Models of quasiprojective homogeneous spaces for Hopf algebras

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Introduction

Given an affine group scheme G of finite type over a field k, a homogeneous space for G is a scheme X over k containing a rational point x such that G operates "transitively" on X. Assuming that G operates on the right, we may identify X with the quotient $K\backslash G$ of G by the left action of the stabilizer K of x in G. The representation-theoretic significance of $K\backslash G$ is that the induction functor ind_K^G from K-modules to G-modules factors as a category equivalence

$$K\text{-modules} \approx G\text{-linearized quasicoherent sheaves on } K\backslash G \qquad (*)$$

followed by the functor of global sections [11, Ch. 5]. If $K\backslash G$ is affine then $(*)$ can be written as $\mathcal{M}^C \approx \mathcal{M}_A^H$ where A, C, H denote the coordinate algebras of $K\backslash G, K, G$, respectively, $\mathcal{M}^C$ is the category of right C-comodules and $\mathcal{M}_A^H$ is the category of right (H, A)-Hopf modules introduced in [36], [6]. In this case H is a commutative Hopf algebra, C a factor Hopf algebra, and A a subalgebra stable under the action of G on H by right translations, i.e. A is a right coideal subalgebra of H. The equivalence $\mathcal{M}^C \approx \mathcal{M}_A^H$ was generalized by Takeuchi [36] to the case where H is in general a noncommutative Hopf algebra and C is only a *left H-module factor coalgebra* of H, which means that the kernel of the projection $H \to C$ is a coideal and a left ideal of H. Takeuchi's result was obtained under the assumption that there exists a faithfully A-flat left H-module.

In general $K\backslash G$ is only a quasiprojective scheme, and the equivalence $(*)$ cannot be interpreted this simply. Although there are serious difficulties in introducing spaces as objects of noncommutative geometry, the categories of sheaves can be defined purely algebraically as quotient categories of the categories of graded modules by certain localizing subcategories. This idea goes back to Serre [28] who characterized in this way the category of coherent sheaves on a projective variety. A general categorical approach to localization was developed by Gabriel [10]. Passing to an open subscheme of a noetherian scheme involves again such a localization on the level of quasicoherent sheaves. There were several attempts to investigate categories representing sheaves on a noncommutative projective scheme [1], [25], [38], [39].

Let H be any Hopf algebra over the ground field k with the comultiplication Δ, the counit ε and the antipode S. For each ring R we denote by $R[t, t^{-1}]$ the ring of Laurent polynomials with coefficients in R equipped with the grading $\bigoplus_{i\in\mathbb{Z}} Rt^i$. We will view $H[t, t^{-1}]$ as either right or left H-comodule algebra with respect to the comodule structure given by Δ on each homogeneous component Ht^i so that

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the action of t commutes with the coaction of H. We may now speak about right or left H-costable subspaces of $H[t,t^{-1}]$. Let

$$A=\bigoplus_{i\in\mathbb{Z}}A_it^i\subset H[t,t^{-1}]$$

be a graded (i.e. homogeneous with respect to the grading) right H-costable subalgebra so that each A_i is a right coideal of H, $1\in A_0$, and $A_iA_j\subset A_{i+j}$ for all i,j. For each vector subspace $V\subset H$ denote $V^+=V\cap H^+$ where $H^+=\operatorname{Ker}\varepsilon$. Put

$$A^+=\bigoplus A_i^+t^i\subset A\quad\text{and}\quad A^\diamond=\sum A_i^+\subset H.$$

It is checked easily that $HA^\diamond$ is a coideal of H. Hence $H/HA^\diamond$ is a left H-module factor coalgebra of H. We view A as a model of a noncommutative homogeneous space. Graded comodule algebras were used in [14], [17], [34] to define quantum flag varieties. Recall that H is *residually finite dimensional* [21] if its ideals of finite codimension have zero intersection.

Theorem 0.1. *Let H be a residually finite dimensional Hopf algebra, $A\subset H[t,t^{-1}]$ a graded right H-costable subalgebra such that $A_1\neq 0$ and $A^\diamond\subset S(HA^\diamond)$. Denote $C=H/HA^\diamond$. If A and H have right artinian classical right quotient rings $Q(A)$ and $Q(H)$, respectively, then $H[t,t^{-1}]$ is left A-flat and*

$$\mathcal{M}^C\approx\text{gr-}\mathcal{M}_A^H/\,\text{gr-}\mathcal{T}_A^H$$

where gr-$\mathcal{M}_A^H$ *is the category of graded right (H,A)-Hopf modules and* gr-$\mathcal{T}_A^H$ *is its localizing subcategory consisting of all objects M such that $M\otimes_A Q(A)=0$.*

The condition $A^\diamond\subset S(HA^\diamond)$ is not really restrictive since it holds automatically in several cases listed in Lemma 2.7 (e.g. when H is noetherian). The existence of a right artinian classical right quotient ring $Q(A)$ for a right noetherian A is supported by the main result of [33], although it is yet unclear whether we need any further restrictions on H for this to be true. When A is noetherian and module-finite over its center the claim about $Q(A)$ was proved in [32]. Thus the hypotheses of Theorem 0.1 are satisfied when A and H are both noetherian and module-finite over their centers. By Goldie's Theorem another such a case occurs when A and H are semiprime right noetherian. The existence of a right artinian classical right quotient ring $Q(H)$ implies that $S:H\to H$ is bijective by [31, Th. A].

Each homogeneous space can be modelled by different algebras. In other words, the correspondence between the graded right H-costable subalgebras of $H[t,t^{-1}]$ and the left H-module factor coalgebras of H is not one-to-one. Under the hypotheses of Theorem 0.1 it will be shown that A also has a graded classical right quotient ring $Q_{\text{gr}}(A)$. Its homogeneous component $Q_0(A)$ of degree 0 presents an analog of the ring of rational functions on a quasiprojective variety. We may identify $Q_0(A)$ with a subring of $Q(H)$. The next result expresses the idea that a homogeneous space can be characterized in terms of birational algebraic geometry:

Theorem 0.2. *Let $A,B\subset H[t,t^{-1}]$ be two graded right H-costable subalgebras both of which satisfy the hypotheses of Theorem* 0.1*. Then $HB^\diamond=HA^\diamond$ if and only if $Q_0(B)=Q_0(A)$.*

A fundamental fact in the theory of group schemes states that the quotient $K\backslash G$ exists as a quasiprojective scheme for every group subscheme K of G [5]. In our interpretation we would like to know whether any given left H-module factor coalgebra C of H corresponds to some graded right H-costable subalgebra of $H[t, t^{-1}]$. It will be seen from Corollary 2.5 and Lemma 5.1 that this question is related to the existence of ample grouplikes in C. The notion of ampleness for grouplikes corresponds to that for equivariant line bundles on a homogeneous space. To what extent homogeneous spaces can be useful for noncommutative Hopf algebras depends on the answer to the following question which is not investigated in the present paper:

Question. *Under what conditions on H does every left H-module factor coalgebra of H has an ample grouplike*?

It was proved by Cline, Parshall, Scott [3] and Oberst [24] that the induction functor ind_K^G is exact if and only if the scheme $K\backslash G$ is affine. In the language of Hopf algebras ind_K^G is nothing but the cotensor product functor $?\,\Box_C\, H : \mathcal{M}^C \rightsquigarrow \mathcal{M}^H$. When the latter functor is (faithfully) exact H is called *left* (*faithfully*) *C-coflat*. The next theorem generalizes the above mentioned results on the exactness of ind_K^G:

Theorem 0.3. *Under the hypotheses of Theorem* 0.1 *H is left C-coflat if and only if H has a right coideal subalgebra B such that $C = H/HB^+$ and H is left faithfully B-flat. In this case H is left faithfully C-coflat.*

Certainly, the "if" part of Theorem 0.3 is covered by the already mentioned result of Takeuchi. I do not know whether the "only if" part holds for an arbitrary left H-module factor coalgebra C of H. When C is a factor Hopf algebra of an arbitrary Hopf algebra H and the antipode of C is bijective, the conclusion of Theorem 0.3 is a special case of Schneider's result [26, Th. I]. If S is bijective and $C = H/HR^+$ where R is a right coideal subalgebra of H, then the left C-coflatness of H implies that H is left faithfully flat over a possibly larger right coideal subalgebra B with $HB^+ = HR^+$ by a result of Masuoka and Wigner [19, Th. 2.1].

The group scheme G may be regarded as the total space of a principal bundle with base $K\backslash G$ and structure group K. This viewpoint leads to another equivalence:

$$K\text{-linearized quasicoherent sheaves on } G \approx \text{quasicoherent sheaves on } K\backslash G. \quad (**)$$

When $K\backslash G$ is affine, $(**)$ can be written as an equivalence ${}^C_H\mathcal{M} \approx {}_A\mathcal{M}$ between the category of left (C, H)-Hopf modules and the category of left A-modules. In this form the second equivalence was established by Takeuchi [36] for an arbitrary Hopf algebra H under the assumption that there exists a faithfully C-coflat right H-comodule. The noncommutative versions of principal bundles with affine total and base spaces are known as faithfully flat H-Galois extensions. The category equivalences for H-Galois extensions are discussed by Doi and Takeuchi [8]; the most comprehensive results are due to Schneider [26]. We do not attempt to develop the theory of noncommutative principal bundles in the nonaffine case. Nevertheless we are able to generalize $(**)$ as follows:

Theorem 0.4. *Let H be a residually finite dimensional Hopf algebra, $A \subset H[t, t^{-1}]$ a graded right H-costable subalgebra such that $A_1 \neq 0$ and $S(A^\diamond) \subset A^\diamond H$. Denote $C = H/HA^\diamond$. If A and H have left artinian classical left quotient rings, then*

$${}^C_H\mathcal{M} \approx \text{gr-}_A\mathcal{M}/\text{gr-}_A\mathcal{T}$$

where $\text{gr-}_A\mathcal{M}$ is the category of graded left A-modules and $\text{gr-}_A\mathcal{T}$ is its localizing subcategory consisting of modules whose elements are annihilated by a nonzero H-costable left ideal of A.

There is little hope for the existence of affine coverings of noncommutative homogeneous spaces in general, although in case of quantum flag varieties such coverings can be constructed [17], [29].

Terminology and Notation

For a ring R denote by $\mathcal{C}(R)$ the set of all regular elements, i.e. nonzerodivisors, of R. A multiplicatively closed subset $\Sigma \subset \mathcal{C}(R)$ is called a *right Ore set* if Σ satisfies the right Ore condition, i.e., if for each $s \in \Sigma$ and each $a \in R$ there exists $u \in \Sigma$ such that $au \in sR$. If Σ is a right Ore set, then the right ring of fractions $R\Sigma^{-1}$ is defined (see, e.g., [20, Ch. 2]). In the special case when $\Sigma = \mathcal{C}(R)$, the ring $Q(R) = R\Sigma^{-1}$ is called the *classical right quotient ring* of R. A right R-module M is *torsion* if $M \otimes_R Q(R) = 0$.

Unless specified otherwise, the graded rings and modules are graded by the group $\mathbb{Z}$ of integers, and the indices run over $\mathbb{Z}$. If $R = \bigoplus R_i$ is a graded ring, we denote by $\mathcal{C}_{\rm gr}(R)$ the set of all homogeneous regular elements of R. When $\mathcal{C}_{\rm gr}(R)$ is a right Ore set, the corresponding right ring of fractions $Q_{\rm gr}(R)$ will be called the *graded classical right quotient ring* of R. For $i \in \mathbb{Z}$ the ith homogeneous component $Q_i(R)$ of $Q_{\rm gr}(R)$ consists of all elements which can be written as as^{-1} for some $a \in R_{i+j}$ and $s \in \mathcal{C}(R) \cap R_j$ with an arbitrary $j \in \mathbb{Z}$.

The notion of *Hopf modules* was generalized by Doi in [7]. There the category $\mathcal{M}_A^C$ was defined when A is any *right H-comodule algebra* and C is any *right H-module coalgebra*. This assumption about A and C means that A is equipped with a right H-comodule structure given by an algebra homomorphism $A \to A \otimes H$, while C is equipped with a right H-module structure given by a coalgebra homomorphism $C \otimes H \to C$. An object $M \in \mathcal{M}_A^C$ is a right A-module and a right C-comodule with the provision that

$$\sum (ma)_{(0)} \otimes (ma)_{(1)} = \sum m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)}$$

for all $m \in M$ and $a \in A$. Morphisms in $\mathcal{M}_A^C$ are A-linear C-colinear maps.

If, moreover, $A = \bigoplus A_i$ is a graded algebra with H-costable homogeneous components, we say that A is a *graded right H-comodule algebra*. In this case we define the category gr-$\mathcal{M}_A^C$ of *graded right (C, A)-Hopf modules* whose objects are right (C, A)-Hopf modules M equipped with a grading $M = \bigoplus M_i$ such that M is a graded A-module and each homogeneous component M_i is C-costable.

There are modifications of the previous definitions which make use of left module or comodule structures instead of right ones. For example, the category ${}_A^C\mathcal{M}$ is defined when A is a left H-comodule algebra and C is a left H-module coalgebra. We denote by $\mathcal{M}_A$ and ${}_A\mathcal{M}$ the categories of right and left A-modules, by $\mathcal{M}^C$ and ${}^C\mathcal{M}$ the categories of right and left C-comodules. These are special cases of either $\mathcal{M}_A^C$ or ${}_A^C\mathcal{M}$ when one of the arguments is k with the trivial action or coaction of H. When we take $A = H$, the H-comodule structure is assumed to be given by Δ; when $C = H$, the action of H on itself is given by right or left multiplications. The category $\mathcal{M}_H^H$ is equivalent to the category of vector spaces [35, Th. 4.1.1]. In particular, H is a simple object of $\mathcal{M}_H^H$.

Details on the quotient categories can be found in [10] or, e.g., in [9, Ch. 15]. Suppose that $\mathcal{A}$ is a locally small abelian category and $\mathcal{B}$ is an arbitrary abelian category. If an exact functor $\Phi : \mathcal{A} \rightsquigarrow \mathcal{B}$ admits a fully faithful right adjoint functor

$\Psi : \mathcal{B} \rightsquigarrow \mathcal{A}$, then $\operatorname{Ker}\Phi$ is a localizing subcategory of $\mathcal{A}$, and Φ induces an equivalence $\mathcal{A}/\operatorname{Ker}\Phi \approx \mathcal{B}$ [10, Ch. III, Prop. 5]. Conversely, any localizing subcategory and any quotient category of $\mathcal{A}$ can be characterized in this way. All equivalences in the present paper are obtained by applying the previous criterion to a suitable pair of adjoint functors. When $\mathcal{A}$ is a Grothendieck category, its full subcategory $\mathcal{L}$ is localizing if and only if $\mathcal{L}$ is closed under subobjects, factor objects, extensions and small direct limits [10, Ch. III, Prop. 8]. Each category $\mathcal{M}_A^C$ is Grothendieck (cf. [2, p. 78, Th. 19]). This result extends to graded Hopf modules.

Throughout the paper $\otimes$ and Hom mean $\otimes_k$ and Hom_k unless the base ring is indicated explicitly.

1. An equivalence associated with a coideal subalgebra

The main work in this paper will be done in the setup of graded rings. In the first section we illustrate the proof of Theorem 0.1 by giving the ungraded version of this result. Here we deal with a right coideal subalgebra A of H, which corresponds to the special case of Theorem 0.1 when we take the subalgebra of $H[t,t^{-1}]$ to be $A[t]$ or $A[t,t^{-1}]$.

With A one associates the coideal HA^+ and the left H-module factor coalgebra $C = H/HA^+$ of H. There is a pair of adjoint functors

$$\Phi : \mathcal{M}_A^H \rightsquigarrow \mathcal{M}^C \quad \text{and} \quad \Psi : \mathcal{M}^C \rightsquigarrow \mathcal{M}_A^H$$

introduced by Takeuchi [36]. For an object $M \in \mathcal{M}_A^H$ we have $\Phi(M) = M/MA^+$ with the C-comodule structure induced by the H-comodule structure on M.

The construction of Ψ involves the cotensor product $\square_C$. Given two comodules $V \in \mathcal{M}^C$ and $W \in {}^C\mathcal{M}$, we have

$$V \square_C W = \operatorname{Ker}(V \otimes W \xrightarrow{\mu\otimes\mathrm{id}-\mathrm{id}\otimes\lambda} V \otimes C \otimes W)$$

where $\mu : V \to V \otimes C$ and $\lambda : W \to C \otimes W$ are the C-comodule structure maps. Properties of cotensor products are discussed, e.g., in [4]. In particular, $? \square_C H$ is an additive functor $\mathcal{M}^C \rightsquigarrow \mathcal{M}^H$, right adjoint to the functor $\mathcal{M}^H \rightsquigarrow \mathcal{M}^C$ obtained by regarding each H-comodule as a C-comodule via the projection $H \to C$. The adjunction

$$\varepsilon_V : V \square_C H \to V$$

coincides with the restriction of the map $\mathrm{id} \otimes \varepsilon : V \otimes H \to V$. Therefore for each C-colinear map $\psi : N \to V$ with $N \in \mathcal{M}^H$ there exists a unique H-colinear map $\varphi : N \to V \square_C H$ satisfying $\psi = \varepsilon_V \circ \varphi$; we say that φ *coextends* ψ.

The functor Ψ is defined as $\Psi(V) = V \square_C H$ with the action of A by right multiplications on the second tensorand of $V \otimes H$. Hence $\Phi\Psi(V)$ is a factor space of $V \square_C H$. The adjunction $\eta_V : \Phi\Psi(V) \to V$ is induced by ε_V.

