{\tilde{\to}} \mathbb{Q}$$

(the Poincaré duality theorem) implies the nondegeneracy of the canonical pairing

$$H^2(C, R^i\pi_*\mathbb{Q}) \times H^0(C, R^{2d-i}\pi_*\mathbb{Q}) \to H^2(C, R^{2d}\pi_*\mathbb{Q}) \overset{\mathrm{tr}}{\tilde{\to}} \mathbb{Q}.$$

In particular,

$$H^2(C, R^i\pi_*\mathbb{Q})^\wedge \tilde{\to} H^0(C, R^{2d-i}\pi_*\mathbb{Q}).$$

Therefore, by (7.1),

$$\dim H^2(C, R^i\pi_*\mathbb{Q}) = \dim H^2(C, R^i\pi_*\mathbb{Q})^\wedge =$$
$$\dim H^0(C, R^{2d-i}\pi_*\mathbb{Q}) = \dim H^0(C, R^i\pi_*\mathbb{Q}),$$

and an injective morphism (7.2) is bijective:

$$H^0(C, R^i\pi_*\mathbb{Q}) \overset{\cup\,\mathrm{cl}_X(X_s)}{\dashrightarrow\!\!\widetilde{}} H^2(C, R^i\pi_*\mathbb{Q}). \tag{7.3}$$

7.3. In the situation under consideration,

$$H^2(C, R^{i-2}\pi_*\mathbb{Q}) = \mathrm{cl}_X(X_s) \cup H^0(C, R^{i-2}\pi_*\mathbb{Q}) = H^2(C, \mathbb{Q}) \cup H^0(C, R^{i-2}\pi_*\mathbb{Q}),$$

and the isomorphism

$$H^0(C, R^{i-2}\pi_*\mathbb{Q}) \overset{\cup\,\mathrm{cl}_X(X_s)}{\dashrightarrow\!\!\widetilde{}} H^2(C, R^{i-2}\pi_*\mathbb{Q})$$

is induced by an algebraic class $\Delta_*\mathrm{cl}_X(X_s)$, where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding ([14], Proposition 1.3.5).

The composition of the algebraic isomorphisms

$$\varphi_i : H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) \xrightarrow{[j_*v^{i-2}]^*} H^0(C, R^{i-2}\pi_*\mathbb{Q}) \xrightarrow{[\Delta_*\mathrm{cl}_X(X_s)]^*}$$
$$H^2(C, R^{i-2}\pi_*\mathbb{Q})$$

is an algebraic isomorphism, defined by an algebraic class

$$[\Delta_*\mathrm{cl}_X(X_s)] \circ [j_*v^{i-2}] =$$
$$\mathrm{pr}_{13*}([j_*v^{i-2}] \otimes 1_X \,\cup\, 1_X \otimes [\Delta_*\mathrm{cl}_X(X_s)]) \in H^{2i}(X \times X, \mathbb{Q}).$$

Indeed, it is evident that

$$j_*v^{i-2} \in H^{2i-2}(X \times X, \mathbb{Q}),$$
$$\Delta_*\mathrm{cl}_X(X_s) \in H^{2+2\cdot\dim X}(X \times X, \mathbb{Q}),$$
$$[j_*v^{i-2}] \otimes 1_X \,\cup\, 1_X \otimes [\Delta_*\mathrm{cl}_X(X_s)] \in H^{2i+2\cdot\dim X}(X \times X \times X, \mathbb{Q}),$$
$$\mathrm{pr}_{13*}([j_*v^{i-2}] \otimes 1_X \,\cup\, 1_X \otimes [\Delta_*\mathrm{cl}_X(X_s)]) \in H^{2i}(X \times X, \mathbb{Q}).$$

We transform the diagram (6.3) into the new diagram

$$\begin{array}{ccccc} H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) & \bigoplus & H^1(C, R^{2d+1-i}\pi_*\mathbb{Q}) & \bigoplus & H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \\ \simeq \downarrow{\scriptstyle\varphi_i} & & \simeq \downarrow{\scriptstyle [j_*v^{i-1}]^*} & & \simeq \downarrow{\scriptstyle [j_*v^i]^*} \\ H^2(C, R^{i-2}\pi_*\mathbb{Q}) & \bigoplus & H^1(C, R^{i-1}\pi_*\mathbb{Q}) & \bigoplus & H^2(C, R^i\pi_*\mathbb{Q}), \end{array} \tag{7.4}$$

where all vertical arrows are algebraic isomorphisms.

It remains to correct the right vertical arrow in (7.4).

§ 8. On the components of j_*v^i

8.1. Lemma. *We have*

$$j_*v^i \in [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^2(C, R^i\pi_*\mathbb{Q})] \bigoplus [H^2(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})]$$
$$\bigoplus [H^1(C, R^i\pi_*\mathbb{Q}) \otimes H^1(C, R^i\pi_*\mathbb{Q})].$$

Proof. Let Θ_n be the endomorphism of X induced by multiplication by n in the fibers of $\pi : X \to C$, for any integer $n > 1$. It is well known that $H^a(C, R^b\pi_*\mathbb{Q})$ coincides with the n^b-eigenspace $H^{<a,b>}(X, \mathbb{Q})$ of Θ_n^* in $H^r(X, \mathbb{Q})$ ([14], Lemma 2A3).

First of all note that the map $\Theta_n : X \to X$ is a morphism of schemes over C, and so it determines the C-morphism $\Theta_n \times_C \mathrm{id}_X : X \times_C X \to X \times_C X$ and the $C \times C$-morphism $\Theta_n \times \mathrm{id}_X : X \times X \to X \times X$. Moreover,

$$j \circ [\Theta_n \times_C \mathrm{id}_X] = [\Theta_n \times \mathrm{id}_X] \circ j.$$

Therefore

$$j_*[\Theta_n \times_C \mathrm{id}_X]_* v^i = [\Theta_n \times \mathrm{id}_X]_* j_* v^i.$$

Since $v^i \in H^0(C, R^i\pi_*\mathbb{Q} \otimes R^i\pi_*\mathbb{Q})$, we have

$$[\Theta_n \times_C \mathrm{id}_X]^* v^i = n^i \cdot v^i.$$

On the other hand, $\deg [\Theta_n \times_C \mathrm{id}_X] = n^{2d}$. Therefore the projection formula

$$[\Theta_n \times_C \mathrm{id}_X]_*[\Theta_n \times_C \mathrm{id}_X]^* = \deg [\Theta_n \times_C \mathrm{id}_X]$$

gives the relations

$$n^{2d} \cdot v^i = [\Theta_n \times_C \mathrm{id}_X]_*[\Theta_n \times_C \mathrm{id}_X]^* v^i =$$
$$[\Theta_n \times_C \mathrm{id}_X]_*(n^i \cdot v^i) = n^i \cdot [\Theta_n \times_C \mathrm{id}_X]_* v^i.$$

We see that

$$[\Theta_n \times_C \mathrm{id}_X]_* v^i = n^{2d-i} \cdot v^i.$$

Consequently,

$$n^{2d-i} \cdot j_*v^i = j_*(n^{2d-i} \cdot v^i) = j_*[\Theta_n \times_C \mathrm{id}_X]_* v^i = [\Theta_n \times \mathrm{id}_X]_* j_* v^i.$$

Let $u = j_*v^i$. We denote by $u_{pqrt} \in H^p(C, R^q\pi_*\mathbb{Q}) \otimes H^r(C, R^t\pi_*\mathbb{Q})$ the corresponding component of u. In virtue of (3.7),

$$[\Theta_n \times \mathrm{id}_X]^* u_{pqrt} = n^q \cdot u_{pqrt}.$$

On the other hand, $\deg [\Theta_n \times \mathrm{id}_X] = n^{2d}$, and the projection formula

$$[\Theta_n \times \mathrm{id}_X]_*[\Theta_n \times \mathrm{id}_X]^* = \deg [\Theta_n \times \mathrm{id}_X]$$

gives the relations

$$n^{2d}\cdot u_{pqrt}=[\Theta_n\times \mathrm{id}_X]_*[\Theta_n\times \mathrm{id}_X]^*u_{pqrt}=$$
$$[\Theta_n\times \mathrm{id}_X]_*(n^q\cdot u_{pqrt})=n^q\cdot[\Theta_n\times \mathrm{id}_X]_*u_{pqrt}.$$

Therefore

$$[\Theta_n\times \mathrm{id}_X]_*u_{pqrt}=n^{2d-q}\cdot u_{pqrt}.$$

Since

$$\sum n^{2d-q}\cdot u_{pqrt}=[\Theta_n\times \mathrm{id}_X]_*\sum u_{pqrt}=$$
$$[\Theta_n\times \mathrm{id}_X]_*j_*v^i=n^{2d-i}\cdot j_*v^i=n^{2d-i}\cdot\sum u_{pqrt},$$

one has the equality $q=i$.

Similarly, using $\mathrm{id}_X\times\Theta_n$ instead of $\Theta_n\times \mathrm{id}_X$, we arrive to the relation $t=i$.

