Positivity of Schur function expansions of Thom polynomials

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Abstract

Combining the Kazarian approach to Thom polynomials via classifying spaces of singularities with the Fulton-Lazarsfeld theory of numerical positivity for ample vector bundles, we show that the coefficients of Schur function expansions of the Thom polynomials of stable singularities are nonnegative.

1 Introduction

The global behavior of singularities[1] is governed by their *Thom polynomials* (cf. [13], [1], [6], [12]). As these polynomials are quite complicated even for "simplest" singularities (cf., e.g., [12], [10]), it is important to study their structure. In the present note, following [10], we study *Schur function expansions* of these polynomials from a "qualitive" point of view. Contrary to [12] and [10], where the Szücs-Rimanyi approach via symmetries of singularities was used, we follow here the Kazarian approach [6] to Thom polynomials. This approach relies on suitable "classifying spaces of singularities",

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[1]In the present note we study complex singularities.

and combined with the Fulton-Lazarsfeld theory of numerical positivity of polynomials in the Chern classes of ample vector bundles [4], leads to *non-negativity* of the coefficients in the Schur function expansions of the Thom polynomials of the singularities stable under suspension.

This positivity was previously checked for a number of singularities: by Thom [13] for $A_1(r)$, by Feher and Komuves [3] for some second order Thom-Boardman singularities, by the first author [10] for $I_{2,2}(r)$, $A_3(r)$, and for $A_i(r)$ under the aditional assumption that $\Sigma^j = \emptyset$ for $j \geq 2$, by the first author and Ozturk for Thom polynomials from [12], by the second author for the Thom polynomials (from [6]) of singularities of functions, and by Ozturk [9] for $A_4(3)$, $A_4(4)$. [2]

2 Thom polynomials

Fix $m, n, k \in \mathbf{N}$. We denote by Aut_n the group of k-jets of automorphisms of $(\mathbf{C}^n, 0)$, and by $\mathcal{J} = \mathcal{J}(m,n)$ the space of k-jets of functions $(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ [3].

Moreover, we set

$$G := \mathrm{Aut}_m \times \mathrm{Aut}_n .$$

Consider the classifying principal G-bundle $EG \to BG$, i.e. a contractible space EG with a free action of the group G, and define

$$\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}(m,n) = EG \times_G \mathcal{J} .$$

Let $\Sigma \subset \mathcal{J}$ be an analytic closed G-invariant subset, which we shall call a "class of singularities". For a given class of singularities Σ, set

$$\widetilde{\Sigma} = EG \times_G \Sigma \subset \widetilde{\mathcal{J}}$$

and denote by $\mathcal{T}^\Sigma \in H^{2\,\mathrm{codim}\,\Sigma}(\widetilde{\mathcal{J}}; \mathbf{Z})$ the (Poincaré) dual class of $[\widetilde{\Sigma}]$.

Since

$$H^\bullet(\widetilde{\mathcal{J}}; \mathbf{Z}) \simeq H^\bullet(BG; \mathbf{Z}) \simeq H^\bullet(BGL_m \times BGL_n; \mathbf{Z}) ,$$

$\mathcal{T}^\Sigma$ is identified with a polynomial in $c_1, \ldots, c_m$ and $c'_1, \ldots, c'_n$ which are the Chern classes of universal bundles R_m and R_n on BGL_m and BGL_n. This is a classical *Thom polynomial*. Speaking slightly informally, given a general map $f : M \to N$ of smooth varieties of corresponding dimensions m and n, the Thom polynomial

$$\mathcal{T}^\Sigma(c_1(M), \ldots, c_m(M), c_1(N), \ldots, c_n(N))$$

[2]We use here the notation from [10]. The calculations in the last three cases used extensively ACE [15].

[3]Though these objects depend on k, we omit "k" in the notation. This will happen also to other objects introduced later.

evaluates the dual class of the set where f has singularity "of the class Σ". A precise version of this statement is a content of the Thom theorem [13] (see also [6], Theorem 1 and [12], Section 6).

The *suspension*

$$\mathcal{S} : \mathcal{J}(m,n) \hookrightarrow \mathcal{J}(m+1,n+1)$$

allows one to increase the dimension of the source and the target simultaneously: with the local coordinates $x_1, x_2, \dots$ for the source and a function $f = f(x_1, \dots, x_m)$, the jet $(\mathcal{S}f) \in \mathcal{J}(m+1,n+1)$ is defined by

$$(\mathcal{S}f)(x_1, \dots, x_m, x_{m+1}) := (f(x_1, \dots, x_m), x_{m+1}) .$$

Suppose that the class of singularities Σ is *stable under suspension.* By this we mean that it is a member $\Sigma_0 = \Sigma$ of a family

$$\{\Sigma_r \subset \mathcal{J}(m+r, n+r)\}_{r\ge 0}$$

such that

$$\Sigma_{r+1} \cap \mathcal{J}(m+r, n+r) = \Sigma_r$$

and

$$\mathcal{T}^{\Sigma_{r+1}}{}_{|H^\bullet(BGL_{m+r}\times BGL_{n+r};\mathbf{Z})} = \mathcal{T}^{\Sigma_r} .$$

This means that if we specialize

$$c_{m+r+1} = c'_{n+r+1} = 0$$

in the polynomial $\mathcal{T}^{\Sigma_{r+1}}$, we obtain the polynomial $\mathcal{T}^{\Sigma_r}$. The usual notion of stable equivalence fits into our setup.