Lemma 1.1. *The adjunction $\eta_E : \Phi\Psi(E) \to E$ is an isomorphism whenever E is an injective in $\mathcal{M}_C$. If Φ is exact, then Ψ is fully faithful.*

Proof. Regarded as a right comodule with respect to the comultiplication, C is an injective cogenerator in $\mathcal{M}^C$. Therefore the injectives in $\mathcal{M}^C$ are precisely the direct summands of direct sums of copies of C, and it suffices to prove that η_E is an isomorphism when $E = C$. Now $\Psi(C) = C \square_C H \cong H$ and $\Phi(H) = H/HA^+ = C$. It is easy to see that η_C coincides with the resulting isomorphism $\Phi\Psi(C) \cong C$.

For an arbitrary right C-comodule V there is an exact sequence $0 \to V \to E \to E'$ in $\mathcal{M}^C$ with injective E, E'. By a general property of right adjoint functors Ψ is left exact. If Φ is exact, then $\Phi\Psi$ takes the above exact sequence to an exact sequence $0 \to \Phi\Psi(V) \to \Phi\Psi(E) \to \Phi\Psi(E')$. Since both η_E and $\eta_{E'}$ are isomorphisms, so too is $\eta_V : \Phi\Psi(V) \to V$. Thus all adjunctions $\Phi\Psi(V) \to V$ are isomorphisms, which is a necessary and sufficient condition for Ψ to be fully faithful [18, p. 88, Th. 1]. □

The next lemma is similar to a result of Schneider [26, Cor. 4.2].

Lemma 1.2. *If H is left (faithfully) A-flat then Φ is (faithfully) exact. The converse is true when S is bijective.*

Proof. For objects $M \in \mathcal{M}_A^H$ there are natural isomorphisms of H-modules

$$\xi : M \otimes_A H \cong \Phi(M) \otimes H$$

defined by Takeuchi [36, p. 456]. Explicitly, $\xi(m \otimes h) = \sum (m_{(0)} + MA^+) \otimes m_{(1)}h$ for $m \in M$ and $h \in H$. If H is left A-flat, then $? \otimes_A H$ is exact, and it follows that Φ is exact. For each $M \in \operatorname{Ker}\Phi$ we have $M \otimes_A H$; so $M = 0$ when H is left faithfully A-flat.

For each $V \in \mathcal{M}_A$ we may regard $V \otimes H$ as an object of $\mathcal{M}_A^H$ with respect to the action of A and the coaction of H given by the formulas

$$(v \otimes h) \cdot a = \sum va_{(1)} \otimes ha_{(2)} \quad \text{and} \quad v \otimes h \mapsto v \otimes \Delta(h)$$

where $v \in V$, $h \in H$ and $a \in A$. Denote by ζ the linear transformation $\mathrm{id} \otimes S$ of $V \otimes H$ and by $K \subset V \otimes H$ the linear span of elements $va \otimes h - v \otimes ah$ with v, h, a as above. Then

$$\zeta\big((v \otimes h) \cdot a\big) = \sum va_{(1)} \otimes S(a_{(2)})S(h) \equiv \sum v \otimes a_{(1)}S(a_{(2)})S(h) \equiv \varepsilon(a)\zeta(v \otimes h)$$

modulo K. Hence $\zeta\big((V \otimes H) \cdot A^+\big) \subset K$. If S is bijective, so is ζ, in which case

$$\begin{aligned}\zeta^{-1}(va \otimes h) = va \otimes S^{-1}(h) &= \sum va_{(1)} \otimes S^{-1}(a_{(3)}h)a_{(2)} \\ &\equiv \sum v \otimes S^{-1}(a_{(2)}h)\varepsilon(a_{(1)}) \\ &\equiv \zeta^{-1}(v \otimes ah)\end{aligned}$$

modulo $(V \otimes H) \cdot A^+$. This shows that $\zeta^{-1}(K) \subset (V \otimes H) \cdot A^+$, and therefore ζ induces an isomorphism

$$\Phi(V \otimes H) \cong (V \otimes H) \otimes_A A/A^+ \cong (V \otimes H)/K \cong V \otimes_A H.$$

If Φ is exact, then $V \mapsto \Phi(V \otimes H)$ is exact, whence so is $V \mapsto V \otimes_A H$. If Φ is faithfully exact, we have $V \otimes H = 0$, and therefore $V = 0$, whenever $V \otimes_A H = 0$. □

It is not easy to verify the flatness of H over A directly. Passing to the quotient ring $Q(A)$ provides a crucial link. As a rule the comodule structure does not extend from A to $Q(A)$. At this point we have to switch to the module structure over the finite dual H° of H and invoke several earlier results proved for module algebras.

Recall that a *left H-module algebra* is an algebra A equipped with a left H-module structure such that $h1_A = \varepsilon(h)1_A$ and

$$h(ab) = \sum (h_{(1)}a)(h_{(2)}b) \quad \text{for all } h \in H \text{ and } a, b \in A.$$

A left H-module algebra A is called *H-semiprime* if A has no nonzero nilpotent H-stable ideals. Moreover, A is *H-prime* if $A \neq 0$ and $IJ \neq 0$ for any two nonzero H-stable ideals I, J of A. A left H-module algebra Q is called *H-simple* if $Q \neq 0$ and Q has no H-stable ideals other than 0 and Q itself. A left H-module algebra is *H-semisimple* if it is a finite direct product of H-simple H-module algebras. Now we recall [33, Th. 2.2]:

Theorem 1.3. *Suppose that a left H-module algebra A has a right artinian classical right quotient ring $Q(A)$. Then the H-module structure on A has a unique extension to $Q(A)$ with respect to which $Q(A)$ becomes a left H-module algebra.*

Lemma 1.4. *Under the hypotheses of Theorem* 1.3 *$Q(A)$ is H-semisimple when A is H-semiprime, and $Q(A)$ is H-simple when A is H-prime.*

Proof. The H-semiprimeness and the H-primeness pass from A to Q. Now the conclusion follows from [33, Lemma 4.2]. □

For a left H-module algebra A denote by ${}_H\mathcal{M}_A$ the category whose objects are right A-modules equipped with a left H-module structure such that

$$h(va) = \sum (h_{(1)}v)(h_{(2)}a) \quad \text{for all } h \in H,\ a \in A,\ v \in M.$$

The morphisms in ${}_H\mathcal{M}_A$ are maps which are A-linear and H-linear simultaneously. An object $M \in {}_H\mathcal{M}_A$ is called *A-finite* if it is finitely generated as an A-module; M is *locally A-finite* if M is a directed union of A-finite subobjects. Given a homomorphism of left H-module algebras $A \to B$, there is a well-defined left H-module structure which makes $M \otimes_A B$ an object of ${}_H\mathcal{M}_B$ for any $M \in {}_H\mathcal{M}_A$. The next result is [31, Th. 7.6] stated in a slightly different form:

Theorem 1.5. *Suppose Q is a semilocal H-simple left H-module algebra. Denote by l the greatest common divisor of the lengths of simple factor rings of Q. Then M^l is a free Q-module for each locally A-finite object $M \in {}_H\mathcal{M}_Q$.*

Lemma 1.6. *If Q is a semilocal H-simple H-stable subalgebra of a left H-module algebra Q', then the functor $? \otimes_Q Q' : {}_H\mathcal{M}_Q \rightsquigarrow {}_H\mathcal{M}_{Q'}$ is faithfully exact on the full subcategory $\mathcal{L}$ of locally Q-finite objects of ${}_H\mathcal{M}_Q$.*

Proof. By Theorem 1.5 all objects of $\mathcal{L}$ are projective in $\mathcal{M}_Q$. Hence any exact sequence in $\mathcal{L}$ splits in $\mathcal{M}_Q$, and the exactness of $? \otimes_Q Q'$ on $\mathcal{L}$ is immediate. If F is any nonzero free right Q-module, then $F \otimes_Q Q' \neq 0$. Applying this to $F = M^l$ with $M \in \mathcal{L}$, we deduce that $M \otimes_Q Q' = 0$ entails $M = 0$. □

Lemma 1.7. *Let $\varphi : A \to B$ be a homomorphism of left H-module algebras. Suppose that A, B both have right artinian classical right quotient rings and $\varphi(I)B$ intersects $\mathcal{C}(B)$ for each $S(H)$-stable ideal I of A satisfying $I \cap \mathcal{C}(A) \neq \varnothing$. Then φ extends to a homomorphism of H-module algebras $Q(A) \to Q(B)$.*

Proof. Since $\mathcal{C}(A)$ satisfies the right Ore condition, the set $\{sA \mid s \in \mathcal{C}(A)\}$ of right ideals of A is directed by inverse inclusion. The same holds then for the set

$$\{\varphi(s)Q(B) \mid s \in \mathcal{C}(A)\}$$

of right ideals of $Q(B)$. The latter set must have a smallest element since $Q(B)$ is right artinian. In other words, there exists $u \in \mathcal{C}(A)$ such that $\varphi(u) \in \varphi(s)Q(B)$ for each $s \in \mathcal{C}(A)$. Put

$$L = \{q \in Q(B) \mid q\varphi(s) = 0 \text{ for some } s \in \mathcal{C}(A)\},$$

which coincides with the kernel of the map $\iota : Q(B) \to Q(B) \otimes_A Q(A)$ defined by the rule $q \mapsto q \otimes 1$. Clearly L is a left ideal of $Q(B)$ and $L\varphi(A) \subset L$, which implies that $I = \{a \in A \mid L\varphi(a) = 0\}$ is an ideal of A. The right annihilator in $Q(B)$ of any element of L is a right ideal which intersects $\varphi\big(\mathcal{C}(A)\big)$ and therefore must contain $\varphi(u)$. Thus $u \in I$. There is a well-defined H-module structure on $Q(B) \otimes_A Q(A)$ such that

$$h(x \otimes y) = \sum h_{(1)}x \otimes h_{(2)}y \qquad \text{for } h \in H,\ x \in Q(B),\ y \in Q(A).$$

Since ι is an H-linear map, L is H-stable. Since

$$b\varphi\big((Sh)a\big) = \sum (Sh_{(1)})\big((h_{(2)}b) \cdot \varphi(a)\big) = 0$$

for all $b \in L$, $a \in I$ and $h \in H$, the ideal I is $S(H)$-stable. By the hypothesis $\varphi(I)B$ contains a regular element, say v, of B. Since $Lv = 0$ and v is invertible in $Q(B)$, we conclude that $L = 0$. This shows that each element in $\varphi\big(\mathcal{C}(A)\big)$ has zero left annihilator in $Q(B)$. If $x \in Q(B)$ is any element, then for a sufficiently large integer $n > 0$ the right ideal $x^nQ(B)$ is generated by an idempotent, say e; when x has zero left annihilator the equality $(1-e)x^n = 0$ entails $e = 1$, in which case x has to be invertible in $Q(B)$. It follows that $\varphi\big(\mathcal{C}(A)\big) \subset \mathcal{C}(B)$. By the universality property of the rings of fractions, φ extends to a homomorphism of algebras $\psi : Q(A) \to Q(B)$.

For each H-module algebra R we consider $\mathrm{Hom}(H, R)$ as an algebra with respect to the convolution multiplication. There is a homomorphism of algebras $\tau : R \to \mathrm{Hom}(H, R)$ defined by the rule $\tau(a)(h) = ha$ for $a \in R$ and $h \in H$. The diagram

$$\begin{array}{ccc} Q(A) & \xrightarrow{\tau} & \mathrm{Hom}\big(H, Q(A)\big) \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(H,\psi)} \\ Q(B) & \xrightarrow{\tau} & \mathrm{Hom}\big(H, Q(B)\big) \end{array}$$

commutes since the two composite maps $Q(A) \to \mathrm{Hom}\big(H, Q(B)\big)$ are ring homomorphisms which agree on A. This means that ψ intertwines the action of H on $Q(A)$ and on $Q(B)$. □

Theorem 1.8. *Let A be a right coideal subalgebra of a residually finite dimensional Hopf algebra H. If A and H both have right artinian classical right quotient rings, then H is left A-flat and $\Phi : \mathcal{M}_A^H \rightsquigarrow \mathcal{M}^C$ induces an equivalence $\mathcal{M}_A^H/\,\mathcal{T}_A^H \approx \mathcal{M}^C$ where $\mathcal{T}_A^H = \{M \in \mathcal{M}_A^H \mid M \otimes_A Q(A) = 0\}$.*

Proof. By [30, Th. A] the antipode of H is bijective, whence so is the antipode of H°. We may regard A and H as left H°-module algebras. Since H is residually finite dimensional, the H°-submodules of H coincide with the right coideals. Therefore VH is an $\mathcal{M}_H^H$-subobject of H for any H°-submodule V. It follows that either

$V = 0$ or $VH = H$. Hence $VWH = H$, and therefore $VW \neq 0$, for any two nonzero H°-submodules. In particular, A and H are H°-prime H°-module algebras. By Theorem 1.3 $Q(A)$ is H°-simple. By Lemma 1.7 the inclusion $A \to H$ extends to a homomorphism of H°-module algebras $Q(A) \to Q(H)$.

Each $M \in \mathcal{M}_A^H$ may be regarded as an object of ${}_{H^\circ}\mathcal{M}_A$. We have $M = \bigcup VA$ where V runs over the finite dimensional H-subcomodules of M. Hence M is locally A-finite, and therefore $M \otimes_A Q(A)$ is a locally $Q(A)$-finite object of ${}_{H^\circ}\mathcal{M}_{Q(A)}$. Since $Q(A)$ is left A-flat, it follows from Lemma 1.6 that the functor

$$? \otimes_A Q(H) \cong \bigl(? \otimes_A Q(A)\bigr) \otimes_{Q(A)} Q(H)$$

is exact on $\mathcal{M}_A^H$. On the other hand, Takeuchi's isomorphism ξ recalled in Lemma 1.2 yields

$$M \otimes_A Q(H) \cong (M \otimes_A H) \otimes_H Q(H) \cong \Phi(M) \otimes Q(H).$$

Then Φ has to be exact. By Lemma 1.1 Ψ is fully faithful. Now we conclude that $\operatorname{Ker}\Phi$ is a localizing subcategory of $\mathcal{M}_A^H$, and $\mathcal{M}_A^H / \operatorname{Ker}\Phi \cong \mathcal{M}^C$. Moreover, $\Phi(M) = 0$ if and only if $M \otimes_A Q(H) = 0$, which is equivalent to $M \otimes_A Q(A) = 0$ by Lemma 1.6. Hence $\operatorname{Ker}\Phi = \mathcal{T}_A^H$. The flatness of H follows from Lemma 1.2. □

We can also characterize the objects of $\mathcal{T}_A^H$ as those $M \in \mathcal{M}_A^H$ whose elements are annihilated by a nonzero H-costable right ideal of A (cf. Lemma 4.7). In particular, $\mathcal{T}_A^H = 0$, i.e. Φ is an equivalence, if and only if A is a simple object of $\mathcal{M}_A^H$. The latter is always true when A is a Hopf subalgebra.

Takeuchi's result [36, Th. 1] says that Φ is an equivalence whenever H is left faithfully A-flat without any other assumptions about the Hopf algebra H and its right coideal subalgebra A. When S is bijective the converse is also true, and there are several other equivalent conditions as was established by Masuoka and Wigner [19, Th. 2.1]. In particular, Φ is an equivalence if and only if H is left A-flat and A is a simple object of $\mathcal{M}_A^H$, if and only if H is a projective generator in ${}_A\mathcal{M}$. When S is bijective and A is a Hopf subalgebra even more was shown in [19, Th. 2.9] and [27, Cor. 1.8]. Projectivity in this case was first proved by Schneider. We obtain

Corollary 1.9. *If in Theorem* 1.8 *A is a simple object of $\mathcal{M}_A^H$, then H is a projective generator in ${}_A\mathcal{M}$. In particular, this holds when A is a Hopf subalgebra, and then H is a projective generator also in $\mathcal{M}_A$.*

Wu and Zhang [40, Th. 0.2] proved that any finitely generated noetherian PI Hopf algebra has a quasi-Frobenius classical quotient ring. Projectivity over Hopf subalgebras was established under restrictive finiteness assumptions [40, Th. 0.3]. Corollary 1.9 yields a better result when the Hopf algebra is assumed to be residually finite dimensional (probably this holds automatically):

Corollary 1.10. *Suppose that A is a Hopf subalgebra of H. If both A and H are residually finite dimensional, finitely generated, noetherian and satisfy a polynomial identity, then H is a projective generator in $\mathcal{M}_A$ and in ${}_A\mathcal{M}$.*

2. Grouplikes and ampleness

Let C be a left H-module factor coalgebra of H and $\pi : H \to C$ the projection, so that $\operatorname{Ker}\pi$ is a coideal and a left ideal of H. The element $1_C = \pi(1)$ is a grouplike of C. Denote by $\operatorname{Aut} C$ the group of automorphisms of C as a left H-module coalgebra. Our convention is that $\operatorname{Aut} C$ operates on the right. Put

$$X(C) = \{1_C\theta \mid \theta \in \operatorname{Aut} C\} \subset C.$$

Clearly $X(C)$ is a subset of grouplikes. Since $C = H1_C$, the map $\theta \mapsto 1_C\theta$ is a bijection $\operatorname{Aut} C \to X(C)$. There is a group structure on $X(C)$ with respect to which that bijection is a group isomorphism. We may regard C as a right module coalgebra over the group Hopf algebra $kX(C)$. Also, C is an $\big(H, kX(C)\big)$-bimodule. If $x \in H$ is any element such that $\pi(x) = \chi \in X(C)$, then

$$\pi(h)\chi = (h1_C)\chi = h(1_C\chi) = h\chi = \pi(hx)$$

for $h \in H$. When C is a factor Hopf algebra of H, all grouplikes of C belong to $X(C)$. There appears to be no clear reason why this should be true in general.

Given $V \in \mathcal{M}^C$ and $U \in \mathcal{M}^{kX(C)}$, we define a right C-comodule structure on $V \otimes U$ by the rule $v \otimes u \mapsto \sum (v_{(0)} \otimes u_{(0)}) \otimes v_{(1)}u_{(1)}$ where $v \in V$ and $u \in U$. We will use a shorter notation $V \otimes \chi$ for the right C-comodule $V \otimes k\chi$ where $k\chi$ is a simple onedimensional $kX(C)$-comodule corresponding to some $\chi \in X(C)$ (so that $\sum u_{(0)} \otimes u_{(1)} = u \otimes \chi$). The natural isomorphisms $(V \otimes \chi) \otimes \chi^{-1} \cong V$ show that the functor $? \otimes \chi$ is an autoequivalence of $\mathcal{M}^C$. There is an autoequivalence of ${}^C\mathcal{M}$ defined similarly.