We see that $j_*v^i\in H^{2i+2}(X\times X,\mathbb{Q})$ belongs to

$$[H^0(C,R^i\pi_*\mathbb{Q})\otimes H^2(C,R^i\pi_*\mathbb{Q})]\bigoplus[H^2(C,R^i\pi_*\mathbb{Q})\otimes H^0(C,R^i\pi_*\mathbb{Q})]$$
$$\bigoplus[H^1(C,R^i\pi_*\mathbb{Q})\otimes H^1(C,R^i\pi_*\mathbb{Q})].$$

Lemma 8.1 is proved.

8.2. Corollary. *For $i=0$ we have:*

$$j_*1=\mathrm{cl}_{X\times X}(X\times_C X)\in$$
$$[H^0(C,\mathbb{Q})\otimes H^2(C,\mathbb{Q})]\bigoplus[H^2(C,\mathbb{Q})\otimes H^0(C,\mathbb{Q})]\bigoplus[H^1(C,\mathbb{Q})\otimes H^1(C,\mathbb{Q})].$$

8.3. In virtue of Lemma 8.1, $j_*v^{i-1}\in H^{2i}(X\times X,\mathbb{Q})$ belongs to

$$[H^0(C,R^{i-1}\pi_*\mathbb{Q})\otimes H^2(C,R^{i-1}\pi_*\mathbb{Q})]\bigoplus[H^2(C,R^{i-1}\pi_*\mathbb{Q})\otimes H^0(C,R^{i-1}\pi_*\mathbb{Q})]$$
$$\bigoplus[H^1(C,R^{i-1}\pi_*\mathbb{Q})\otimes H^1(C,R^{i-1}\pi_*\mathbb{Q})].$$

The latter algebraic class defines an algebraic isomorphism

$$H^1(C,R^{2d+1-i}\pi_*\mathbb{Q})\ \overset{[j_*v^{i-1}]^*}{-\,-\,-\widetilde{\to}}\ H^1(C,R^{i-1}\pi_*\mathbb{Q}),$$

which is the composition

$$H^1(C,R^{2d+1-i}\pi_*\mathbb{Q})\overset{\mathrm{pr}_1^*}{\widetilde{\to}}H^1(C,R^{2d+1-i}\pi_*\mathbb{Q})\otimes H^0(C,\mathbb{Q})\ \overset{\cup j_*v^{i-1}}{-\,-\,-\to}$$
$$[H^1(C,R^{2d}\pi_*\mathbb{Q})\otimes H^2(C,R^{i-1}\pi_*\mathbb{Q})]\bigoplus[H^2(C,R^{2d}\pi_*\mathbb{Q})\otimes H^1(C,R^{i-1}\pi_*\mathbb{Q})]$$
$$\overset{\mathrm{pr}_{2*}}{\to}\ H^1(C,R^{i-1}\pi_*\mathbb{Q}).$$

Finally, $j_*v^{i-2}\in H^{2i-2}(X\times X,\mathbb{Q})$ belongs to

$$[H^0(C,R^{i-2}\pi_*\mathbb{Q})\otimes H^2(C,R^{i-2}\pi_*\mathbb{Q})]\bigoplus[H^2(C,R^{i-2}\pi_*\mathbb{Q})\otimes H^0(C,R^{i-2}\pi_*\mathbb{Q})]$$
$$\bigoplus[H^1(C,R^{i-2}\pi_*\mathbb{Q})\otimes H^1(C,R^{i-2}\pi_*\mathbb{Q})].$$

This algebraic class defines an algebraic isomorphism

$$H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) \overset{[j_*v^{i-2}]^*}{\dashrightarrow} H^0(C, R^{i-2}\pi_*\mathbb{Q}),$$

which is the composition

$$H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) \overset{\mathrm{pr}_1^*}{\widetilde{\rightarrow}} H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) \otimes H^0(C,\mathbb{Q}) \overset{\cup j_*v^{i-2}}{\dashrightarrow}$$
$$[H^0(C, R^{2d}\pi_*\mathbb{Q}) \otimes H^2(C, R^{i-2}\pi_*\mathbb{Q})] \bigoplus [H^2(C, R^{2d}\pi_*\mathbb{Q}) \otimes H^0(C, R^{i-2}\pi_*\mathbb{Q})]$$
$$\bigoplus [H^1(C, R^{2d}\pi_*\mathbb{Q}) \otimes H^1(C, R^{i-2}\pi_*\mathbb{Q})] \overset{\mathrm{pr}_{2*}}{\rightarrow} H^0(C, R^{i-2}\pi_*\mathbb{Q}).$$

§ 9. Restriction of line bundles

9.1. Lemma. *If $\pi : X \to C$ has no nontrivial constant Abelian subschemes, then the restriction morphism*

$$j^* : \mathrm{Pic}^0(X \times X) \to \mathrm{Pic}^0(X \times_C X)$$

of the Picard varieties is surjective.

Proof. Recall that $H^1(X \times X, \mathcal{O}_{X\times X})$ is a tangent space at the origin of the Picard variety $\mathrm{Pic}^0(X \times X)$, and $H^1(X \times_C X, \mathcal{O}_{X\times_C X})$ is a tangent space at the origin of the Picard variety $\mathrm{Pic}^0(X \times_C X)$. By the Hodge theory,

$$H^1(X \times X, \mathbb{Q}) \otimes \mathbb{C} = H^1(X \times X, \mathcal{O}_{X\times X}) \bigoplus \overline{H^1(X \times X, \mathcal{O}_{X\times X})};$$
$$H^1(X \times_C X, \mathbb{Q}) \otimes \mathbb{C} = H^1(X \times_C X, \mathcal{O}_{X\times_C X}) \bigoplus \overline{H^1(X \times_C X, \mathcal{O}_{X\times_C X})}.$$

Consequently, it remains to check the surjectivity of the restriction map

$$j^* : H^1(X \times X, \mathbb{Q}) \to H^1(X \times_C X, \mathbb{Q}).$$

The Leray spectral sequence

$$E_2^{p,q} = H^p(C \times C, R^q(\pi\times\pi)_*\mathbb{Q}) \Rightarrow H^*(X \times X, \mathbb{Q})$$

induces a canonical exact sequence $0 \to E_2^{1,0} \to E^1 \to E_2^{0,1}$ ([21], Appendix B). It has the form

$$0 \to H^1(C \times C, \mathbb{Q}) \to H^1(X \times X, \mathbb{Q}) \to H^0(C \times C, R^1(\pi\times\pi)_*\mathbb{Q}).$$

Similarly we arrive to the exact sequence

$$0 \to H^1(C, \mathbb{Q}) \to H^1(X \times_C X, \mathbb{Q}) \to H^0(C, R^1\tau_*\mathbb{Q}),$$

where $\tau : X \times_C X \to C$ is the structure morphism.

In virtue of Lemma 2.6, we have the following relations:

$$H^0(C \times C, R^1(\pi\times\pi)_*\mathbb{Q}) = H^0(C, R^1\pi_*\mathbb{Q}) \bigoplus H^0(C, R^1\pi_*\mathbb{Q}) = (0);$$
$$H^0(C, R^1\tau_*\mathbb{Q}) = H^0(C, R^1\pi_*\mathbb{Q} \otimes \pi_*\mathbb{Q}) \bigoplus H^0(C, \pi_*\mathbb{Q} \otimes R^1\pi_*\mathbb{Q}) =$$
$$H^0(C, R^1\pi_*\mathbb{Q}) \bigoplus H^0(C, R^1\pi_*\mathbb{Q}) = (0).$$

Let $\Delta_C : C \to C \times C$ be the diagonal embedding. The commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & H^1(C \times C, \mathbb{Q}) & \to & H^1(X \times X, \mathbb{Q}) & \to & 0 \\ & & \downarrow^{[\Delta_C]^*} & & \downarrow^{j^*} & & \\ 0 & \to & H^1(C, \mathbb{Q}) & \to & H^1(X \times_C X, \mathbb{Q}) & \to & 0 \end{array}$$

shows that it remains to prove the surjectivity of the canonical morphism

$$[\Delta_C]^* : H^1(C \times C, \mathbb{Q}) \to H^1(C, \mathbb{Q}).$$

Since

$$H^1(C \times C, \mathbb{Q}) = [H^1(C, \mathbb{Q}) \otimes H^0(C, \mathbb{Q})] \bigoplus [H^0(C, \mathbb{Q}) \otimes H^1(C, \mathbb{Q})],$$

and (by definition of the cup product), for any $a, b \in H^*(C, \mathbb{Q})$,

$$[\Delta_C]^*(a \otimes b) = a \cup b,$$

we have a surjective morphism $[\Delta_C]^* : H^1(C, \mathbb{Q}) \otimes H^0(C, \mathbb{Q}) \to H^1(C, \mathbb{Q})$. Lemma 9.1 is proved.

