COHOMOLOGY THEORIES

by

E. Spanier*

University of California
at Berkeley

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
5300 Bonn 1

MPI/SFB 84 - 35

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1. Introduction:

This paper is a study of cohomology theories of various types on various categories. Basically it is a study of axiomatics, their interrelationships, and their applications, and in particular of the type of axiom system considered in [1,13,21].

For our purposes a cohomology theory on a space X is a continuous exact contravariant functor from the category of closed subsets of X to the category of graded abelian groups. A homomorphism between two cohomology theores on the same space is a natural transformation between the contravariant functors which maps the exact sequences of one into the exact sequences of the other. We are interested in uniqueness theorems asserting that a homomorphism which is an isomorphism for every point of the space as an isomorphism for every closed subset of the space. Such theorems were proved in [13,21] for nonneqative cohomology theories. These theorems have as consequences many results in ordinary cohomology theory previously obtained using sheaf theory.

In the present paper the condition of nonnegativity is dropped so that the theory may be applied to extraordinary cohomology theories. In this case the uniqueness theorem is not valid without some additional assuption. We prove it is valid for finite-dimensional spaces. This form of the uniqueness theorem has some interesting applications to cohomology of manifolds. It can also be applied to prove a similar uniqueness theorem for cohomology theories defined on larger categories such as the category of all compact spaces and continuous functions or the category of all locally compact spaces and proper continuous functions.

On the category of compact spaces we prove that our cohomology theories are equivalent to continuous theories satisfying the Eilenberg-Steenrod axioms except the dimension axiom. As an example, a spectrum of ANR defines a cohomology theory on the category of locally compact spaces. Application of the uniqueness theorem to such cohomology theories provides another proof of the fact that the Chern character is an isomorphism of $K(X) \otimes \mathbb{Q}$ with $\check{H}_{ev}(X;\mathbb{Q})$ for every compact space X. We also use the uniqueness theorem to give another proof of the duality in stable homotopy theory for compact subsets of S^n.

The rest of the paper is divided into seven Sections of which the first three are devoted to cohomology theories on a single space and the last four are devoted to cohomology theories on categories of spaces.

Section 2 contains the definition of a cohomology theory on a space and some of its elementary properties. Section 3 contains a proof that the cohomology theories considered by Lawson [13] are essentially the same as ours and that ES theories (which satisfy Eilenberg-Steenrod axioms) determine cohomology theories. Section 4 is devoted to a proof of the uniqueness theorem for finite dimensional spaces and to applications of this result to manifolds.

In Section 5 we consider cohomology theories on the category $\mathcal{C}_{comp}$ of compact spaces and continuous functions and on the category $\mathcal{C}_{loc\ comp}$ of locally compact spaces and proper continuous functions. In each case there is a uniqueness theorem. We also show that cohomology theories on $\mathcal{C}_{\text{comp}}$ are equivalent to compactly supported cohomology theories on $\mathcal{C}_{\text{loc comp}}$. Section 6 is devoted to ES theories on $\mathcal{C}_{\text{comp}}$. There is a corresponding uniqueness theorem for these, and we also prove the equivalence of ES theories and cohomology theories on $\mathcal{C}_{\text{comp}}$.

In Section 7 it is shown that a spectrum of ANR defines a compactly supported ES theory on $\mathcal{C}_{\text{loc comp}}$. As a consequence we obtain the theorem concerning the Chern character previously mentioned. In Section 8 these ideas are applied to stable homotopy theory. The uniqueness theorem implies the duality theorem in stable homotopy theory.

2. Cohomology on a space

In this section we present the definition of a cohomology theory on a space X and some elementary properties of such theories. The present definition differs from that in [21] in that nonnegativity is not assumed.

All topological spaces will be assumed to be normal Hausdorff spaces. If H is a contravariant functor from a category of subsets (and inclusion maps between them) of such a space to the category of graded abelian groups (and homomorphisms of degree zero between them) we use the following notation. If H is defined for an inclusion map $i:B\subset A$ and $u\in H(A)$ then $u|B\in H(B)$ is defined by $u|B=H(i)(u)$. With this notation the statement that H is a contravariant functor is equivalent to the two conditions:

i) for $u\in H(A)$, $u|A=u$, and ii) for $C\subset B\subset A$ and $u\in H(A)$ then $(u|B)|C=u|C$. In general we will use ρ to denote a homomorphism induced by an inclusion map (i.e. $\rho=H(i):H(A)\to H(B)$ for $i:B\subset A$). Given a topological space X let $cl(X)$ denote the category of all closed subsets of X and all inclusion maps between them. A <u>cohomology theory</u> H,δ on X consists of i) a contravariant functor H from $cl(X)$ to the category of graded abelian groups $(H(X)=\{H^q(X)\}_{q\in\mathbb{Z}})$ such that $H(\emptyset)=0$ and ii) a natural transformation δ assigning to every two closed subsets $A,B\subset X$ a homomorphism of degree 1

$$\delta : H(A\cap B)\longrightarrow H(A\cup B)$$

such that the following are satisfied:

<u>Continuity</u> For every closed $A\subset X$ there is an isomorphism

$$\rho : \varinjlim \{ H^q(N) | N \text{ a closed nbhd of } A \text{ in } X\} \approx H^q(A)$$

where $\rho\{u\} = u|A$ for $u\in H^q(N)$.

<u>MV exactness</u> For every two closed sets A,B in X there is an exact sequence

$$\cdots \xrightarrow{\delta} H^q(A\cup B) \xrightarrow{\alpha} H^q(A)\oplus H^q(B) \xrightarrow{\beta} H^q(A\cap B) \xrightarrow{\delta} H^{q+1}(A\cup B) \xrightarrow{\alpha} \cdots$$

where $\alpha(u) = (u|A, u|B)$ for $u\in H^q(A\cup B)$ and $\beta(u,v) = u|A\cap B - v|A\cap B$ for $u\in H^q(A), v\in H^q(B)$.

<u>Remarks (2.1)</u> This definition differs from that in [21] in that it has <u>not</u> been assumed that $H^q(A) = 0$ for $q<0$ for all closed $A\subset X$. This is a substantial difference in that the uniqueness theorem [21, Theorem 2.20] is only proved for finite dimensional spaces X and, in fact, is false for arbitrary compact spaces (see Section 4 below). Two other minor differences with the definition in [21] are to be noted. One is that we have here used the term "continuity" for what was previously called "tautness". The other is that we have here used "α,β" for homomorphisms previously denoted by "J,I".

(2.2) The continuity property is equivalent to the following two conditions. A contravariant functor H on $cl(X)$ is said to be <u>extensive</u> if given $u\in H^q(A)$ there is a closed nbhd N of A in X and $v\in H^q(N)$ such that $v|A=u$. Then H is extensive is equivalent to the assertion that the homomorphism ρ of the continuity property is an epimorphism.

A contravariant functor H on $cl(X)$ is said to be <u>reductive</u> if given N a closed nbhd of A in X and $u \in H^q(N)$ such that $u|A=0$ then there is a closed nbhd M of A in N such that $u|M=0$. Then H is reductive is equivalent to the assertion that the homomorphism ρ of the continuity property is a monomorphism.

The terms "extensive" and "reductive" were introduced by Wallace [23] . Note that the continuity property does not involve the natural transformation δ . Thus, we speak of a continuous H and an exact H,δ . Clearly H is continuous if and only if it is extensive and reductive.

(2.3) If H,δ is a cohomology theory on X, then its p^{th} <u>suspension</u> $\sigma^p H,\delta$ where $(\sigma^p H)^q(A)=H^{p+q}(A)$ and $\delta : H^{p+q}(A\cap B) \longrightarrow H^{p+q+1}(A\cup B)$ is also a cohomology theory on X for every $p \in \mathbb{Z}$.

(2.4) If $\{H_j,\delta_j\}_{j\in J}$ is a family of cohomology theories on X indexed by an arbitrary set J then the <u>direct sum</u> $\bigoplus_{j\in J} H_j, \bigoplus_{j\in J}\delta_j$ where

$$\left(\bigoplus_{j\in J} H_j\right)^q(A) = \bigoplus_{j\in J} H_j^q(A) \quad \text{and} \quad \bigoplus_{j\in J}\delta_j : \bigoplus_{j\in J} H_j^q(A\cap B) \longrightarrow \bigoplus_{j\in J} H_j^{q+1}(A\cup B)$$

is also a cohomology theory on X (because direct sums commute with direct limits and preserve exactness).

(2.5) If H,δ is a cohomology theory on X and G is a torsion-free abelian group the <u>tensor product</u> $H \otimes G$, $\delta \otimes 1_G$ where $(H \otimes G)^q(A) = H^q(A) \otimes G$ and $\delta \otimes 1_G : H^q(A \cap B) \otimes G \longrightarrow H^{q+1}(A \cup B) \otimes G$ is also a cohomology theory on X (because tensor product with G commutes with direct limits and, because G is torsion-free, preserves exactness).

(2.6) Let $f: X \longrightarrow Y$ be a closed continuous map and let H, δ be a cohomology theory on X. The <u>direct image</u> $f_* H$, where $(f_* H)^q(A) = H^q(f^{-1}(A))$ for $A \subset Y$ and $\delta : H^q(f^{-1}(A) \cap f^{-1}(B)) \longrightarrow H^{q+1}(f^{-1}(A) \cup f^{-1}(B))$ is a cohomology theory on Y (closedness of f and normality of Y imply that the collection $\{f^{-1}(N) \mid N$ a closed nbhd of A in $Y\}$ is cofinal in the collection of closed nbhds of $f^{-1}(A)$ in X).

The following proposition concerns two consequences of the continuity property of cohomology theories.

<u>Proposition (2.7)</u> Let H be a continuous contravariant functor on $cl(X)$.

1) If $A \subset A'$ are closed subsets of X, there is an isomorphism

$$\rho : \varinjlim \{H^q(N) \mid N \text{ a closed nbhd of } A \text{ in } A'\} \approx H^q(A)$$

where $\rho\{u\} = u|A$ for $u \in H^q(N)$.

2) If $\{A_j\}_{j \in J}$ is a family of compact subsets of X directed downward by inclusion, there is an isomorphism

$$\rho : \varinjlim \{H^q(A_j)\}_{j \in J} \approx H^q(\bigcap_{j \in J} A_j)$$

where $\rho\{u\} = u|\bigcap_{j \in J} A_j$ for $u \in H^q(A_j)$.

Proof. 1) follows as did the corresponding property for the cohomology theories previously defined [21, Lemma 2.13] 2) follows from continuity using the fact that compactness of A_j for each $j\in J$ implies that if N is any nbhd of $\bigcap_{j\in J} A_j$ there is $j\in J$ such that $A_j\subset N$. □

A cohomology theory H,δ is said to be nonnegative if $H^q(A) = 0$ for $q<0$ and all closed $A\subset X$. The cohomology theories considered in [21] were nonnegative cohomology theories.

A cohomology theory H,δ on X is said to be compactly supported (or to have compact supports) if given $u\in H^q(A)$ there is a decomposition $A=B\cup C$ where B is closed, C is compact and $u|B=0$.

A cohomology theory H,δ is said to be additive if given a discrete* family $\{A_j\}_{j\in J}$ of closed sets there is an isomorphism $H^q(\bigcup_{j\in J} A_j)\approx \prod_{j\in J} H^q(A_j)$ sending $u\in H^q(\bigcup_{j\in J} A_j)$ to the family $\{u|A_j\}_{j\in J}$.

In none of the above definitions does the natural transformation δ enter.

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A family $\{A_j\}_{j\in J}$ of subsets of a topological space is said to be discrete if every point of X has a nbhd meeting at most one member of the family. This implies the members of the family are pairwise disjoint and, since a discrete family is obviously locally finite, if each is closed in X, then $\bigcup_{j\in J} A_j$ is also closed in X.

If H,δ and H',δ' are cohomology theories on the same space X, a homomorphism φ from H,δ to H',δ' is a natural transformation from H to H' commuting up to sign with δ,δ'

Recall [21] that a homomorphism $\varphi:G \longrightarrow G'$ of degree 0 between graded abelian groups is on n-equivalence if $\varphi:G^q \longrightarrow G'^q$ is an isomorphism for all $q<n$ and a momomorphism for $q=n$. We prove two results about homomorphisms between cohomology theories on the same space X which are assumed to be n-equivalences for certain subsets of X. (Actually neither result requires the continuity property.)

Proposition (2.8) Let $\varphi:H,\delta \longrightarrow H',\delta'$ be a homomorphism between two compactly supported cohomology theories on X and suppose there is $n\in\mathbb{Z}$ such that $\varphi_C:H(C)\longrightarrow H'(C)$ is an n-equivalence for every compact $C\subset X$. Then $\varphi_A:H(A)\longrightarrow H'(A)$ is an n-equivalence for every closed $A\subset X$.

Proof. 1) We prove $\varphi_A:H^q(A)\longrightarrow H'^q(A)$ is an epimorphism for $q<n$ and A closed in X. Let $u\in H'^q(A)$ where $q<n$. Since H' is compactly supported, $A=B\cup C$ where B is closed, C is compact and $u|B=0$. The following diagram has exact rows and commutes up to sign

$$\begin{array}{ccccccc}
H^{q-1}(B\cap C) & \xrightarrow{\delta} & H^q(A) & \xrightarrow{\alpha} & H^q(B)\oplus H^q(C) & \xrightarrow{\beta} & H^q(B\cap C) \\
\varphi\downarrow\approx & & \varphi\downarrow & & \downarrow\varphi & & \approx\downarrow\varphi \\
H'^{q-1}(B\cap C) & \xrightarrow{\delta'} & H'^q(A) & \xrightarrow{\alpha'} & H'^q(B)\oplus H'^q(C) & \xrightarrow{\beta'} & H'^q(B\cap C)
\end{array}$$

and the vertical maps on the two ends are isomorphisms because $B\cap C$ is compact and $q<n$. By [21, Lemma 2,19 part 2], $\alpha'^{-1}(\operatorname{im}\varphi)\subset \operatorname{im}\varphi$.

Since $\varphi_C : H^q(C)\approx H'^q(C)$ there is $v\in H^q(C)$ such that $\varphi(v)=u|C$. Then $(0,v)\in H^q(B)\oplus H^q(C)$ is such that $\varphi(0,v)=(0,\varphi(v))=(0,u|C)=$ $=\alpha'(u)$ so $\alpha'(u)\in \operatorname{im}\varphi$ and, hence, $u\in \operatorname{im}\varphi$.

2) We prove $\varphi_A : H^q(A)\longrightarrow H'^q(A)$ is a monomorphism for $q\leq n$. Assume $u\in H^q(A)$, $q\leq n$ is such that $\varphi(u)=0$. Because H has compact supports, $A=B\cup C$ where B is closed, C is compact and $u|B=0$. The following diagram has exact rows and commutes up to sign

$$\begin{array}{ccccccc}
H^{q-1}(B)\oplus H^{q-1}(C) & \xrightarrow{\beta} & H^{q-1}(B\cap C) & \xrightarrow{\delta} & H^q(A) & \xrightarrow{\alpha} & H^q(B)\oplus H^q(C)\\
\varphi\downarrow & & \varphi\downarrow\approx & & \downarrow\varphi & & \downarrow\varphi\\
H'^{q-1}(B)\oplus H'^{q-1}(C) & \xrightarrow{\beta'} & H'^{q-1}(B\cap C) & \xrightarrow{\delta'} & H'^q(A) & \xrightarrow{\alpha'} & H'^q(B)\oplus H'^q(C)
\end{array}$$

and the first vertical map is an epimorphism by 1) above and the second vertical map is an isomorphism because $B\cap C$ is compact and $q-1<n$. By [21, Lemma 2,19 part 1] $\ker\alpha\cap\ker\varphi = 0$. Since $\varphi_C : H^q(C)\longrightarrow H'^q(C)$ is a monomorphism for $q\leq n$ it follows that $u|C=0$ (since $\varphi(u)=0$). Therefore, $\alpha(u)=0$ so $u\in\ker\varphi\cap\ker\alpha$ and so $u=0$.