Definition. *A grouplike $\chi \in X(C)$ is* $\mathcal{M}^C$-ample (*resp.* ${}^C\mathcal{M}$-ample) *if for each finite dimensional right* (*resp. left*) *C-comodule V there exist an integer $n > 0$, a right* (*resp. left*) *H-comodule T and an epimorphism $T \to V \otimes \chi^n$ in $\mathcal{M}^C$* (*resp. in* ${}^C\mathcal{M}$).

Note that for $V \in \mathcal{M}^C$ there exists an $\mathcal{M}^C$-epimorphism $T \to V$ with $T \in \mathcal{M}^H$ if and only if $\varepsilon_V : V \,\square_C\, H \to V$ is surjective. Thus $\chi \in X(C)$ is $\mathcal{M}^C$-ample if and only if for each finite dimensional $V \in \mathcal{M}^C$ the map $\varepsilon_{V \otimes \chi^n}$ is surjective for sufficiently large n. There is a similar characterization of ${}^C\mathcal{M}$-ampleness.

For any grouplike $\eta \in C$ and objects $V \in \mathcal{M}^C$ and $W \in {}^C\mathcal{M}$ put

$$V_\eta = \{v \in V \mid \sum v_{(0)} \otimes v_{(1)} = v \otimes \eta\} \cong V \,\square_C\, k\eta.$$

$${}_\eta W = \{w \in W \mid \sum w_{(-1)} \otimes w_{(0)} = \eta \otimes w\} \cong k\eta \,\square_C\, W.$$

In particular, H_η is a left coideal and ${}_\eta H$ a right coideal of H. If $\chi \in X(C)$ is an $\mathcal{M}^C$-ample grouplike, then for each nonzero $V \in \mathcal{M}^C$ there exists $n > 0$ such that $(V \otimes \chi^n) \,\square_C\, H \neq 0$. When applied to $V = k\eta$, this shows that ${}_{\eta\chi^n}H \neq 0$ for some $n > 0$.

Lemma 2.1. *If $\eta \in C$ is any grouplike and $\chi \in X(C)$, then*:

(i) $\pi(h) = \varepsilon(h)\eta$ *for all* $h \in {}_\eta H$ *and for all* $h \in H_\eta$.

(ii) *A right coideal* $V \neq 0$ *of* H *satisfies* $\pi(V) = k\eta$ *if and only if* $V \subset {}_\eta H$.

(iii) ${}_\eta H \cdot {}_\chi W \subset {}_{\eta\chi} W$ *when* $W \in {}^C_H\mathcal{M}$.

(iv) $A(\chi) = \bigoplus {}_{\chi^i}Ht^i$ *is a graded right H-costable subalgebra of* $H[t, t^{-1}]$.

(v) $S(H_\chi) \subset {}_{\chi^{-1}}H$. *Moreover,* ${}_{\chi^{-1}}H = S(H_\chi)$ *when S is bijective.*

Proof. (i) If $h \in {}_\eta H$, then $\sum \pi(h_{(1)}) \otimes h_{(2)} = \eta \otimes h$, and it remains to apply $\mathrm{id} \otimes \varepsilon$ to both sides of this equality. The second case is similar.

(ii) We have $\Delta(V) \subset V \otimes H$. Suppose $\pi(V) \subset k\eta$. Then there is a map $\lambda : V \to k$ such that $\pi(h) = \lambda(h)\eta$ for $h \in V$. Since the counit $\varepsilon_C : C \to k$ satisfies $\varepsilon_C \circ \pi = \varepsilon$

and $\varepsilon_C(\eta) = 1$, we must have $\lambda = \varepsilon|_V$. Hence $\sum \pi(h_{(1)}) \otimes h_{(2)} = \lambda(h_{(1)})\eta \otimes h_{(2)} = \eta \otimes h$ for all $h \in V$, i.e. $V \subset {}_\eta H$. The converse is clear from (i). Note that $\varepsilon(V) \neq 0$ in view of the identity $h = \sum \varepsilon(h_{(1)})h_{(2)}$.

(iii) For $h \in {}_\eta H$ and $w \in {}_\chi W$ the comodule structure map $W \to C \otimes W$ sends

$$hw \mapsto \sum h_{(1)}\chi \otimes h_{(2)}w = \sum \pi(h_{(1)})\chi \otimes h_{(2)}w = \eta\chi \otimes hw.$$

(iv) This follows from (iii) applied to $W = H$.

(v) We may assume that $H_\chi \neq 0$. Then $\varepsilon(H_\chi) \neq 0$, and we can find $x \in H_\chi$ with $\varepsilon(x) = 1$. By (i) $\pi(x) = \chi$. If $h \in H_\chi$ is an arbitrary element, then $\Delta(h) \in H \otimes H_\chi$. Since

$$(\mathrm{id} \otimes \varepsilon)\Delta(h) = h = (\mathrm{id} \otimes \varepsilon)(h \otimes x),$$

we deduce that $\Delta(h) - h \otimes x \in H \otimes H_\chi^+$, whence

$$\varepsilon(h)1 = \sum S(h_{(1)})h_{(2)} \equiv S(h)x \pmod{HH_\chi^+}.$$

By (i) $\pi(H_\chi^+) = 0$, and so $HH_\chi^+ \subset \operatorname{Ker}\pi$. Hence $\varepsilon(h)1_C = \pi\big(S(h)x\big) = \pi\big(S(h)\big)\chi$, which yields $\pi\big(S(H_\chi)\big) = k\chi^{-1}$. Since $S(H_\chi)$ is a right coideal of H, the inclusion in (v) follows from (ii). When S is bijective we also obtain $S^{-1}({}_{\chi^{-1}}H) \subset H_\chi$ passing to the left H^{cop}-module factor coalgebra C^{cop} of the Hopf algebra H^{cop}. □

Lemma 2.2. *If $W \in {}^C\mathcal{M}$ and $\chi \in X(C)$, then ${}_{1_C}(W \otimes \chi^{-1}) = {}_\chi W \otimes \chi^{-1}$.*

The verification of Lemma 2.2 is straightforward from the definitions.

Recall that $H \,\square_C\, ? : {}^C\mathcal{M} \rightsquigarrow {}^H\mathcal{M}$ is right adjoint to the functor ${}^H\mathcal{M} \rightsquigarrow {}^C\mathcal{M}$ induced by π. The adjunction $\varepsilon'_W : H \,\square_C\, W \to W$ coincides with the restriction of $\varepsilon \otimes \mathrm{id} : H \otimes W \to W$.

Lemma 2.3. *If $W \in {}^C_H\mathcal{M}$ then $\operatorname{Im}\varepsilon'_W = H \cdot {}_{1_C}W$.*

Proof. Put $\widetilde{W} = H \,\square_C\, W$. Letting $\mu = (\mathrm{id}\otimes\pi)\Delta : H \to H \otimes C$ and $\lambda : W \to C \otimes W$ denote the C-comodule structures involved, we have

$$\widetilde{W} = \operatorname{Ker}(H \otimes W \xrightarrow{\mu\otimes\mathrm{id}-\mathrm{id}\otimes\lambda} H \otimes C \otimes W).$$

Each of the tensor products appearing here is a left H-module with respect to the tensor product of module structures (see [21, 1.8]). Since λ and μ are H-linear, so too is $\mu \otimes \mathrm{id} - \mathrm{id} \otimes \lambda$. The latter map is also H-colinear for the left H-comodule structures on $H \otimes W$ and $H \otimes C \otimes W$ given by Δ on the first tensorand. Hence $\widetilde{W}$ is an ${}^H_H\mathcal{M}$-subobject of $H \otimes W$. By the left hand version of [35, Th. 4.1.1] $\widetilde{W} = H \cdot {}_1\widetilde{W}$ where

$${}_1\widetilde{W} = \{w \in \widetilde{W} \mid \sum w_{(-1)} \otimes w_{(0)} = 1 \otimes w\}.$$

Since $\varepsilon'_W : \widetilde{W} \to W$ is C-colinear, we have $\varepsilon'_W({}_1\widetilde{W}) \subset {}_{1_C}W$. On the other hand, each $w \in {}_{1_C}W$ is the image of $1 \otimes w \in {}_1\widetilde{W}$ under ε'_W. Hence $\varepsilon'_W({}_1\widetilde{W}) = {}_{1_C}W$. Since the map $\varepsilon \otimes \mathrm{id} : H \otimes W \to W$ is H-linear, so is ε'_W. Applying ε'_W to both sides of the equality $\widetilde{W} = H \cdot {}_1\widetilde{W}$, we arrive at the desired conclusion. □

Lemma 2.4. *If χ^{-1} is ${}^C\mathcal{M}$-ample then $W = H \cdot \sum_{i>0} {}_{\chi^i}W$ for each $W \in {}^C_H\mathcal{M}$.*

Proof. It suffices to check that each finite dimensional C-subcomodule V of W is contained in the right hand side of the equality. There exist an integer $n > 0$ and a ${}^C\mathcal{M}$-epimorphism $\psi : T \to V \otimes \chi^{-n}$ with $T \in {}^H\mathcal{M}$. Then ψ factors as

$$T \xrightarrow{\varphi} H \,\square_C\, (V \otimes \chi^{-n}) \xrightarrow{\varepsilon'_{V\otimes\chi^{-n}}} V \otimes \chi^{-n}$$

where φ is the H-colinear map coextending ψ. Hence

$$V \otimes \chi^{-n} = \operatorname{Im}\psi \subset \operatorname{Im}\varepsilon'_{V\otimes\chi^{-n}} \subset \operatorname{Im}\varepsilon'_{W\otimes\chi^{-n}} = H \cdot {}_{1_C}(W \otimes \chi^{-n})$$

where the last equality follows from Lemma 2.3 since the left C-comodule $W \otimes \chi^{-n}$ is an object of ${}^C_H\mathcal{M}$ with respect to the action of H on the first tensorand. Lemma 2.2 yields ${}_{1_C}(W \otimes \chi^{-n}) = {}_{\chi^n}W \otimes \chi^{-n}$, and we conclude that $V \subset H \cdot {}_{\chi^n}W$. □

Remark. Applying Lemma 2.4 to $W \otimes \chi^{-r}$, we deduce that $W = H \cdot \sum_{i>r} {}_{\chi^i}W$ for any $r \in \mathbb{Z}$.

Corollary 2.5. *Let* $A = k \oplus \left(\bigoplus_{i>0} {}_{\chi^i}Ht^i\right)$. *If* χ^{-1} *is* ${}^C\mathcal{M}$*-ample then* $C = H/HA^\diamond$. *If* χ *is* $\mathcal{M}^C$*-ample and* S *is bijective then* $C = H/HS^{-1}(A^\diamond)$.

Proof. We may view $W = \operatorname{Ker}\pi$ as a ${}^C_H\mathcal{M}$-subobject of H since π is H-linear and C-colinear. Then ${}_{\chi^i}W = {}_{\chi^i}H \cap W$, which equals ${}_{\chi^i}H^+$ by Lemma 2.1(i). When χ^{-1} is ${}^C\mathcal{M}$-ample, Lemma 2.4 yields $W = HA^\diamond$. When χ is $\mathcal{M}^C$-ample and S is bijective we can apply this equality replacing H, C, χ with H^{cop}, C^{cop}, χ^{-1} and making use of Lemma 2.1(v). □

Let further $A = \bigoplus A_it^i \subset H[t, t^{-1}]$ be an arbitrary graded right H-costable subalgebra, and let $C = H/HA^\diamond$. Note that

$$N = \{g \in H \mid A^\diamond g \subset HA^\diamond\}$$

is a subalgebra of H in which $HA^\diamond$ is an ideal. We have $A_i \subset N$ for all i since A^+ is an ideal of A. Clearly C is an (H, N)-bimodule.

Lemma 2.6. (i) *C is a left H-module factor coalgebra of H.*

(ii) *For each $i \in \mathbb{Z}$ with $A_i \neq 0$ there is a grouplike $\chi_i \in C$ which spans $\pi(A_i)$.*

(iii) *$\chi_i\chi_j = \chi_{i+j}$ in the factor algebra $N/HA^\diamond$ when $j \in \mathbb{Z}$ also satisfies $A_j \neq 0$.*

Assuming S to be bijective, we also have

(iv) *$\chi_i \in X(C)$ for all i with $A_i \neq 0$ if and only if $A^\diamond \subset S(HA^\diamond)$.*

Proof. Note that $HA^\diamond \subset H^+$. On the other hand, $A_i \not\subset H^+$ since H^+ contains no nonzero right coideals of H. It follows that $\pi(A_i) \cong A_i/A_i^+$ is onedimensional. Choose any $x \in A_i$ such that $\varepsilon(x) = 1$. Given an arbitrary $a \in A_i$, we have

$$\Delta(a) - x \otimes a \in A_i^+ \otimes H$$

since $\Delta(a) \in A_i \otimes H$ and $(\varepsilon \otimes \mathrm{id})\Delta(a) = a = (\varepsilon \otimes \mathrm{id})(x \otimes a)$. Hence that A_i^+ is a coideal of H. It follows that so are $A^\diamond$ and $HA^\diamond$, which proves (i).

Taking $a = x$ in the displayed inclusion, we deduce that $\chi_i = \pi(x)$ is grouplike. If $y \in A_j$ is any element satisfying $\varepsilon(y) = 1$, then $xy \in A_{i+j}$ and $\varepsilon(xy) = 1$, whence $\chi_{i+j} = \pi(xy) = \pi(x)\pi(y) = \chi_i\chi_j$ in $\pi(N) = N/HA^\diamond$.

The right multiplication by an element $g \in N$ induces an H-linear endomorphism θ_g of C. Moreover, if $\pi(g)$ is grouplike, then

$$\Delta_C\bigl(\pi(hg)\bigr) = \sum h_{(1)}\pi(g) \otimes h_{(2)}\pi(g) = \sum \pi(h_{(1)}g) \otimes \pi(h_{(2)}g)$$

for all $h \in H$, i.e. θ_g is a coalgebra endomorphism. If $\pi(g)$ is invertible in $\pi(N)$, then $\theta_g \in \operatorname{Aut} C$, and therefore $\pi(g) = 1_C\theta_g \in X(C)$. To show that $\chi_i \in X(C)$ we need only to check that $\pi(x)$ is invertible in $\pi(N)$.

Suppose further that S is bijective. Then the formulas $\sum S^{-1}(a_{(2)})a_{(1)} = \varepsilon(a)1$ and $\sum a_{(2)}S^{-1}(a_{(1)}) = \varepsilon(a)1$ yield

$$S^{-1}(a)x \equiv \varepsilon(a)1 \pmod{HA_i^+}, \qquad aS^{-1}(x) \equiv \varepsilon(a)1 \pmod{HS^{-1}(A_i^+)}$$

for all $a \in A_i$. Hence

$$S^{-1}(x)x \equiv 1 \pmod{HA_i^+}, \qquad xS^{-1}(x) \equiv 1 \pmod{HS^{-1}(A_i^+)} \qquad (*)$$

and

$$S^{-1}(A_i^+)x \subset HA_i^+, \qquad A_i^+S^{-1}(x) \subset HS^{-1}(A_i^+). \qquad (**)$$

Similarly, $S^{-1}(y)y - 1 \in HA_j^+$ and $A_j^+S^{-1}(y) \subset HS^{-1}(A_j^+)$ where $y \in A_j$ is such that $\varepsilon(y) = 1$. Since $xy - yx \in A_{i+j}^+$, we have

$$A_j^+S^{-1}(x)S^{-1}(y) \subset A_j^+S^{-1}(y)S^{-1}(x) + HS^{-1}(A_{i+j}^+) \subset HS^{-1}(A_{i+j}^+),$$

whence $A_j^+S^{-1}(x) \subset A_j^+S^{-1}(x)S^{-1}(y)y + HA_j^+ \subset HS^{-1}(A^\diamond)N + HA^\diamond$.

The inclusion $A^\diamond \subset S(HA^\diamond)$ is equivalent to $S^{-1}(A^\diamond) \subset HA^\diamond$. If it holds, we get $A_j^+S^{-1}(x) \subset HA^\diamond$ for all $j \in \mathbb{Z}$, i.e. $S^{-1}(x) \in N$; furthermore, equations $(*)$ show that $\pi S^{-1}(x) = \pi(x)^{-1}$ in $\pi(N)$.

Conversely, if $\chi_i \in X(C)$, then θ_x is bijective since θ_x gives the action of χ_i on C. Since $\pi\bigl(S^{-1}(A_i^+)\bigr) \subset \operatorname{Ker}\theta_x$ by the first inclusion in $(**)$, we conclude that $S^{-1}(A_i^+) \subset \operatorname{Ker}\pi = HA^\diamond$. When this holds for all i, we get $S^{-1}(A^\diamond) \subset HA^\diamond$. □

Lemma 2.7. *Assume S to be bijective. The equality $HS^{-1}(A^\diamond) = HA^\diamond$ equivalent to $A^\diamond H = S(HA^\diamond)$ is fulfilled in the following cases*:

(a) *all surjective endomorphisms of cyclic left H-modules are bijective* (*e.g., this holds when H is either left noetherian or left module-finite over a commutative subring*),

(b) *the set $\{j \in \mathbb{Z} \mid A_j \ne 0\}$ contains both a positive integer and a negative one,*

(c) *there exists $j \ne 0$ such that A_j contains an element $y \equiv 1 \pmod{A^\diamond}$* (*e.g., this holds when $0 \ne A_j \subset A_0$*).