9.2. Lemma. *If $\pi : X \to C$ has no nontrivial constant Abelian subschemes, then the restriction morphism*

$$j^* : H^1(X \times X, \mathcal{O}^\times_{X \times X}) \to H^1(X \times_C X, \mathcal{O}^\times_{X \times_C X})$$

of the Picard groups is not surjective; *moreover,*

$$\mathcal{P} \notin j^* \left(H^1(X \times X, \mathcal{O}^\times_{X \times X})\right).$$

Proof. Assume, for a moment, that there exists a line bundle $\mathcal{L}$ on $X \times X$ such that $j^*(\mathcal{L}) = \mathcal{P}$. Let $i_s : X_s \hookrightarrow X$ be the canonical embedding. Then we have canonical embeddings

$$X_s \times X_s \overset{i_s \times i_s}{\hookrightarrow} X \times_C X \overset{j}{\hookrightarrow} X \times X.$$

By the Deligne theorem ([6], 4.1.2), the canonical restriction map

$$H^2(X \times X, \mathbb{Q}) \to H^2(X_s \times X_s, \mathbb{Q})$$

is a composition of the canonical surjective edge morphism

$$H^2(X \times X, \mathbb{Q}) \to H^0(C \times C, R^2(\pi \times \pi)_* \mathbb{Q}))$$

of the Leray spectral sequence $E_2^{p,q} = H^p(C \times C, R^q(\pi \times \pi)_* \mathbb{Q}) \Rightarrow H^*(X \times X, \mathbb{Q})$ and the canonical maps

$$H^0(C \times C, R^2(\pi \times \pi)_* \mathbb{Q}) \widetilde{\to} H^2(X_s \times X_s, \mathbb{Q})^{\pi_1(C \times C, s \times s)} \hookrightarrow H^2(X_s \times X_s, \mathbb{Q}).$$

It is evident that $\mathrm{cl}_{X \times X}(\mathcal{L}) \in H^2(X \times X, \mathbb{Q})$ maps to an element

$$v_s = \mathrm{cl}_{X_s \times X_s}(\mathcal{P}_s) \in H^2(X_s \times X_s, \mathbb{Q})^{\pi_1(C \times C, s \times s)}$$

of Künneth type (1,1) because $j^*(\mathrm{cl}_{X\times X}(\mathcal{L})) = \mathrm{cl}_{X\times_C X}(j^*(\mathcal{L})) = \mathrm{cl}_{X\times_C X}(\mathcal{P})$ ([14], 1.2.C), $[i_s \times i_s]^*\mathrm{cl}_{X\times_C X}(\mathcal{P}) = \mathrm{cl}_{X_s\times X_s}([i_s\times i_s]^*(\mathcal{P}))$ ([14], 1.2.C) and $[i_s \times i_s]^*(\mathcal{P}) = \mathcal{P}_s$. Since $v_s = \mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)$ is of Künneth type (1,1), we see that the image of $\mathrm{cl}_{X\times X}(\mathcal{L})$ under the canonical edge morphism

$$H^2(X\times X,\mathbb{Q}) \to H^0(C\times C, R^2(\pi\times\pi)_*\mathbb{Q})$$

belongs to

$$H^0(C,R^1\pi_*\mathbb{Q}) \bigotimes H^0(C,R^1\pi_*\mathbb{Q}) \widetilde{\to} \left[H^1(X_s,\mathbb{Q})\otimes H^1(X_s,\mathbb{Q})\right]^{\pi_1(C\times C, s\times s)} = \left[H^1(X_s,\mathbb{Q})\right]^{\pi_1(C,s)} \otimes \left[H^1(X_s,\mathbb{Q})\right]^{\pi_1(C,s)}.$$

In virtue of Lemma 2.6, $H^0(C,R^1\pi_*\mathbb{Q}) = 0$. Therefore $v_s = \mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s) = 0$ contradicting to (6.1). Lemma 9.2 is proved.

9.3. Lemma. *If $\pi : X \to C$ has no nontrivial constant Abelian subschemes, then the restriction morphism*

$$j^* : H^2_{\mathrm{alg}}(X\times X,\mathbb{Q}) \to H^2_{\mathrm{alg}}(X\times_C X,\mathbb{Q})$$

is not surjective; *moreover,*

$$v \notin j^*\left(H^2_{\mathrm{alg}}(X\times X,\mathbb{Q})\right).$$

Proof. Assume, for a moment, that there exists a line bundle $\mathcal{L}$ on $X\times X$ such that $j^*(\mathrm{cl}_{X\times X}(\mathcal{L})) = v$.

It is evident that the restriction of $\mathrm{cl}_{X\times X}(\mathcal{L}) \in H^2(X\times X,\mathbb{Q})$ equals to an element

$$v_s = \mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s) \in H^2(X_s\times X_s,\mathbb{Q})^{\pi_1(C\times C, s\times s)}$$

because $j^*(\mathrm{cl}_{X\times X}(\mathcal{L})) = v$. Since $v_s = \mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s)$ is of Künneth type (1,1), we see that the image of $\mathrm{cl}_{X\times X}(\mathcal{L})$ under the canonical edge morphism

$$H^2(X\times X,\mathbb{Q}) \to H^0(C\times C, R^2(\pi\times\pi)_*\mathbb{Q})$$

belongs to

$$H^0(C,R^1\pi_*\mathbb{Q}) \bigotimes H^0(C,R^1\pi_*\mathbb{Q}) \widetilde{\to} \left[H^1(X_s,\mathbb{Q})\otimes H^1(X_s,\mathbb{Q})\right]^{\pi_1(C\times C, s\times s)} = \left[H^1(X_s,\mathbb{Q})\right]^{\pi_1(C,s)} \otimes \left[H^1(X_s,\mathbb{Q})\right]^{\pi_1(C,s)}.$$

In virtue of Lemma 2.6, $H^0(C,R^1\pi_*\mathbb{Q}) = 0$. Therefore $v_s = \mathrm{cl}_{X_s\times X_s}(\mathcal{P}_s) = 0$ contradicting to (6.1). Lemma 9.3 is proved.

§ 10. Hodge cycles and a conditional correction of the right vertical arrow

10.1. Theorem. *Let $\pi : X \to C$ be a principally polarized complex Abelian scheme of relative dimension d over a smooth projective curve C. Then for any natural number $i \geq 2$, there exists the unique Hodge cycle*

$$\alpha_i = \alpha_i(X/C) \in [H^0(C,R^i\pi_*\mathbb{Q})\otimes H^0(C,R^i\pi_*\mathbb{Q})] \cap H^{i,i}(X\times X,\mathbb{C}),$$

characterized by the following condition: $\alpha_i \cup \mathrm{cl}_{X\times X}(X \times X_s)$ *determines an isomorphism*

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\rightarrow} H^2(C, R^i\pi_*\mathbb{Q}),$$

which coincides with an algebraic isomorphism

$$[j_*v^i]^* : H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\rightarrow} H^2(C, R^i\pi_*\mathbb{Q}).$$

If α_i is algebraic, then the algebraic class

$$[\Delta_*\mathrm{cl}_X(X_s)] \circ [j_*v^{i-2}] + j_*v^{i-1} + \alpha_i$$

induces an algebraic isomorphism $H^{2d+2-i}(X, \mathbb{Q}) \tilde{\rightarrow} H^i(X, \mathbb{Q})$, and the conjecture $B^i(X)$ holds.

Proof. By Lemma 8.1,

$$j_*v^i \in [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^2(C, R^i\pi_*\mathbb{Q})] \bigoplus [H^2(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})]$$
$$\bigoplus [H^1(C, R^i\pi_*\mathbb{Q}) \otimes H^1(C, R^i\pi_*\mathbb{Q})].$$

In virtue of (7.3),

$$H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^2(C, R^i\pi_*\mathbb{Q}) =$$
$$[H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cup \mathrm{cl}_{X\times X}(X \times X_s);$$
$$[H^2(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] =$$
$$[H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cup \mathrm{cl}_{X\times X}(X_s \times X).$$

Consequently, there exist uniquely determined elements

$$\alpha_i, \beta_i \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q}),$$
$$\delta_i \in H^1(C, R^i\pi_*\mathbb{Q}) \otimes H^1(C, R^i\pi_*\mathbb{Q})$$

such that

$$j_*v^i = \alpha_i \cup \mathrm{cl}_{X\times X}(X \times X_s) + \beta_i \cup \mathrm{cl}_{X\times X}(X_s \times X) + \delta_i.$$

Let $V_\mathbb{Q}$ be a rational Hodge structure. From the general theory of groups of multiplicative type it follows that to define a Hodge decomposition

$$V_\mathbb{Q} \otimes \mathbb{C} = \bigoplus V_\mathbb{C}^{p,q}$$

is the same as to define a morphism of real algebraic groups $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{G}_m) \to \mathrm{GL}(V_\mathbb{Q} \otimes \mathbb{R})$, where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{G}_m)$ is the real algebraic group obtained from the multiplicative group G_m by restricting the field of scalars from $\mathbb{C}$ to $\mathbb{R}$ ([10], Definition B.42); an element $z \in \mathbb{C}^\times = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{G}_m)(\mathbb{R})$ acts on $V_\mathbb{C}^{p,q}$ as the multiplication by $z^p\bar{z}^q$. Let $U^1 = \{z \in \mathbb{C}^\times = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{G}_m)(\mathbb{R}) \mid |z| = 1\}$. The smallest $\mathbb{Q}$-subgroup in $\mathrm{GL}(V_\mathbb{Q})$ whose group of $\mathbb{R}$-points contains $h(U^1)$ is called the *Hodge* group of a Hodge structure $V_\mathbb{Q}$ and is denoted by $\mathrm{Hg}(V_\mathbb{Q})$. The smallest $\mathbb{Q}$-subgroup in

$\mathrm{GL}(V_{\mathbb{Q}})$ whose group of $\mathbb{R}$-points contains $h(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{G}_m))$ is called the *Mumford – Tate* group of a Hodge structure $V_{\mathbb{Q}}$ and is denoted by $\mathrm{MT}(V_{\mathbb{Q}})$ ([10], Definition B.51).