□

Proposition (2.9). Let $\varphi: H,\delta \longrightarrow H',\delta'$ be a homomorphism between two additive cohomology theories on a paracompact space X. Suppose there is $n \in \mathbb{Z}$ and an open covering $\mathcal{U}$ of X such that $\varphi_A: H(A) \longrightarrow H'(A)$ is an n-equivalence for every A contained in some element of $\mathcal{U}$. Then $\varphi_A: H(A) \longrightarrow H'(A)$ is an n-equivalence for every closed $A \subset X$.

Proof. Let $\mathcal{C}$ be the collection of closed subsets A of X such that $\varphi_B: H(B) \longrightarrow H'(B)$ is an n-equivalence for every closed $B \subset A$. By the hypothesis of the Proposition every point of X has a closed nbhd in $\mathcal{C}$. From the definition of $\mathcal{C}$ it is clear that if A', A are closed sets with $A' \subset A$ and $A \in \mathcal{C}$ then $A' \in \mathcal{C}$. It follows from the exactness of H,δ and of H',δ' that if A, B are in $\mathcal{C}$ then $A \cup B \in \mathcal{C}$. It follows from the additivity of H,δ and H',δ' that if $\{A_j\}_{j \in J}$ is a discrete family in $\mathcal{C}$ then $\bigcup_{j \in J} A_j$ is also in $\mathcal{C}$. Hence, $\mathcal{C}$ satisfies the hypotheses of [17, Theorem 5.5] so $X \in \mathcal{C}$. □

3. Other theories

In [13] Lawson considered cohomology theories satisfying axioms similar to, but somewhat different from, those defined in Section 2. We prove that his definition is essentially equivalent to ours. We also consider theories defined on $cl(X)^2$ satisfying axioms similar to those of Eilenberg and Steenrod [5] and show that they define cohomology theories.

An L theory H,Δ on X is defined to be a pointwise taut cohomology theory on X in the sense of [13]. Thus, it consists of:

i) A contravariant functor H from $cl(X)$ to nonnegative graded abelian groups such that $H(\emptyset)=0$, and

ii) A natural transformation of degree 1, $\Delta : H^q(A\cap B) \longrightarrow H^{q+1}(A\cup B)$ defined for every two closed subsets A,B of X such that $\mathrm{int}_{A\cup B}A \cup \mathrm{int}_{A\cup B}B = A\cup B$ (in which case we say A,B are an excisive couple in X), satisfying

1) For every $x\in A$ where A is closed in X there is an isomorphism

$$\rho : \varinjlim \{H^q(N) \mid N \text{ a closed nbhd of } x \text{ in } A\} \approx H^q(x)$$

such that $\rho(u) = u|x$ for $u \in H^q(N)$.

2) For every excisive couple A,B in X there is an exact sequence

$$\ldots \xrightarrow{\Delta} H^q(A\cup B) \xrightarrow{\alpha} H^q(A)\oplus H^q(B) \xrightarrow{\beta} H^q(A\cap B) \xrightarrow{\Delta} H^{q+1}(A\cup B) \xrightarrow{\alpha} \ldots$$

where $\alpha(u)=(u|A,u|B)$ and $\beta(u,v)=u|A\cap B-v|A\cap B$.

Thus, an L theory differs from a cohomology theory in two respects. Firstly, in an L theory the continuity property 1) is for a point x in a closed set A rather than for a closed set A in X. Secondly, in an L theory the natural transformation Δ is defined and the exactness property 2) of H,Δ is required only for excisive couples in X whereas in a cohomology theory δ is defined and exactness of H,δ is required for every two closed sets in X.

An L theory is <u>additive</u> if it satisfies the same additivity property defined in Section 2 for cohomology theories. The following shows that on paracompact spaces additive L theories are essentially the same as additive nonnegative cohomology theories.

<u>Theorem (3.1).</u> Every nonnegative cohomology theory on X is an L theory on X. On a paracompact space X every additive L theory is isomorphic to one obtained in this way from an additive nonnegative cohomology theory on X.

<u>Proof.</u> If H,δ is a nonnegative cohomology theory on X, then 1) above follows from 1) of Proposition (2.7) and 2) above is MV exactness for excisive couples in X which is a consequence of MV exactness for every two closed sets A,B in X. Thus, H,δ is an L theory on X.

Conversely, given H,Δ an additive L theory on X and given A closed in X define

$$\bar{H}^q(A) = \varinjlim \{H^q(N) \mid N \text{ a closed nbhd of } A \text{ in } X\} .$$

Then $\bar{H}$ is obviously a contravariant functor from $cl(X)$ to nonnegative graded abelian groups such that $\bar{H}(\emptyset)=0$ and satisfying the continuity property.

To define a natural transformation of degree 1 $\bar{\delta} : \bar{H}^q(A\cap B) \longrightarrow \bar{H}^{q+1}(A\cup B)$ such that $\bar{H},\bar{\delta}$ satisfy MV exactness for every two closed sets $A,B\subset X$ we first show that if A,B are closed sets in X and U,V,W are open nbhds of $A,B,A\cap B$, respectively, in X, there exist closed nbhds M,N of A,B, respectively, such that M,N is an excisive couple and $M\subset U$, $N\subset V$, $M\cap N\subset W$. Since $U\cap V\cap W$ is a nbhd of $A\cap B$ and X is normal there exists an open nbhd W' of $A\cap B$ with $\bar{W}'\subset U\cap V\cap W$. Then $A-W',B-W'$ are disjoint closed subsets of X so there exist disjoint open nbhds U' of $A-W'$ and V' of $B-W'$ and, without loss of generality, it can be assumed also that $\bar{U}'\subset U,\bar{V}'\subset V$ and $\bar{U}'\cap\bar{V}'=\emptyset$. Then $M=\bar{U}'\cup\bar{W}'$ is a closed nbhd of A contained in U, $N=\bar{V}'\cup\bar{W}'$ is a closed nbhd of B contained in V, and

$$\begin{aligned} \operatorname{int}_{M\cup N} M &= M-\overline{(M\cup N)-M} = M-\overline{\bar{U}'\cup\bar{W}'\cup\bar{V}'-\bar{U}'\cup\bar{W}'} \\ &= M-\overline{\bar{V}'-\bar{W}'} = \bar{U}'\cup\bar{W}'-(\overline{\bar{V}'-\bar{W}'})\supset\bar{U}'\cup\bar{W}'-\bar{V}' \end{aligned}$$

Similarly $\operatorname{int}_{M\cup N} N = \bar{V}'\cup\bar{W}'-(\overline{\bar{U}'-\bar{W}'})\supset\bar{V}'\cup\bar{W}'-\bar{U}'$ so that

$$(\bar{U}'\cup\bar{W}'-\bar{V}')\cup(\bar{V}'\cup\bar{W}'-\bar{U}')\subset \operatorname{int}_{M\cup N} M\cup\operatorname{int}_{M\cup N} N\subset M\cup N$$

Since $M\cup N = \bar{U}'\cup\bar{W}'\cup\bar{V}' = (\bar{U}'\cup\bar{W}'-\bar{V}')\cup(\bar{V}'\cup\bar{W}'-\bar{U}')$, it follows that $\operatorname{int}_{M\cup N} M\cup\operatorname{int}_{M\cup N} N = M\cup N$. Therefore M,N are closed nbhds of A,B, respectively, with the requisite properties.

Thus, as M,N vary over closed nbhds of A,B, respectively, such that M,N is an excisive couple in X, it follows that $M,N,M\cap N,M\cup N$ vary over a cofinal family of closed nbhds $A,B,A\cap B,A\cup B$, respectively. The direct limit of the exact sequences

$$\ldots\xrightarrow{\Delta} H^q(M\cup N) \xrightarrow{\alpha} H^q(M)\oplus H^q(N) \xrightarrow{\beta} H^q(M\cap N) \xrightarrow{\Delta} H^{q+1}(M\cup N) \longrightarrow\ldots$$

is an exact sequence

$$\ldots\xrightarrow{\bar\delta} \bar H^q(A\cup B) \xrightarrow{\bar\alpha} \bar H^q(A)\oplus\bar H^q(B) \xrightarrow{\bar\beta} \bar H^q(A\cap B) \xrightarrow{\bar\delta} \bar H^{q+1}(A\cup B) \longrightarrow\ldots$$

and defines the natural transformation

$$\bar\delta : \bar H^q(A\cap B) \longrightarrow \bar H^{q+1}(A\cup B) \quad .$$

Therefore, $\bar H,\bar\delta$ is a nonnegative cohomology theory on X. We show it is additive. Let $\{A_j\}_{j\in J}$ be a discrete family of closed subsets of X. For each j' let $U_{j'} = X-\bigcup_{j\neq j'} A_j$. Then $\mathcal{U} = \{U_{j'}\}_{j'\in J}$ is an open covering of X by sets each of which meets at most one member of $\{A_j\}$. Let $\mathcal{V}$ be an open star refinement of $\mathcal{U}$ (which exists because X is paracompact [4]) and, for each $j\in J$, let

$$V_j = \cup\,\{V\in\mathcal{V} \mid V\cap A_j \neq\emptyset\}$$

Then $A_j\subset V_j\subset U_j$. Furthermore, if V is any element of $\mathcal{V}$ and

$V \cap V_j \neq \emptyset$, $V \cap V_k \neq \emptyset$ for $j,k \in J$ there are $V', V'' \in \mathcal{V}$ with $V \cap V' \neq \emptyset, V' \cap A_j \neq \emptyset$ and $V \cap V'' \neq \emptyset$, $V'' \cap A_k \neq \emptyset$. Then $V', V'' \subset V^* \subset$ some element of $\mathcal{U}$. Since no element of $\mathcal{U}$ meets more than one member of $\{A_j\}$, it follows that $j=k$. Hence, every element of $\mathcal{U}$ meets at most one member of $\{V_j\}_{j \in J}$ so $\{V_j\}_{j \in J}$ is a discrete family of open nbhds of $\{A_j\}_{j \in J}$, respectively. If U is any open nbhd of $A = \cup A_j$, then $\{U \cap V_j\}_{j \in J}$ is a discrete family of open nbhds of $\{A_j\}$, respectively, contained in U. For each $j \in J$ let N_j be a closed nbhd of A_j contained in $U \cap V_j$. Then $\{N_j\}$ is a discrete family of closed nbhds of $\{A_j\}$, respectively, whose union is contained in U. This implies that the collection of unions of discrete families $\{N_j\}$ where N_j is a closed nbhd of A_j for each $j \in J$ is cofinalin the family of all closed nbhds of A in X. Therefore,

$$\begin{aligned}\bar{H}(A) &\approx \underset{\longrightarrow}{\lim} \{H(N) \mid N = \cup N_j, \{N_j\} \text{ discrete, } N_j \text{ a closed nbhd of } A_j \text{ in } X\} \\ &\approx \underset{\longrightarrow}{\lim} \{\prod_{j \in J} H(N_j) \mid \{N_j\} \text{ discrete, } N_j \text{ a closed nbhd of } A_j \text{ in } X\} \\ &\approx \prod_{j \in J} \bar{H}(A_j)\end{aligned}$$

so $\bar{H}$ is additive.

Clearly there is a natural homomorphism $\varphi: \bar{H}, \bar{\delta} \longrightarrow H, \Delta$ defined by $\varphi\{u\} = u|A$ for $u \in H(N)$, N a closed nbhd of A. Then φ is a homomorphism between two additive L theories on the paracompact space X. The hypothesis 1) above implies that φ_x is an isomorphism for every $x \in X$. By [13, Theorem 3.2] φ is an isomorphism of L theories. Therefore, H,Δ is isomorphic to the L theory determined by the cohomology theory $\overline{H},\overline{\delta}$.

□

Cohomology theories on X frequently arise from suitable contravariant functors defined on $cl(X)^2$, the category of pairs of closed subsets of X. We formalize this using the following Eilenberg-Steenrod axioms. An ES theory H,δ^* on X consists of:

i) a contravariant functor H from $cl(X)^2$ to the category of graded abelian groups, and

ii) a natural transformation of degree 1

$$\delta^* : H^q(B,\phi) \longrightarrow H^{q+1}(A,B)$$

for every (A,B) in $cl(X)^2$ such that the following hold:

Continuity. For every closed A in X there is an isomorphism

$$\rho : \underset{\longrightarrow}{\lim} \{H^q(N,\emptyset) \mid N \text{ a closed nbhd of } A \text{ in } X\} \approx H^q(A,\emptyset)$$

where $\rho\{u\} = u|(A,\emptyset)$ for $u \in H^q(N,\emptyset)$.

Exactness. For every closed pair (A,B) in X the following sequence is exact

$$\cdots \xrightarrow{\delta^*} H^q(A,B) \xrightarrow{H(j)} H^q(A,\emptyset) \xrightarrow{H(i)} H^q(B,\emptyset) \xrightarrow{\delta^*} H^{q+1}(A,B) \longrightarrow \cdots$$

where $i:(B,\emptyset)\subset(A,\emptyset)$ and $j:(A,\emptyset)\subset(A,B)$.

Excision. For closed sets A,B in X there is an isomorphism

$$\rho : H(A\cup B,B) \approx H(A,A\cap B) \quad .$$

Thus, an ES theory satisfies some of the Eilenberg-Steenrod axioms [5] on $cl(X)^2$. It need not satisfy the homotopy axiom nor the dimension axiom.

Proposition (3.2). If H,δ^* is an ES theory on X there is a cohomology theory H',δ' on X such that $H'(A)=H(A,\emptyset)$ and $\delta'=H'(A\cap B) \longrightarrow H'(A\cup B)$ is suitably defined.

Proof. It is standard [5] that the exactness property of H implies that $H'(\emptyset)=H(\emptyset,\emptyset)=0$ and the exactness and excision properties of H,δ^* imply the exactness of Mayer-Vietoris sequences with δ' suitably defined. The continuity of H' follows from that of H.

□

In general we do not have a way of associating to a cohomology theory on X on ES theory on X. With suitable definitions of cohomology theories and ES theories on larger categories we will show in Section 6 that the two theories are equivalent on the category of all compact spaces.

The concepts of <u>nonnegativity</u>, <u>compactly supported</u> and <u>additivity</u> are defined for ES theories to correspond to the same properties of the associated cohomology theories.

Most of the cohomology theories on X we consider will be obtained from an ES theory on X using Proposition (3.2).

<u>Example. (3.3)</u> Define ${}^{\Delta}H$ on $cl(X)^2$ by ${}^{\Delta}H^q(A,B)=H_{-q}(X-B\ ,X-A)$ (singular homology with an arbitrary but fixed coefficient group) and define $\delta^*:{}^{\Delta}H^q(B,\emptyset) \longrightarrow H^{q+1}(A,B)$ to equal the connecting homomorphism $\partial:H_{-q}(X,X-B)\longrightarrow H_{-q-1}(X-B,X-A)$ of the triple $(X,X-B,X-A)$. Exactness of the homology sequence of the triple $(X,X-B,X-A)$ yields exactness of ${}^{\Delta}H,\delta^*$. The excision property for ${}^{\Delta}H$ follows from the fact that A,B closed in X imply that X-B,X-A are open in X so the inclusion map

$$(X-B,\ (X-A)\cap(X-B)) \subset ((X-A)\cup(X-B),\ X-A)$$

incluces isomorphisms of singular homology.