The theorem of Thom has the following refinement due to Damon [2] for a class of singularities Σ which is stable under suspension: $\mathcal{T}^{\Sigma}$ is *supersymmetric*, i.e. is a polynomial in

$$c_i(TM - f^*TN) = [c(TM)/c(f^*TN)]_i \quad \text{where} \quad i = 1, 2, \dots .$$

Cf. also [6, Theorem 2].

3 Schur functions expansions

Instead of using *Chern monomial expansions* to express Thom polynomials (cf., e.g., [12] and the references therein), there is a recent (initiated in 2005) attempt to use *Schur function expansions* $\sum_I \alpha_I S_I$ (cf. [3] and [10]).

Recall that given a partition $I = (0 \le i_1 \le i_2 \le \dots \le i_l)$, and vector bundles E and F, the *Schur function* $S_I(E - F)$ is

$$S_I(E-F) := \Big| S_{i_p+p-q}(E-F) \Big|_{1\le p,q\le l} ,$$

where
$$\sum S_i(E-F) = \prod_b (1-b) / \prod_a (1-a)\,,$$

and the a's and b's are the Chern roots of E and F. We have for any partition I,
$$S_I(E^* - F^*) = S_{I^\sim}(F-E) \tag{1}$$

where $I^\sim$ denotes the *dual partition* of I. In particular, $S_i(E^* - F^*) = c_i(F-E)$ for any i. We refer to [7] and [11] for the theory of Schur functions (a brief account of Schur functions applied to Thom polynomials, can be found in [10, Section 3]).

Using the theory of supersymmetric functions (cf., e.g., [11]), the Thom-Damon theorem can be rephrased by saying that there exist $\alpha_I \in \mathbf{Z}$ such that
$$\mathcal{T}^\Sigma = \sum_I \alpha_I S_I(R_m^* - R_n^*)\,, \tag{2}$$

the sum is over partitions I with $|I| = \operatorname{codim}(\Sigma)$. Here, and in the following, we omit pull back indices. The expression in Eq. (2) is unique (loc.cit.).

4 Positive polynomials for ample vector bundles

In the proof of our main result, we shall use the following two results of Fulton and Lazarsfeld from [4]. Recall that by a *cone* in a vector bundle E we mean a subvariety of E which is stable under the natural $\mathbf{C}^*$ action on E. The first result is

Proposition 1 ([4, Theorem 2.1]) *Let E be an ample vector bundle of rank e on a projective variety X of dimension e, and let $C \subset E$ be a cone of pure dimension e. Then the cone class $[C] \in A_0(X)$ (the Chow group) has strictly positive degree.*

The second result characterizes *polynomials numerically positive for ample vector bundles*, cf. [4, p. 35] for this last notion.

Theorem 2 ([4, Theorem I]) *A homogeneous polynomial*
$$\sum_I \beta_I S_I\,,$$

where $\beta_I \in \mathbf{Z}$, is numerically positive for ample vector bundles iff it is nonzero and for any partition I, $\beta_I \geq 0$.

5 Main result

The main result of the present note, suggested/conjectured in [10] and in [3] for Thom-Boardman singularities, is

Theorem 3 *Let Σ be a stable class of singularities. Then for any partition I, the coefficient α_I occuring in the Schur function expansion of the Thom polynomial $\mathcal{T}^{\Sigma}$ (cf. Eq. (2)), is nonnegative.*

Proof. To prove the theorem, we first pull back the bundle $\widetilde{\mathcal{J}}$ from BG to $BGL_m \times BGL_n$ via the map induced by the embedding

$$GL_m \times GL_n \hookrightarrow \mathrm{Aut}_m \times \mathrm{Aut}_n \,.$$

Since $GL_m \times GL_n$ acts linearly on $\mathcal{J}$, the obtained pullback bundle is now the vector bundle on $BGL_m \times BGL_n$ associated with the representation of $Gl_m \times GL_n$ on $\mathcal{J}$:

$$\mathcal{J}(R_m, R_n) := \left(\oplus_{i=1}^{k} \mathrm{Sym}^i(R_m^*)\right) \otimes R_n \,.$$

The bundle $\mathcal{J}(R_m, R_n)$ contains the preimage of $\widetilde{\Sigma}$, denoted by $\Sigma(R_m, R_n)$, whose dual class is given by the RHS of Eq. (2).