It is well known that there exists an isogeny $\mathrm{G}_m \times \mathrm{Hg}(V_{\mathbb{Q}}) \to \mathrm{MT}(V_{\mathbb{Q}})$ ([10], Lemma B.53); if $V_{\mathbb{Q}}$ is a polarizable rational Hodge structure, then $\mathrm{Hg}(V_{\mathbb{Q}})$ and $\mathrm{MT}(V_{\mathbb{Q}})$ are reductive $\mathbb{Q}$-groups acting in a natural way in the space $V_{\mathbb{Q}}$ ([10], Proposition B.57).

The vital role of Hodge groups in analizing Hodge structures comes from the fact, that the rational $\mathrm{Hg}(V_{\mathbb{Q}})$-subrepresentations in $V^{\otimes r} \otimes (V^{\wedge})^{\otimes s}$ are precisely the rational sub-Hodge structures of $V^{\otimes r} \otimes (V^{\wedge})^{\otimes s}$ ([10], Proposition B54).

Assume that $V_{\mathbb{Q}}$ is a rational Hodge structure of even weight $2r$. Then

$$V_{\mathbb{Q}}^{\mathrm{Hg}(V_{\mathbb{Q}})} = V_{\mathbb{Q}} \cap V_{\mathbb{C}}^{r,r}, \tag{10.1}$$

where $V_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus V_{\mathbb{C}}^{p,q}$ is the Hodge decomposition ([10], Corollary B.55). By definition, elements of $V_{\mathbb{Q}}^{\mathrm{Hg}(V_{\mathbb{Q}})} = V_{\mathbb{Q}} \cap V_{\mathbb{C}}^{r,r}$ are called *Hodge cycles.*

In the situation under consideration, the components

$$\begin{gathered} H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^2(C, R^i\pi_*\mathbb{Q}) = \\ [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cup \mathrm{cl}_{X\times X}(X \times X_s), \\ [H^2(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] = \\ [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cup \mathrm{cl}_{X\times X}(X_s \times X), \\ H^1(C, R^i\pi_*\mathbb{Q}) \otimes H^1(C, R^i\pi_*\mathbb{Q}) \end{gathered}$$

are rational Hodge substructures of $H^{2i+2}(X \times X, \mathbb{Q})$ (recall that $H^0(C, R^i\pi_*\mathbb{Q})$ is a rational Hodge substructure of $H^i(X, \mathbb{Q})$ which is canonically identified (by (3.2)) with the n^i-eigenspace of the operator $[\Theta_n]^*$ on $H^i(X, \mathbb{Q})$; similarly, $H^1(C, R^i\pi_*\mathbb{Q})$ is canonically identified with the n^i-eigenspace of the operator $[\Theta_n]^*$ on $H^{i+1}(X, \mathbb{Q})$). Consequently, they are stable under the canonical action of $\mathrm{Hg}(H^{2i+2}(X \times X, \mathbb{Q}))$.

By the Lefschetz theorem,

$$\mathrm{NS}(X \times X) \otimes \mathbb{Q} \xrightarrow{\sim} H^2(X \times X, \mathbb{Q}) \cap H^{1,1}(X \times X, \mathbb{C}).$$

In virtue of (10.1),

$$\mathrm{NS}(X \times X) \otimes \mathbb{Q} \xrightarrow{\sim} H^2(X \times X, \mathbb{Q})^{\mathrm{Hg}(H^2(X\times X, \mathbb{Q}))}.$$

Therefore $\mathrm{cl}_{X\times X}(X \times X_s)$ and $\mathrm{cl}_{X\times X}(X_s \times X)$ are fixed under the canonical action of U^1 on $H^2(X \times X, \mathbb{C})$. Similarly, the algebraic class j_*v^i belongs to

$$\begin{gathered} H^{2i+2}_{\mathrm{alg}}(X \times X, \mathbb{Q}) \hookrightarrow H^{2i+2}(X \times X, \mathbb{Q}) \cap H^{i+1,i+1}(X \times X, \mathbb{C}) = \\ H^{2i+2}(X \times X, \mathbb{Q})^{\mathrm{Hg}(H^{2i+2}(X\times X, \mathbb{Q}))} \end{gathered}$$

and is fixed under the action of U^1 on $H^{2i+2}(X \times X, \mathbb{C})$. Since this action is compatible with the cup product, the elements

$$\alpha_i, \beta_i \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$$

are fixed under the action of U^1 on $H^{2i}(X \times X, \mathbb{C})$. Consequently, α_i and β_i are fixed under the action of the Hodge group $\mathrm{Hg}(H^{2i}(X \times X, \mathbb{Q}))$. In other words, α_i and β_i are Hodge cycles,

$$\alpha_i, \beta_i \in [H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})] \cap H^{i,i}(X \times X, \mathbb{C}).$$

We claim that the Hodge cycle $\alpha_i \cup \mathrm{cl}_{X\times X}(X \times X_s)$ induces *an algebraic isomorphism*

$$[j_*v^i]^* : H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^2(C, R^i\pi_*\mathbb{Q})$$

and the Hodge cycle α_i induces an isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^0(C, R^i\pi_*\mathbb{Q}),$$

which is not assumed to be algebraic.

Indeed, in virtue of (6.2), an algebraic class j_*v^i induces an algebraic isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^2(C, R^i\pi_*\mathbb{Q}), \tag{10.2}$$

determined by the composition

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \overset{\mathrm{pr}_1^*}{\tilde{\to}} H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q}) \xrightarrow{\cup j_*v^i} H^{2d+4+i}(X \times X, \mathbb{Q}) \overset{\mathrm{pr}_{2*}}{\tilde{\to}} H^{i+2}(X, \mathbb{Q}).$$

Since an element $\beta_i \cup \mathrm{cl}_{X\times X}(X_s \times X) + \delta_i$ annihilates $H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q})$, an algebraic isomorphism (10.2) coincides with the composition

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \overset{\mathrm{pr}_1^*}{\tilde{\to}} H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q}) \xrightarrow{\cup \alpha_i \cup \mathrm{cl}_{X\times X}(X\times X_s)} H^2(C, R^{2d}\pi_*\mathbb{Q}) \otimes H^2(C, R^i\pi_*\mathbb{Q}) \overset{\mathrm{pr}_{2*}}{\tilde{\to}} H^2(C, R^i\pi_*\mathbb{Q}).$$

In particular, the map

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q}) \overset{\cup \alpha_i}{\to} H^2(C, R^{2d}\pi_*\mathbb{Q}) \bigotimes H^0(C, R^i\pi_*\mathbb{Q}) \tag{10.3}$$

is injective.

On the other hand, (7.3) and (10.2) give the relations

$$\dim H^2(C, R^{2d-i}\pi_*\mathbb{Q}) = \dim H^2(C, R^i\pi_*\mathbb{Q}) = \dim H^0(C, R^i\pi_*\mathbb{Q}).$$

Consequently, the map (10.3) is bijective, and the claim is proved.

If α_i is algebraic, then it induces an algebraic isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^0(C, R^i\pi_*\mathbb{Q}),$$

therefore there exists an algebraic correction

$$\begin{array}{ccccc} H^0(C,R^{2d+2-i}\pi_*\mathbb{Q}) & \bigoplus & H^1(C,R^{2d+1-i}\pi_*\mathbb{Q}) & \bigoplus & H^2(C,R^{2d-i}\pi_*\mathbb{Q}) \\ \simeq\downarrow^{\varphi_i} & & \simeq\downarrow^{[j_*v^{i-1}]^*} & & \simeq\downarrow^{[\alpha_i]^*} \\ H^2(C,R^{i-2}\pi_*\mathbb{Q}) & \bigoplus & H^1(C,R^{i-1}\pi_*\mathbb{Q}) & \bigoplus & H^0(C,R^{i}\pi_*\mathbb{Q}), \end{array}$$

of the right vertical arrow of the diagram (7.4), and the conjecture $B^i(X)$ holds. Theorem 10.1 is proved.

10.2. Corollary. *If the Hodge cycles $\alpha_i(X/C)$ are algebraic for all natural numbers $i \geq 2$ and for all complex principally polarized Abelian schemes $\pi : X \to C$ without nontrivial constant Abelian subschemes over smooth projective curves, then the Hodge conjecture holds for all Abelian varieties.*

Proof. In virtue of Theorem 10.1, the conjecture $B^i(X)$ holds for all $i \geq 2$. Since $B^0(X)$ and $B^1(X)$ hold by the results of § 6.7, the conjecture $B^i(X)$ holds for all $i \geq 0$. It remains to apply the Y.André theorem 2.1, Corollary 2.3 and Remark 2.7.