The continuity property for ${}^{\Delta}H$ follows from the fact that singular homology is carried by compact sets and the fact that, as N varies over closed nbhds of A in X,X-N varies over a collection of open subsets of X directed upward by inclusion whose union equals $X-A$ and this implies [20, Theorem 4.4.6]

$$\varinjlim \{H_{-q}(X,X-N) \mid N \text{ a closed nbhd of } A \text{ in } X\} \approx H_{-q}(X,X-A) .$$

Therefore, H,δ^* is an ES theory on X .

In case X is locally compact this theory is compactly supported. In fact, if $z \in {}^{\Delta}H^q(A) = H_{-q}(X,X-A)$ there is an open set U with compact closure $\bar{U}$ such that z is in the image of $H_{-q}(U \cup (X-A),X-A) \longrightarrow H_{-q}(X,X \cup A)$. Then $A=(A-U) \cup \overline{A \cap \bar{U}}$ where $A-U=B$ is closed, $A \cap \bar{U} = C$ is compact and $z|B=0$ because $X-B=U \cup (X-A)$ so the composite

$$H_{-q}(U \cup (X-A),X-A) \longrightarrow H_{-q}(X,X-A) \longrightarrow H_{-q}(X,X-B)$$

is zero.

4. Finite dimensional spaces

First we present an example to show that the main uniqueness theorem [21, Theorem 2.20] generally fails if H, H' are not nonnegative cohomology theories on X. Then we prove that the uniqueness theorem is valid for arbitrary H, H' if X is assumed to be a finite dimensional space (finite dimensional will mean a finite dimensional separable metric space, as in [8]). We then present some applications of this uniqueness theorem to manifolds.

We consider the cohomology theory $K(X)$ defined for a compact Hausdorff space using complex vector bundles over X (see Section 7 below). It is well known [10] that K is a cohomology theory on every compact Hausdorff space X and that it is periodic of order 2 (i.e. $K^{q+2}(A) \approx K^q(A)$ for all q). Since $K^0(A) \approx \mathbb{Z} \oplus \tilde{K}(A)$, it follows that K is not a nonnegative cohomology theory.

In [22] there is given an example* of a continuous map $f: X \longrightarrow Y$ between compact Hausdorff spaces such that for every $y \in Y, f|f^{-1}y: f^{-1}y \longrightarrow y$ induces an isomorphism of $K(y)$ with $K(f^{-1}y)$ but $K(Y)$ is not isomorphic to $K(X)$. There is a homomorphism f from the cohomology theory K on Y to the direct image (as in Example 2.6) $f_* K$ (which is also a cohomology theory on Y). This homomorphism is an isomorphism for every $y \in Y$ but is not an isomorphism for Y itself. Thus, the uniqueness theorem of [21] is not true for arbitrary cohomology theories on a compact Hausdorff space.

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* The author is indebted to S. Ferry for pointing out this example to him.

In order to extend the uniqueness theorem to arbitrary cohomology theories we need an inductive argument based on something other than degree of cohomology. Such an argument is possible based on dimension of the subset $A \subset X$. First we establish the following property of finite dimensional spaces needed for the inductive proof.

Lemma 4.1 Let A be a compact metric space of dimension q. Given $\varepsilon > 0$ there exists a finite number of closed sets $A_1, \dots, A_m$ such that:

1) $A = A_1 \cup \dots \cup A_m$
2) $\operatorname{diam} A_i \le \varepsilon$ for $1 \le i \le m$
3) $\dim (A_i \cap A_j) < q$ for $1 \le i \ne j \le m$

Proof. Because A is a compact metric space of dimension q there is a finite open covering $A = U_1 \cup \dots \cup U_m$ such that $\operatorname{diam} U_i < \varepsilon$ for $1 \le i \le m$ and $\dim \operatorname{bdry} U_i < q$ for $1 \le i \le q$. Such a covering of A will be called ε-thin. We prove by induction on m that if A has an ε-thin open covering by m sets, then A has a closed covering by m sets $A_1, \dots, A_m$ satisfying 1), 2), 3) of the Lemma.

In case there is an ε-thin covering of A by one set U, we take $A_1 = A$ and observe that 1), 2) 3) are satisfied (3) is vacuously satisfied). Now assume $m > 1$ and that the result is valid for compact sets having an ε-thin open covering of $m-1$ sets. Suppose $A = U_1 \cup \dots \cup U_m$ is an ε-thin covering of A by m sets. Let $A_1 = \bar{U}_1$ and consider $B = A - U_1$. Then B is a compact metric space of dimension q with the ε-thin open covering $V_2=B\cap U_2,\ldots,V_m=B\cap U_m$ by $m-1$ sets (note that diam $V_j \le \text{diam } U_j < \varepsilon$ for $2\le j\le m$ and, since $B\cap\overline{U}_j\cap(B-U_j)$ is a closed subset of $\overline{U}_j\cap(A-U_j)$,

$$\begin{aligned}\dim \text{bdry } V_j &= \dim\,[B\cap\overline{U}_j\cap(B-U_j)] \le \dim[\overline{U}_j\cap(A-U_j)] \\ &= \dim \text{bdry } U_j<q) .\end{aligned}$$

By the inductive hypothesis $B=B_2\cup\ldots\cup B_m$ where B_i is closed, diam $B_i\le\varepsilon$ for $2\le i\le m$, and $\dim B_i\cap B_j<q$ for $2\le i\ne j\le m$. Define $A_i=B_i$ for $2\le i\le m$. Then $A=A_1\cup\ldots\cup A_m$, A_i is closed, and diam $A_i\le\varepsilon$ for $1\le i\le m$. Clearly
$\dim A_i\cap A_j<q$ if $2\le i\ne j\le m$. For $2\le i\le m$

$$\begin{aligned}\dim\,(A_1\cap A_i) = \dim\,(\overline{U}_1\cap B_i) &= \dim\,[(U_1\cup \text{bdry } U_1)\cap B_i] \\ &= \dim\,(\text{bdry } U_1\cap B_i) \quad \text{because } U_1\cap B_i=\emptyset \\ &\le \dim \text{bdry } U_1<q\end{aligned}$$

so $A_1,\ldots,A_m$ have all the requisite properties. □

We now prove the uniqueness theorem for finite dimensional spaces.

Theorem (4.2). Let $\varphi:H \longrightarrow H'$ be a homomorphism of cohomology theories on a metric space X and let $n\in\mathbb{Z}$ be such that $\varphi_x:H(x) \longrightarrow H'(x)$ is an n-equivalence for every $x\in X$. Then φ_A is an n-equivalence for every compact finite dimensional subset $A\subset X$.

Proof. We prove the theorem by induction on the dimension of the subset A. If A has dimension -1, then $A=\emptyset$ and $\varphi_\emptyset$ is an isomorphism because $H(\emptyset)=0=H'(\emptyset)$ so $\varphi_\emptyset$ is an n-equivalence. Assume A is compact, $\dim A = q > -1$ and the result is valid for all compact subsets of dimension $< q$.

1) We prove $\varphi_A:H^k(A) \longrightarrow H'^k(A)$ is an epimorphism for $k<n$. Let $u\in H'^k(A)$, $k<n$ and assume $u\notin \operatorname{im}\varphi_A$. By Lemma 4.1 we have closed sets $B_1,\ldots,B_m$ such that $A=B_1\cup\ldots\cup B_m$, $\operatorname{diam} B_i \le 1$ for $1\le i\le m$ and $\dim B_i\cap B_j<q$ for $1\le i\ne j\le m$. We claim there is some i with $1\le i\le m$ such that $u|B_i\notin \operatorname{im}\varphi$. In fact, if there is no such i let $C_j=B_1\cup\ldots\cup B_j$ for $1\le j\le m$. We prove by induction on j that $u|C_j\in \operatorname{im}\varphi$, which will give a contradiction because $C_m=A$ and $u|C_m=u\notin \operatorname{im}\varphi$ by hypothesis. For $j=1$ we know by hypothesis on $u|B_i$ that $u|C_1=u|B_1\in \operatorname{im}\varphi$. Assume $j>1$ and $u|C_{j-1}\in \operatorname{im}\varphi$. The following diagram has exact rows and commutes up to sign

$$\begin{array}{ccccccc}
H^{k-1}(C_{j-1}\cap B_j) & \xrightarrow{\delta} & H^k(C_j) & \xrightarrow{\alpha} & H^k(C_{j-1})\oplus H^k(B_j) & \xrightarrow{\beta} & H^k(C_{j-1}\cap B_j) \\
\varphi\downarrow\approx & & \varphi\downarrow & & \varphi\downarrow & & \approx\downarrow\varphi \\
H'^{k-1}(C_{j-1}\cap B_j) & \xrightarrow{\delta'} & H'^k(C_j) & \xrightarrow{\alpha'} & H'^k(C_{j-1})\oplus H'^k(B_j) & \xrightarrow{\beta'} & H'^k(C^k_{j-1}\cap B_j)
\end{array}$$

and the first and last vertical homomorphisms are isomorphisms because $k<n$ and

$$\dim (C_{j-1}\cap B_j) \le \dim ((B_1\cup\ldots\cup B_{j-1})\cap B_j) \le \dim (B_1\cap B_j\cup\ldots\cup B_{j-1}\cap B_j)<q$$

By [21, Lemma 2.19 part 2] $\alpha'^{-1}(\mathrm{im}\,\varphi)\subset \mathrm{im}\,\varphi$. The inductive hypothesis on $u|C_{j-1}$ implies $u|C_{j-1}\in \mathrm{im}\,\varphi$. Also we have supposed $u|B_j \in \mathrm{im}\,\varphi$. Therefore, $\alpha'|u(C_j)\in \mathrm{im}\,\varphi$ so $u|C_j\in \mathrm{im}\,\varphi$. This completes the induction. Thus, the assumption that $u|B_i\in \mathrm{im}\,\varphi$ for $1\le i\le m$ leads to a contradiction so there is some i such that $u|B_i\notin \mathrm{im}\,\varphi$.

Choose A_1 to be a B_i such that $u|A_1\notin \mathrm{im}\,\varphi$. Then $A\supset A_1$, diam $A_1\le 1$ and $u|A_1\notin \mathrm{im}\,\varphi$. Repeat the argument to obtain $A_1\supset A_2$, A_2 closed, diam $A_2\le\frac{1}{2}$ and $u|A_2\notin \mathrm{im}\,\varphi$. Continue to obtain a decreasing sequence of closed sets

$$A\supset A_1\supset A_2\supset\ldots$$

such that diam $A_i\le 1/2^{i-1}$ and $u|A_i\notin \mathrm{im}\,\varphi$. Then $\cap A_i$ is a single point, say x, and by continuity of H,H' it follows that $u|x\notin \mathrm{im}\,\varphi_x$ contradicting the hypothesis that φ_x is an epimorphism in dimensions $<n$. Therefore, $u\in \mathrm{im}\,\varphi_A$ so φ_A is an epimorphism for dimensions $<n$.

2) We prove $\varphi_A : H^k(A) \longrightarrow H'^k(A)$ is a monomorphism for $k\le n$. Assume $u\in H^k(A), k\le n$ is such that $\varphi_A(u) = 0$. We want to prove $u = 0$. Assume $u\neq 0$ and let $A = B_1\cup\ldots\cup B_m$ be such that B_i is closed and diam $B_i\le 1$ for $1\le i\le m$ and $\dim (B_i\cap B_j) <q$ for $1\le i\neq j\le m$. We claim $u|B_i\neq 0$ for some

i with $1 \le i \le m$. Otherwise let $C_j = B_1 \cup \ldots \cup B_{j-1}$ for $1 \le j \le m$. We prove by induction on j that $u|C_j = 0$ (this would give a contradiction because $0 = u|C_m = u|A = u \neq 0$). For $j = 1$ we know by hypothesis on $u|B_i$ that $u|C_1 = u|B_1 = 0$. Assume $j > 1$ and that $u|C_{j-1} = 0$.

The following diagram has exact rows and commutes up to sign

$$\begin{array}{ccccccc}
H^{k-1}(C_{j-1}) \oplus H^{k-1}(B_j) & \xrightarrow{\beta} & H^{k-1}(C_{j-1} \cap B_j) & \xrightarrow{\delta} & H^k(C_j) & \xrightarrow{\alpha} & H^k(C_{j-1}) \oplus H^k(B_j) \\
\varphi \downarrow & & \varphi \downarrow \approx & & \downarrow \varphi & & \downarrow \varphi \\
H'^{k-1}(C_{j-1}) \oplus H'^{k-1}(B_j) & \xrightarrow{\beta'} & H'^{k-1}(C_{j-1} \cap B_j) & \xrightarrow{\delta'} & H'^k(C_j) & \xrightarrow{\alpha'} & H'^k(C_{j-1}) \oplus H'^k(B_j)
\end{array}$$

and the first vertical homomorphism is an epimorphism by 1) above and the second vertical homomorphism is an isomorphism because $\dim (C_{j-1} \cap B_j) < q$. By [21, Lemma 2.19 part 1] , $\ker \alpha \cap \ker \varphi = 0$. Since

$$\alpha (u|C_j) = (u|C_{j-1}, u|B_j) = (0,0) = 0$$

$u|C_j \in \ker \alpha \cap \ker \varphi$ so $u|C_j = 0$. This completes the induction. Thus, the hypothesis that $u|B_i = 0$ for all $1 \le i \le m$ leads to a contradiction so there is some i such that $u|B_i \neq 0$.

Choose A_1 to be a B_i such that $u|A_1 \neq 0$, continue as above to obtain a decreasing sequence of closed sets

$$A \supset A_1 \supset A_2 \supset \ldots$$

such that diam $A_i \leq 1/2^{i-1}$ and $u|A_i \neq 0$. Then $\cap A_i$ is a point $x \in X$ and by continuity of H and H', $u|x \neq 0$ contradicting the hypothesis that $\varphi_x : H^k(x) \longrightarrow H'^k(x)$ is a monomorphism for $k \leq n$. Therefore, $u = 0$ so φ_A is a monomorphism for dimensions $\leq n$.

□

Corollary (4.3) Let $\varphi : H \longrightarrow H'$ be a homomorphism of cohomology theories with compact supports on a finite dimensional space X and let n be such that $\varphi_x : H(x) \longrightarrow H'(x)$ is an n-equivalence for every $x \in X$. Then φ_A is an n-equivalence for every closed $A \subset X$.

Proof. By Theorem 4.2. φ_A is an n-equivalence for every compact subset $A \subset X$. The corollary follows from this and Proposition (2.8).

□

Corollary (4.4). Let H be a cohomology theory with compact supports on a finite dimensional space such that H(x) is nonnegative for every $x \in X$. Then H(A) is nonnegative for all closed subsets $A \subset X$.

Proof. Let O be the trivial cohomology theory on X (i.e. $O(A) = 0$ for closed $A \subset X$). Clearly O has compact supports. There is a homomorphism $\varphi : O \longrightarrow H$, and the hypothesis on H implies that $\varphi_x : O(x) \longrightarrow H(x)$ is a 0-equivalence for all $x \in X$. By Corollary (4.3), φ_A is a 0-equivalence for every closed $A \subset X$, which implies H(A) is nonnegative for every closed $A \subset X$.