Proof. Put $C' = H/HS^{-1}(A^\diamond)$. Let $x \in A_i$ be such that $\varepsilon(x) = 1$. The right multiplications by x and $S^{-1}(x)$ in H induce H-linear endomorphisms, respectively, $\theta : C \to C$ and $\theta' : C' \to C'$. Since

$$hS^{-1}(x)x \equiv h \pmod{HA^\diamond} \quad \text{and} \quad hxS^{-1}(x) \equiv h \pmod{HS^{-1}(A^\diamond)}$$

for $h \in H$ in view of relations $(*)$ from the proof of Lemma 0.1, both θ and θ' are surjective. If θ and θ' are injective, then $S^{-1}(A_i^+) \subset HA^\diamond$ and $A_i^+ \subset HS^{-1}(A^\diamond)$ in view of $(**)$. When this holds for each i with $A_i \neq 0$, we get $S^{-1}(A^\diamond) \subset HA^\diamond$ and $A^\diamond \subset HS^{-1}(A^\diamond)$.

This proves the conclusion of the lemma in case (a) since both C and C' are cyclic left H-modules. By Fitting's Lemma all surjective endomorphisms of finitely generated left R-modules are bijective when R is a left noetherian ring. The same is true when R is a commutative ring, and this immediately extends to the case when R is a ring left module-finite over a commutative subring [15, Ex. 20.9].

Suppose $i \geq 0$ and $A_j \neq 0$ for some $j < 0$. Pick any $y \in A_j$ with $\varepsilon(y) = 1$, and denote $n = -j$. Since $x^n y^i \equiv 1$ and $y^i x^n \equiv 1$ modulo A_0^+, the transformation of C induced by the right multiplication by y^i is the inverse of θ^n. Hence θ is bijective. Since $S^{-1}(y^i)S^{-1}(x^n) \equiv 1$ and $S^{-1}(x^n)S^{-1}(y^i) \equiv 1$ modulo $S^{-1}(A_0^+)$, we deduce that θ' is also bijective. By symmetry the same conclusion is valid when $i < 0$ and $A_j \neq 0$ for some $j > 0$, which completes the proof in case (b).

In case (c) we may assume $j > 0$, reversing the grading if necessary. If $i \geq 0$, then $x^j \equiv y^i \equiv 1 \pmod{A^\diamond}$ since x^j and y^i are both in A_{ij} with $\varepsilon(x^j) = 1 = \varepsilon(y^i)$. In this case θ^j and $(\theta')^j$ are identity transformations, and therefore θ and θ' are again bijective. If $i < 0$, we are done by (b). If $A_j \subset A_0$ and $y \in A_j$ is such that $\varepsilon(y) = 1$, then $y - 1 \in A_0^+$, so that the hypothesis is satisfied. □

Note that $S(A_i^+)H = A_i^+ H$ whenever $1 \in A_i$ by [13, Lemma 3.1].

3. The graded quotient ring

Let $R = \bigoplus R_i$ be a $\mathbb{Z}$-graded ring. Denote by $\mathcal{C}_+(R) \subset \mathcal{C}_{\mathrm{gr}}(R)$ the subset of homogeneous regular elements of positive degree.

Lemma 3.1. *Suppose that R has a right artinian classical right quotient ring $Q(R)$ and $\mathcal{C}(R) \bigcap \sum_{i>0} R_i R \neq \varnothing$. Then for a graded right ideal $I = \bigoplus I_i$ of R we have:*

(i) *If $\bigcup_{i>0} I_i$ is a nil subset, then I is nilpotent.*

(ii) *If $I \cap \mathcal{C}(R) \neq \varnothing$, then $I \cap \mathcal{C}_+(R) \neq \varnothing$.*

Proof. As is well known, any nil multiplicatively closed subset of a right artinian ring is nilpotent [16]. In particular, any nil right ideal J of R is nilpotent. Moreover, since RJ is a nilpotent ideal of R and $RJQ(R)$ is an ideal of $Q(R)$ by [20, Prop. 2.1.16(vi)], the latter is also nilpotent.

(i) Put $I_+ = \sum_{i>0} I_i$ and $F_j = \sum_{i \geq j} R_i$ for each $j \in \mathbb{Z}$. Let $x \in I$. We claim that for each integer $n \geq 0$ there exists j such that

$$(xF_jR)^n \subset I_+^n Q(R).$$

For $n = 0$ the claim is obvious. Suppose that $n > 0$ and $(xF_lR)^{n-1} \subset I_+^{n-1}Q(R)$ for some $l \in \mathbb{Z}$. Since $Q(R)$ is right noetherian by the Hopkins-Levitzki Theorem, there exists $r \in \mathbb{Z}$ such that $RI_+^{n-1}Q(R) = F_r I_+^{n-1}Q(R)$. For all sufficiently large j we have $xF_{j+r} \subset I_+$. If we choose such a j with $j \geq l$, then $F_j \subset F_l$, and therefore

$$(xF_jR)^n \subset xF_jRI_+^{n-1}Q(R) \subset xF_{j+r}I_+^{n-1}Q(R) \subset I_+^n Q(R).$$

Induction on n proves the claim.

The nil multiplicatively closed subset $\bigcup_{i>0} I_i$ has to be nilpotent. Hence $I_+^n = 0$ for sufficiently large n, which shows that xF_jR is a nilpotent right ideal of R for some j. By an observation at the beginning of the proof $xF_jQ(R)$ is also nilpotent.

The hypothesis of the lemma implies that $F_1Q(R) = Q(R)$. Noting that $F_1^j \subset F_j$ when $j > 0$ and $F_1 \subset F_j$ otherwise, we get $F_jQ(R) = Q(R)$. Hence $x \in xF_jQ(R)$. We conclude that I is nil, and therefore I is nilpotent.

(ii) Put $X = \{x \in \bigcup_{i>0} I_i \mid \operatorname{rann}(x) = \operatorname{rann}(x^2)\}$ where rann denotes the right annihilator in $Q(R)$. If $x \in X$, then $Q(R) = xQ(R) \oplus \operatorname{rann}(x)$. For each $q \in Q(R)$ there exists $s \in \mathcal{C}(R)$ such that $qs \in I$; when $q \in \operatorname{rann}(x)$, we have $qs \in \operatorname{rann}(x)$. This shows that

$$\operatorname{rann}(x) = \big(I \cap \operatorname{rann}(x)\big) \cdot Q(R).$$

Clearly $I \cap \operatorname{rann}(x)$ is a graded right ideal of R. The right ideal $\operatorname{rann}(x)$ of $Q(R)$ is generated by an idempotent. It follows that $I \cap \operatorname{rann}(x)$ cannot be nilpotent when $\operatorname{rann}(x) \neq 0$. By (i) $I \cap \operatorname{rann}(x)$ contains in this case a nonnilpotent homogeneous element v of positive degree. Replacing v with a suitable power of v, we may assume that $v \in X$. Let $x \in R_i$ and $v \in R_j$ with $i, j > 0$. Consider $y = x^j + v^i \in R_{ij}$. Since $xv = 0$, we have $y^2 = x^{2j} + v^ix^j + v^{2i}$ where the last two summands are in $\operatorname{rann}(x)$. If $y^2q = 0$ for some $q \in Q(R)$, we must have $x^{2j}q = 0$, whence $xq = 0$, and then $v^{2i}q = 0$, whence $vq = 0$. This yields

$$\operatorname{rann}(y^2) = \operatorname{rann}(y) = \operatorname{rann}(x) \cap \operatorname{rann}(v).$$

Thus $y \in X$ and $\operatorname{rann}(y)$ is properly contained in $\operatorname{rann}(x)$ since $v \notin \operatorname{rann}(v)$.

The previous argument shows that $\operatorname{rann}(x) = 0$ when x is an element of X for which $\operatorname{rann}(x)$ is minimal possible. In this case x is invertible in $Q(R)$ since $Q(R)$ is right artinian [20, Prop. 3.1.1]. Hence $x \in I \cap \mathcal{C}_+(R)$. □

Lemma 3.2. *Under the hypotheses of Lemma* 3.1 *both $\mathcal{C}_{\rm gr}(R)$ and $\mathcal{C}_+(R)$ are right Ore sets, and $Q_{\rm gr}(R)$ is isomorphic with the ring of fractions $R\mathcal{C}_+(R)^{-1}$. Moreover*:

(i) *$Q_{\rm gr}(R)$ satisfies both ACC and DCC on graded right ideals.*

(ii) *Each $Q_i(R)$ is a right $Q_0(R)$-module of finite length.*

(iii) *A graded right R-module M is torsion if and only if $M \otimes_R Q_{\rm gr}(R) = 0$.*

Proof. Suppose that M is a torsion graded right R-module. For $i \in \mathbb{Z}$ denote by m_i the ith homogeneous component of an element $m \in M$. The annihilator of m_i in R intersects $\mathcal{C}(R)$. The graded right ideal $I = \bigcap \operatorname{Ann}(m_i)$ of R annihilates m. Since $\mathcal{C}(R)$ satisfies the right Ore condition and $m_i \neq 0$ for only a finite number of indices, we have $I \cap \mathcal{C}(R) \neq \varnothing$. Hence $I \cap \mathcal{C}_+(R) \neq \varnothing$ by Lemma 3.1. Thus each element of M is annihilated by some element of $\mathcal{C}_+(R)$.

If $s \in \mathcal{C}_{\rm gr}(R)$, then R/sR is a torsion graded right R-module. It follows that for each $a \in R$ there exists $u \in \mathcal{C}_+(R)$ satisfying $au \in sR$. This verifies the right Ore condition for $\mathcal{C}_{\rm gr}(R)$ and for $\mathcal{C}_+(R)$. Thus $Q_{\rm gr}(R)$ is defined, and (iii) is now proved. Given $s \in \mathcal{C}_{\rm gr}(R)$, there exists $u \in \mathcal{C}_+(R) \cap sR$, and we have $s^{-1}u \in R$. This shows that any element of $Q_{\rm gr}(R)$ can be written as bu^{-1} for some $b \in R$ and $u \in \mathcal{C}_+(R)$.

Suppose that $J \subset K$ are two graded right ideals of $Q_{\rm gr}(R)$. We may regard $Q_{\rm gr}(R)$ as a subring of $Q(R)$. If $JQ(R) = KQ(R)$, then K/J is a torsion graded right R-module, whence $K/J = 0$ by (iii). This means that the lattice of graded right ideals of $Q_{\rm gr}(R)$ embeds in the lattice of right ideals of $Q(R)$, and (i) is immediate.

If V is a right $Q_0(R)$-submodule of $Q_i(R)$, then $VQ_{\rm gr}(R) \cap Q_i(R) = V$. It follows that the lattice of right $Q_0(R)$-submodules of $Q_i(R)$ embeds in the lattice of graded right ideals of $Q_{\rm gr}(R)$. Therefore $Q_i(R)$ satisfies ACC and DCC on right submodules, which yields (ii). □

The ACC in Lemma 3.2(i) and Lemma 3.3 is actually a consequence of DCC by [22, Ch. A, Th. I.7.7].

Lemma 3.3. *If Q is a graded ring satisfying ACC and DCC on graded right ideals, then each homogeneous element $x \in Q$ with zero left or right annihilator is invertible.*

Proof. There exists an integer $n > 0$ such that $x^i Q = x^n Q$ and $\mathrm{rann}(x^i) = \mathrm{rann}(x^n)$ for all integers $i > n$ where rann stands for the right annihilator in Q. Replacing x with x^n, we may assume that $n = 1$. Then $Q = xQ \oplus \mathrm{rann}(x)$, whence there exists an idempotent $e \in Q$ such that $xQ = eQ$ and $\mathrm{rann}(x) = (1-e)Q$. Since $x \in eQ$, we have $ex = x$. If x has zero left annihilator in Q, the last equality yields $e = 1$, and therefore $\mathrm{rann}(x) = 0$. If $\mathrm{rann}(x) = 0$, then $e = 1$ as well; since $xy = e$ for some $y \in Q$, it follows from the equality $x(yx-1) = 0$ that $yx = 1$, i.e. $y = x^{-1}$. □

A *graded left H-module algebra* is a left H-module algebra A equipped with a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$ such that each homogeneous component of A is H-stable.

Lemma 3.4. *Let A be a graded left H-module algebra. Suppose that A has a right artinian classical right quotient ring $Q(A)$ and $\mathcal{C}(A) \bigcap \sum_{i>0} A_i A \neq \varnothing$. Then all homogeneous components of $Q_{\mathrm{gr}}(A)$ are stable under the action of H on $Q(A)$ given by Theorem* 1.3. *If A is H-prime, then*:

(i) *$Q_{\mathrm{gr}}(A)$ has no nonzero proper H-stable graded ideals.*

(ii) *$Q_0(A)$ is an H-semisimple H-module algebra.*

Proof. For each subcoalgebra C of H we consider $\mathrm{Hom}\big(C, Q(A)\big)$ as an algebra with respect to the convolution multiplication. There is an algebra homomorphism

$$\tau : Q(A) \to \mathrm{Hom}\big(C, Q(A)\big)$$

defined by the rule $\tau(q)(c) = cq$ for $q \in Q(A)$ and $c \in C$. We may identify the graded algebra

$$T = \mathrm{Hom}\big(C, Q_{\mathrm{gr}}(A)\big) = \bigoplus \mathrm{Hom}\big(C, Q_i(A)\big)$$

with a subalgebra of $\mathrm{Hom}\big(C, Q(A)\big)$. Suppose that $s \in \mathcal{C}_{\mathrm{gr}}(A)$ has degree n. Then $s^{-1} \in Q_{-n}(A)$, and therefore $Q_{-n}(A) = s^{-1}Q_0(A)$ is a cyclic free right module over $Q_0(A)$. Then also $T_{-n} \cong T_0$ as right T_0-modules. Since $Cs \subset A_n$, we have $\tau(s) \in T_n$, whence $\tau(s)T_{-n} \subset T_0$. Since $\tau(s)$ has an inverse $\tau(s^{-1})$ in $\mathrm{Hom}\big(C, Q(A)\big)$, the left multiplication by $\tau(s)$ induces a monomorphism of right T_0-modules $T_{-n} \to T_0$. If $\dim C < \infty$, then the ring $T_0 \cong Q_0(A) \otimes C^*$ is right artinian, and a comparison of the composition series lengths yields $\tau(s)T_{-n} = T_0$. The last equality means that $\tau(s^{-1}) \in T_{-n}$, i.e. $Cs^{-1} \subset Q_{-n}(A)$. Since H coincides with the union of its finite dimensional subcoalgebras, we get $Hs^{-1} \subset Q_{-n}(A)$, and then $H(A_{i+n}s^{-1}) \subset Q_i(A)$ for all $i \in \mathbb{Z}$.

(i) Let J be any graded H-stable ideal of $Q_{\mathrm{gr}}(A)$. We have $J = IQ_{\mathrm{gr}}(A)$ where $I = J \cap A$ is a graded H-stable ideal of A. By [20, Prop. 2.1.16] $IQ(A)$ is an ideal of $Q(A)$, obviously H-stable. Since $Q(A)$ is H-simple by Lemma 1.4, there are two possibilities. If $IQ(A) = 0$, then $I = 0$, and $J = 0$. Otherwise $IQ(A) = Q(A)$, which means that $I \cap \mathcal{C}(A) \neq \varnothing$; in this case $I \cap \mathcal{C}_{\mathrm{gr}}(A) \neq \varnothing$ by Lemma 3.1, and therefore $J = Q_{\mathrm{gr}}(A)$.

(ii) Suppose that I is any nilpotent H-stable ideal of $Q_0(A)$. Then I generates a graded H-stable ideal J of $Q_{\mathrm{gr}}(A)$ whose homogeneous component in degree 0 equals $J_0 = \sum_{j\in\mathbb{Z}} I_j$ where $I_j = Q_j(A)IQ_{-j}(A)$. Note that each I_j is a nilpotent ideal of $Q_0(A)$ since $I_j^n \subset Q_j(A)I^nQ_{-j}(A)$ for each integer $n > 0$. Hence J_0 is nil. In particular, $1 \notin J_0$, and so $J \neq Q_{\mathrm{gr}}(A)$. Now (i) yields $J = 0$, whence $I = 0$. This shows that $Q_0(A)$ is H-semiprime. Since $Q_0(A)$ is right artinian by Lemma 3.2, the conclusion in (ii) follows from [33, Lemma 4.2]. □

It is in general not true that $Q_0(A)$ is H-simple in Lemma 3.4, even when $H = k$ is the trivial Hopf algebra. For example, if A is the algebra of $n \times n$-matrices with entries in k, then A has a grading in which A_0 is the subalgebra of diagonal matrices. However, in some cases we can prove a stronger conclusion.

Lemma 3.5. *Let A be a graded left H-module algebra. Suppose that A has a right artinian classical right quotient ring and $I \cap \mathcal{C}(A) \neq \varnothing$ for each nonzero H-stable right ideal I of A. Then A is H-prime and $Q_{\mathrm{gr}}(A)$ exists. Moreover:*

(i) *The set $\Gamma = \{i \in \mathbb{Z} \mid Q_i(A) \neq 0\}$ is a subgroup of $\mathbb{Z}$.*

(ii) *$Q_i(A)$ is a simple object of ${}_H\mathcal{M}_{Q_0(A)}$ for each $i \in \Gamma$.*

(iii) *Each component $Q_i(A)$ with $i \in \Gamma$ contains an element invertible in $Q_{\mathrm{gr}}(A)$.*

Proof. The H-primeness of A is obvious. If $A_i = 0$ for all $i \neq 0$ then $Q_{\mathrm{gr}}(A) \cong Q(A)$ is a graded ring concentrated in degree 0. Otherwise we may assume that $A_i \neq 0$ for some $i > 0$, reversing the grading if necessary. In this case A satisfies the hypotheses of Lemma 3.1, and the existence of $Q_{\mathrm{gr}}(A)$ has been established in Lemma 3.2.