§ 11. The proof of $\mathbf{B}^i(\mathbf{X})$ for some Abelian schemes

11.1. Let $\pi : X \to C$ be a complex Abelian scheme of relative dimension d over a smooth projective curve. There exists a finitely generated extension k of $\mathbb{Q}$ such that $X,\ C,\ \pi$ are defined over $k \hookrightarrow \mathbb{C}$:

$$X = X_0 \otimes_k \mathbb{C}, \quad C = C_0 \otimes_k \mathbb{C}, \quad \pi = \pi_0 \otimes_k \mathbb{C}.$$

Let $X_{0\eta}$ be a generic fiber of an Abelian scheme $\pi_0 : X_0 \to C_0$, and let $X_{0\overline{\eta}} = X_{0\eta} \otimes_{k(\eta)} \mathbb{C}$ for some embedding $k(\eta) \hookrightarrow \mathbb{C}$ which is a prolongation of the embedding $k \hookrightarrow \mathbb{C}$.

We call $X_{0\overline{\eta}}$ the generic geometric fiber of an Abelian scheme $\pi_0 : X_0 \to C_0$.

For any complex Abelian variety X_s, we denote by $\mathrm{Hg}(X_s) \stackrel{\mathrm{def}}{=} \mathrm{Hg}(H^1(X_s,\mathbb{Q}))$ *the Hodge group* of X_s. It is a reductive connected algebraic $\mathbb{Q}$-group with a canonical action on $H^*(X_s,\mathbb{Q}) = \bigwedge^* H^1(X_s,\mathbb{Q})$. In virtue of (10.1),

$$H^{2i}(X_s,\mathbb{Q})^{\mathrm{Hg}(X_s)} = H^{2i}(X_s,\mathbb{Q}) \cap H^{i,i}(X_s,\mathbb{C}).$$

11.2. Theorem. *Assume that for a complex Abelian scheme $\pi : X \to C$ of relative dimension d over a smooth projective curve C,* Lie $\mathrm{Hg}(X_{0\overline{\eta}})$ *is a simple Lie algebra of type C_d.*

Then the conjecture $B(X)$ holds. In particular, the operators ${}^{\mathrm{c}}{*},\ *,\ {}^{\mathrm{c}}\Lambda,\ \Lambda$ on $H^*(X,\mathbb{Q})$ are algebraic, the numerical equivalence of algebraic cycles on X coincides with the homological equivalence, the Künneth components of an arbitrary algebraic cycle on $X \times X$ are algebraic.

Proof. Let

$$\rho : \pi_1(C,s) \to \mathrm{GL}(H^1(X_s,\mathbb{Q}))$$

be the canonical monodromy representation. We denote by G the Zariski closure of $\rho(\pi_1(C,s))$ in $\mathrm{GL}(H^1(X_s,\mathbb{Q}))$. By definition,

$$H^i(X_s,\mathbb{Q})^G = H^i(X_s,\mathbb{Q})^{\pi_1(C,s)}$$

([34], § 7.0).

In virtue of Theorem 1.6, we may consider a ramified covering $C' \to C$ and an Abelian scheme $X' = X \times_C C' \to C'$ instead of $X \to C$. So we may assume that G is a connected $\mathbb{Q}$-group. It is a semisimple group ([6], Corollary 4.2.9).

By the Deligne theorem, for all s outside a countable subset of C, the group G is a normal subgroup of the Hodge group $\mathrm{Hg}(X_s)$ ([7], Proposition 7.5; [34], Theorem 7.3). In particular, there exists an embedding $G \hookrightarrow \mathrm{Hg}(X_{0\overline{\eta}})$.

If $G = 1$, then an Abelian scheme $\pi : X \to C$ is constant ([6], 4.1.3.3; [7], § 5.2): there exists an Abelian variety J such that $X \widetilde{\to} J \times C$. Since $B(C)$ is true and $B(J)$ holds by the D.Lieberman theorem (Corollary 1.8), the conjecture $B(X)$ holds by the property (1.10).

Consequently, we may assume that G is a nontrivial normal algebraic subgroup of a simple algebraic group $\mathrm{Hg}(X_{0\overline{\eta}})$, and therefore

$$\mathrm{Lie}\ G = \mathrm{Lie}\ \mathrm{Hg}(X_{0\overline{\eta}})$$

is a simple Lie algebra of type C_d.

We have:

$$H^0(C, R^i\pi_*\mathbb{Q}) \widetilde{\to} H^i(X_s, \mathbb{Q})^{\pi_1(C,s)} = H^i(X_s, \mathbb{Q})^G =$$
$$H^i(X_s, \mathbb{Q})^{\mathrm{Lie}\ G} = H^i(X_s, \mathbb{Q})^{\mathrm{sp}(H^1(X_s,\mathbb{Q}))} = \left[\bigwedge^i H^1(X_s, \mathbb{Q})\right]^{\mathrm{sp}(H^1(X_s,\mathbb{Q}))}.$$

On the other hand, let Γ be a generator of the 1-dimensional space $\left[\bigwedge^2 H^1(X_s, \mathbb{Q})\right]^{\mathrm{sp}(H^1(X_s,\mathbb{Q}))}$. Then

$$\left[\bigwedge^i H^1(X_s, \mathbb{Q})\right]^{\mathrm{sp}(H^1(X_s,\mathbb{Q}))} = \begin{cases} 0, & \text{for } i \text{ odd,} \\ \mathbb{Q} \cdot \Gamma^{\wedge i/2} & \text{for } i \text{ even} \end{cases}$$

([28], Lemma 4.8.4).

Consequently,

$$H^0(C, R^i\pi_*\mathbb{Q}) = \begin{cases} 0, & \text{for } i \text{ odd,} \\ \mathbb{Q} \cdot h^{\cup i/2} & \text{for } i \text{ even,} \end{cases}$$

where h is an image of the class $\mathrm{cl}_X(H)$ of a hyperplane section under the canonical surjective edge morphism $H^2(X, \mathbb{Q}) \to H^0(C, R^2\pi_*\mathbb{Q})$ of the Leray spectral sequence $E_2^{p,q} = H^p(C, R^q\pi_*\mathbb{Q}) \Rightarrow H^*(X, \mathbb{Q})$.

We see that $H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ is trivial or else is generated by an algebraic class $h^{\cup i/2} \otimes h^{\cup i/2}$, so

$$\alpha_i(X/C) \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$$

is algebraic, and Theorem 11.2 follows from Theorem 10.1.

11.3. Corollary. *Assume that* $\mathrm{End}(X_s) = \mathbb{Z}$ *for some geometric fiber of a complex Abelian scheme* $\pi : X \to C$ *of odd relative dimension* d *over a smooth projective curve* C.

Then the conjecture $B(X)$ holds.

Indeed,

$$\mathrm{End}(X_{0\overline{\eta}}) \hookrightarrow \mathrm{End}(X_s) = \mathbb{Z}.$$

Therefore $\mathrm{End}(X_{0\overline{\eta}}) = \mathbb{Z}$. Since d is odd, $\mathrm{Hg}(X_{0\overline{\eta}})$ is a simple algebraic $\mathbb{Q}$-group of type C_d ([28], Theorem 4.7; [29], Theorem 6.2).

11.4. Corollary. *Let*

$$Ex(1) = \{4, 10, 16, 32, 64, 108, 126, 256, 500, 512, 864, 1024, \dots\}$$

be the set of exceptional numbers in the sense of [31] *for simple Abelian varieties of type* I *by Albert's classification. If* $\mathrm{End}(X_s) = \mathbb{Z}$ *for some geometric fiber of a complex Abelian scheme* $\pi : X \to C$ *of relative dimension* $d \notin Ex(1)$ *over a smooth projective curve* C*, then the conjecture* $B(X)$ *holds.*

Indeed, it is known that in the case under consideration, $\mathrm{Hg}(X_{0\overline{\eta}})$ is a simple algebraic $\mathbb{Q}$-group of type C_d ([31], Theorem 1.1).

11.5. Corollary. *If* $\mathrm{End}(X_s) = \mathbb{Z}$ *for some geometric fiber of a complex Abelian scheme* $\pi : X \to C$ *of relative dimension* $d \in \{4, 10\}$ *over a smooth projective curve* C*, then the conjecture* $B(X)$ *holds.*

Indeed, it is known that in the case under consideration, $\mathrm{Hg}(X_{0\overline{\eta}})$ is a simple algebraic $\mathbb{Q}$-group of type C_d or else of type $A_1 \times A_1 \times A_1$ $(d = 4)$ (over $\overline{\mathbb{Q}}$), A_5 $(d = 10)$. Therefore $G = 1$ or else G is a simple algebraic $\mathbb{Q}$-group of the same type as $\mathrm{Hg}(X_{0\overline{\eta}})$, and

$$H^0(C, R^i\pi_*\mathbb{Q}) = \begin{cases} (0), & \text{for } i \text{ odd}, \\ \mathbb{Q} \cdot h^{\cup\, i/2} & \text{for } i \text{ even} \end{cases}$$

([31], Lemma 2.6), $H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ is trivial or else is generated by an algebraic class $h^{\cup\, i/2} \otimes h^{\cup\, i/2}$. It remains to apply on Theorem 10.1.