□

Examples. (4.5) Let X be a paracompact n-manifold with or without boundary and consider the cohomology ${}^{\Delta}H$ defined in Example (3.3) on X. This is a cohomology on X with compact supports such that ${}^{\Delta}H^q(x)=H_{-q}(X,X-x)=0$ for $q \neq -n$ so $\sigma^{-n\Delta}H(x)$ is nonnegative for every $x \in X$ (where $\sigma^{-n\Delta}H$ is as defined in Remark (2.3)). By Corollary (4.4) $\sigma^{-n\Delta}H(A)$ is nonnegative for all closed A or, equivalently, $H_q(X,X-A)=0$ for all $q > n$ and all closed $A \subset X$.

(4.6) Let X be a paracompact n-manifold with bdry $\dot{X}$. We prove that for singular homology $H_q(X-\dot{X}) \approx H_q(X)$ for all q (see also [16]). It suffices to prove $H_q(X,X-\dot{X}) = 0$ for all q. Let ${}^{\Delta}H$ be the compactly supported cohomology theory on X defined in Example (3.3) and consider its restriction H' to $\dot{X}$ (i.e. for A closed in $\dot{X}$ we have $H'^q(A) = H_{-q}(X,X-A)$). Then $H'(x) = 0$ for all $x \in \dot{X}$. The unique homomorphism $\varphi : 0 \longrightarrow H'$ of the trivial cohomology to H' is such that φ_x is an isomorphism for all $x \in \dot{X}$. By Corollary (4.3), φ_A is an isomorphism for all closed $A \subset \dot{X}$. Taking $A = \dot{X}$ we see that $H_q(X,X-\dot{X}) = H'^{-q}(\dot{X}) = 0$ for all q.

Corollary (4.7). Let $\varphi : H \longrightarrow H'$ be a homomorphism of additive cohomology theories on a locally compact finite dimensional space X such that, for some n, φ_x is an n-equivalence for every $x \in X$. Then φ_A is an n-equivalence for every closed $A \subset X$.

Proof. By Theorem (4.2), φ_A is an n-equivalence for every compact $A \subset X$. Since X is locally compact, X has an open covering $\mathcal{U}$ such that every closed subset contained in some element of $\mathcal{U}$ is compact. The Corollary follows from this and Proposition (2.9). □

<u>Example (4.8)</u> Let X be a paracompact n-manifold with bdry $\dot{X}$. We show that for Čech cohomology $\check{H}^q(X) \approx \check{H}^q(X,X-\dot{X})$ for all q (see also [16]). It suffices to prove $\check{H}^q(X,X-\dot{X})=0$ for all q. There is an additive cohomology theory H on $\dot{X}$ with $H(A)=\check{H}(X,X-A)$ for closed $A \subset X$. The unique homomorphism $\varphi : O \longrightarrow H$ of the trivial cohomology theory (which is additive) into H has the property that $\varphi_x : O(x) \longrightarrow H(x)$ is an isomorphism for all $x \in X$. It follows from Corollary (4.7) that φ_A is an isomorphism for all closed $A \subset X$. In particular taking $A = X$ we have

$$O = H(\dot{X}) = \check{H}(X,X-\dot{X}) .$$

5. Cohomology theories on categories of spaces

In this section we consider cohomology theories defined on categories of topological spaces and continuous functions. These consist of contravariant functors and natural transformations on the category whose restriction to $cl(X)$ is a cohomology theory on X for every object X in the category. The categories of interest are the category of all compact spaces and continuous functions and the category of all locally compact spaces and proper continuous functions. We prove the uniqueness theorem for cohomology theories on these two categories. We also show that cohomology theories on the category of compact spaces are equivalent to compactly supported cohomology theories on the category of locally compact spaces.

We begin by considering cohomology theories on cubes. By a <u>cube</u> we mean a product space $\prod_{j\in J} I_j$ where I_j is a closed interval of $\mathbb{R}$ for each $j\in J$. Given a cube $C(J)=\prod_{j\in J} I_j$ and given a subset $J'\subset J$ let $C(J')=\prod_{j\in J'} I_j$. There is a canonical projection map $p_{J'}:C(J)\longrightarrow C(J')$.

<u>Lemma (5.1).</u> Let $\varphi : H \longrightarrow H'$ be a homomorphism of cohomology theories on a cube $C(J)$ and let $n\in\mathbb{Z}$ be such that $\varphi_A : H(A) \longrightarrow H'(A)$ is an n-equivalence for every A of the form $A = p_F^{-1}(x)$ where F is a finite subset of J and $x\in C(F)$. Then φ_A is an n-equivalence for every closed $A\subset C(J)$.

<u>Proof.</u> 1) Let F be an arbitrary finite subset of J and let H_F, H'_F be the cohomology theories on $C(F)$ which equal the direct images (in the sense of Example (2.6)) of the cohomology theories H, H' on $C(J)$ under $p_F : C(J) \longrightarrow C(F)$ (so $H_F(B) = H(p_F^{-1}(B))$ and $H'_F(B) = H'(p_F^{-1}(B))$ for every closed $B \subset C(F)$). The hypotheses on φ imply that φ induces a homomorphism $\varphi_F : H_F \longrightarrow H'_F$ of cohomology theories on $C(F)$ which is an n-equivalence for every $x \in C(F)$. Since $C(F)$ is a finite dimensional compact space, it follows from Theorem (4.2) that φ_F is an n-equivalence for every closed $B \subset C(F)$.

2) Let A be a closed subset of $C(J)$ and for F a finite subset of J let $A_F = p_F^{-1}\, p_F(A)$. If $F \subset F'$ are finite subsets of J, there is a projection $p : C(F') \longrightarrow C(F)$ such that $p_F = p \circ p_{F'}$. It follows that $p_F(A) = pp_{F'}(A)$ so $p_{F'}(A) \subset p^{-1} p_F(A)$ and

$$A_{F'} = p_{F'}^{-1}(p_{F'}(A)) \subset p_{F'}^{-1}(p^{-1} p_F(A)) = p_F^{-1} p_F(A) = A_F \quad .$$

Hence, the collection $\{A_F \mid F \text{ finite} \subset J\}$ is a family of closed subsets of $C(J)$ directed downward by inclusion. Clearly $A \subset A_F$ for every F so $A \subset \bigcap_F A_F$.

We show $A = \bigcap_F A_F$. If $y \in C(J) - A$ there is a nbhd of y disjoint from A (because A is closed). Every nbhd of y contains a subset of the form $p_F^{-1}(N)$ where F is a finite subset of J and N is a closed nbhd of $p_F(y)$ in $C(F)$. Clearly $p_F^{-1}(N)$ is disjoint from A if and only if N is disjoint from $p_F(A)$. This implies $p_F(y) \notin p_F(A)$ and so $y \notin p_F^{-1}\, p_F(A) = A_F$. Therefore, $\bigcap_F A_F = A$.

It follows from Proposition (2.7) part 2) that $\varinjlim \{H(A_F) \mid F \text{ finite} \subset J\} \approx H(A)$ and $\varinjlim \{H'(A_F) \mid F \text{ finite} \subset J\} \approx H'(A)$. By 1) above, $\varphi_J : H_F(p_F(A)) \longrightarrow H'_F(p_F(A))$ is an n-equivalence for every finite $F \subset J$. This is equivalent to the assertion that $\varphi : H(A_F) \longrightarrow H'(A_F)$ is an n-equivalence for every finite $F \subset J$. Passing to the direct limit we see that $\varphi : H(A) \longrightarrow H'(A)$ is an n-equivalence for an arbitrary closed $A \subset C(J)$.

□

Let C be a category of topological spaces and continuous functions such that if X is an object of C then $cl(X)$ is a subcategory of C. A <u>cohomology theory</u> H, δ on C consists of:

i) A contravariant functor H from C to the category of graded abelian groups such that $H(\emptyset) = 0$,

and

ii) A natural transformation $\delta : H^q(A \cap B) \longrightarrow H^{q+1}(A \cup B)$ for every triad $(X;A,B)$ in C (by such a triad we mean X is an object of C and A,B are closed subsets of X)

such that for every object X in C the restriction of H, δ to $cl(X)$ is a cohomology theory on X.

A cohomology theory on C is <u>nonnegative</u>, <u>compactly supported</u> or <u>additive</u>, respectively, if its restriction to $cl(X)$ has the corresponding property for every object X in C.

A cohomology theory H,δ is <u>invariant under homotopy</u> if for every $f_0,f_1 : X \longrightarrow Y$ which are homotopic in C (i.e. there is a continuous map $F : X\times I \longrightarrow Y$ in C such that $F(x,0)=f_0(x)$, $F(x,1)=f_1(x)$ for all $x\in X$) then $H(f_0)=H(f_1) : H(Y) \longrightarrow H(X)$.

Of primary interest are the categories C_{comp} of all compact spaces and continuous functions and $C_{loc\ comp}$ of all locally compact spaces and proper continuous funtions.

<u>Proposition (5.2).</u> Every cohomology theory on C_{comp} is invariant under homotopy.

<u>Proof.</u> It is shown in [11] that every contravariant functor H on C_{comp} whose restriction to $cl(X)$ is continuous for every compact space X is invariant under homotopy.

□

A homomorphism $\varphi:H,\delta \longrightarrow H',\delta'$ between two cohomology theories on the same category C is a natural transformation of degree 0 from H to H' commuting up to sign with δ,δ' for every triad $(X;A,B)$ in C . We have the following extension of the uniqueness theorem.

<u>Theorem (5.3).</u> Let $\varphi:H,\delta \longrightarrow H',\delta'$ be a homomorphism between cohomology theories on C_{comp} such that for some one-point space $P,\varphi_P : H(P) \longrightarrow H'(P)$ is an n-equivalence for some $n\in\mathbb{Z}$. Then $\varphi_X : H(X) \longrightarrow H'(X)$ is an n-equivalence for every compact space X .

Proof. Because H, H' are contravariant functors on C_{comp} it follows that for every one-point space Q, $\varphi_Q : H(Q) \longrightarrow H'(Q)$ is an n-equivalence. Consider a cube $C(J) = \prod_{j\in J} I_j$ and let $y \in C(F)$ for F a finite subset of J. Since the projection map $p_F : p_F^{-1}(y) \longrightarrow y$ is a homotopy equivalence, there is a commutative square whose vertical maps are isomorphisms by Proposition (5.2)

$$\begin{array}{ccc} H(y) & \xrightarrow{\varphi_y} & H'(y) \\ H(p_F)\downarrow\approx & & \approx\downarrow H'(p_F) \\ H(p_F^{-1}(y)) & \xrightarrow{\varphi} & H\ (p_F^{-1}(y)) \end{array}$$

It follows that $\varphi : H(p_F^{-1}(y)) \longrightarrow H'(p_F^{-1}(y))$ is an n-equivalence for every finite $F \subset J$ and every $y \in C(F)$. By Lemma (5.1), φ_A is an n-equivalence for every $A \subset C(J)$. Since every compact X is homomorphic to a closed subset of some cube, $\varphi_X : H(X) \longrightarrow H'(X)$ is an n-equivalence for every X.

□

Corollary (5.4). Let $\varphi : H, \delta \longrightarrow H', \delta'$ be a homomorphism between compactly supported cohomology theories on $C_{loc\ comp}$ such that for some one-point space $P, \varphi_p : H(P) \longrightarrow H'(P)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then $\varphi_X : H(X) \longrightarrow H'(X)$ is an n-equivalence for every locally compact space X.

<u>Proof.</u> Since C_{comp} is a subcategory of $C_{loc\ comp}$, we can apply Theorem (5.3) to deduce that $\varphi_X : H(X) \longrightarrow H'(X)$ is an n-equivalence for every compact X. The Corollary follows from this and Proposition (2.8).

□

In the above Corollary we used the fact that C_{comp} is a subcategory of $C_{loc\ comp}$. Therefore, every cohomology theory on $C_{loc\ comp}$ defines by restriction a cohomology theory on C_{comp}. We now present a way of obtaining a compactly supported cohomology theory on $C_{loc\ comp}$ from a cohomology theory on C_{comp}.

For a subset $A \subset X$ we say A is <u>cobounded</u> in X if $\overline{X - A}$ is compact. We need the following lemma.

<u>Lemma (5.5).</u> Let H be a contravariant functor from $cl(X)$ to graded abelian groups such that $H(\emptyset) = 0$ and such that for every $A \subset X$

$$(*) \qquad \rho : \varinjlim \{H(N) \mid N \text{ a closed cobounded nbhd of } A \text{ in } X\} \to H(A)$$

Then H is continuous and compactly supported.

<u>Proof.</u> We first show H is continuous. Clearly (*) implies it is extensive. To show it reductive assume N is a closed nbhd of A in X and $u \in H(N)$ is such that $u|A = 0$. By (*) there is a closed cobounded nbhd $\overline{N}$ of N in X and $\overline{u} \in H(\overline{N})$ such that $u = \overline{u}|N$. Then $\overline{u}|A = 0$ so, again by (*), there is a closed cobounded nbhd M of A in $\overline{N}$ such that $\overline{u}|M = 0$. Then $N \cap M$ is a closed nbhd of A in N and

$u|N\cap M = \bar{u}|N\cap M = 0$ proving H is reductive. Therefore, H is continuous.

To show H is compactly supported let $u\in H(A)$. By (*) there is a closed cobounded nbhd N of A in X and $v\in H(N)$ such that $v|A = u$. Since $v|\emptyset = 0$ because $H(\emptyset) = 0$ it follows from (*) again that there is a closed cobounded M in N such that $v|M = 0$. Let $B = A\cap M$ and $C = \overline{A - M}$. Then $A = B\cup C$ where B is closed, C is compact and $u|B = 0$

□

Given an arbitrary locally compact space X let X^+ be the compact space consisting of X together with exactly one more point ∞ such that $X^+ - \{\infty\} = X$. In case X is compact, X^+ is the topological sum of X and $\{\infty\}$. In case X is non-compact, X^+ is the one-point compactification of X . Note that $A\subset X \Rightarrow A^+\subset X^+$, $\emptyset^+ = \{\infty\}$, and $(A\cap B)^+ = A^+ \cap B^+$, $(A\cup B)^+ = A^+ \cup B^+$ for A,B closed in X .

<u>Proposition (5.6).</u> Given a cohomology theory H,δ on C_{comp} there is a compactly supported cohomology theory $\tilde{H},\tilde{\delta}$ on $C_{loc\ comp}$ where $\tilde{H}(X) = \ker\,[H(X^+) \xrightarrow{\rho} H(\infty)]$ for a locally compact space X .

<u>Proof.</u> $\tilde{H}$ is defined by the above. To define $\tilde{\delta}$ note that if A is a closed subset of X and $c : A^+ \longrightarrow \infty$ is the constant map, then the composite

$$H(\infty) \xrightarrow{H(c)} H(A^+) \xrightarrow{\rho} H(\infty)$$

is the identity. Therefore, in the following commutative diagram with exact rows the vertical maps are epimorphisms

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\beta} & H^{q-1}(A^+\cap B^+) & \xrightarrow{\delta} & H^q(A^+\cup B^+) & \xrightarrow{\alpha} & H^q(A^+) \oplus H^q(B^+) & \xrightarrow{\beta} & H^q(A^+\cap B^+) & \xrightarrow{\delta} \cdots \\
 & \rho\downarrow & & \rho\downarrow & & \rho\downarrow & & \rho\downarrow & \\
\cdots \xrightarrow{\beta} & H^{q-1}(\infty) & \xrightarrow{0} & H^q(\infty) & \xrightarrow{\alpha} & H^q(\infty) \oplus H^q(\infty) & \xrightarrow{\beta} & H^q(\infty) & \xrightarrow{0} \cdots
\end{array}$$

It follows that there is an exact sequence of the kernels of ρ

$$\cdots \xrightarrow{\tilde{\beta}} \tilde{H}^{q-1}(A\cap B) \xrightarrow{\tilde{\delta}} \tilde{H}^q(A\cup B) \xrightarrow{\tilde{\alpha}} \tilde{H}^q(A) \oplus \tilde{H}^q(B) \xrightarrow{\tilde{\beta}} \tilde{H}^q(A\cap B) \xrightarrow{\tilde{\delta}} \cdots$$

This defines the natural transformation $\tilde{\delta}$ and shows that $\tilde{H}, \tilde{\delta}$ satisfy MV exactness.