Let J be any H-stable graded right ideal of $Q_{\mathrm{gr}}(A)$. We have $J = IQ_{\mathrm{gr}}(A)$ where $I = J \cap A$ is an H-stable graded right ideal of A. If $J \neq 0$, then $I \neq 0$. In this case $I \cap \mathcal{C}(A) \neq \varnothing$, whence $I \cap \mathcal{C}_{\mathrm{gr}}(A) \neq \varnothing$ in view of Lemma 3.1, which yields $J = Q_{\mathrm{gr}}(A)$. Since the lattice of H-stable right $Q_0(A)$-submodules of $Q_i(A)$ embeds in the lattice of H-stable graded right ideals of $Q_{\mathrm{gr}}(A)$ (cf. the proof of Lemma 3.2), (ii) is now clear.

The right ideal of $Q_{\mathrm{gr}}(A)$ generated by a nonzero component $Q_i(A)$ must coincide with $Q_{\mathrm{gr}}(A)$, i.e. $Q_i(A)Q_{j-i}(A) = Q_j(A)$ for all $j \in \mathbb{Z}$. It follows that $j - i \in \Gamma$ for all $i, j \in \Gamma$, which proves (i). Moreover, the previous equalities mean that $Q_{\mathrm{gr}}(A)$ is a strongly Γ-graded ring. Then the multiplication maps

$$Q_i(A) \otimes_{Q_0(A)} Q_j(A) \to Q_{i+j}(A)$$

have to be bijective for all $i, j \in \Gamma$ [37, Lemma 1.1] (also [23, Cor. 3.1.2]). In view of Lemma 3.2 each $Q_i(A)$ is finitely generated as a right $Q_0(A)$-module. Hence $Q_i(A)$ is a $Q_0(A)$-finite object of ${}_H\mathcal{M}_{Q_0(A)}$. By (ii) $Q_0(A)$ is an H-simple H-module algebra. Denote by l the greatest common divisor of the lengths of simple factor rings of $Q_0(A)$. By Theorem 1.5 for each $i \in \Gamma$ there exists an integer $n_i > 0$ such that $Q_i(A)^l \cong Q_0(A)^{n_i}$ as right $Q_0(A)$-modules. Tensoring with $Q_j(A)$, we deduce that $Q_{i+j}(A)^l \cong Q_j(A)^{n_i}$, whence

$$Q_0(A)^{ln_{i+j}} \cong Q_{i+j}(A)^{l^2} \cong Q_j(A)^{ln_i} \cong Q_0(A)^{n_i n_j}$$

as right $Q_0(A)$-modules. It follows that $ln_{i+j} = n_i n_j$ for all $i, j \in \Gamma$. This can be rewritten as $r_{i+j} = r_i r_j$ where we put $r_i = n_i/l \in \frac{1}{l}\mathbb{Z}$. Hence $r_{pi} = r_i^p$ for all integers $p > 0$. If $r_i < 1$ for some $i \in \Gamma$, then $r_{pi} < 1/l$ for sufficiently large p, which is impossible. If $r_i > 1$ for some i, then $r_{pi} \to \infty$ as $p \to \infty$. This means that the length of the right $Q_0(A)$-module $Q_{pi}(A)$ becomes arbitrarily large. But we have seen that the lattice of right $Q_0(A)$-submodules of $Q_i(A)$ embeds in the lattice of right ideals of the right artinian ring $Q(A)$. Therefore there is a bound on the length of $Q_i(A)$ which does not depend on i. Thus we must have $r_i = 1$ for all $i \in \Gamma$.

It follows that $Q_i(A)^l \cong Q_0(A)^l$ as right $Q_0(A)$-modules. By the Krull-Schmidt Theorem $Q_i(A)$ is a cyclic free right $Q_0(A)$-module for any $i \in \Gamma$. Let u_i be any generator for this module. The left multiplication by u_i induces bijective maps $Q_j(A) \to Q_{i+j}(A)$. In particular, $u_i v_i = 1$ for some $v_i \in Q_{-i}(A)$. As $u_i(v_i u_i - 1) = 0$, we get $v_i u_i = 1$ as well. Thus $v_i = u_i^{-1}$. Now (iii) is also proved. □

Lemma 3.5(iii) means that $Q_{\mathrm{gr}}(A)$ is a crossed product $Q_0(A) * \Gamma$.

Lemma 3.6. *Denote* $\Gamma' = \{i \in \mathbb{Z} \mid A_i \cap \mathcal{C}_{\mathrm{gr}}(A) \neq \varnothing\}$. *In addition to the hypotheses of Lemma* 3.5 *assume that* $A_j \neq 0$ *for some* $j > 0$. *Then there exists an integer* $n > 0$ *such that* $\{i \in \Gamma \mid i > n\} \subset \Gamma'$. *If* $A_l \neq 0$ *also for some* $l < 0$, *then* $\Gamma' = \Gamma$.

Proof. Clearly Γ' is a subsemigroup of Γ. If $i \in \Gamma$, then $Q_i(A)$ contains an invertible element which can be written as as^{-1} for some $a, s \in \mathcal{C}_{\mathrm{gr}}(A)$. This shows that Γ is generated by Γ' as a group. If Γ' contains both a positive integer and a negative one, then $\Gamma' = \Gamma$. Indeed, if i is the largest negative integer in Γ', then i must divide all positive integers in Γ'. In this case Γ' must contain $-i$ as the smallest positive integer, and $-i$ must divide all negative integers in Γ', so that $\Gamma' = \mathbb{Z}i$ is a group.

Since the graded H-stable right ideal $\sum_{i>0} A_i A$ is nonzero, it intersects $\mathcal{C}(A)$. Hence $\mathcal{C}_+(A) \neq \varnothing$ by Lemma 3.1, which means that Γ' contains a positive integer. By symmetry Γ' contains a negative integer when $\sum_{i<0} A_i \neq 0$. In this case Γ' is a group as we have seen, whence $\Gamma' = \Gamma$.

Suppose that $A_i = 0$ for all $i < 0$. Let $\Gamma = \mathbb{Z}d$ where $d > 0$. Since d can be written as a $\mathbb{Z}$-linear combination of integers in Γ', there exists $m \in \Gamma'$ such that $m + d \in \Gamma'$. We have $m = qd$ for some integer $q > 0$. Put $n = qm$. If $i \in \Gamma$ and $i > n$, then $i = pm + rd$ for some $p, r \in \mathbb{Z}$ such that $p \geq q$ and $0 \leq r < q$. Since $i = (p-r)m + r(m+d)$ with $p - r > 0$, we get $i \in \Gamma'$. □

Lemma 3.7. *Let B be a graded left H-module algebra, A a graded H-stable subalgebra. Suppose that $S(H) = H$, both A and B have right artinian classical right quotient rings and $J \cap \mathcal{C}(B) \neq \varnothing$ for each nonzero H-stable right ideal J of B. Then*:

(i) $I \cap \mathcal{C}(A) \neq \varnothing$ *for each nonzero H-stable right ideal I of A.*

(ii) $Q_{\mathrm{gr}}(A)$ *embeds in* $Q_{\mathrm{gr}}(B)$ *as a graded H-stable subalgebra.*

Proof. (i) We have $IB \cap \mathcal{C}(B) \neq \varnothing$ for each nonzero H-stable right ideal I of A. This implies that A is H-prime, and therefore $Q(A)$ is H-simple by Lemma 1.4. By Lemma 1.7 the inclusion $A \to B$ extends to a homomorphism of H-module algebras $\psi : Q(A) \to Q(B)$ which has to be injective. We may regard A/I as an A-finite object of ${}_H\mathcal{M}_A$. Since $1 \in IQ(B)$, we have $A/I \otimes_A Q(B) = 0$, whence $A/I \otimes_A Q(A) = 0$ by Lemma 1.6, i.e. $1 \in IQ(A)$. This yields the conclusion.

(ii) The graded quotient rings exist by Lemma 3.5. The existence of ψ means that $\mathcal{C}(A) \subset \mathcal{C}(B)$. Hence $\mathcal{C}_{\mathrm{gr}}(A) \subset \mathcal{C}_{\mathrm{gr}}(B)$, and therefore ψ maps $Q_{\mathrm{gr}}(A)$ into $Q_{\mathrm{gr}}(B)$. □

An ascending *filtration* in a ring R is a collection of additive subgroups $(F_i)_{i\in\mathbb{Z}}$ such that $1 \in F_0$, $F_iF_j \subset F_{i+j}$ for all $i, j \in \mathbb{Z}$, and $F_i \subset F_j$ whenever $i < j$. With F one associates the Rees ring $R(F) = \bigoplus F_i t^i$ which is a graded subring of $R[t, t^{-1}]$. If $\bigcup F_i = R$, the filtration is called *exhaustive*. The rings $R[t]$ and $R[t, t^{-1}]$ are special cases of $R(F)$. Thus Lemma 3.8 describes their quotient rings.

Lemma 3.8. *Let (F_i) be an exhaustive filtration in a ring R. If $\mathcal{C}(R)$ satisfies the right Ore condition, then so does $\mathcal{C}_{\mathrm{gr}}\big(R(F)\big)$, and $Q_{\mathrm{gr}}\big(R(F)\big) \cong Q(R)[t, t^{-1}]$.*

Moreover, if $Q(R)$ is right artinian, then $R(F)$ has a right artinian classical right quotient ring.

Lemma 3.9. *Each right H-costable right ideal J of $H[t,t^{-1}]$ is generated by a right ideal of $k[t,t^{-1}]$. If J is graded, then either $J=0$ or $J=H[t,t^{-1}]$.*

Proof. We may regard $H[t,t^{-1}]$ as an object of $\mathcal{M}_H^H$ and J as its subobject. It is clear that $k[t,t^{-1}]=\{v\in H[t,t^{-1}]\mid \sum v_{(0)}\otimes v_{(1)}=v\otimes 1\}$. The conclusion follows from Sweedler's structure theorem for objects of $\mathcal{M}_H^H$. □

Proposition 3.10. *Let H be a residually finite dimensional Hopf algebra, and let $A=\bigoplus A_it^i\subset H[t,t^{-1}]$ be a graded right H-costable subalgebra such that $A_1\ne 0$. If A and H have right artinian classical right quotient rings, then:*

(i) *$I\cap\mathcal{C}(A)\ne\varnothing$ for each nonzero H-costable right ideal I of A.*

(ii) *$I\cap\mathcal{C}_+(A)\ne\varnothing$ for each nonzero H-costable graded right ideal I of A.*

(iii) *$Q_{\rm gr}(A)$ embeds in $Q(H)[t,t^{-1}]$ as a graded H°-stable subalgebra.*

(iv) *$Q(A)$ and $Q_0(A)$ are right artinian H°-simple H°-module algebras.*

(v) *$Q_i(A)$ is a simple object of ${}_{H^\circ}\mathcal{M}_{Q_0(A)}$ for each $i\in\mathbb{Z}$.*

(vi) *Each $Q_i(A)$ contains an element invertible in $Q_{\rm gr}(A)$.*

(vii) *$A_i\cap\mathcal{C}(H)\ne\varnothing$ for all sufficiently large $i>0$.*

(viii) *If $A_l\ne 0$ for at least one $l<0$, then $A_i\cap\mathcal{C}(H)\ne\varnothing$ for all $i\in\mathbb{Z}$.*

Proof. We may regard $B=H[t,t^{-1}]$ as a graded left H°-module algebra and A as a graded H°-stable subalgebra. By Lemma 3.8 B has a right artinian classical right quotient ring and $Q_{\rm gr}(B)\cong Q(H)[t,t^{-1}]$. Since H is residually finite dimensional, the H°-submodules of B coincide with the right H-costable subspaces. Since nonzero elements of $k[t,t^{-1}]$ are regular in B, we deduce from Lemma 3.9 that $J\cap\mathcal{C}(B)\ne\varnothing$ for each nonzero H°-stable right ideal J of B. We know that H and H° have bijective antipode. Hence Lemmas 3.1, 3.2, 3.5, 3.6, 3.7 yield all conclusions. □

4. The first equivalence

We will assume that $S:H\to H$ is bijective and $A=\bigoplus A_it^i$ is a graded right H-costable subalgebra of $H[t,t^{-1}]$ such that $A_1\ne 0$ and $A^\diamond\subset S(HA^\diamond)$. Denote $C=H/HA^\diamond$, and let $\pi:H\to C$ be the projection. By Lemma 2.6 there is a grouplike $\chi\in X(C)$ such that $\pi(A_i)=k\chi^i$ for all $i\in\mathbb{Z}$ with $A_i\ne 0$. Let $x_i\in A_i$ be any element satisfying $\varepsilon(x_i)=1$, so that $\chi^i=\pi(x_i)$. This notation will be kept unchanged throughout the rest of the paper.

The counit $\varepsilon:H\to k$ induces an isomorphism of A/A^+ onto a subalgebra of $k[t,t^{-1}]$ containing t. Hence $A/A^+\cong k[t]$ when $A_i=0$ for all $i<0$ and $A/A^+\cong k[t,t^{-1}]$ otherwise. In any case $Q_{\rm gr}(A/A^+)\cong k[t,t^{-1}]$. Let $\bar x=x_1t+A^+\in A/A^+$, so that $\bar x\mapsto t$ under the previous isomorphism. Thus the elements $\bar x^i$ with $i\in\mathbb{Z}$ form a basis for $Q_{\rm gr}(A/A^+)$ over k.

We will regard A/A^+ and $Q_{\rm gr}(A/A^+)$ as graded right $kX(C)$-comodule algebras with respect to the comodule structure defined by the rule $\bar x^i\mapsto\bar x^i\otimes\chi^i$. Since C is a right $kX(C)$-module coalgebra, we may consider right $\big(C,Q_{\rm gr}(A/A^+)\big)$-Hopf modules defined with respect to the Hopf algebra $kX(C)$.

Lemma 4.1. *There is an equivalence* gr-$\mathcal{M}^C_{Q_{\rm gr}(A/A^+)} \rightsquigarrow \mathcal{M}^C$.

Proof. If $W = \bigoplus W_i$ is an object of gr-$\mathcal{M}^C_{Q_{\rm gr}(A/A^+)}$, then $W_0 \in \mathcal{M}^C$ and the map $W_0 \otimes Q_{\rm gr}(A/A^+) \to W$ afforded by the module structure on W is bijective. Hence $W \mapsto W_0$ is the required equivalence with the inverse equivalence $\mathcal{M}^C \rightsquigarrow$ gr-$\mathcal{M}^C_{Q_{\rm gr}(A/A^+)}$ given by $V \mapsto V \otimes Q_{\rm gr}(A/A^+)$. □

Let $M \in$ gr-$\mathcal{M}^H_A$, and let $\rho : M \to M \otimes H$ denote the comodule structure map. If $A_i \ne 0$ and $a \in A_i$, then $\Delta(a) - x_i \otimes a \in A_i^+ \otimes H$. Hence

$$\rho\big(m(at^i)\big) = \rho(m)\big(\sum a_{(1)}t^i \otimes a_{(2)}\big) \equiv \rho(m)(x_i t^i \otimes a) \pmod{MA^+ \otimes H}$$

for all $m \in M$. In particular, $\rho(MA^+) \subset MA^+ \otimes H + M \otimes A^\diamond$, and therefore ρ induces a right C-comodule structure $\mu : M/MA^+ \to M/MA^+ \otimes C$. Clearly each homogeneous component of M/MA^+ is stable under μ. Taking $a = x_i$, we deduce from the displayed formula that $\mu\big(v(x_i t^i)\big) = \mu(v)(x_i t^i \otimes \chi^i)$ for all $v \in M/MA^+$. Thus M/MA^+ is an object of gr-$\mathcal{M}^C_{A/A^+}$ and

$$M \otimes_A Q_{\rm gr}(A/A^+) \cong M/MA^+ \otimes_{A/A^+} Q_{\rm gr}(A/A^+)$$

is an object of gr-$\mathcal{M}^C_{Q_{\rm gr}(A/A^+)}$. Denote by $\Phi(M) \in \mathcal{M}^C$ the degree 0 homogeneous component of the latter. By Lemma 4.1

$$M \otimes_A Q_{\rm gr}(A/A^+) \cong \Phi(M) \otimes Q_{\rm gr}(A/A^+).$$

Lemma 4.2. *The functor* $\Phi :$ gr-$\mathcal{M}^H_A \rightsquigarrow \mathcal{M}^C$ *admits a right adjoint* Ψ.

Proof. For $V \in \mathcal{M}^C$ put $\widetilde V = V \otimes Q_{\rm gr}(A/A^+)$, $\widetilde V_i = V \otimes Q_i(A/A^+)$, and

$$\Psi(V) = \widetilde V \,\square_C\, H = {\rm Ker}(\widetilde V \otimes H \xrightarrow{\mu\otimes{\rm id}-{\rm id}\otimes\lambda} \widetilde V \otimes C \otimes H)$$

where $\mu : \widetilde V \to \widetilde V \otimes C$ and $\lambda = (\pi \otimes {\rm id})\Delta : H \to C \otimes H$ are C-comodule structure maps. We will regard $\widetilde V \otimes H = \bigoplus (\widetilde V_i \otimes H)$ as an object of gr-$\mathcal{M}^H_{H[t,t^{-1}]}$ with respect to the right comodule structure ${\rm id} \otimes \Delta$ and the right module structure defined by the formula $(v \otimes h) \cdot gt^i = v\bar x^i \otimes hg$ for $v \in \widetilde V$, $h, g \in H$ and $i \in \mathbb{Z}$.