11.6. Theorem. *Assume that for a complex Abelian scheme* $\pi : X \to C$ *of relative dimension* d *over a smooth projective curve* C*,* $\mathrm{End}(X_{0\overline{\eta}}) \otimes \mathbb{Q}$ *is an imaginary quadratic extension of* $\mathbb{Q}$ *and the semisimple part* $[\mathrm{Lie\ Hg}(X_{0\overline{\eta}})]^{\mathrm{ss}}$ *of a reductive Lie algebra* $\mathrm{Lie\ Hg}(X_{0\overline{\eta}})$ *is a simple Lie algebra of type* A_{d-1}*.*

Then for any $i \neq d$*, the isomorphism*

$$\Lambda^{d+1-i} : H^{2d+2-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X, \mathbb{Q})$$

is algebraic.

Proof. Since G is a normal semisimple subgroup of $\mathrm{Hg}(X_{0\overline{\eta}})$, it is a normal subgroup of $[\mathrm{Hg}(X_{0\overline{\eta}})]^{\mathrm{ss}}$. Therefore $G = 1$ (in this situation an Abelian scheme $\pi : X \to C$ is constant and $B(X)$ holds), or else G is a simple $\mathbb{Q}$-group of type A_{d-1}. In the latter case,

$$H^1(X_s, \mathbb{Q}) \otimes \overline{\mathbb{Q}} = E(\omega_1) \bigoplus E(\omega_1)^* = E(\omega_1) \bigoplus E(\omega_{d-1})$$

is the sum of irreducible representations $V = E(\omega_1)$ and $V^* = E(\omega_{d-1})$ with highest weights ω_1 and ω_{d-1} of a simple Lie algebra $g = [\mathrm{Lie}\ G] \otimes \overline{\mathbb{Q}}$ of type A_{d-1} ([31], § 2.9).

If $i \leq d - 1$, then

$$[H^i(X_s, \mathbb{Q}) \otimes \overline{\mathbb{Q}}]^g = \wedge^i (V \oplus V^*)^g = \wedge^i (V \oplus V^*)^{sl(V)} =$$

$$\bigoplus_{m=0}^{i} [\wedge^m V \otimes \wedge^{i-m} V^*]^{sl(V)} = \bigoplus_{m=0}^{i} \mathrm{Hom}_{sl(V)}(\wedge^{i-m} V, \wedge^m V).$$

Recall that
$$\wedge^m V = \wedge^m E(\omega_1) = E(\omega_m)$$
is an irreducible representation in the case $d-1 \geq m \geq 1$, and
$$\wedge^{i-m} V = \wedge^{i-m} E(\omega_1) = E(\omega_{i-m})$$
is an irreducible representation in the case $d-1 \geq i-m \geq 1$ ([3], Chapter 8, § 13.1); hence by Schur's lemma,
$$\mathrm{Hom}_{sl(V)}(\wedge^{i-m}V, \wedge^m V) \neq (0) \ \Leftrightarrow \ i-m=m \quad (\text{for} \quad d-1 \geq i \geq i-m \geq 1)$$
with
$$\dim [\wedge^m V \otimes \wedge^m V^*]^{sl(V)} = 1.$$
If $d-1 \geq i \geq i-m = 0$, then
$$\dim \mathrm{Hom}_{sl(V)}(\wedge^{i-m}V, \wedge^m V) = \dim \mathrm{Hom}_{sl(V)}(\wedge^0 V, \wedge^i V) = \begin{cases} 0, & \text{for} \quad i \geq 1, \\ 1, & \text{for} \quad i = 0. \end{cases}$$

In particular, for all $i \leq d-1$,
$$\dim [H^i(X_s, \mathbb{Q})]^G = \dim [H^i(X_s, \mathbb{Q})]^{\pi_1(C,s)} = \begin{cases} 0, & \text{for } i \text{ odd}, \\ 1, & \text{for } i \text{ even}. \end{cases}$$

The case $i > d$ reduces to the previous case with the aid of the strong Lefschetz theorem.

We see that for $i \neq d$,
$$H^0(C, R^i\pi_*\mathbb{Q}) = \begin{cases} (0), & \text{for} \quad i \quad \text{odd}, \\ \mathbb{Q} \cdot h^{\cup\, i/2} & \text{for} \quad i \quad \text{even}, \end{cases}$$
$H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ is trivial or else is generated by an algebraic class $h^{\cup\, i/2} \otimes h^{\cup\, i/2}$. It remains to apply on Theorem 10.1.

Finally, for $i = d$ we have:
$$[H^d(X_s, \mathbb{Q}) \otimes \overline{\mathbb{Q}}]^g = \wedge^d (V \oplus V^*)^g = \wedge^d (V \oplus V^*)^{sl(V)} =$$
$$\bigoplus_{m=1}^{d-1} [\wedge^m V \otimes \wedge^{d-m} V^*]^{sl(V)} \bigoplus [\wedge^0 V \otimes \wedge^d V^*] \bigoplus [\wedge^d V \otimes \wedge^0 V^*] =$$
$$\bigoplus_{m=1}^{d-1} \mathrm{Hom}_{sl(V)}(\wedge^{d-m} V, \wedge^m V) \bigoplus [\wedge^0 V \otimes \wedge^d V^*] \bigoplus [\wedge^d V \otimes \wedge^0 V^*],$$
and
$$\dim [H^d(X_s, \mathbb{Q})]^G = \dim [H^d(X_s, \mathbb{Q})]^{\pi_1(C,s)} = \begin{cases} 2, & \text{for } d \text{ odd}, \\ 3, & \text{for } d \text{ even}. \end{cases}$$
Theorem 11.6 is proved.

11.7. Corollary. *Let*

$$Ex(4) = \{9, 10, 15, 16, 21, 25, 27, 28, 35, 36, 45, 49, \\ 55, 56, 64, 66, 78, 81, 84, 91, 100, \dots\}$$

be the set of exceptional numbers in the sense of [31] *for simple Abelian varieties of type* IV *by Albert's classification. If* $\mathrm{End}_C(X)\otimes\mathbb{Q}$ *and* $\mathrm{End}(X_s)\otimes\mathbb{Q}$ *are imaginary quadratic extensions of* $\mathbb{Q}$, *where* X_s *is some geometric fiber of a complex Abelian scheme* $\pi : X \to C$ *of relative dimension* $d \notin Ex(4)$ *over a smooth projective curve* C, *then for any* $i \neq d$, *the isomorphism*

$$\Lambda^{d+1-i} : H^{2d+2-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X, \mathbb{Q})$$

is algebraic.

Indeed, it is known that in the case under consideration, $\mathrm{Hg}(X_{0\overline{\eta}})$ is a reductive algebraic $\mathbb{Q}$-group with a semisimple part $[\mathrm{Hg}(X_{0\overline{\eta}})]^{\mathrm{ss}}$ of type A_{d-1} ([31], Theorem 1.1).

11.8. Theorem. *Let* $\pi : X \to C$ *be an Abelian scheme over a smooth projective curve* C. *Assume that some geometric fiber* X_s *is a simple Abelian variety of prime dimension* d.

Then for any $i \neq d$, *the isomorphism*

$$\Lambda^{d+1-i} : H^{2d+2-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X, \mathbb{Q})$$

is algebraic.

Proof. In virtue of Remark 2.7, we may assume that an Abelian scheme $\pi : X \to C$ contains no nontrivial constant Abelian subschemes. Since X_s is a simple Abelian variety, the fiber $X_{0\overline{\eta}}$ is a simple Abelian variety of prime dimension d.

It is well known that $\mathrm{End}(X_{0\overline{\eta}}) \otimes \mathbb{Q}$ is isomorphic to a totally real extension K_0 of $\mathbb{Q}$ of degree $[K_0 : \mathbb{Q}] \in \{1, d\}$ or else to an imaginary quadratic extension K of a totally real field K_0 of degree $[K : \mathbb{Q}] \in \{2, 2d\}$ ([29], 6.1).

Assume that $\mathrm{End}(X_{0\overline{\eta}}) \otimes \mathbb{Q} = K_0$. If $K_0 = \mathbb{Q}$, then we have a situation of Corollaries 11.3 – 11.4, and $B(X)$ holds.