Clearly $\tilde{H}(\emptyset) = \ker [H(\infty) \xrightarrow{\rho} H(\infty)] = 0$ and the closed nbhds of A^+ in X^+ are precisely the sets N^+ where N is a closed cobounded nbhd of A in X. Hence, the continuity of H on X^+ implies

$$\rho : \varinjlim \{\tilde{H}(N) \mid N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \tilde{H}(A)$$

It follows from Lemma (5.5) that $\tilde{H}$ is continuous and compactly supported on X.

□

__Theorem (5.7).__ The map H,δ to $\tilde{H},\tilde{\delta}$ is an equivalence between cohomology theories on C_{comp} and compactly supported cohomology theories on $C_{\text{loc comp}}$.

__Proof.__ Given H,δ on C_{comp} let $\tilde{H},\tilde{\delta}$ be the compactly supported cohomology theory on $C_{\text{loc comp}}$ defined by it as in Proposition (5.6). If X is any compact space, the exactness of $0 = H(\emptyset) \longrightarrow H(X^+) \xrightarrow{\alpha} H(X) \oplus H(\infty) \longrightarrow H(\emptyset) = 0$ implies that $\tilde{H}(X) = \ker\ [H(X^+) \xrightarrow{\rho} H(\infty)] \approx H(X)$. This shows that $\tilde{H}$ when restricted to C_{comp} is isomorphic to H . Similarly $\tilde{\delta}$ restricted to compact triads is isomorphic to δ . Thus, the restriction of $\tilde{H},\tilde{\delta}$ to C_{comp} is isomorphic to H,δ .

Conversely, let H',δ' be a compactly supported cohomology theory on $C_{\text{loc comp}}$ and consider $\tilde{H}',\tilde{\delta}'$ where $\tilde{H}'(X) = \ker\ [H'(X^+) \xrightarrow{\rho} H'(\infty)]$ for a locally compact space X . From the commutative diagram (where $X = B\cup C, B$ closed, C compact)

$$\begin{array}{ccccccc}
H'(B^+\cap C) & \xrightarrow{\delta'} & H'(X^+) & \xrightarrow{\alpha} & H'(B^+)\oplus H'(C) & \longrightarrow & H'(B^+\cap C) \\
\approx\downarrow & & & & & & \downarrow\approx \\
H'(B\cap C) & \xrightarrow{\delta'} & H'(X) & \xrightarrow{\alpha} & H'(B)\oplus H'(C) & \longrightarrow & H'(B\cap C)
\end{array}$$

we see there is a well defined isomorphism

$$\theta : \tilde{H}'(X^+) \approx H'(X)$$

such that $u\in H'(X)$ with $X = B\cup C, B$ closed, C compact and $u|B = 0$ equals $\theta(u^+)$ where $u^+\in H'(X^+)$ is such that $u^+|B^+=0$

and $u^{+}|C = u|C$. Then θ induces an isomorphism of $\tilde{H}',\tilde{\delta}'$ with H',δ' on $C_{\text{loc comp}}$.

□

6. ES theories on categories of spaces

We consider ES theories on a category of topological spaces and continuous mappings. Since ES theories define cohomology theories, the uniqueness theorem is valid for ES theories. We also show that there is an equivalence between ES theories and cohomology theories on the category of all compact spaces and continuous functions. Thus, cohomology theories are "single space" equivalents to ES theories. Other single space equivalents to ES theories have been given in [3,7,12].

Let C be a category of topological spaces and continuous functions such that if X is an object of C then $cl(X)$ is a subcategory of C. An ES theory H,δ^* on C consists of:

i) A contravariant functor H from C^2 (the category of closed pairs in C) to the category of graded abelian groups,

and

ii) A natural transformation $\delta^* : H^q(B,\emptyset) \longrightarrow H^{q+1}(A,B)$ for every (A,B) in C^2

such that for every object X in C the restriction of H,δ^* to $cl(X)^2$ is an ES theory on X.

Since ES theories are continuous, the result in [11] implies they are invariant under homotopy. Therefore, they are continuous extraordinary cohomology theories because they satisfy all of the Eilenberg-Steenrod axioms [5] except the dimension axiom and are continuous.

As in Proposition (3.2) every ES theory on C determines a cohomology theory on C . The concepts of <u>nonnegativity</u>, <u>compactly supported</u>, and <u>additivity</u> for ES theories on C are defined to correspond to the same properties of the associated cohomology theories.

A homomorphism $\varphi : H,\delta^* \longrightarrow H',\delta'^*$ between ES theories on C is a natural transformation of degree 0 from H to H' commuting up to sign with δ^*,δ'^* for every pair (X,A) in C^2. The following uniqueness theorem for ES theories is valid.

<u>Theorem (6.1).</u> Let $\varphi : H,\delta^* \longrightarrow H',\delta'^*$ be a homomorphism between two compactly supported ES theories on $C_{\text{loc comp}}$ such that for some one-point space P ,$\varphi_P : H(P,\emptyset) \longrightarrow H'(P,\emptyset)$ is an n-equivalence for some $n \in \mathbb{Z}$. Then $\varphi : H(X,A) \longrightarrow H'(X,A)$ is an n-equivalence for every locally compact pair.

<u>Proof.</u> The homomorphism φ determines a homomorphism $\bar{\varphi} : \bar{H},\bar{\delta} \longrightarrow \bar{H}',\bar{\delta}'$ between the cohomology theories on $C_{\text{loc comp}}$ defined by the ES theories H,δ^* and H',δ'^* respectively. Since these are compactly supported and $\bar{\varphi}$ is an n-equivalence for the one-point space P, it follows from Corollary (5.4) that $\bar{\varphi}$ is an n-equivalence for every locally compact space X . This is equivalent to the assertion that $\varphi : H(X,\emptyset) \longrightarrow H'(X,\emptyset)$ is an n-equivalence for every locally compact space X . Then the "five-lemma" shows that $\varphi : H(X,A) \longrightarrow H'(X,A)$ is an n-equivalence for every locally compact pair.

□

The next result asserts the equivalence between cohomology theories and ES theories on $\mathcal{C}_{comp}$.

Theorem (6.2). The assigment of a cohomology theory to an ES theory is an equivalence on $\mathcal{C}_{comp}$.

Proof. We have already seen that for an ES theory H,δ^* on $\mathcal{C}_{comp}$ there is associated a cohomology theory H',δ' on $\mathcal{C}_{comp}$ with $H'(X) = H(X,\emptyset)$ for every compact space X .

For the converse we consider for every compact space X the cone CX over X with vertex v (so CX is the join of X with a point v not in X). Given a cohomology theory H',δ' on $\mathcal{C}_{comp}$ define a contravariant functor H on $\mathcal{C}^2_{comp}$ by

$$H(X,A) = \ker [\rho : H'(X \cup CA) \longrightarrow H'(CA)]$$

(in case $A = \emptyset$, $CA = \{v\}$). Then

$$H(X,\emptyset) = \ker [\rho : H'(X \cup \{v\}) \longrightarrow H'(v)]$$

and by the exactness of

$$H'(\emptyset) \xrightarrow{\delta'} H'(X \cup \{v\}) \xrightarrow{\alpha} H'(X) \oplus H'(v) \xrightarrow{\beta} H'(\emptyset) ,$$

there is an isomorphism $\alpha' : H(X,\emptyset) \approx H'(X)$. Therefore, continuity of H' implies continuity of H.

To define δ^* and verify exactness for H, δ^* note that $\delta': H'(A) \longrightarrow H'(X \cup CA)$ for the triad $(CX;X,CA)$ has image lying in ker $[\rho : H'(X \cup CA) \longrightarrow H'(CA)]$ by exactness of the MV sequence of X,CA. Therefore,

$$\text{im } [\delta': H'(A) \longrightarrow H'(X \cup CA)] \subset \ker [\rho: H'(X \cup CA) \longrightarrow H'(CA)] = H(X,A) .$$

Thus, there is a unique homomorphism $\delta^*: H(A,\emptyset) \longrightarrow H(X,A)$ such that there is a commutative square

$$\begin{array}{ccc} H(A,\emptyset) & \xrightarrow[\approx]{\alpha'} & H'(A) \\ \delta^* \downarrow & & \downarrow \delta' \\ H(X,A) & \subset & H'(X \cup CA) \end{array}$$

Then δ^* is a natural transformation of degree 1 defined for every (X,A) in C^2_{comp}. Furthermore, if $c: X \cup CA \longrightarrow v$ is the constant map, the composite

$$H'(v) \xrightarrow{H'(c)} H'(X \cup CA) \xrightarrow{\rho} H'(CA)$$

is an isomorphism (by Proposition (5.2) since $c|CA: CA \longrightarrow v$ is a homotopy equivalence). Therefore, in the commutative diagram with exact rows all vertical maps are epimorphisms

$$\begin{array}{ccccccccc} \ldots \longrightarrow & H'(A) & \xrightarrow{\delta'} & H'(X \cup CA) & \longrightarrow & H'(X) \oplus H'(CA) & \longrightarrow & H'(A) & \xrightarrow{\delta'} \ldots \\ & \downarrow & & \downarrow \rho & & \downarrow \rho & & & \\ \ldots \longrightarrow & 0 & \longrightarrow & H'(CA) & \approx & 0 \oplus H'(CA) & \longrightarrow & 0 & \longrightarrow \ldots \end{array}$$

Hence, there is an exact sequence of kernels

$$\cdots \longrightarrow H'(A) \longrightarrow H(X,A) \longrightarrow H'(X) \longrightarrow H'(A) \longrightarrow \cdots$$

Replacing $H'(A)$ by $H(A,\emptyset)$ and $H'(X)$ by $H(X,\emptyset)$ we obtain the exact sequence

$$\cdots \longrightarrow H(A,\emptyset) \xrightarrow{\delta^*} H(X,A) \xrightarrow{H(j)} H(X,\emptyset) \xrightarrow{H(i)} H(A,\emptyset) \xrightarrow{\delta^*} \cdots$$

where $i:(A,\emptyset) \subset (X,\emptyset)$ and $j:(X,\emptyset) \subset (X,A)$. Therefore, H, δ^* satisfy exactness.

To prove the excision property let $(X;A,B)$ be a triad in C_{comp} and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H'(A) \oplus H'(CB) & \xrightarrow{\beta} & H'(A\cap B) & \xrightarrow{\delta'} & H'(A\cup CB) & \xrightarrow{\alpha} & H'(A) \oplus H'(CB) & \xrightarrow{\beta} & H'(A\cap B) \\ \rho \downarrow \approx & & \rho \downarrow \approx & & \rho \downarrow & & \approx \downarrow \rho & & \approx \downarrow \rho \\ H'(A) \oplus H'(C(A\cap B)) & \xrightarrow{\beta} & H'(A\cap B) & \xrightarrow{\delta'} & H'(A\cup C(A\cap B)) & \xrightarrow{\alpha} & H'(A) \oplus H'(C(A\cap B)) & \xrightarrow{\beta} & H'(A\cap B) \end{array}$$

From the "five lemma" the middle vertical map is an isomorphism and there is a commutative square

$$\begin{array}{ccc} H'(A\cup CB) & \xrightarrow{\rho} & H'(CB) \\ \rho \downarrow \approx & & \approx \downarrow \rho \\ H'(A\cup C(A\cap B)) & \xrightarrow{\rho} & H'(C(A\cap B)) \end{array}$$

Therefore, $H(A\cup B,B) = \ker\ [\rho : H'(A\cup CB) \longrightarrow H'(CB)]$ is isomorphic to $H(A,A\cap B) = \ker\ [\rho : H'(A\cup C(A\cap B)) \longrightarrow H'(C(A\cap B))]$ by the restriction map.

Thus, H,δ^* is an ES theory on $\mathcal{C}_{comp}$. It is clear that α' induces an isomorphism of $H(A,\emptyset)$ with $H'(A)$ so the cohomology theory induced on $\mathcal{C}_{comp}$ by the ES theory H,δ^* is isomorphic to the original cohomology theory H',δ' on $\mathcal{C}_{comp}$.

Conversely, if H,δ^* is an ES theory on $\mathcal{C}_{comp}$ and H',δ' is the corresponding cohomology theory on $\mathcal{C}_{comp}$ let H'',δ'' be the ES theory on $\mathcal{C}_{comp}$ constructed as above from H',δ'. Then

$$\begin{aligned} H''(X,A) &= \ker\,[H'(X \cup CA) \longrightarrow H'(CA)] \\ &= \ker\,[H(X \cup CA,\emptyset) \longrightarrow H(CA,\emptyset)] \end{aligned}$$

From the exact sequence

$$\xrightarrow{0} H(X \cup CA, CA) \longrightarrow H(X \cup CA,\emptyset) \longrightarrow H(CA,\emptyset) \xrightarrow{0}$$

we see that $\ker\,[H(X \cup CA,\emptyset) \longrightarrow H(CA,\emptyset)] \approx H(X \cup CA, CA)$. Since there is an excision isomorphism

$$H(X \cup CA, CA) \approx H(X,A)$$

we finally obtain $H''(X,A) \approx H(X,A)$. This isomorphism carries δ'' to δ^* and completes the proof. □

7. Cohomology defined by spectra

In this section we show that a spectrum of ANR defines an ES theory on the category $\mathcal{C}_{\text{loc comp}}$. In particular, K-theory, which is defined by such a spectrum, is an ES theory. The uniqueness theorem of the preceding section yields another proof of the fact that the Chern character is an isomorphism of $K \otimes \mathbb{Q}$ with $\check{H}_{ev}$ (rational Čech cohomology with compact supports) on $\mathcal{C}_{\text{loc comp}}$.