Let us check that $\Psi(V)$ is stable under the action of A. Suppose that $A_i \ne 0$ and $a \in A_i$. Since $\Delta(ha) = \Delta(h)\Delta(a) \equiv \Delta(h)(x_i \otimes a) \pmod{HA^\diamond \otimes H}$, we have

$$(\mu \otimes {\rm id})\big((v \otimes h) \cdot at^i\big) = \mu(v\bar x^i) \otimes ha = \big(\mu(v) \otimes h\big) \cdot (\bar x^i \otimes \chi^i \otimes a),$$
$$({\rm id} \otimes \lambda)\big((v \otimes h) \cdot at^i\big) = v\bar x^i \otimes \lambda(ha) = \big(v \otimes \lambda(h)\big) \cdot (\bar x^i \otimes \chi^i \otimes a)$$

where $\widetilde V \otimes C \otimes H$ is regarded as a right $Q_{\rm gr}(A/A^+) \otimes kX(C) \otimes H$-module. Hence the equality

$$(\mu \otimes {\rm id} - {\rm id} \otimes \lambda)(u \cdot at^i) = (\mu \otimes {\rm id} - {\rm id} \otimes \lambda)(u) \cdot (\bar x^i \otimes \chi^i \otimes a)$$

must hold for all $u \in V \otimes H$, and it follows that the kernel of $\mu \otimes {\rm id} - {\rm id} \otimes \lambda$ is stable under the action of at^i, as claimed. The right A-module $\Psi(V)$ has a grading with homogeneous components $\widetilde V_i \square_C H$. Since $\Psi(V)$ is an H-subcomodule of $\widetilde V \otimes H$, it is an object of gr-$\mathcal{M}^H_A$.

Let $M \in \text{gr-}\mathcal{M}_A^H$. As we recalled in section 1 there is a bijective correspondence between the H-colinear maps $\varphi : M \to \Psi(V)$ and the C-colinear maps $\psi : M \to \widetilde{V}$. Precisely, φ corresponds to ψ if and only if $\psi = \varepsilon_{\widetilde{V}} \circ \varphi$ where $\varepsilon_{\widetilde{V}}$ is the restriction to $\Psi(V)$ of the map $\mathrm{id} \otimes \varepsilon : \widetilde{V} \otimes H \to \widetilde{V}$. Given φ, define $\alpha, \beta : M \otimes A \to \Psi(V)$ by the formulas

$$\alpha(m \otimes a) = \varphi(ma) \quad \text{and} \quad \beta(m \otimes a) = \varphi(m)a$$

for $m \in M$ and $a \in A$. Thus φ is A-linear if and only if $\alpha = \beta$. We equip $M \otimes A$ with the tensor product of H-comodule structures. Since α, β are both H-colinear, the equality $\alpha = \beta$ is equivalent to $\varepsilon_{\widetilde{V}} \circ \alpha = \varepsilon_{\widetilde{V}} \circ \beta$. Assuming that $A_i \neq 0$ and $a \in A_i$, we have

$$\varepsilon_{\widetilde{V}}\alpha(m \otimes at^i) = \psi(m \cdot at^i) \quad \text{and} \quad \varepsilon_{\widetilde{V}}\beta(m \otimes at^i) = \psi(m) \cdot \varepsilon(a)\bar{x}^i.$$

The equality $\psi(m \cdot at^i) = \psi(m) \cdot \varepsilon(a)\bar{x}^i$ holds for all m, a, i if and only if $\psi(MA^+) = 0$ and ψ induces an A/A_+-linear map $M/MA^+ \to \widetilde{V}$. It follows that there are natural bijections between the sets of morphisms

$$\begin{aligned} \text{gr-}\mathcal{M}_A^H\big(M, \Psi(V)\big) &\cong \text{gr-}\mathcal{M}_{A/A^+}^C(M/MA^+, \widetilde{V}) \\ &\cong \text{gr-}\mathcal{M}_{Q_{\text{gr}}(A/A^+)}^C\big(M \otimes_A Q_{\text{gr}}(A/A^+), \widetilde{V}\big) \\ &\cong \mathcal{M}^C\big(\Phi(M), V\big). \end{aligned}$$

□

Lemma 4.3. *If $M = H[t, t^{-1}]$, then $\Phi(M) \cong C$ and the adjunction $M \to \Psi\Phi(M)$ is an isomorphism.*

Proof. Clearly $MA^+ = (HA^\diamond)[t, t^{-1}]$, and therefore $M/MA^+ \cong C[t, t^{-1}]$. The right C-comodule structure on each homogeneous component Ct^i is given by the comultiplication of C; the action of A/A_+ commutes with the action of t and $c\bar{x} = (c\chi)t$ for all $c \in C$. Since χ operates on C as an invertible transformation, so does $\bar{x}$ on $C[t, t^{-1}]$. Hence $M \otimes_A Q_{\text{gr}}(A/A^+) \cong M/MA^+$, which restricts to $\Phi(M) \cong C$ on homogeneous components of degree 0. Now

$$\Psi\Phi(M) \cong \big(M \otimes_A Q_{\text{gr}}(A/A^+)\big) \,\square_C\, H \cong C[t, t^{-1}] \,\square_C\, H \cong H[t, t^{-1}] \qquad (*)$$

since $C \,\square_C\, H \cong H$. The H-colinear adjunction $\gamma_M : M \to \Psi\Phi(M)$ coextends the C-colinear map $M \to M \otimes_A Q_{\text{gr}}(A/A^+)$ given by the rule $m \mapsto m \otimes 1$. Since the latter map may be identified with the projection $M \to C[t, t^{-1}]$, it is clear that γ_M coincides with the inverse of $(*)$. □

Lemma 4.4. *The adjunction $\eta_E : \Phi\Psi(E) \to E$ is an isomorphism whenever E is an injective in $\mathcal{M}^C$. If Φ is exact, then Ψ is fully faithful.*

Proof. By a general property of adjoint functors $\eta_{\Phi(M)} : \Phi\Psi\Phi(M) \to \Phi(M)$ is a right inverse of the morphism $\Phi(M) \to \Phi\Psi\Phi(M)$ obtained by applying Φ to the adjunction $\gamma_M : M \to \Psi\Phi(M)$. When γ_M is an isomorphism, so too is $\eta_{\Phi(M)}$. In particular, η_E is an isomorphism for $E = C$ by Lemma 4.3. The proof is completed as in Lemma 1.1. □

Lemma 4.5. *For objects $M \in \text{gr-}\mathcal{M}_A^H$ there are natural isomorphisms of graded $H[t, t^{-1}]$-modules $M \otimes_A H[t, t^{-1}] \cong \Phi(M) \otimes H[t, t^{-1}]$.*

Proof. We may regard $N = M \otimes_A H[t,t^{-1}]$ as an object of gr-$\mathcal{M}^H_{H[t,t^{-1}]}$. Each homogeneous component N_i of N is an object of $\mathcal{M}^H_H$, and the action of t gives isomorphisms $N_i \to N_{i+1}$. Hence $N \cong V \otimes H[t,t^{-1}]$ for some vector space V. Considering the algebra homomorphism $H[t,t^{-1}] \to k[t,t^{-1}]$ induced by ε, we get

$$V \otimes k[t,t^{-1}] \cong N \otimes_{H[t,t^{-1}]} k[t,t^{-1}] \cong M \otimes_A k[t,t^{-1}] \cong \Phi(M) \otimes k[t,t^{-1}]$$

since $k[t,t^{-1}] \cong Q_{\mathrm{gr}}(A/A^+)$. It follows that $V \cong \Phi(M)$. □

We will denote by $Q_i(M)$, $i \in \mathbb{Z}$, the homogeneous components of the graded $Q_{\mathrm{gr}}(A)$-module $Q_{\mathrm{gr}}(M) = M \otimes_A Q_{\mathrm{gr}}(A)$.

Lemma 4.6. *Let $M \in$ gr-$\mathcal{M}^H_A$. If the hypotheses of Theorem* 0.1 *are fulfilled, then*:
(i) $Q_{\mathrm{gr}}(M) \cong Q_0(M) \otimes_{Q_0(A)} Q_{\mathrm{gr}}(A)$ *as graded right $Q_{\mathrm{gr}}(A)$-modules.*
(ii) $Q_0(M) \otimes_{Q_0(A)} Q(H) \cong \Phi(M) \otimes Q(H)$ *as right $Q(H)$-modules.*
(iii) *For each $i \in \mathbb{Z}$ the $Q_0(A)$-module $Q_i(M)$ is free of rank equal to* $\dim \Phi(M)$.

The isomorphisms in (i), (ii) *are natural in M.*

Proof. By Proposition 3.10 for each i there exists $u_i \in Q_i(A)$ with $u_i^{-1} \in Q_{-i}(A)$. The action of u_i on $Q_{\mathrm{gr}}(M)$ gives a bijection $Q_0(M) \to Q_i(M)$. Since u_i is a free generator for $Q_i(A)$ as a left $Q_0(A)$-module, the canonical map

$$Q_0(M) \otimes_{Q_0(A)} Q_i(A) \to Q_i(M)$$

is bijective. This yields (i). Now

$$\begin{aligned} M \otimes_A Q(H)[t,t^{-1}] &\cong Q_{\mathrm{gr}}(M) \otimes_{Q_{\mathrm{gr}}(A)} Q(H)[t,t^{-1}] \\ &\cong Q_0(M) \otimes_{Q_0(A)} Q(H)[t,t^{-1}]. \end{aligned}$$

On the other hand, Lemma 4.5 yields

$$\begin{aligned} M \otimes_A Q(H)[t,t^{-1}] &\cong \big(M \otimes_A H[t,t^{-1}]\big) \otimes_{H[t,t^{-1}]} Q(H)[t,t^{-1}] \\ &\cong \Phi(M) \otimes Q(H)[t,t^{-1}]. \end{aligned}$$

Comparing the homogeneous components of degree 0 in the two expressions above, we deduce (ii).

By Proposition 3.10 $Q_0(A)$ is a right artinian H°-simple H°-module algebra. We may regard $Q_0(M)$ as an object of ${}_{H^\circ}\mathcal{M}_{Q_0(A)}$. If M is A-finite, then $Q_{\mathrm{gr}}(M)$ is $Q_{\mathrm{gr}}(A)$-finite. In this case (i) shows that there exists a finite set of generators for $Q_{\mathrm{gr}}(M)$ contained in $Q_0(M)$, which implies that $Q_0(M)$ is $Q_0(A)$-finite. In general $Q_{\mathrm{gr}}(M) = \bigcup Q_{\mathrm{gr}}(VA)$ where V runs over the finite dimensional graded subcomodules of M. Hence $Q_0(M)$ is always locally $Q_0(A)$-finite. By Theorem 1.5 $Q_0(M)^l$ is a free right $Q_0(A)$-module for some integer $l > 0$. Now $\Phi(M)^l \otimes Q_0(A)$ is another free right $Q_0(A)$-module whose extension to $Q(H)$ is isomorphic with $Q_0(M)^l \otimes_{Q_0(A)} Q(H)$. Since $Q(H)$ is right artinian, any two bases of a free $Q(H)$-module have the same cardinality. It follows that

$$Q_0(M)^l \cong \Phi(M)^l \otimes Q_0(A) \quad \text{as right } Q_0(A)\text{-modules.}$$

The Krull-Schmidt Theorem yields $Q_0(M) \cong \Phi(M) \otimes Q_0(A)$ (we do not claim that it is possible to obtain this naturally in M). Hence $Q_i(M) \cong \Phi(M) \otimes Q_i(A)$ as right

$Q_0(A)$-modules. Since $Q_i(A)$ is a cyclic free right $Q_0(A)$-module, (iii) is now clear. $\square$

Lemma 4.7. *If the hypotheses of Theorem* 0.1 *are fulfilled, then* Φ *is exact and* Ψ *is fully faithful. For an object* $M \in \text{gr-}\mathcal{M}_A^H$ *the following conditions are equivalent*:

(i) $\Phi(M) = 0$,

(ii) $Q_0(M) = 0$,

(iii) $M \otimes_A Q(A) = 0$,

(iv) *each element of* M *is annihilated by a nonzero* H*-costable right ideal of* A.

Proof. The localization at a right Ore set is an exact functor. In particular, $Q_{\text{gr}}(?)$ is exact, whence so too is $Q_0(?)$. As we know, $Q(H)$ is a left H°-module algebra containing $Q_0(A)$ as an H°-stable subalgebra. Since $Q_0(A)$ is H°-simple and right artinian, in view of Lemma 1.6 the functor $?\otimes_{Q_0(A)} Q(H)$ is faithfully exact on the full subcategory of ${}_{H^\circ}\mathcal{M}_{Q_0(A)}$ consisting of locally $Q_0(A)$-finite objects. Therefore $Q_0(?) \otimes_{Q_0(A)} Q(H)$ is exact on gr-$\mathcal{M}_A^H$. It follows from Lemma 4.6 that Φ is exact and (i)$\Leftrightarrow$(ii). Also, (ii) holds if and only if $Q_{\text{gr}}(M) = 0$, which is equivalent to (iii) by Lemma 3.2. Any element of M is contained in a finite dimensional subcomodule of M. The annihilator of the latter in A is an H-costable right ideal which is nonzero when $M \otimes_A Q(A) = 0$. Hence (iii)$\Rightarrow$(iv). The implication (iv)$\Rightarrow$(iii) follows from Proposition 3.10(i). By Lemma 4.4 Ψ is fully faithful. $\square$

Lemma 4.8. *If the hypotheses of Theorem* 0.1 *are fulfilled, then* $H[t,t^{-1}]$ *is left* A*-flat. For a graded right* A*-module* V *the following conditions are equivalent*:

(i) $V \otimes_A H[t,t^{-1}] = 0$,

(ii) *each element of* V *is annihilated by a nonzero* H*-costable right ideal of* A.

Proof. For each graded right A-module V we may regard $V \otimes H = \bigoplus (V_i \otimes H)$ as an object of gr-$\mathcal{M}_A^H$ with the module and comodule structures as in Lemma 1.2. There is a linear map

$$\zeta : (V \otimes H) \otimes_A Q_{\text{gr}}(A/A^+) \to V \otimes_A H[t,t^{-1}], \qquad (v \otimes h) \otimes \bar{x}^i \mapsto v \otimes S(h)t^i.$$

To show that ζ is well-defined note that for $a \in A_j$ we have

$$(v \otimes h) \cdot at^j = \sum v(a_{(1)}t^j) \otimes ha_{(2)}, \quad at^j \cdot \bar{x}^i = \varepsilon(a)\bar{x}^{i+j}, \quad \text{and}$$

$$\sum v(a_{(1)}t^j) \otimes S(ha_{(2)})\bar{x}^i = \sum v \otimes a_{(1)}t^j S(a_{(2)})S(h)t^i = v \otimes \varepsilon(a)S(h)t^{i+j}$$

in $V \otimes_A H[t,t^{-1}]$. Moreover, the rule $v \otimes ht^i \mapsto \big(v \otimes S^{-1}(h)\big) \otimes \bar{x}^i$ gives a well-defined inverse of ζ (cf. Lemma 1.2). Thus ζ is bijective. In view of Lemma 4.1 we arrive at a linear bijection

$$\Phi(V \otimes H) \otimes Q_{\text{gr}}(A/A^+) \cong V \otimes_A H[t,t^{-1}].$$

Since Φ is exact by Lemma 4.7, so is the functor $?\otimes_A H[t,t^{-1}]$ on the category of graded right A-modules. By [22, Ch. A, Prop. I.2.18] this suffices to deduce the A-flatness of $H[t,t^{-1}]$. Furthermore, V satisfies (i) if and only if $\Phi(V \otimes H) = 0$. Suppose that $v \in V$ is any element and $I \neq 0$ an H-costable right ideal of A. If I annihilates $v \otimes 1 \in V \otimes H$, then

$$va \otimes 1 = \sum va_{(1)} \otimes a_{(2)} S(a_{(3)}) = \sum \big((v \otimes 1) \cdot a_{(1)}\big) \cdot \big(1 \otimes S(a_{(2)})\big) = 0$$

for all $a \in I$, i.e. $vI = 0$. Conversely, the equality $vI = 0$ implies that I annihilates $v \otimes h$ for any $h \in H$; since $IQ(A) = Q(A)$, the torsion A-submodule of $V \otimes H$ contains $v \otimes H$. Thus (i)⇔(ii) in Lemma 4.8 follows from (i)⇔(iii)⇔(iv) in Lemma 4.7. □

Theorem 0.1 is immediate from Lemmas 4.7 and 4.8.

5. Dependence on the algebra

We retain the assumptions from section 4 and continue to use the same notation. In particular, $\Phi : \text{gr-}\mathcal{M}_A^H \rightsquigarrow \mathcal{M}^C$ and $\Psi : \mathcal{M}^C \rightsquigarrow \text{gr-}\mathcal{M}_A^H$ are a pair of adjoint functors introduced in section 4.