If $K_0 \neq \mathbb{Q}$, then K_0 is a totally real field of degree $[K_0 : \mathbb{Q}] = d$. Since the monodromy representation

$$\rho : \pi_1(C, s) \to \mathrm{GL}(H^1(X_s, \mathbb{Q}))$$

is semisimple ([6], 4.2.6), the $\mathbb{Q}$-algebra $H = \mathrm{End}(R^1\pi_*\mathbb{Q})$ is isomorphic to a direct sum of algebras of the form $M_{n_\alpha}(D_\alpha)$, where D_α is a division algebra over $\mathbb{Q}$. On the other hand, the center Z of the algebra H is of Hodge type $(0, 0)$ and hence is contained in the center of the division algebra $\mathrm{End}_C(X)\otimes\mathbb{Q} \hookrightarrow \mathrm{End}(X_{0\overline{\eta}})\otimes\mathbb{Q} = K_0$ ([6], 4.4.7). Therefore $H \xrightarrow{\sim} M_n(D)$, where D is a central division algebra over a totally real field $Z \hookrightarrow K_0$. Let $e = \dim_{\mathbb{Q}} Z$ and $f^2 = \dim_Z D$. The space $H^1(X_s, \mathbb{Q})$ is decomposed into a direct sum of irreducible $M_n(D)$-modules each of which is a linear space over $\mathbb{Q}$ of dimension enf^2 (this is a well-known fact from the theory of simple central algebras). Hence

$$enf^2 | 2d. \tag{11.1}$$

We claim that
$$\mathrm{End}_C(X)\otimes\mathbb{Q}\xrightarrow{\sim}\mathrm{End}(R^1\pi_*\mathbb{Q}). \tag{11.2}$$

The case $d\leq 3$ is considered in ([6], 4.4.13). Hence we may assume that $d\geq 5$. From the relation (11.1) it follows that $f=1$. Consequently, $D=Z$, and for any embedding $\varphi: Z\hookrightarrow\mathbb{R}$, the algebra $H\otimes_Z\mathbb{R}_{(\varphi)}$ is a matrix algebra over $\mathbb{R}$. Therefore (11.2) follows from ([6], 4.4.11).

Replacing $X\to C$ by $X\times_C C'\to C'$ for some finite covering $C'\to C$, we may assume that
$$K_0=\mathrm{End}_C(X)\otimes\mathbb{Q}\xrightarrow{\sim}\mathrm{End}(R^1\pi_*\mathbb{Q}).$$

In this situation,
$$K_0=\mathrm{End}(R^1\pi_*\mathbb{Q})=\mathrm{End}_{\pi_1(C,s)}(H^1(X_s,\mathbb{Q}))=\mathrm{End}_G(H^1(X_s,\mathbb{Q})),$$

and by the Schur lemma,
$$H^1(X_s,\mathbb{Q})\otimes\overline{\mathbb{Q}}\xrightarrow{\sim}V_1\oplus\cdots\oplus V_d$$

is a direct sum of mutually nonisomorphic $[\mathrm{Lie}\,G]\otimes\overline{\mathbb{Q}}$-modules of degree 2 over $\overline{\mathbb{Q}}$. Therefore $[\mathrm{Lie}\,G]\otimes\overline{\mathbb{Q}}$ is a semisimple Lie algebra of type $A_1\times\cdots\times A_1=A_1^d$.

On the other hand, it is known that $\mathrm{Hg}(X_{0\overline{\eta}})$ is a semisimple $\mathbb{Q}$-group of type $A_1\times\cdots\times A_1=A_1^d$, so the geometric analog of the Mumford – Tate conjecture [23] holds. In particular, the Hodge conjecture for $X_{0\overline{\eta}}$ is equivalent to the geometric analog of the Tate conjecture for X_η, where X_η is the generic fiber of π. Since the Hodge conjecture holds for all simple Abelian varieties of prime dimension (moreover,
$$H^{2i}(X_{0\overline{\eta}},\mathbb{Q})\cap H^{i,i}(X_{0\overline{\eta}},\mathbb{C})$$

is spanned by classes of intersections of divisors [30]), it is easy to verify that $H^0(C,R^i\pi_*\mathbb{Q})$ is trivial for odd i; for i even, it is generated by cup products of elements of $H^0(C,R^2\pi_*\mathbb{Q})$. Moreover, elements of $H^0(C,R^2\pi_*\mathbb{Q})$ are algebraic (they come from classes of divisors on the generic fiber). Therefore $H^0(C,R^i\pi_*\mathbb{Q})\otimes H^0(C,R^i\pi_*\mathbb{Q})$ is generated by algebraic classes, and the conjecture $B(X)$ holds.

If K is an imaginary quadratic extension of $\mathbb{Q}$, then we have the situation of Corollary 11.7, and for any $i\neq d$, the isomorphism
$$\Lambda^{d+1-i}: H^{2d+2-i}(X,\mathbb{Q})\xrightarrow{\sim}H^i(X,\mathbb{Q})$$

is algebraic.

If $[K:\mathbb{Q}]=2d$, then $X_{0\overline{\eta}}$ is an Abelian variety of CM-type, the Hodge group $\mathrm{Hg}(X_{0\overline{\eta}})$ is a torus [24], so the relation $G\hookrightarrow\mathrm{Hg}(X_{0\overline{\eta}})$ and semisimplicity of G imply the relation $G=1$. It follows that an Abelian scheme $\pi: X\to C$ is potentially constant, so $B(X)$ holds. Theorem 11.8 is proved.

§ 12. Variations

12.1. Theorem. *Assume that for some natural number $i\geq 2$, there exists an algebraic cycle Z_i with rational coefficients of codimension i on $X\times X$ such that*
$$j_*v^i=\mathrm{cl}_{X\times X}(Z_i)\cup j_*1.$$

Then the algebraic class $[\mathrm{cl}_{X\times X}(Z_i)]_{0i0i}$ *induces an algebraic isomorphism*

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^0(C, R^i\pi_*\mathbb{Q}).$$

Proof. For $u = \mathrm{cl}_{X\times X}(Z_i)$, we have the decomposition $u = \sum u_{pqrt}$, where $u_{pqrt} \in H^p(C, R^q\pi_*\mathbb{Q}) \otimes H^r(C, R^t\pi_*\mathbb{Q})$. In virtue of Lemma 3.3, u_{0i0i} is an algebraic class.

We claim that u_{0i0i} induces an algebraic isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^0(C, R^i\pi_*\mathbb{Q}).$$

Indeed, consider the composition

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \overset{\mathrm{pr}_1^*}{\tilde{\to}} H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q}) \overset{\cup u_{0i0i}}{\dashrightarrow}$$
$$H^2(C, R^{2d}\pi_*\mathbb{Q}) \bigotimes H^0(C, R^i\pi_*\mathbb{Q}) = H^{2d+2}(X, \mathbb{Q}) \bigotimes H^0(C, R^i\pi_*\mathbb{Q}) \overset{\mathrm{pr}_{2*}}{\tilde{\to}}$$
$$H^0(C, R^i\pi_*\mathbb{Q}).$$

In virtue of (6.2), an algebraic class j_*v^i induces an algebraic isomorphism

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \tilde{\to} H^2(C, R^i\pi_*\mathbb{Q}). \tag{12.1}$$

Therefore (7.3) gives the relations

$$\dim\, H^2(C, R^{2d-i}\pi_*\mathbb{Q}) = \dim\, H^2(C, R^i\pi_*\mathbb{Q}) = \dim\, H^0(C, R^i\pi_*\mathbb{Q}),$$

and it remains to prove the injectivity of the map

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q}) \overset{\cup u_{0i0i}}{\dashrightarrow}$$
$$H^2(C, R^{2d}\pi_*\mathbb{Q}) \bigotimes H^0(C, R^i\pi_*\mathbb{Q}). \tag{12.2}$$

We have

$$\mathrm{cl}_{X\times X}(Z_i) = u = u_{0i0i} + \sum_{p\neq 0} u_{pqrt} + \sum_{r\neq 0} u_{0qrt} + \sum_{q\neq i} u_{0q0t}.$$

Since $\sum_{p\neq 0} u_{pqrt}$ annihilates $H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C, \mathbb{Q})$, an isomorphism (12.1) is determined by the correspondence

$$\left[u_{0i0i} + \sum_{r\neq 0} u_{0qrt} + \sum_{q\neq i} u_{0q0t}\right] \cup j_*1.$$

By Corollary 8.2,

$$j_*1 = \mathrm{cl}_{X\times X}(X \times_C X) \in$$
$$[H^0(C, \mathbb{Q}) \otimes H^2(C, \mathbb{Q})] \bigoplus [H^2(C, \mathbb{Q}) \otimes H^0(C, \mathbb{Q})] \bigoplus [H^1(C, \mathbb{Q}) \otimes H^1(C, \mathbb{Q})].$$

Therefore $\sum_{r\neq 0} u_{0qrt} \cup j_*1$ annihilates $H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C,\mathbb{Q})$, and an isomorphism (12.1) is determined by the correspondence

$$\left[u_{0i0i} + \sum_{q\neq i} u_{0q0t}\right] \cup j_*1.$$

Finally,

$$\mathrm{pr}_{2*}\left(\left[H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C,\mathbb{Q})\right] \cup \left[\sum_{q\neq i} u_{0q0t} \cup j_*1\right]\right) = (0),$$

therefore an isomorphism (12.1) is determined by the algebraic correspondence $u_{0i0i} \cup j_*1$. In particular, the map

$$H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \bigotimes H^0(C,\mathbb{Q}) \xrightarrow{\cup u_{0i0i} \cup j_*1} H^2(C, R^{2d}\pi_*\mathbb{Q}) \bigotimes H^2(C, R^i\pi_*\mathbb{Q})$$

is injective. Therefore the map (12.2) is injective. Theorem 12.1 is proved.