For a pointed space X we let C_0X denote the reduced cone over X (so $C_0X = I \wedge X$ where $0 \in I$ is the base point of I) and $\Sigma_0 X = S^1 \wedge X$ the reduced suspension of X. If (X,A) is a compact pair with base point $x_0 \in A$ and Y is a pointed space there is an exact sequence of based homotopy classes [20]

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A;Y] \xrightarrow{\bar{k}^\#} [X \cup C_0 A;Y] \xrightarrow{\bar{i}^\#} [X;Y] \xrightarrow{i^\#} [A;Y]$$

where $i : A \subset X$, $\bar{i} : X \subset X \cup C_0 A$ and $\bar{k} : X \cup C_0 A \longrightarrow \Sigma_0 A$ is the map collapsing X to the base point. In case Y is an ANR (absolute neighborhood retract for normal spaces) the contractibility of $C_0 A$ implies that the quotient map $q : X \cup C_0 A \longrightarrow X/A$ induces, for every $n \geq 0$, a bijection $(\Sigma_0^n q)^\# : [\Sigma_0^n (X/A);Y] \approx [\Sigma_0^n (X \cup CA);Y]$. Since $q \circ \bar{i} = k : X \longrightarrow X/A$ where k is the quotient map, there is an exact sequence

$$\cdots \xrightarrow{(\Sigma_0 i)^\#} [\Sigma_0 A;Y] \xrightarrow{\delta} [X/A;Y] \xrightarrow{k^\#} [X;Y] \xrightarrow{i^\#} [A;Y]$$

where $\delta:[\Sigma_0^{n+1} A;Y] \longrightarrow [\Sigma_0^n(X/A);Y]$ is defined to equal the composite

$$\left(\Sigma_0^n q\right)^{\#-1}\left(\Sigma_0^n \bar{k}\right)^{\#} : \quad [\Sigma_0^{n+1} A;Y] \longrightarrow [\Sigma_0^n(X \cup C_0 A);Y] \xrightarrow[\approx]{} [\Sigma_0^n(X/A);Y]$$

Now suppose $\textcircled{Y} = \{ Y_k, \varepsilon_k : \Sigma_0 Y_k \longrightarrow Y_{k+1}\}$ is a spectrum of pointed ANR's. Define

$$\{ X,A;\textcircled{Y}\}^q = \xrightarrow[n]{\lim} \{[\Sigma_0^n(X/A);Y_{n+q}]\}$$

where the direct limit is with respect to the maps

$$[\Sigma_0^n(X/A);Y_{n+q}] \xrightarrow{\Sigma_0} [\Sigma_0^{n+1}(X/A);\Sigma_0 Y_{n+q}] \xrightarrow{(\varepsilon_{n+q})_\#} [\Sigma_0^{n+1}(X/A);Y_{n+q+1}]$$

Taking the direct limit of the exact sequences above we obtain an exact sequence

$$\cdots \xrightarrow{\delta^*} \{X,A;\textcircled{Y}\}^q \xrightarrow{j^\#} \{X,x_0;\textcircled{Y}\}^q \xrightarrow{i^\#} \{A,x_0;\textcircled{Y}\}^q \xrightarrow{\delta^*} \{X,A;\textcircled{Y}\}^{q+1} \longrightarrow \cdots$$

where $i:(A,x_0) \subset (X,x_0)$, $j:(X,x_0) \subset (X,A)$.

We consider the category $C^2_{\text{loc comp}}$. As in Section 5 to every locally compact space X there is associated a compact space X^+ with base point ∞ such that $X^+ - \{\infty\} = X$. Define a contravariant functor $H_{\textcircled{Y}}$ on $C^2_{\text{loc comp}}$ by $H^q_{\textcircled{Y}}(X,A) = \{X^+,A^+;\textcircled{Y}\}^q$ and define a natural transformation $\delta^*: H^q_{\textcircled{Y}}(A,\emptyset) \longrightarrow H^{q+1}_{\textcircled{Y}}(X,A)$ to be the homomorphism δ^* in the the exact sequence above for the pair of pointed spaces (X^+,A^+).

<u>Theorem (7.1)</u> For every spectrum $\textcircled{Y}$ of ANR there is a compactly supported ES theory $H_{\textcircled{Y}}, \delta^*$ on $C_{loc\ comp}$.

<u>Proof.</u> From the exact sequence above for the pair (X^+,A^+) it is clear that $H_{\textcircled{Y}}, \delta^*$ satisfies the exactness property for the pair (X,A). It also satisfies excision because if $(X;A,B)$ is a locally compact triad then $(X^+;A^+,B^+)$ is a compact pointed triad with $A^+ \cup B^+ = (A\cup B)^+$ and $A^+ \cap B^+ = (A\cap B)^+$. Therefore, there are isomorphisms

$$H^q_{\textcircled{Y}}(A\cup B,B) = \{(A\cup B)^+,B^+;\textcircled{Y}\}^q = \{A^+\cup B^+, B^+;\textcircled{Y}\}^q$$
$$\approx \{A^+,A^+\cap B^+;\textcircled{Y}\}^q = \{A^+,(A\cap B)^+;\textcircled{Y}\}^q = H^q_{\textcircled{Y}}(A,A\cap B),$$

If (X,A) is a locally compact pair, then (X^+,A^+) and $(\Sigma_0^n X^+, \Sigma_0^n A^+)$ are compact pairs for every n. Since Y_{n+q} is an ANR it follows that

$$\rho : \underrightarrow{\lim}\{[\Sigma_0^n B;\ Y_{n+q}] \mid B \text{ a closed nbhd of } A^+ \text{in } X^+\} \approx [\Sigma_0^n A^+;Y_{n+q}]$$

Taking direct limits with respect to n we obtain an isomorphism

$$\rho : \underrightarrow{\lim}\ \{\{B;Y\}^q \mid B \text{ a closed nbhd of } A^+ \text{ in } X^+\} \approx \{A^+;Y\}^q$$

Since the closed nbhds B of A^+ in X^+ are exactly the sets of the form $B = N^+$ where N is a closed cobounded nbhd of A in X , it follows that

$$\rho : \underrightarrow{\lim}\{H^q_{\mathbb{Y}}(N) \mid N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx H^q_{\mathbb{Y}}(A) .$$

Since $H_{\mathbb{Y}}(\emptyset) = 0$ it follows from Lemma (5.5) that $H_{\mathbb{Y}}$ is continuous and compactly supported.

□

We consider a spectrum of ANR that defines the ES theory known as K-theory. Let $G_n(\mathbb{C}^m)$ be the complex Grassmannian of n-dimensional (complex) subspaces of $\mathbb{C}^m$ and let $U(m)$ denote the group of $m \times m$ unitary matrices. In [2] there is defined a continuous map $\nu_n : \Sigma G_n(\mathbb{C}^{2n}) \longrightarrow U(2n)$ (where $\Sigma G_n(\mathbb{C}^{2n})$ is the unreduced suspension of $G_n(\mathbb{C}^{2n})$) such that the composite

$$\pi_i (G_n(\mathbb{C}^{2n})) \xrightarrow{\Sigma} \pi_{i+1}(\Sigma G_n(\mathbb{C}^{2n})) \xrightarrow{\nu_{n\#}} \pi_{i+1}(U(2n))$$

is an isomorphism for $0 < i \leq 2n$. Since the reduced suspension $\Sigma_0 G_n(\mathbb{C}^{2n})$ (where $G_n(\mathbb{C}^{2n})$ is given a base point) is obtained from the unreduced suspension by collapsing the closed interval equal to the suspension of the base point of $G_n(\mathbb{C}^{2n})$, it follows that ν_n also defines a map of the reduced suspension

$$\nu'_n : \Sigma_0 G_n(\mathbb{C}^{2n}) \longrightarrow U(2n)$$

such that the composite

$$\pi_i(G_n(\mathbb{C}^{2n}))\xrightarrow{\Sigma_0} \pi_{i+1}(\Sigma_0 G_n(\mathbb{C}^{2n}))\xrightarrow{\nu'_n{}^{\#}} \pi_{i+1}(U(2n))$$

is an isomorphism for $0 < i \leq 2n$. There is also a fibration

$$U(2n+2n^2)/U(2n^2) \longrightarrow U(2n+2n^2)/U(2n)\times U(2n^2) = G_{2n}(\mathbb{C}^{2n+2n^2}) \ .$$

in which $\pi_i(U(2n+2n^2)/U(2n^2)) = 0$ for $i \leq 2(2n^2) = (2n)^2$.
Since the fiber of this fibration is $U(2n)$ and $\dim U(2n) = (2n)^2$, it follows that there is a map

$$\mu_n : (C_0U(2n), U(2n)) \longrightarrow (U(2n+2n^2)/U(2n^2),\ U(2n))$$

(where $U(2n)$ has been given an arbitrary base point) hence a map $\mu'_n : \Sigma_0 U(2n) \longrightarrow G_{2n}(\mathbb{C}^{2n+2n^2})$ such that the composite

$$\pi_i(U(2n))\xrightarrow{\Sigma_0} \pi_{i+1}(\Sigma_0 U(2n))\xrightarrow{\mu'_n{}^{\#}} \pi_{i+1}(G_{2n}(\mathbb{C}^{2n+2n^2}))$$

is an isomorphism for $i < (2n)^2$. Since the canonical map $G_{2n}(\mathbb{C}^{2n+2n^2})) \longrightarrow G_{2n^2}(\mathbb{C}^{4n^2})$ induces a homomorphism $\pi_i(G_{2n}(\mathbb{C}^{2n+2n^2})) \longrightarrow \pi_i(G_{2n^2}(\mathbb{C}^{4n^2}))$ which is an isomorphism for $i < 4n$, composing μ'_n with this canonical map gives a map

$$\mu''_n : \Sigma_0 U(2n) \longrightarrow G_{2n^2}(\mathbb{C}^{4n^2})$$

which induces an isomorphism for $i < 4n$. Thus, we have a spectrum $\mathbb{K}$ of ANR

$$G_1(\mathbb{C}^2),\ U(2),\ G_2(\mathbb{C}^4),\ U(4),\ G_8(\mathbb{C}^{16}),\ U(16),\ G_{128}(\mathbb{C}^{256}),\ldots$$

whose maps are $\nu_1', \mu_1'', \nu_2', \mu_2'', \nu_8', \mu_8'', \nu'_{128}, \ldots$ The correspon-ding ES theory is denoted by K and is known [10] to be periodic of period 2 (i.e. $K^{q+2}(X,A) \approx K^q(X,A)$) and that for a one-point space P, $K^q(P) \approx \begin{cases} \mathbb{Z} & \text{even} \\ 0 & \text{odd} \end{cases}$.

Let $\check{H}_{ev}$ be the ES theory defined in terms of rational Čech cohomology with compact supports by

$$\check{H}^q_{ev}(X,A) = \begin{cases} \bigoplus\limits_{i \text{ even}} \check{H}^i_c(X,A;\mathbb{Q}) & q \text{ even} \\ \bigoplus\limits_{i \text{ odd}} \check{H}^i_c(X,A;\mathbb{Q}) & q \text{ odd} \end{cases}$$

This is an ES theory on $\mathcal{C}_{\text{loc comp}}$ (by analogues of Remarks (2.3) and (2.4) for ES theories). By an analogue of Remark (2.5) for ES theories there is also an ES theory $K \otimes \mathbb{Q}$ on $\mathcal{C}_{\text{loc comp}}$.

The Chern character [10] $Ch : K \otimes \mathbb{Q} \longrightarrow \check{H}_{ev}$ is a homomorphism of ES theories on $\mathcal{C}_{\text{loc comp}}$.

<u>Theorem (7.2).</u> For every locally compact space X, $\mathrm{Ch} : K^q(X) \otimes \mathbb{Q} \approx \check{H}^q_{ev}(X)$.

<u>Proof.</u> Since $K \otimes \mathbb{Q}$ and $\check{H}_{ev}$ are both compactly supported ES theories on $C_{\text{loc comp}}$ and Ch is an isomorphism for a one-point space [10], the result follows from Theorem (6.1).

□

If $\mathbb{Y} = \{Y_k, \varepsilon_k\}$ is a spectrum its <u>homotopy groups</u> $\pi_q(\mathbb{Y})$ are defined by $\pi_q(\mathbb{Y}) = \underset{n}{\underrightarrow{\lim}}\{\pi_{n+q}(Y_n)\}$. If $\mathbb{Y}$ is an ANR spectrum, it is clear that if P is a one-point space, then $H^q_{\mathbb{Y}}(P,\emptyset) \approx \pi_{-q}(\mathbb{Y})$ for all q.

If $\mathbb{Y}, \mathbb{Y}'$ are spectra, a <u>map</u> $g : \mathbb{Y} \to \mathbb{Y}'$ between them is defined to be a sequence of pointed continuous functions $g_k : Y_k \to Y'_k$ for each k such that the following square is homotopy commutative for every k

$$\begin{array}{ccc} \Sigma_0 Y_k & \xrightarrow{\varepsilon_k} & Y_{k+1} \\ \Sigma_0 g_k \downarrow & & \downarrow g_{k+1} \\ \Sigma_0 Y'_k & \xrightarrow{\varepsilon'_k} & Y'_{k+1} \end{array}$$

Such a map g induces a homomorphism $g_{\#} : \pi_q(Ⓨ) \longrightarrow \pi_q(Ⓨ')$ and, in case Y, Y' are ANR spectra, it induces a homomorphism $g_* : H_{Ⓨ} \longrightarrow H_{Ⓨ'}$ of the corresponding ES theories.

<u>Theorem (7.3).</u> Let $g : Ⓨ \longrightarrow Ⓨ'$ be a map between ANR spectra which induces an isomorphism $g_{\#} : \pi_q(Ⓨ) \approx \pi_q(Ⓨ')$ for all q. Then for every locally compact pair (X,A),

$$g_* : H_{Ⓨ}(X,A) \approx H_{Ⓨ'}(X,A) .$$

<u>Proof.</u> Since $g_{\#}$ is an isomorphism, it follows that if P is a one-point space, then $g_* : H_{Ⓨ}(P,\emptyset) \approx H_{Ⓨ'}(P,\emptyset)$. Since $H_{Ⓨ}, H_{Ⓨ'}$ are compactly supported ES theories on $C_{\text{loc comp}}$, the result follows from Theorem (6.1).

□

8. Duality in $\mathbb{R}^n$

We introduce the functional spectrum $\mathbb{F}(A,B)$ whose k^{th} term consists of the space of continuous functions $(A^+,B^+) \longrightarrow (S^k,\infty)$ topologized with the compact-open topology. For (A,B) a closed pair in $\mathbb{R}^n$ we show that the n^{th} suspension of this functional spectrum is equivalent to a spectrum defined by $(\mathbb{R}^n-B, \mathbb{R}^n-A)$. This leads to a duality theorem a corollary of which is the result [14] that for A,B compact subsets of S^n there is an isomorphism of stable homotopy classes

$$\{A;S^n-B\}^q \approx \{B;S^n-A\}^q \quad .$$

We regard S^k as $(\mathbb{R}^k)^+ = \mathbb{R}^k \cup \{\infty\}$ with ∞ as base point. Given a locally compact pair (A,B) let $F(A,B;S^k)$ be the space of all continuous functions $(A^+,B^+) \longrightarrow (S^k,\infty)$ in the compact-open topology with the constant map $A^+ \longrightarrow \infty$ as base point. Clearly $F(A,B;S^k)$ can also be regarded as the space of pointed continuous functions $A^+/B^+ \longrightarrow S^k$ in the compact-open topology. It follows from the exponential theorem [4], the compactness of A^+ and the fact that S^k is an ANR that $F(A,B;S^k)$ is an ANR (the proof of Theorem 4 on p. 38 of [15] applies to our case as well).

It is known [6] that if (A,B) is a locally compact pair the restriction map $F(A,\emptyset;S^k) \longrightarrow F(B,\emptyset;S^k)$ is a fibration with fiber $F(A,B;S^k)$. Therefore, there is an exact sequence of homotopy groups

$$(*)\ \ldots \longrightarrow \pi_{q+k}(F(A,B;S^k)) \longrightarrow \pi_{q+k}(F(A;\emptyset;S^k)) \longrightarrow \pi_{q+k}(F(B,\emptyset;S^k))$$

$$\longrightarrow \pi_{q+k-1}(F(A,B;S^{k-1})) \longrightarrow \ldots$$

Define $\varepsilon_k : \Sigma_0 F(A,B;S^k) \longrightarrow F(A,B;S^{k+1})$ by $(\varepsilon_k(t \wedge f))(a) = t \wedge f(a)$ for $t \wedge f \in \Sigma_0 F(A,B;S^k) = S^1 \wedge F(A,B;S^k)$, $a \in A$ and $\Sigma_0 S^k = S^1 \wedge S^k \approx S^{k+1}$. Let $\mathbb{F}(A,B)$ be the spectrum $\{F(A,B;S^k), \varepsilon_k\}$. The direct limit over k of the exact sequence (*) using the maps ε_k is an exact sequence

$$\cdots \longrightarrow \pi_q(\mathbb{F}(A,B)) \longrightarrow \pi_q(\mathbb{F}(A,\emptyset)) \longrightarrow \pi_q(\mathbb{F}(B,\emptyset)) \longrightarrow \pi_{q-1}(\mathbb{F}(A,B)) \longrightarrow .$$

extending indefinitely on both ends.