Lemma 5.1. *If Ψ is faithful, then the grouplike χ associated with A is $\mathcal{M}^C$-ample.*

Proof. The faithfulness of Ψ means that the adjunction $\eta_V : \Phi\Psi(V) \to V$ is an epimorphism in $\mathcal{M}^C$ for each right C-comodule V [18, p. 88, Th. 1]. Recall that $\Psi(V) = \widetilde{V} \,\square_C\, H$ where $\widetilde{V} = V \otimes Q_{\text{gr}}(A/A^+)$ and η_V is the restriction to the degree 0 homogeneous components of the gr-$\mathcal{M}^C_{Q_{\text{gr}}(A/A^+)}$-morphism

$$\Psi(V) \otimes_A Q_{\text{gr}}(A/A^+) \to \widetilde{V}$$

which is the $Q_{\text{gr}}(A/A^+)$-linear extension of the map $\varepsilon_{\widetilde{V}} : \Psi(V) \to \widetilde{V}$ defined in section 1. Let $\widetilde{V}_i = V \otimes \bar{x}^i$ denote the ith homogeneous component of $\widetilde{V}$. Since the image of $\varepsilon_{\widetilde{V}}$ is an A/A^+-submodule of $\widetilde{V}$, we have $(\operatorname{Im} \varepsilon_{\widetilde{V}_i})\bar{x} \subset \operatorname{Im} \varepsilon_{\widetilde{V}_{i+1}}$ for each $i \in \mathbb{Z}$. The image of η_V coincides with the union of the ascending chain of subcomodules $(\operatorname{Im} \varepsilon_{\widetilde{V}_i})\bar{x}^{-i} \subset V$. If $\dim V < \infty$, then the surjectivity of η_V implies that $V = (\operatorname{Im} \varepsilon_{\widetilde{V}_n})\bar{x}^{-n}$ for sufficiently large $n > 0$. Hence $\varepsilon_{\widetilde{V}_n} : \widetilde{V}_n \,\square_C\, H \to \widetilde{V}_n$ is surjective. It remains to note that $\widetilde{V}_n \cong V \otimes \chi^n$ in $\mathcal{M}^C$. □

Lemma 5.2. *If $M = \bigoplus M_i t^i$ is a gr-$\mathcal{M}_A^H$-subobject of $H[t, t^{-1}]$ and Φ is exact, then $\Phi(M)$ is isomorphic with the right coideal $\bigcup \pi(M_i)\chi^{-i}$ of C.*

Proof. By Lemma 4.3 $\Phi(H[t, t^{-1}]) \cong C$. If Φ is exact, the map $\Phi(M) \to \Phi(H[t, t^{-1}])$ afforded by functoriality is injective. So one needs only to compute the image of $\Phi(M)$, which is straightforward. Note that the sequence of right coideals $\pi(M_i)\chi^{-i}$ of C is ascending since $\pi(M_i)\chi^{-i} = \pi(M_i x_1)\chi^{-i-1} \subset \pi(M_{i+1})\chi^{-i-1}$ for each $i \in \mathbb{Z}$. □

Lemma 5.3. *Let M, M' be two gr-$\mathcal{M}_A^H$-subobjects of $H[t, t^{-1}]$. Assuming that the hypotheses of Theorem 0.1 are satisfied, we have $M' \subset MQ_{\text{gr}}(A)$ if and only if $\Phi(M') \subset \Phi(M)$ as right coideals of C.*

Proof. It was proved in section 4 that Φ is exact, so Lemma 5.2 applies. Since $Q_{\text{gr}}(A)$ is a subring of $Q(H)[t, t^{-1}]$ by Proposition 3.10, there is an injective canonical map

$$Q_{\text{gr}}(M) = M \otimes_A Q_{\text{gr}}(A) \to Q(H)[t, t^{-1}]$$

We may identify $Q_{\text{gr}}(M)$ with the image $MQ_{\text{gr}}(A)$ of that map. Similarly, we have $Q_{\text{gr}}(M') \cong M'Q_{\text{gr}}(A)$. Note that $\Phi(M + M') = \Phi(M) + \Phi(M')$ as right coideals of

C. This allows us to replace M' with $M+M'$ and so assume that $M \subset M'$. Then $\Phi(M) \subset \Phi(M')$, and the equality $\Phi(M) = \Phi(M')$ holds if and only if $\Phi(M'/M) = 0$, which is equivalent to $Q_{\rm gr}(M'/M) = 0$ by Lemma 4.7. Since $Q_{\rm gr}(?)$ is an exact functor, the last equality can be rewritten as $Q_{\rm gr}(M') = Q_{\rm gr}(M)$. □

Lemma 5.4. *Let V be a right coideal of H. Under the hypotheses of Theorem* 0.1 *we have*:

(i) *$VQ_0(A)$ is a free $Q_0(A)$-module of rank equal to $\alpha = \dim \pi(V)$.*

(ii) *If $V \neq 0$ and $V \subset {}_\eta H$ for some grouplike $\eta \in C$, then $\alpha = 1$.*

(iii) *If $V \neq 0$ and $V \subset {}_{\chi^n} H$ for some $n \in \mathbb{Z}$, then $VQ_0(A)t^n = Q_n(A)$.*

Proof. Consider the gr-$\mathcal{M}_A^H$-subobject $M = VA = \bigoplus VA_i t^i$ of $H[t,t^{-1}]$. We have $Q_{\rm gr}(M) \cong VQ_{\rm gr}(A)$. Since $\pi(VA_i) = \pi(V)\chi^i$, Lemma 5.2 yields $\Phi(M) \cong \pi(V)$. Now (i) follows from Lemma 4.6. In (ii) $\pi(V) = k\eta$ by Lemma 2.1. Note also that $\Phi(Mt^n) = \pi(V)\chi^{-n}$. If $\eta = \chi^n$, then $\Phi(Mt^n) = k = \Phi(A)$, whence $VQ_{\rm gr}(A)t^n = Q_{\rm gr}(A)$ by Lemma 5.3. □

Theorem 5.5. *Let $A, B \subset H[t,t^{-1}]$ be two graded right H-costable subalgebras, both satisfying the hypotheses of Theorem* 0.1*. The following conditions are equivalent*:

(i) *$B^\diamond \subset HA^\diamond$,*

(ii) *for each $j \in \mathbb{Z}$ there exist $i \in \mathbb{Z}$ and $s \in \mathcal{C}(H)$ such that $B_j s \subset {}_{\chi^i} H$,*

(iii) *for each finite dimensional subspace V of a homogeneous component of B there exists $s \in \mathcal{C}(H)$ such that Vs is contained in a homogeneous component of A,*

(iv) *$Q_0(B) \subset Q_0(A)$.*

Proof. If $a \in A_i$ is any element such that $at^i \in \mathcal{C}_{\rm gr}(A)$, then at^i is invertible in $Q_{\rm gr}(A) \subset Q(H)[t,t^{-1}]$, whence $a \in \mathcal{C}(H)$. A similar characterization of regularity is valid for elements of B_i.

(i)⇒(ii) Denote $C' = H/HB^\diamond$, and let $\pi' : H \to C'$ be the projection. By the hypothesis $HB^\diamond \subset HA^\diamond$, whence $\pi : H \to C$ factors through C'. By Lemma 2.6 $\pi'(B_j)$ is spanned by a grouplike, say $\eta \in X(C')$. We have $\eta^{-1} \in X(C')$. If $h \in H$ is any element such that $\pi'(h) = \eta^{-1}$, then the right multiplication by h in H induces the action of η^{-1} on C'. It follows that $\pi'(B_j h) = k\eta\eta^{-1} = k1_{C'}$, and therefore $\pi(B_j h) = k1_C$. The image of η^{-1} in C, that is $\pi(h)$, is again grouplike. Since χ is $\mathcal{M}^C$-ample by Lemma 5.1, there exists $n \in \mathbb{Z}$ such that ${}_{\pi(h)\chi^n} H \neq 0$. Let U be any nonzero right coideal of H contained in ${}_{\pi(h)\chi^n} H$. By Lemma 2.1 we have $\pi(U) = k\pi(h)\chi^n = \pi(khx_n)$, whence $U \subset khx_n + HA^\diamond$. It follows that $B_j U \subset B_j hx_n + HA^\diamond$, which yields $\pi(B_j U) \subset \pi(B_j h)\chi^n = k\chi^n$. Since $B_j U$ is a right coideal of H, Lemma 2.1 entails $B_j U \subset {}_{\chi^n} H$.

By Lemma 5.4 $UQ_0(A)$ is a cyclic free $Q_0(A)$-module. Each element of $UQ_0(A)$ can be written as us^{-1} with $u \in UA_l$ and a regular element $s \in A_l$ for some $l \in \mathbb{Z}$. We can find u and s such that us^{-1} is a free generator of $UQ_0(A)$. Lemma 4.6 applied to $M = UA$ shows that $UQ_{\rm gr}(A) \cong UQ_0(A) \otimes_{Q_0(A)} Q_{\rm gr}(A)$. Hence u is a free generator of that $Q_{\rm gr}(A)$-module. Replacing U with UA_l and n with $n+l$, we may assume that $u \in U$.

Suppose that B_j contains a regular element v. Then vu is a free generator of a right $Q_{\rm gr}(A)$-submodule of $Q(H)[t,t^{-1}]$. By Lemma 5.4 $B_j U t^n \subset Q_n(A)$. In particular, $vut^n \in Q_n(A)$. Since vut^n has zero right annihilator in $Q_{\rm gr}(A)$, this element is invertible in $Q_{\rm gr}(A)$ by Lemma 3.3. Hence vu is invertible in $Q(H)$, and so too is u. In this case $u \in \mathcal{C}(H)$, and $B_j u \subset {}_{\chi^n} H$.

In general $Q_j(B)$ contains an element invertible in $Q_{\rm gr}(B)$ by Proposition 3.10. This element can be written as ab^{-1} with $a, b \in \mathcal{C}_{\rm gr}(B)$. Hence there exists $j' \in \mathbb{Z}$ such that both $B_{j'}$ and $B_{j+j'}$ contain regular elements of H. By the previous step we have $B_{j+j'}w \subset {}_{\chi^i}H$ for some $i \in \mathbb{Z}$ and $w \in \mathcal{C}(H)$. Pick any $z \in B_{j'} \cap \mathcal{C}(H)$. Since $B_j z \subset B_{j+j'}$, we get $B_j s \subset {}_{\chi^i}H$ with $s = zw \in \mathcal{C}(H)$.

(ii)⇒(iii) Let $V \subset B_j$, and let i, s be given by (ii). There exists a finite dimensional right H-subcomodule $W \subset H_{\chi^i}$ such that $Vs \subset W$. By Lemma 5.4 $Wt^i \subset Q_i(A)$, whence there exists a regular element $s' \in A_l$ for some l such that $Ws' \subset A_{i+l}$. We get $Vss' \subset A_{i+l}$ with $ss' \in \mathcal{C}(H)$.

(iii)⇒(iv) Each element of $Q_0(B)$ can be written as bu^{-1} with $b \in B_j$ and a regular element $u \in B_j$ for some j. By (iii) there exists $s \in \mathcal{C}(H)$ such that $bs \in A_i$ and $us \in A_i$ for some i. Then $bu^{-1} = (bs)(us)^{-1} \in Q_0(A)$.

(iv)⇒(i) We have to show that $B_j^+ \subset HA^\diamond$ for each $j \in \mathbb{Z}$. If B_j contains a regular element s, then $s^{-1}B_j \subset Q_0(B) \subset Q_0(A)$, whence $B_j \subset sQ_0(A)$, and therefore $B_jQ_0(A) = sQ_0(A)$. In general there exists $j' \in \mathbb{Z}$ such that both $B_{j'}$ and $B_{j+j'}$ contain regular elements. If $u \in B_{j'}$ is regular, then the left multiplication by u induces an embedding of right $Q_0(A)$-modules $B_jQ_0(A) \to B_{j+j'}Q_0(A)$ where the second module is cyclic free by the previous observation. Hence the length of $B_jQ_0(A)$ does not exceed the length of $Q_0(A)$ as a right module over itself. By Lemma 5.4 $B_jQ_0(A)$ is a free $Q_0(A)$-module of rank equal to $\dim \pi(B_j)$. The only possibility is $\dim \pi(B_j) = 1$. We have $\pi(B_j) \cong B_j/(B_j \cap HA^\diamond)$. Since $B_j \cap HA^\diamond$ is contained in B_j^+ and $\dim B_j/B_j^+ = 1$, we deduce that $B_j \cap HA^\diamond = B_j^+$. The conclusion is now clear. □

Theorem 0.2 follows immediately from Theorem 5.5.

6. Exactness of induction

In this section we meet the situation where the functor $\Phi : \text{gr-}\mathcal{M}_A^H \rightsquigarrow \mathcal{M}^C$ is an equivalence. The full faithfulness of Φ can be expressed by saying that the adjunctions $\gamma_M : M \to \Psi\Phi(M)$ are isomorphisms for all $M \in \text{gr-}\mathcal{M}_A^H$. To achieve the bijectivity of γ_M for $M = A$ we have to replace A with a possibly larger graded right H-costable subalgebra $A(\chi) = \bigoplus {}_{\chi^i}Ht^i$ of $H[t, t^{-1}]$ (see Lemma 2.1). Another observation is that γ_M behaves well under tensoring operations.

For objects $U \in \mathcal{M}^H$, $V \in \mathcal{M}^C$ and $M \in \text{gr-}\mathcal{M}_A^H$ we will view $U \otimes V$ and $U \otimes M$ as objects, respectively, of $\mathcal{M}^C$ and $\text{gr-}\mathcal{M}_A^H$. The comodule structures are defined by the rule

$$u \otimes v \mapsto \sum (u_{(0)} \otimes v_{(0)}) \otimes u_{(1)}v_{(1)}$$

for $u \in U$ and $v \in V$ (resp. $v \in M$), while A acts on the second tensorand of $U \otimes M$.

Lemma 6.1. *There are canonical natural isomorphisms $\Phi(U \otimes M) \cong U \otimes \Phi(M)$ and $\Psi(U \otimes V) \cong U \otimes \Psi(V)$, and there is a commutative diagram*

$$\begin{array}{ccc} U \otimes M & \xrightarrow{U \otimes \gamma_M} & U \otimes \Psi\Phi(M) \\ \| & & \cong \downarrow \text{can.} \\ U \otimes M & \xrightarrow{\gamma_{U \otimes M}} & \Psi\Phi(U \otimes M). \end{array}$$

Proof. Obviously $(U \otimes M) \otimes_A Q_{\rm gr}(A/A^+) \cong U \otimes \big(M \otimes_A Q_{\rm gr}(A/A^+)\big)$. Restriction to homogeneous components of degree 0 yields $\Phi(U \otimes M) \cong U \otimes \Phi(M)$.

We may regard $\mathcal{M}^C$ and gr-$\mathcal{M}_A^H$ as left module categories over the tensor category $\mathcal{M}^H$. If $\dim U < \infty$, then the dual vector space U^* has a right H-comodule structure such that $\sum f_{(0)}(u) f_{(1)} = \sum f(u_{(0)}) S(u_{(1)})$ for $f \in U^*$ and $u \in U$. It makes U^* the left dual of U in $\mathcal{M}^H$ (see [12, Ch. XIV]). Then $U^* \otimes ?$ is left adjoint of $U \otimes ?$ either as a functor $\mathcal{M}^C \rightsquigarrow \mathcal{M}^C$ or as a functor gr-$\mathcal{M}_A^H \rightsquigarrow$ gr-$\mathcal{M}_A^H$. Hence there are natural bijections

$$\begin{aligned}\text{gr-}\mathcal{M}_A^H\big(X, U \otimes \Psi(V)\big) &\cong \text{gr-}\mathcal{M}_A^H\big(U^* \otimes X, \Psi(V)\big)\\ &\cong \mathcal{M}^C\big(\Phi(U^* \otimes X), V\big)\\ &\cong \mathcal{M}^C\big(U^* \otimes \Phi(X), V\big)\\ &\cong \mathcal{M}^C\big(\Phi(X), U \otimes V\big)\end{aligned}$$

where $X \in$ gr-$\mathcal{M}_A^H$. This can be extended to arbitrary U. Indeed,

$$\begin{aligned}\text{gr-}\mathcal{M}_A^H\big(X, U \otimes \Psi(V)\big) &\cong \varprojlim_{X'} \text{gr-}\mathcal{M}_A^H\big(X', U \otimes \Psi(V)\big)\\ &\cong \varprojlim_{X'} \Big(\varinjlim_{U'} \text{gr-}\mathcal{M}_A^H\big(X', U' \otimes \Psi(V)\big)\Big)\end{aligned}$$

where X' runs over A-finite subobjects of X and U' runs over finite dimensional subcomodules of U. For each X' appearing here $X' \otimes_A Q_{\text{gr}}(A/A^+)$ is a finitely generated $Q_{\text{gr}}(A/A^+)$-module, whence $\dim \Phi(X') < \infty$. Since $\Phi(X) \cong \varinjlim \Phi(X')$, we get

$$\mathcal{M}^C\big(\Phi(X), U \otimes V\big) \cong \varprojlim_{X'} \Big(\varinjlim_{U'} \mathcal{M}^C\big(\Phi(X'), U' \otimes V\big)\Big).$$

Hence

$$\text{gr-}\mathcal{M}_A^H\big(X, U \otimes \Psi(V)\big) \cong \mathcal{M}^C\big(\Phi(X), U \otimes V\big) \tag{$*$}$$

in general. Also, there are natural bijections

$$\text{gr-}\mathcal{M}_A^H\big(X, \Psi(U \otimes V)\big) \cong \mathcal{M}^C\big(\Phi(X), U \otimes V\big). \tag{$**$}$$

By Yoneda's Lemma $\Psi(U \otimes V) \cong U \otimes \Psi(V)$.

Now take $X = U \otimes M$ and $V = \Phi(M)$. The canonical isomorphism in the set $\mathcal{M}^C\big(\Phi(X), U \otimes V\big)$ corresponds to $U \otimes \gamma_M$ under $(*)$ and to the composite

$$U \otimes M \xrightarrow{\gamma_{U \otimes M}} \Psi\Phi(U \otimes M) \xrightarrow{\text{can.}} \Psi\big(U \otimes \Phi(M)\big)$$

under $(**)$, which ensures the commutativity of the diagram. □

Lemma 6.2. *The algebra $A(\chi)$ satisfies the hypotheses of Theorem* 0.1 *provided so does A. Replacing A with $A(\chi)$ does not affect the coalgebra $C = H/HA^\diamond$ and the quotient rings $Q(A)$, $Q_{\text{gr}}(A)$.*

Proof. We have $A \subset A(\chi) \subset Q_{\text{gr}}(A)$ by Lemmas 2.1(ii) and 5.4(iii). Hence $\mathcal{C}_{\text{gr}}(A)$, $\mathcal{C}(A)$ are right Ore subsets of $A(\chi)$ and $Q_{\text{gr}}(A)$, $Q(A)$ are right Ore localizations of $A(\chi)$. It follows that all regular elements of $A(\chi)$ have zero right annihilator in $Q(A)$. Since $Q(A)$ is right artinian, such elements are invertible in $Q(A)$. Therefore $Q(A)$ is the classical right quotient ring of $A(\chi)$. Each homogeneous regular element of $A(\chi)$ is invertible in $Q_{\text{gr}}(A)$ by Lemma 3.3, whence $Q_{\text{gr}}\big(A(\chi)\big) \cong Q_{\text{gr}}(A)$.