12.2. Under the assumption of Theorem 12.1, the diagram (7.4) can be transformed into a new diagram

$$\begin{array}{ccccc} H^0(C, R^{2d+2-i}\pi_*\mathbb{Q}) & \bigoplus & H^1(C, R^{2d+1-i}\pi_*\mathbb{Q}) & \bigoplus & H^2(C, R^{2d-i}\pi_*\mathbb{Q}) \\ \simeq \downarrow^{\varphi_i} & & \simeq \downarrow^{[j_*v^{i-1}]^*} & & \simeq \downarrow^{[u_{0i0i}]^*} \\ H^2(C, R^{i-2}\pi_*\mathbb{Q}) & \bigoplus & H^1(C, R^{i-1}\pi_*\mathbb{Q}) & \bigoplus & H^0(C, R^i\pi_*\mathbb{Q}), \end{array} \tag{12.3}$$

where $u = \mathrm{cl}_{X\times X}(Z_i)$, and all vertical arrows are algebraic isomorphisms. We arrive to the following result:

12.3. Theorem. *Let $\pi : X \to C$ be a principally polarized complex Abelian scheme of relative dimension d over a smooth projective curve C. Assume that for some natural number $i \geq 2$, there exists an algebraic cycle Z_i with rational coefficients of codimension i on $X \times X$ such that*

$$j_*v^i = \mathrm{cl}_{X\times X}(Z_i) \cup j_*1.$$

Let $u = \mathrm{cl}_{X\times X}(Z_i)$ and let $u_{0i0i} \in H^0(C, R^i\pi_*\mathbb{Q}) \otimes H^0(C, R^i\pi_*\mathbb{Q})$ be the corresponding component of u.

Then the algebraic class

$$[\Delta_*\mathrm{cl}_X(X_s)] \circ [j_*v^{i-2}] + j_*v^{i-1} + u_{0i0i}$$

determines an algebraic isomorphism

$$H^{2d+2-i}(X,\mathbb{Q}) \xrightarrow{\sim} H^i(X,\mathbb{Q}).$$

In particular, the conjecture $B^i(X)$ holds.

References

[1] Y.André, *"Pour une théorie inconditionnelle des motifs"*, Publ. Math. IHES **83** (1996), 5-49.

[2] S.Yu.Arakelov, *"Families of algebraic curves with fixed degeneracies"*, Izv. Akad. Nauk SSSR. Ser. Mat. **35** (1971), 1269-1293; English transl., Math. USSR Izv., **5, n. 6** (1971), 1277-1302.

[3] N.Bourbaki, *Groupes et algèbres de Lie, Chaps.* $1-8$, Actualités Sci. Indust., nos. 1285, 1349, 1337, 1364, Hermann, Paris, 1971, 1972, 1968, 1975; Russian transl., Mir, Moscow 1976, 1972, 1978.

[4] S.S.Chern, *Complex manifolds*, Lectures at the University of Chicago Autumn 1955 – Winter 1956; Russian transl., Moscow, 1961.

[5] P.Deligne, *"Théorème de Lefschetz et critères de dégénérescence de suites spectrales"*, Publ. Math. IHES **35** (1968), 259-278.

[6] P.Deligne, *"Théorie de Hodge II"*, Publ. Math. IHES **40** (1971), 5-57; Russian transl., Matematika **17:5** (1973), 3-56.

[7] P.Deligne, *"La conjecture de Weil pour les surfaces* $K3$*"*, Invent. Math. **15** (1972), 206-226.

[8] G.Faltings, Ch.-L.Chai, *Degeneration of Abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **22** , Springer-Verlag, Berlin – Heidelberg – New York – London – Paris – Tokyo – Hong Kong – Barselona, 1990, 316 pp.

[9] W.Fulton, *Intersection theory*, Springer-Verlag, Berlin – Heidelberg – New York – Tokyo, 1984; Russian translation, Mir, Moscow, 1989.

[10] B.B.Gordon, *"A survey of the Hodge conjecture for Abelian varieties"*, in: J.D.Lewis, *A survey of the Hodge conjecture*, second edition, CRM Monograph Series, Centre de Recherches Mathématiques Université de Montréal, vol. 10 (1999), 297-356.

[11] Ph.Griffiths and J.Harris, *Principles of algebraic geometry*, A Wiley-Interscience Publication John Wiley and Sons, New York – Chichester – Brisbane – Toronto, 1978; Russian transl., Mir, Moscow, 1982.

[12] A.Grothendieck, *"Standard conjectures on algebraic cycles"*, Algebr. Geom., Oxford Univ. Press, London (1969), 193-199.

[13] W.V.D.Hodge, *"The topological invariants of algebraic varieties"*, Proc. Int. Congr. Math. (Cambridge, MA, 1950), Amer. Math. Soc., Providence, R.I., **1** (1952), 182-192.

[14] S.L.Kleiman, *"Algebraic cycles and the Weil conjectures"*, Dix exposés sur la cohomologie des schémas , North - Holland, Amsterdam (1968), 359-386.

[15] S.L.Kleiman, *"The standard conjectures"*, Proc. Symp. Pure Math. **55 (Part I)** (1994), 3-20.

[16] S.Lang, *Abelian varieties*, Interscience, New York, 1959.

[17] S.Lang, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York – Berlin – Heidelberg – Tokyo, 1983.

[18] H.Lange, C.Birkenhake, *Complex Abelian varieties*, Grundlehren der Mathematischen Wissenschaften, 302, Springer-Verlag, Berlin, 1992, 435 pp.

[19] D.Lieberman, *"Numerical and homological equivalence of algebraic cycles on Hodge manifolds"*, Amer. J. Math. **90** (1968), 366-374.

[20] Yu.I.Manin, *"Correspondences, motifs and monoidal transformations"*, Math. Sb. (N.S.) **77(119)** (1968), 475-507; English transl., Mathematics of the USSR-Sbornik, **6**, 1968, 439-470.

[21] J.S.Milne, *Etale cohomology*, Princeton Univ. Press, Princeton, 1980; Russian transl., Mir, Moscow, 1983.

[22] J.S.Milne, *"Lefschetz motives and the Tate conjecture"*, Compositio Math. **117** (1999), 45-76.

[23] D.Mumford, *"Families of abelian varieties"*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, **9** (1966), 347-352.

[24] D.Mumford, "A note on Shimura's paper "Discontinuous groups and abelian varieties"" *Math. Ann.* **181, n. 4** (1969), 345-351.

[25] H.J.Pohlmann, "Algebraic cycles on abelian varieties of complex multiplication type", Ann. Math. **88, n. 1** (1968), 161-180.

[26] J.-P.Serre, *Groupes algébriques et corps de classes*, Actualités Sci. Indust., no. 1264, Hermann, Paris, 1959; Russian transl., Mir, Moscow, 1968.
[27] T.Shioda, *"What is known about the Hodge conjecture?"*, Adv. Stud. Pure Math. **1** (1983), 55-68.
[28] S.G.Tankeev, *"On algebraic cycles on surfaces and abelian varieties"*, Izv. Akad. Nauk SSSR. Ser. Mat. **45, n. 2** (1981), 398-434; English transl., Math. USSR-Izv. **18, n. 2** (1982), 349-380.
[29] S.G.Tankeev, *"Cycles on simple* 5*-dimensional abelian varieties"*, Izv. Akad. Nauk SSSR. Ser. Mat. **45, n. 4** (1981), 793-823; English transl., Math. USSR-Izv. **19, n. 1** (1982), 95-123.
[30] S.G.Tankeev, *"Cycles on simple abelian varieties of prime dimension"*, Izv. Akad. Nauk SSSR Ser. Mat. **46, n. 1** (1982), 155-170; English transl., Math. USSR-Izv. **20, n. 1** (1983), 157-171.
[31] S.G.Tankeev, *"Cycles on Abelian varieties and exceptional numbers"*, Izv. Ross. Akad. Nauk. Ser. Mat. **60, n. 2** (1996), 159-194; English transl., Izv. Math. **60, n. 2** (1996), 391-424.
[32] J.Tate, *"Algebraic cycles and poles of zeta functions"*, Arithmetical Algebraic Geometry (Proc. Conf., Purdue Univ., Lafayette, IN, 1963), Harper and Row, New York, 1965, pp. 93-110.
[33] J.Tate, *"Conjectures on algebraic cycles in l-adic cohomology"*, Proc. Symposia in Pure Math. **55, Part 1** (1994), 71-83.
[34] Yu.G.Zarhin,, *"Weights of simple Lie algebras in cohomology of algebraic varieties"*, Izv. Akad. Nauk SSSR Ser. Mat. **48, n. 2** (1984), 264-304; English transl., Math. USSR-Izv. **24, n. 2**, 245-281 (1985).

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