If we define a contravariant functor $\mathbb{F}^*$ on $C^2_{\text{loc comp}}$ by $\mathbb{F}^q(A,B) = \pi_{-q}(\mathbb{F}(A,B))$ and a natural transformation $\delta^* : \mathbb{F}^q(B,\emptyset) \longrightarrow \mathbb{F}^{q+1}(A,B)$ to correspond to ∂ in the exact sequence above, we see that $\mathbb{F}^*, \delta^*$ satisfy the exactness property of ES theories. We shall prove that $\mathbb{F}^*, \delta^*$ is an ES theory on $C_{\text{loc comp}}$, but first we establish the following.

<u>Lemma (8.1).</u> Let (K',K) be a compact pair, N a closed cobounded nbhd of A in X, and $\lambda : K \longrightarrow F(N,\emptyset;S^k)$, $\mu : K' \longrightarrow F(A,\emptyset;S^k)$ continuous functions such that $\mu|K$ is the composite $K \xrightarrow{\lambda} F(N,\emptyset;S^k) \xrightarrow{\rho} F(A,\emptyset;S^k)$. Then there is a closed cobounded nbhd N' of A in N and a map $\lambda' : K' \longrightarrow F(N,\emptyset;S^k)$ such that $\lambda'|K$ is the composite

$K \xrightarrow{\lambda} F(N',\emptyset;S^k) \xrightarrow{\rho'} F(N',\emptyset;S^k)$ and μ is the composite $K' \xrightarrow{\lambda'} F(N',\emptyset;S^k) \xrightarrow{\rho''} F(A,\emptyset;S^k)$.

Proof. By the exponential theorem the functions λ and μ correspond to continuous functions $\overline{\lambda} : (K \times N^+, K \times \infty) \longrightarrow (S^k,\infty)$ and $\overline{\mu} : (K' \times A^+, K' \times \infty) \longrightarrow (S^k,\infty)$ such that $\overline{\lambda}|K \times A^+ = \overline{\mu}|K \times A^+$. Therefore, there is a continuous function

$$\overline{f} : (K \times N^+ \cup K' \times A^+, K' \times \infty) \longrightarrow (S^k,\infty)$$

such that $\overline{f}|K \times N^+ = \overline{\lambda}$ and $\overline{f}|K' \times A^+ = \overline{\mu}$. Since $K' \times N^+$ is compact, $K \times N^+ \cup K' \times A^+$ is closed in $K' \times N^+$ and S^k is an ANR, there is a nbhd U of $K \times N^+ \cup K' \times A^+$ in $K' \times N^+$ and an extension $\overline{f}' : U \longrightarrow S^k$ of $\overline{f}$. U contains a subset of the form $K' \times N'^+$ where N'^+ is a closed cobounded nbhd of A in N and $\overline{f}' \mid K' \times N'^+$ corresponds by the exponential theorem to a map $\lambda' : K' \longrightarrow F(N,\emptyset;S^k)$ having all the requisite properties.

□

Theorem (8.2). The pair $\mathbb{F}^*, \delta^*$ is a compactly supported ES theory on $C_{\text{loc comp}}$.

Proof. We have already seen that exactness is satisfied. The excision property is also satisfied because if $(X;A,B)$ is a locally compact triad then there are homeomorphisms for each k

$$F(A \cup B, B; S^k) \approx F((A^+ \cup B^+)/B^+; S^k) \approx F(A^+/(A^+ \cap B^+); S^k) \approx F(A, A \cap B; S^k)$$

so the map of spectra $\mathbb{F}(A\cup B,B) \longrightarrow \mathbb{F}(A,A\cap B)$ induces an isomorphism of homotopy groups. Therefore, $\mathbb{F}^q(A\cup B,B) =$ $= \pi_{-q}(\mathbb{F}(A\cup B,B)) \approx \pi_{-q}(\mathbb{F}(A,A\cap B)) \approx \mathbb{F}^q(A,A\cap B)$ for all q. To complete the proof it suffices to show that for every locally compact space X the restriction of the functor $\mathbb{F}^*(\cdot,\emptyset)$ to $cl(X)$ satisfies Lemma (5.5). Clearly $\mathbb{F}^*(\emptyset,\emptyset) = 0$ so we need only verify that the following homomorphism is an isomorphism

$$\rho : \underrightarrow{\lim} \{\mathbb{F}^*(N,\emptyset) \mid N \text{ a closed cobounded nbhd of } A \text{ in } X\} \approx \mathbb{F}^*(A,\emptyset)$$

To show ρ is an epimorphism let $\mu : S^{k+q} \longrightarrow F(A,\emptyset;S^k)$ represent an element $\{[\mu]\} \in \pi_q(\mathbb{F}(A,\emptyset)) = \mathbb{F}^{-q}(A,\emptyset)$ where $A \subset X$. By Lemma (8.1) with $K' = S^{k+q}, K = \infty$, $N = X$, $\lambda : \infty \longrightarrow F(X,\emptyset;S^m)$ the unique pointed map, $\mu : S^{k+q} \longrightarrow F(A,\emptyset;S^k)$ we obtain a closed cobounded nbhd N' of A in X and a map $\lambda' : S^{k+q} \longrightarrow F(N',\emptyset;S^k)$ representing an element $\{[\lambda]\} \in \pi_q(\mathbb{F}(N',\emptyset))$ whose restriction to $\mathbb{F}(A,\emptyset)$ equals $\{[\mu]\}$. This implies ρ is an epimorphism.

To show ρ is a monomorphism let $\lambda : S^{k'+q} \longrightarrow F(N,\emptyset;S^{k'})$ represent an element $\{[\lambda]\} \in \pi_q(N,\emptyset)$ whose restriction to $\mathbb{F}(A,\emptyset)$ is 0 (where N is a closed cobounded nbhd of A in X). Then there is a map $\mu : C_0S^{k+q} \longrightarrow F(A,\emptyset;S^k)$ for some $k \geq k'$ such that $\mu | S^{k+q} = \rho \circ (\Sigma_0^{k-k'}\lambda)$. By Lemma (8.1) with $K' = C_0S^{k+q}, K = \Sigma_0^{k-k'} S^{k'+q} = S^{k+q}$ and the maps $\Sigma_0^{k-k'}\lambda : S^{k+q} \longrightarrow F(N,\emptyset;S^k)$, $\mu : C_0S^{k+q} \longrightarrow F(A,\emptyset;S^k)$ there is a closed cobounded nbhd N' of A in N and a map

$\lambda' : C_0S^{k+q} \longrightarrow F(N',\emptyset;S^k)$ such that $\lambda'|S^{k+q} = \rho' \circ \lambda$. Therefore, $\{[\lambda]\}$ maps to 0 in $\mathbb{F}(N',\emptyset)$ proving ρ is a monomorphism.

□

Let T be an arbitrary but fixed triangulation of S^n with ∞ as vertex and let $T^{(k)}$ be the k^{th} barycentric subdivision of T for $k \geq 0$. For a closed subset $A \subset \mathbb{R}^n = S^n - \{\infty\}$ let $T_k(A)$ be the compact polyhedron equal to the union of all closed simplexes of $T^{(k)}$ disjoint from $A^+ = A \cup \{\infty\}$. Clearly $T_k(A) \subset T_{k+1}(A)$ for all $k \geq 0$. We also have:

<u>Lemma (8.3).</u> (1) $A \subset B \subset \mathbb{R}^n \Rightarrow T_k(B) \subset T_k(A)$ for all $k \geq 0$

(2) $A,B \subset \mathbb{R}^n \Rightarrow T_k(A \cup B) = T_k(A) \cap T_k(B)$ for all $k \geq 0$

(3) $A,B \subset \mathbb{R}^n \Rightarrow$ for every $k \geq 0$ there is N_k such that if $k' \geq N_k$ then $T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$.

(4) $A \subset N \subset \mathbb{R}^n$ where N is a closed cobounded nbhd of A in $\mathbb{R}^n$ $\Rightarrow$ for every $k \geq 0$ there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $N' \subset N$ and $T_k(N') = T_k(N)$.

<u>Proof.</u> (1) If $A \subset B$ and s is a closed simplex of $T^{(k)}$ disjoint form B, s is disjoint form A. Hence, $T_k(B) \subset T_k(A)$.

□

(2) By (1) $T_k(A \cup B) \subset T_k(A)$ and $T_k(A \cup B) \subset T_k(B)$ so $T_k(A \cup B) \subset T_k(A) \cap T_k(B)$. Given x let s be the unique closed simplex of $T^{(k)}$ containing x in its interior. Then $x \in T_k(A) \cap T_k(B)$ if and only if $s \subset T_k(A) \cap T_k(B)$, but this implies s is disjoint from A and from B so is disjoint from $A \cup B$. Therefore, $T_k(A) \cap T_k(B) \subset T_k(A \cup B)$.

□

(3) $T_k(A \cap B)$ is the union of a finite number of closed simplexes, say $T_k(A \cap B) = s_1 \cup \ldots \cup s_r$. For each j, s_j is disjoint from $A \cap B$ so $s_j \cap A$ and $s_j \cap B$ are disjoint compact sets. Let $d_j > 0$ be the distance between them (in some metric on S^n) and let $d = \min\{d_1, \ldots d_r\}$, choose N_k so that $k' \geq N_k$ implies that the diameter of every closed simplex of $T^{(k')}$ is less than d. If s' is any closed simplex of $T^{(k')}$ contained in s_j for some $1 \leq j \leq r$, then $\operatorname{diam} s' < d \leq d_j$ so s' cannot meet both A and B. Therefore, either $s' \in T_{k'}(A)$ or $s' \in T_{k'}(B)$. Hence, for $k' \geq N_k$, $T_k(A \cap B) \subset T_{k'}(A) \cup T_{k'}(B)$.

□

(4) For $A \subset \mathbb{R}^n$, $S^n - T_k(A)$ is an open nbhd of A^+. Let M be a closed nbhd of A^+ contained in $S^n - T_k(A)$. Then $M - \{\infty\}$ is a closed cobounded nbhd of A in $\mathbb{R}^n$. Since $T_k(A) \subset S^n - M = S^n - (M - \{\infty\})^+$, it follows that $T_k(A) \subset T_k(M - \{\infty\})$. If N is any closed cobounded nbhd of A in $\mathbb{R}^n$, then

$N' = N \cap (M-\{\infty\})$ is a closed cobounded nbhd of A in $\mathbb{R}^n$ such that $N' \subset N$ and since $N' \subset M-\{\infty\}$, it follows from (1) that

$$T_k(A) \subset T_k(M-\{\infty\}) \subset T_k(N') \subset T_k(A)$$

so that $T_k(N') = T_k(A)$.

□

Given a closed pair (A,B) in $\mathbb{R}^n$ we define a spectrum $\mathbb{C}(A,B)$ whose k^{th} term is $\Sigma_0^k(T_k(B)^+/T_k(A)^+)$ with continuous maps

$$\Sigma_0(\Sigma_0^k(T_k(B)^+/T_k(A)^+)) = \Sigma_0^{k+1}(T_k(B)^+/T_k(A)^+) \longrightarrow \Sigma_0^{k+1}(T_{k+1}(B)^+/T_{k+1}(A)^+)$$

where the last map is the $(k+1)$st reduced suspension of the map $(T_k(B)^+/T_k(A)^+) \rightarrowtail (T_{k+1}(B)^+/T_{k+1}(A)^+)$ induced by the inclusion $(T_k(B)^+, T_k(A)^+) \subset (T_{k+1}(B)^+, T_{k+1}(A)^+)$.

We define a contravariant functor $\mathbb{C}^*$ on $cl(\mathbb{R}^n)^2$ by $\mathbb{C}^q(A,B) = \pi_{-q}(\mathbb{C}(A,B)) = \varinjlim_k \{\pi_{-q-k}(\Sigma_0^k(T_k(B)^+/T_k(A)^+))\}$

To define the natural transformation $\delta^*: \mathbb{C}^q(B,\emptyset) \longrightarrow \mathbb{C}^{q+1}(A,B)$ recall [20, Corollary 9.3.6 on p. 487] that the collapsing map

$$(\Sigma_0^k T_k(B)^+, \Sigma_0^k T_k(A)^+) \longrightarrow (\Sigma_0^k (T_k(B)^+/T_k(A)^+), \infty)$$

induces isomorphisms

$$\pi_i\ (\Sigma_0^k T_k(B)^+,\ \Sigma_0^k T_k(A)^+) \approx \pi_i(\Sigma_0^k (T_k(B)^+/T_k(A)^+))$$

for $i \leq 2k-2$ (because the k^{th} reduced suspension of a space, or pair, is $(k-1)$- connected). Hence, the connecting homomorphisms

$$\partial : \pi_{-q+k}(\Sigma_0^k\ T_k(\emptyset)^+, \Sigma_0^k\ T_k(B)^+) \longrightarrow \pi_{-q+k-1}(\Sigma_0^k T_k(B)^+, \Sigma_0^k T_k(A)^+)$$

for various k correspond to a homomorphism

$$\delta^* : \pi_{-q}(\mathbb{C}(B,\emptyset)) \longrightarrow \pi_{-q-1}(\mathbb{C}(A,B))$$

This is a natural transformation of degree 1 from $\mathbb{C}^*(B,\emptyset)$ to $\mathbb{C}^*(A,B)$ for $(A,B) \in cl(\mathbb{R}^n)^2$ such that $\mathbb{C}^*, \delta^*$ satisfy the exactness property of ES theories.

<u>Theorem (8.4)</u> $\mathbb{C}^*, \delta^*$ is a compactly supported ES theory on $\mathbb{R}^n$.

<u>Proof.</u> We have seen above that $\mathbb{C}^*, \delta^*$ satisfy exactness. To verify excision assume A,B are closed subsets of $\mathbb{R}^n$. By (2) of Lemma (8.3)

$$\underset{k}{\underline{\lim}}\{\pi_{k-q}(\Sigma_0^k (T_k(B)^+/(T_k(A)^+ \cap T_k(B^+))))\} \approx \underset{k}{\underline{\lim}}\{\pi_{k-q}(\Sigma_0^k (T_k(B)^+/T_k(A \cup B^+)))\}$$
$$= \mathbb{C}^q(A \cup B, B)\ .$$

From (3) of Lemma (8.3) for given k if $k' \geq N_k$ there are homomorphisms induced by inclusion

$$\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+)) \longrightarrow \pi_{k-q}(\Sigma_0^k(T_k(A \cap B)^+/T_k(A)^+))$$

$$\downarrow$$

$$\pi_{k-q}(\Sigma_0^k((T_{k'}(A)^+ \cup T_{k'}(B)^+)/T_{k'}(A)^+))$$

implying that

$$\underset{k}{\underrightarrow{\lim}}\{\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+))\} \approx \underset{k}{\underrightarrow{\lim}}\{\pi_{k-q}(\Sigma_0^k(T_k(A \cap B)^+/T_k(A)^+))\}$$

$$= \mathbb{C}^q(A, A \cap B)$$

Since $T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+) = (T_k(A)^+ \cup T_k(B^+))/T_k(A)^+$, it follows that $\underset{k}{\underrightarrow{\lim}}\{\pi_{k-q}(\Sigma_0^k(T_k(B)^+/(T_k(A)^+ \cap T_k(B)^+)))\} \approx$

$\underset{k}{\underrightarrow{\lim}}\{\pi_{k-q}(\Sigma_0^k((T_k(A)^+ \cup T_k(B)^+)/T_k(A)^+))\}$ so that $\mathbb{C}^q(A \cup B, B) \approx$ $\mathbb{C}^q(A, A \cap B)$ and excision is satisfied.