Lemma 2.1(i) shows that $\pi({}_{\chi^i}H) \subset k\chi^i$ and ${}_{\chi^i}H^+ \subset \operatorname{Ker}\pi$ for all $i \in \mathbb{Z}$. Hence $HA(\chi)^\diamond \subset HA^\diamond$, while the opposite inclusion is obvious. Thus $C = H/HA(\chi)^\diamond$. Since $\chi^i \in X(C)$ for all i, we have $A(\chi)^\diamond \subset S\big(HA(\chi)^\diamond\big)$ by Lemma 2.6. □

Lemma 6.3. *Suppose that the hypotheses of Theorem* 0.1 *are satisfied. If* ${}_{\chi^i}H \ne 0$ *for at least one* $i < 0$*, then the right coideal subalgebra* $B = {}_{1_C}H$ *has a right artinian classical right quotient ring* $Q(B) \cong Q_0(A)$*, and* $C = H/HB^+$*.*

Proof. We have $B \subset Q_0(A)$ by Lemma 5.4(iii). Proposition 3.10 applied to the algebra $A(\chi)$ shows that ${}_{\chi^i}H \cap \mathcal{C}(H) \ne \varnothing$ for all $i \in \mathbb{Z}$. An arbitrary element of $Q_0(A)$ can be written as $q = as^{-1}$ for some $a, s \in A_j$ with regular s. Choose any regular $u \in {}_{\chi^{-j}}H$. Then $au \in B$, $su \in \mathcal{C}(B)$, and $q = (au)(su)^{-1}$. It follows that each regular element of B has zero right annihilator in $Q_0(A)$; since $Q_0(A)$ is right artinian, this element is invertible in $Q_0(A)$. Hence $Q_0(A)$ is the classical right quotient ring of B.

By Lemma 3.8 $B[t]$ has a right artinian classical right quotient ring and $Q_0(B[t]) = Q(B) = Q_0(A)$. Theorem 0.2 applied to A and $B[t]$ proves the final conclusion. □

Theorem 6.4. *Denote* $B = {}_{1_C}H$*. If the hypotheses of Theorem* 0.1 *are satisfied, then the following conditions are equivalent*:

(i) H *is left faithfully* C*-coflat,*

(ii) H *is left* C*-coflat,*

(iii) $\Phi : \text{gr-}\mathcal{M}^H_{A(\chi)} \rightsquigarrow \mathcal{M}^C$ *is an equivalence,*

(iv) $A(\chi)$ *is a simple object of* $\text{gr-}\mathcal{M}^H_{A(\chi)}$*,*

(v) ${}_{\chi^i}H \ne 0$ *for at least one* $i < 0$ *and* B *is a simple object of* $\mathcal{M}^H_B$*,*

(vi) $C = H/HB^+$ *and* H *is left faithfully* B*-flat.*

Proof. Note that (i)⇒(ii) is obvious and (vi)⇒(i) by [36, Th. 1]. In the proof of the remaining implications Lemma 6.2 allows us to assume that $A = A(\chi)$.

(ii)⇒(iii) The coflatness of H implies that Ψ is exact. We will check that Φ is fully faithful. Recall that $\Phi(A)$ is the degree 0 homogeneous component of $Q_{\mathrm{gr}}(A/A^+)$, i.e. $\Phi(A) = k1_C$. Next, $\Psi(k1_C) = Q_{\mathrm{gr}}(A/A^+) \,\square_C\, H$. For each $i \in \mathbb{Z}$ there is an isomorphism $Q_i(A/A^+) \cong k\chi^i$ in $\mathcal{M}^C$, whence $Q_i(A/A^+) \,\square_C\, H \cong {}_{\chi^i}H = A_i$. Thus γ_A is an isomorphism. By Lemma 6.1 $\gamma_{U\otimes A}$ is an isomorphism for any $U \in \mathcal{M}^H$.

Given an arbitrary object $M \in \text{gr-}\mathcal{M}^H_A$, the map $U \otimes A \to M$ given by the action of A on M is a morphism in $\text{gr-}\mathcal{M}^H_A$ for any subcomodule U of M. Hence there exists an exact sequence $U' \otimes A \to U \otimes A \to M \to 0$ in $\text{gr-}\mathcal{M}^H_A$ with $U, U' \in \mathcal{M}^H$. Since Φ is right exact, so is $\Psi\Phi$. The latter yields an exact sequence

$$\Psi\Phi(U' \otimes A) \to \Psi\Phi(U \otimes A) \to \Psi\Phi(M) \to 0.$$

As we have seen both $\gamma_{U\otimes A}$ and $\gamma_{U'\otimes A}$ are isomorphisms, whence so too is γ_M, as required. Since Ψ is also fully faithful by Lemma 4.7, the functors Φ and Ψ are mutually inverse equivalences.

(iii)⇒(iv) If I is any nonzero graded H-costable right ideal of A, then $IQ(A) = Q(A)$ by Proposition 3.10(i), which means that $A/I \in \text{gr-}\mathcal{T}^H_A = \operatorname{Ker}\Phi = 0$, and therefore $I = A$.

(iv)⇒(v) By assumptions B is the degree 0 homogeneous component of A. Hence $I = IA \cap B$ for any right ideal I of B. Note that IA is a graded right ideal of A.

If I is nonzero and H-costable, then so too is IA, whence $IA = A$, and therefore $I = B$. If we had $A_i = 0$ for all $i < 0$, then $\sum_{i>0} A_i$ would be a nonzero proper H-costable ideal of A, which contradicts (iv).

(v)⇒(vi) By Lemma 6.3 $C = H/HB^+$ and B satisfies the hypotheses of Theorem 1.8. Hence H is left B-flat. That H is left faithfully B-flat follows from [19, Th. 2.1] (alternatively one can observe that the functor $\mathcal{M}_B^H \rightsquigarrow \mathcal{M}^C$ constructed in section 1 is an equivalence and apply Lemma 1.2). □

The reader may note that condition (iv) of Theorem 6.4 implies that $A(\chi)$ is a strongly graded ring. Theorem 6.4 supersedes Theorem 0.3.

7. The second equivalence

Our assumptions about H, A, C are as in section 4. Here we will construct a different pair of adjoint functors $\Phi : \text{gr-}_A\mathcal{M} \rightsquigarrow {}^C_H\mathcal{M}$ and $\Psi : {}^C_H\mathcal{M} \rightsquigarrow \text{gr-}_A\mathcal{M}$.

We will view $H[t, t^{-1}]$ as a left H-comodule algebra with respect to the comodule structure such that $t \mapsto 1 \otimes t$ and $h \mapsto \Delta(h)$ for $h \in H$. Then the category of graded left $(C, H[t, t^{-1}])$-Hopf modules is defined.

Lemma 7.1. *There is an equivalence* $\text{gr-}{}^{\;C}_{H[t,t^{-1}]}\mathcal{M} \rightsquigarrow {}^C_H\mathcal{M}$.

Proof. Each homogeneous component N_i of an object $N \in \text{gr-}{}^{\;C}_{H[t,t^{-1}]}\mathcal{M}$ is an object of ${}^C_H\mathcal{M}$ in a natural way, and the action of t produces isomorphisms $N_i \to N_{i+1}$. Hence $N \mapsto N_0$ is the desired equivalence with the inverse equivalence defined by $W \mapsto W[t, t^{-1}] = \bigoplus_{j\in\mathbb{Z}} Wt^j$. □

For $W \in {}^C_H\mathcal{M}$ put $\Psi(W) = \bigoplus {}_{\chi^j}W$. Since $A_i \subset {}_{\chi^i}H$ and ${}_{\chi^i}H \cdot {}_{\chi^j}W \subset {}_{\chi^{i+j}}W$ for all $i, j \in \mathbb{Z}$ by Lemma 2.1, we may regard $\Psi(W)$ as a graded left A-module.

Lemma 7.2. *The functor* $\Psi : {}^C_H\mathcal{M} \rightsquigarrow \text{gr-}_A\mathcal{M}$ *admits a left adjoint* Φ.

Proof. For each graded left A-module $M = \bigoplus M_j$ denote by $\Phi(M)$ the degree 0 homogeneous component of $\widetilde{M} = H[t, t^{-1}] \otimes_A M$. There is a left C-comodule structure on $\widetilde{M}$ such that

$$ht^i \otimes m \mapsto \sum \pi(h_{(1)})\chi^j \otimes \big(h_{(2)}t^i \otimes m\big) \qquad \text{for } h \in H,\ m \in M_j,\ i, j \in \mathbb{Z}$$

The formula above does define a linear map $\widetilde{M} \to C \otimes \widetilde{M}$ since for $a \in A_l$ we have $(\pi \otimes \text{id})\Delta(ha) = \sum \pi(h_{(1)})\chi^l \otimes h_{(2)}a$, $\quad (at^l)m \in M_{l+j} \quad$ and

$$\sum \pi(h_{(1)})\chi^l\chi^j \otimes \big(h_{(2)}at^{i+l} \otimes m\big) = \sum \pi(h_{(1)})\chi^{l+j} \otimes \big(h_{(2)}t^i \otimes (at^l)m\big).$$

Now $\widetilde{M}$ is an object of $\text{gr-}{}^{\;C}_{H[t,t^{-1}]}\mathcal{M}$ and $\Phi(M)$ is the corresponding object of ${}^C_H\mathcal{M}$. To show that Φ, Ψ is a pair of adjoint functors we have, in view of Lemma 7.1, to construct natural bijections

$$\text{gr-}{}^{\;C}_{H[t,t^{-1}]}\mathcal{M}\big(\widetilde{M}, W[t, t^{-1}]\big) \cong \text{gr-}_A\mathcal{M}\big(M, \Psi(W)\big).$$

The $H[t, t^{-1}]$-linear maps $\varphi : \widetilde{M} \to W[t, t^{-1}]$ are in a canonical bijective correspondence with the A-linear maps $\psi : M \to W[t, t^{-1}]$. In order that φ respect the grading, it is necessary and sufficient that so do ψ. Finally, φ is C-colinear if and only if $\sum \psi(m)_{(-1)} \otimes \psi(m)_{(0)} = \chi^j \otimes \psi(m)$ for $m \in M_j$, which amounts to $\psi(M_j) \subset {}_{\chi^j}Wt^j$. $\square$

Let U be a left H-module. For each $W \in {}^C_H\mathcal{M}$ we may view $W \otimes U$ as an object of ${}^C_H\mathcal{M}$ with respect to the tensor product of H-module structures and the C-comodule structure given on the first tensorand. For each left A-module M we let A operate on $M \otimes U$ via the comodule structure map $A \to A \otimes H$. If $M = \bigoplus M_j$ is a graded left A-module, so too is $M \otimes U$ with homogeneous components $M_j \otimes U$.

Lemma 7.3. *For $U \in {}_H\mathcal{M}$, $W \in {}^C_H\mathcal{M}$ and $M \in \text{gr-}_A\mathcal{M}$ there are canonical natural isomorphisms $\Psi(W \otimes U) \cong \Psi(W) \otimes U$ and $\Phi(M \otimes U) \cong \Phi(M) \otimes U$.*

Proof. The claim concerning Ψ is clear since ${}_\eta(W \otimes U) \cong {}_\eta W \otimes U$ for all grouplikes $\eta \in C$. The second isomorphism is obtained on homogeneous components of degree 0 from the isomorphism of graded $H[t, t^{-1}]$-modules

$$H[t, t^{-1}] \otimes_A (M \otimes U) \to \big(H[t, t^{-1}] \otimes_A M\big) \otimes U \qquad (*)$$

defined by the rule $ht^i \otimes (m \otimes u) \mapsto \sum (h_{(1)}t^i \otimes m) \otimes h_{(2)}u$. It is straightforward to check that this map is well-defined and has an inverse given by $(ht^i \otimes m) \otimes u \mapsto \sum h_{(1)}t^i \otimes \big(m \otimes S(h_{(2)})u\big)$. $\square$

Lemma 7.4. *The adjunctions $\xi_W : \Phi\Psi(W) \to W$ are isomorphisms for all injectives $W \in {}^C_H\mathcal{M}$. If Φ is exact, then Ψ is fully faithful.*

Proof. Under the equivalence of Lemma 7.1 ξ_W corresponds to the morphism

$$\zeta_W : H[t, t^{-1}] \otimes_A \Big(\bigoplus {}_{\chi^j}Wt^j\Big) \to W[t, t^{-1}]$$

in $\text{gr-}_{H[t,t^{-1}]}^{\ C}\mathcal{M}$ given by the action of $H[t, t^{-1}]$ on $W[t, t^{-1}]$. For any $U \in {}_H\mathcal{M}$ there is a commutative diagram

$$\begin{array}{ccc}
H[t, t^{-1}] \otimes_A \Big(\bigoplus ({}_{\chi^j}W \otimes U)t^j\Big) & \xrightarrow{\zeta_{W\otimes U}} & (W \otimes U)[t, t^{-1}] \\
\text{can.}\downarrow & & \\
H[t, t^{-1}] \otimes_A \Big((\bigoplus {}_{\chi^j}Wt^j) \otimes U\Big) & & \downarrow\text{can.} \\
(*)\downarrow & & \\
\Big(H[t, t^{-1}] \otimes_A (\bigoplus {}_{\chi^j}Wt^j)\Big) \otimes U & \xrightarrow{\zeta_W \otimes \text{id}} & W[t, t^{-1}] \otimes U
\end{array}$$

where $(*)$ labels the isomorphism defined in Lemma 7.3. Passing to homogeneous components of degree 0 we deduce commutativity of the diagram

$$\begin{array}{ccc}
\Phi\Psi(W \otimes U) & \xrightarrow{\xi_{W\otimes U}} & W \otimes U \\
\text{can.}\downarrow & & \| \\
\Phi\Psi(W) \otimes U & \xrightarrow{\xi_W \otimes \text{id}} & W \otimes U.
\end{array}$$

Since ${}_\eta C = k\eta$ for each grouplike $\eta \in C$, we have $\Psi(C) = \bigoplus k\chi^j$. In view of Lemma 2.1(i) $(at^i)\chi^j = \pi(a)\chi^j = \varepsilon(a)\chi^{i+j}$ for $a \in A_i$. Thus $\Psi(C) \cong Q_{\text{gr}}(A/A_+)$ as graded left A-modules. Note that

$$H[t, t^{-1}] \otimes_A A/A^+ \cong H[t, t^{-1}]/(HA^\diamond)[t, t^{-1}] \cong C[t, t^{-1}]$$

is torsionfree as a right A/A^+-module, and therefore

$$H[t,t^{-1}]\otimes_A \Psi(C) \cong \big(H[t,t^{-1}]\otimes_A A/A^+\big)\otimes_{A/A^+} Q_{\rm gr}(A/A^+) \cong C[t,t^{-1}].$$

We conclude that ζ_C is an isomorphism, whence so too is ξ_C. It follows that $\xi_{C\otimes U}$ is an isomorphism for any $U \in {}_H\mathcal{M}$.

The functor $C\otimes ?: {}_H\mathcal{M} \rightsquigarrow {}^C_H\mathcal{M}$ is right adjoint to the forgetful functor ${}^C_H\mathcal{M} \rightsquigarrow {}_H\mathcal{M}$ (cf. [2, p. 67, Example 14]). Since the latter is exact, the former preserves injectives. For each $W \in {}^C_H\mathcal{M}$ the comodule structure map $W \to C\otimes W$ is a morphism in ${}^C_H\mathcal{M}$. This map admits a linear retraction induced by the counit $C \to k$. Hence W embeds into an injective object $C\otimes E \in {}^C_H\mathcal{M}$ where E is any injective hull of W in ${}_H\mathcal{M}$. If W is itself injective, it has to be a direct summand of $C\otimes E$. In this case ξ_W has to be an isomorphism. Since the category ${}^C_H\mathcal{M}$ has enough injectives, we can continue as in Lemma 1.1. □

Lemma 7.5. *If the hypotheses of Theorem* 0.4 *are fulfilled, then* Φ *is exact and* Ψ *is fully faithful. Moreover,* $\operatorname{Ker}\Phi = \text{gr-}{}_A\mathcal{T}$*.*

Proof. The hypotheses of Theorem 0.4 imply that the Hopf algebra $H^{\rm op,cop}$ has a right artinian classical right quotient ring. Hence the antipode of $H^{\rm op,cop}$, i.e. S is bijective. It follows that $H^{\rm op}$ is a Hopf algebra with antipode S^{-1}. Now $A^{\rm op}$ is a right $H^{\rm op}$-costable graded subalgebra of $H^{\rm op}[t,t^{-1}]$, and the pair $H^{\rm op}$, $A^{\rm op}$ satisfies the hypotheses of Theorem 0.1. Lemma 4.8 applied to $H^{\rm op}$, $A^{\rm op}$ shows that the functor $H[t,t^{-1}]\otimes_A ?$ is exact on $\text{gr-}{}_A\mathcal{M}$. This entails the exactness of $\Phi: \text{gr-}{}_A\mathcal{M} \rightsquigarrow {}^C_H\mathcal{M}$ by the explicit construction in Lemma 7.2. For $M \in \text{gr-}{}_A\mathcal{M}$ we have $\Phi(M)=0$ if and only if $H[t,t^{-1}]\otimes_A M = 0$, which is equivalent to $M \in \text{gr-}{}_A\mathcal{T}$ by Lemma 4.8. □

The equivalence in Theorem 0.4 is immediate from Lemma 7.5. Replacing A, H with $A^{\rm op}, H^{\rm op}$, we obtain a version of Theorem 0.4 in which the assumptions about A and H are exactly the same as in Theorem 0.1:

Theorem 7.6. *Denote* $D = H/A^\diamond H$*. Under the hypotheses of Theorem* 0.1 *there is a category equivalence* ${}^D\mathcal{M}_H \approx \text{gr-}\mathcal{M}_A/\text{gr-}\mathcal{T}_A$ *where* ${}^D\mathcal{M}_H$ *is the category of left-right* (D,H)*-Hopf modules,* $\text{gr-}\mathcal{M}_A$ *is the category of graded right* A*-modules and* $\text{gr-}\mathcal{T}_A$ *is its localizing subcategory consisting of modules whose elements are annihilated by a nonzero* H*-costable right ideal of* A*.*

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