Consider, more generally, a pair of vector bundles E and F of ranks m and n on a variety X. We define the following vector bundle on X:

$$\mathcal{J}(E, F) := \left(\oplus_{i=1}^{k} \mathrm{Sym}^i(E^*)\right) \otimes F \,.$$

In fact, the pair of bundles (E, F) corresponds to a principal $GL_m \times GL_n$-bundle $P(E, F)$ and

$$\mathcal{J}(E, F) = P(E, F) \times_{GL_m \times GL_n} \mathcal{J}$$

is the bundle associated with the representation. Similarly, we define the singularity set

$$\Sigma(E, F) := P(E, F) \times_{GL_m \times GL_n} \Sigma \subset \mathcal{J}(E, F) \,.$$

The dual class of $[\Sigma(E, F)]$ in

$$H^{2\,\mathrm{codim}(\Sigma)}(\mathcal{J}(E, F); \mathbf{Z}) = H^{2\,\mathrm{codim}(\Sigma)}(X; \mathbf{Z})$$

is equal to

$$\sum_I \alpha_I S_I(E^* - F^*) \,, \tag{3}$$

where the α_I's were defined in Eq. (2). The proof of this fact is fairly standard but one has to pass to topological homotopy theory, where each pair of bundles can be pulled back from the universal pair (R_m, R_n) of bundles on

$BGL_m \times BGL_n$. It is possible to work entirely with the algebraic varieties. One can use the Totaro construction and representability for affine varieties ([14, proof of Theorem 1.3]). Another solution is to pass to homotopy category of algebraic varieties, where the classifying space is defined and has desired properties, cf. [8, §4.2]. (This enormous machinery seems to be "much too heavy" for the needs of our problem.)

Suppose from now on that $\mathrm{rank}(F) = n' = n + r \geq \mathrm{codim}(\Sigma)$ and $E = \mathbf{1}^{m'}$ is a trivial bundle of rank $m' = m + r$. If we check that $\alpha_I \geq 0$ in Eq. (3) for all such pairs of bundles, then by Eq. (2) the assertion of Theorem 3 will follow. By Eq. (3) the dual class of $[\Sigma(\mathbf{1}^{m'}, F)]$ is equal to

$$\sum_I \alpha_I S_I(-F^*) = \sum_I \alpha_I S_{I^\sim}(F) \tag{4}$$

(we use here Eq. (1)). We shall check that if F is ample then the expression (4) is numerically positive. Since a direct sum of ample vector bundles is ample ([5, Proposition 2.2]), the vector bundle $\mathcal{J}(\mathbf{1}^{m'}, F) = F^{\oplus K}$ (for some integer K) is ample. Moreover, $\Sigma(\mathbf{1}^{m'}, F)$ is a cone in $\mathcal{J}(\mathbf{1}^{m'}, F)$ because $\mathbf{C}^* \subset \mathrm{Aut}_{n'}$. Therefore, by Proposition 1 the expression (4) is numerically positive. We conclude that the coefficients α_I in Eq. (4) are nonnegative by Theorem 2.

The theorem has been proved. □

Remark 4 We give here a slightly different argument proving the assertion of the theorem. As above, by [4] we know that the following statement (*) holds: if the dual class of $[\Sigma(\mathbf{1}^{n'}, F)]$ is

$$\sum_I \alpha_I S_{I^\sim}(F)$$

(where the coefficients α_I are universal), then for any partition I we have $\alpha_I \geq 0$.

Let $G = G_m(\mathbf{C}^M)$ be the Grassmannian parametrizing m-dimensional subspaces of $\mathbf{C}^M$. It is endowed with the tautological sequence of vector bundles:

$$0 \to R_{m,M} \to \mathbf{1}_G^m \to Q_{m,M} \to 0\,. \tag{5}$$

Taking another Grassmannian $G_n(\mathbf{C}^N)$, and $M, N >> 0$, to estimate the α_I's from Eq. (2), it is sufficient to estimate the coefficients in the dual class of $[\Sigma(R_{m,M}, R_{n,N})]$, given by

$$\sum_I \alpha_I S_I(R^*_{m,M} - R^*_{n,N}) = \sum_I \alpha_I S_I(-Q^*_{m,M} - R^*_{n,N}) \tag{6}$$

$$= \sum_I \alpha_I S_{I^\sim}(Q_{m,M} + R_{n,N})\,.$$

This equation follows from the sequence (5). Applying (*) to the RHS of Eq. (6), we get that the coefficients on its LHS are nonnegative.

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