To complete the proof we show that the functor $\mathbb{C}^*(\cdot, \emptyset)$ satisfies Lemma (5.5). Clearly $\mathbb{C}^*(\emptyset, \emptyset) = 0$. Hence, we only need verify that

$$\rho : \underrightarrow{\lim}\{\mathbb{C}^*(N, \emptyset) \mid N \text{ a closed cobounded nbhd of } A \text{ in } \mathbb{R}^n\} \approx \mathbb{C}^*(A, \emptyset)$$

To show ρ is an epimorphism let $\mu : S^{k-q} \longrightarrow \Sigma_0^k(T_k(\emptyset)^+/T_k(A)^+)$ represent an element $\omega \in \pi_{-q}(\mathbb{C}(A,\emptyset)) = \mathbb{C}^q(A,\emptyset)$ for some $A \subset \mathbb{R}^n$. By (4) of Lemma (8.3) with $N = \mathbb{R}^n$ there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $T_k(N') = T_k(A)$. Then μ is also a map from S^{k-q} into $\Sigma_0^k(T_k(\emptyset)^+/T_k(N')^+)$ so determines an element $\omega' \in \pi_{-q}(\mathbb{C}(N',\emptyset))$ whose restriction to $\mathbb{C}(A,\emptyset)$ equals ω. Thus, ρ is an epòmorphism.

To show ρ is a monomorphism let $\lambda : S^{k-q} \longrightarrow \Sigma_0^k(T_k(\emptyset)^+/T_k(N)^+)$ represent an element $\omega \in \pi_{-q}(\mathbb{C}(N,\emptyset))$ whose restriction to $\mathbb{C}(A,\emptyset) = 0$ (where N is a closed cobounded nbhd of A in $\mathbb{R}^n$). Then there is a map $\mu : C_0S^{k'-q} \longrightarrow \Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(A)^+)$ for some $k' \geq k$ such that $\mu | S^{k'-q} = \rho' \circ \Sigma_0^{k'-k}(\lambda)$ where $\rho' : \Sigma_0^{k'}(T_k(\emptyset)^+/T_k(N)^+) \longrightarrow \Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(A)^+)$ is induced by inclusions

$$(T_k(\emptyset)^+,\ T_k(N)^+) \subset (T_k(\emptyset)^+,\ T_k(A)^+) \subset (T_{k'}(\emptyset)^+,\ T_{k'}(A)^+)$$

By (4) of Lemma (8.3) there is a closed cobounded nbhd N' of A in $\mathbb{R}^n$ such that $N' \subset N$ and $T_{k'}(N') = T_{k'}(A)$. Then μ is also a map from $C_0S^{k'-q}$ to $\Sigma_0^{k'}(T_{k'}(\emptyset)^+/T_{k'}(N')^+)$ implying that ω restricts to 0 in $\mathbb{C}(N',\emptyset)$. Thus, ρ is a monomorphism.

$\square$

By the last result $\mathbb{C}^*, \delta^*$ is a compactly supported ES theory on $\mathbb{R}^n$. By Theorem (8.2) the restriction of $\mathbb{F}^*, \delta^*$ to $\mathbb{R}^n$ is also a compactly supported ES theory on $\mathbb{R}^n$. This implies that $\sigma^n \mathbb{F}^*, \delta^*$ is also an ES theory on $\mathbb{R}^n$ (where σ^n is defined as in Remark (2.3) so that

$$(\sigma^n \mathbb{F}^*)^q(A,B) = \mathbb{F}^{q+n}(A,B)).$$

Since $\mathbb{F}^*(A,B)$ is the ES theory defined by the spectrum $\mathbb{F}(A,B)$, $\sigma^n \mathbb{F}^*(A,B)$ is the ES theory defined by the spectrum $\Sigma_0^n \mathbb{F}^*(A,B)$ whose m^{th} term is the $(n+m)^{th}$ term of $\mathbb{F}(A,B)$. We shall define a map of spectra

$$\mu : \mathbb{C}(A,B) \longrightarrow \Sigma_0^n \mathbb{F}(A,B)$$

for $(A,B) \in cl(\mathbb{R}^n)^2$. This will induce a homomorphism

$$\mathbb{C}^q(A,B) = \pi_{-q}(\mathbb{C}(A,B)) \longrightarrow \pi_{-q}(\Sigma_0^n \mathbb{F}(A,B)) \approx \pi_{n-q}(\mathbb{F}(A,B))$$
$$= \mathbb{F}^{q+n}(A,B) = (\sigma^n \mathbb{F}^*)^q(A,B)$$

and μ_* will be a homomorphism from $\mathbb{C}^*, \delta^*$ to $\sigma^n \mathbb{F}^*, \delta^*$.

Given a closed pair (A,B) in $\mathbb{R}^n$ for every $k \geq 0$ there is a continuous map

$$\lambda_k : \mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^{n+k}, \mathbb{R}^{n+k} - \{0\})$$

defined by $\lambda_k(x,y,z) = (x,y-z)$ for $x \in \mathbb{R}^k, y \in T_k(B), z \in A$. Since $T_k(B)$ is a compact subset of $\mathbb{R}^n$, λ_k is a proper map so extends to a continuous map

$$\lambda_k^+ : [\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B)]^+ \longrightarrow (\mathbb{R}^{n+k}, \mathbb{R}^{n+k}-\{0\})^+$$

There is a canonical homeomorphism

$$[\mathbb{R}^k \times (T_k(B), T_k(A)) \times (A,B)]^+ \approx S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)$$

Therefore, λ_k^+ can also be regarded as a map

$$\lambda_k^+ : S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+) \longrightarrow (S^{n+k}, S^{n+k}-\{0\}) .$$

This map λ_k^+ has the following two properties.

1) If $(A',B') \subset (A,B)$, then $(T_k(B), T_k(A)) \subset (T_k(B'), T_k(A'))$ and $\lambda'^+_k|[S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)] =$
$= \lambda_k^+|[S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A'^+, B'^+)]$

2) For any $k \geq 0$, $(T_k(B), T_k(A)) \subset (T_{k+1}(B), T_{k+1}(A))$ and
$$\lambda_{k+1}^+|[S^{k+1} \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+, B^+)] = \Sigma_0(\lambda_k^+)$$

Since ∞ is a deformation retract of $S^{n+k}-\{0\}$ if $i : (S^{n+k},\infty)\subset(S^{n+k},S^{n+k}-\{0\})$ is the inclusion map, there is a map

$$\overline{\lambda}_k^+ : S^k \wedge (T_k(B)^+, T_k(A)^+) \wedge (A^+,B^+) \longrightarrow (S^{n+k},\infty)$$

such that $\lambda_k^+ \simeq i \circ \overline{\lambda}_k^+$ and this condition determines $\overline{\lambda}_k^+$ up to homotopy. We may also consider $\overline{\lambda}_k^+$ as a pointed map $\overline{\lambda}_k^+ : S^k \wedge (T_k(B)^+/T_k(A)^+) \wedge (A^+/B^+) \longrightarrow S^{n+k}$. By the exponential theorem [4] $\overline{\lambda}_k^+$ corresponds to a continuous function

$$\mu_k : \Sigma_0^k(T_k(B)^+/T_k(A)^+) \longrightarrow F(A,B;S^{n+k})$$

By 2) above the following square commutes up to homotopy

$$\begin{array}{ccc} \Sigma_0\Sigma_0^k(T_k(B)^+/T_k(A)^+) & \longrightarrow & \Sigma_0^{k+1}(T_{k+1}(B)^+,T_{k+1}(A)^+) \\ \Sigma_0\mu_k \downarrow & & \downarrow \mu_{k+1} \\ \Sigma_0 F(A,B;S^{n+k}) & \xrightarrow{\epsilon_{n+k}} & F(A,B;S^{n+k+1}) \end{array}$$

Therefore, $\{\mu_k\}$ is a map of spectra

$$\mu : \mathbb{C}(A,B) \longrightarrow \Sigma_0^n \mathbb{F}(A,B)$$

and it follows from 1) above that the map

$$\mu_* : \mathbb{C}^q(A,B) \longrightarrow (\sigma^n \mathbb{F}^*)^q\ (A,B)$$

is a natural transformation from $\mathbb{C}^*$ to $\sigma^n\mathbb{F}^*$. It is easy to verify that μ_* commutes with δ^* for the two ES theories so it is a homomorphism of ES theories.

Theorem (8.5). The homomorphism $\mu_* : \mathbb{C}^*,\delta^* \to \sigma^n\mathbb{F}^*,\delta^*$ is an isomorphism of ES theories on $\mathbb{R}^n$.

Proof. Using the "five-lemma" it suffices to prove that μ_* is an isomorphism of the corresponding cohomology theories. Since each is compactly supported and $\mathbb{R}^n$ is finite dimensional, it suffices to prove μ_* is an isomorphism for $(x,\emptyset)$ for every $x \in \mathbb{R}^n$. But $F(x,\emptyset;S^k) \approx S^k$ so $\Sigma_0^n\mathbb{F}\ (x,\emptyset)$ is the spectrum $S^n, S^{n+1}, \ldots$. Also the pair $(T_k(\emptyset), T_k(x))$ for k large is an n-cell together with the complement of an open n-cell inside it and the map $\mu_k : \Sigma_0^k(T_k(\emptyset)^+/T_k(x)^+) \longrightarrow S^{n+k}$ is of degree 1. Therefore,

$$\mu_* : \pi_q(\mathbb{C}(x,\emptyset)) \approx \pi_{-q}(\Sigma_0^n\mathbb{F}\ (x,\emptyset)) \quad \text{for all} \quad q .$$

□

Corollary (8.6). If (A,B), (C,D) are closed pairs in $\mathbb{R}^n$ there is a duality isomorphism

$$\{A^+/B^+; \mathbb{C}(C,D)\}^q \approx \{C^+/D^+; \mathbb{C}(A,B)\}^q \quad \text{for all} \quad q .$$

<u>Proof.</u> If $(C,D) \in \mathrm{cl}\,(\mathbb{R}^n)^2$ then by Theorem (8.5) there is a map $\mu : \mathbb{C}(C,D) \longrightarrow \Sigma_0^n \mathbb{F}(C,D)$ such that

$$\mu_* : \pi_q(\mathbb{C}(C,D)) \approx \pi_q(\Sigma_0^n \mathbb{F}(C,D)) \text{ for all } q.$$

Since both $\mathbb{C}(C,D)$ and $\Sigma_0^n \mathbb{F}(C,D)$ are ANR spectra, it follows from Theorem (7.3) that for every locally compact pair (A,B)

$$\mu_* : H^q_{\mathbb{C}(C,D)}(A,B) \approx H^q_{\Sigma_0^n \mathbb{F}(C,D)}(A,B)$$

or equivalently,

$$\{A^+,B^+;\mathbb{C}(C,D)\}^q \overset{\mu_*}{\approx} \{A^+,B^+;\Sigma_0^n \mathbb{F}(C,D)\}^q$$

$$= \varinjlim_k \{[\Sigma_0^k(A^+/B^+);F(C,D;S^{n+k+q})]\}$$

By the exponential theorem the last limit is isomorphic to $\varinjlim_k \{[\Sigma_0^k(A^+/B^+) \wedge (C^+/D^+);S^{n+k+q}]\}$. There are canonical homeomorphisms

$$\Sigma_0^k(A^+/B^+) \wedge (C^+/D^+) \approx S^k \wedge (A^+/B^+) \wedge (C^+/D^+)$$

$$\approx S^k \wedge (C^+/D^+) \wedge (A^+/B^+) \approx \Sigma_0^k(C^+/D^+) \wedge (A^+/B^+)$$

so that

$$\{A^+,B^+,\mathbb{C}(C,D)\}^q \approx \varinjlim_k \{ [\Sigma_0^k(C^+/D^+) \wedge (A^+/B^+);S^{n+k+q}] \}$$

$$\approx \varinjlim_k \{ [\Sigma_0^k(C^+/D^+);F(A,B;S^{n+k+q})] \}$$

$$\approx \{C^+,D^+;\Sigma_0^n\mathbb{F}(A,B)\}^q$$

In case (A,B) is also a closed pair in $\mathbb{R}^n$ there is also an isomorphism

$$\{C^+,D^+;\mathbb{C}(A,B)\}^q \overset{\mu_*}{\approx} \{C^+,D^+;\Sigma_0^n\mathbb{F}(A,B)\}^q$$

Combining these isomorphisms gives the result.

□

Our final result is due to Lima [14]. In the proof we essentially show that for a compact $A \subset S^n$, the spectrum of S^n-A and $\Sigma_0^{n-1}\mathbb{F}(A,\emptyset)$ are equivalent (compare with [9, Theorem 4.5]).

Corollary (8.7). Let A,B be nonempty proper compact subsets of S^n. Then

$$\{A;S^n-B\}^q \approx \{B;S^n-A\}^q \quad \text{for all } q .$$

Proof. If φ is a homeomorphism of S^n and the result is valid for A,B it is also valid for $\varphi(A),B$ and $A,\varphi(B)$ so, without loss of generality, we can assume $\infty \in A \cap B$.

Then $A = (A')^+$, $B = (B')^+$ for closed subsets $A', B' \subset \mathbb{R}^n$. By Corollary (8.6) there is a duality isomorphism

$$\{A'^+/\emptyset^+; \mathbb{C}(B',\emptyset)\}^q \approx \{B'^+/\emptyset^+; \mathbb{C}(A',\emptyset)\}^q$$

For k large $T_k(\emptyset)$ is a closed n-cell containing $T_k(B')$ and, therefore, $T_k(\emptyset)^+/T_k(B')^+$ has the same homotopy type as $\Sigma_0 T_k(B')$, and we have

$$\{A'^+/\emptyset^+; \mathbb{C}(B',\emptyset)\}^q = \underset{k}{\underrightarrow{\lim}}\{[\Sigma_0^k(A); \Sigma_0^{k+q}(T_{k+q}(\emptyset)^+/T_{k+q}(B')^+)]\}$$

$$\approx \underset{k}{\underrightarrow{\lim}}\{[\Sigma_0^k(A); \Sigma_0^{k+q+1}\ T_{k+q}(B')]\} .$$

(where in the above $T_k(B')$ is given an arbitrary base point for k large enough which is also the base point for $T_{k'}(B')$ for $k' > k$ and for S^n-B). Since $\{T_k(B')\}_k$ is an increasing sequence of subspaces of S^n-B such that $\bigcup_k \operatorname{int} T_k(B') = S^n-B$, it follows that for the compact space A,

$$[A; \Sigma_0^{q+1}\ (S^n-B)] \approx \underset{k}{\underrightarrow{\lim}}\ \{[A; \Sigma_0^{q+1}\ T_k(B')]\} ,$$

and this implies that

$$\{A; S^n-B\}^{q+1} \approx \underset{k}{\underrightarrow{\lim}}\{[\Sigma_0^k(A); \Sigma_0^{k+q+1} T_k(B')]\} \approx \{A'^+/\emptyset^+, \mathbb{C}(B',\emptyset)\}^q .$$

Similarly $\{B; S^n-A\}^{q+1} \approx \{B'^+/\emptyset^+; \mathbb{C}(A',\emptyset)\}^q$. Combining these isomorphisms with the duality isomorphism gives the result.

□

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