Testing S-duality conjecture and exceptional bundles.

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Abstract

In this paper we deal with algebro-geometrical problems connected with testing S-duality conjecture for super-symmetric Yang-Mills quantum field theories in four dimensions. We describe all field configurations such that beta function coefficient for gauge groups of rank 1 and 2 is zero or negative. We also give special series of such field configurations for gauge groups of an arbitrary rank. Realization of one of the series discovers a connection with exceptional bundles. That points again at relation between S-duality and string duality.

Introduction.

Testing S-duality conjecture for super-symmetric Yang-Mills quantum field theories in four dimensions is a geometric problem in a nature (one works as a rule with the case $SYM, N=4$ with or without twisting, or $N=2+$ supermultiplet). Recall that for possibility of calculation of quantum correlation functions in semi-classical limit it is necessary for the value of the coefficient of Gell-Mann — Low β-function in one-loop approximation on field configuration of the theory to be zero (or at least negative). In this case the gauge coupling constant (electro-magnetic e or strong g) is well-defined. Besides, if the beta coefficient is zero, then there is no anomalous $U(1)$ global symmetry, so the physics depends on a vacuum angle θ and our theory is scaling-invariant. Therefore, there appears a constant as a parameter of our theory

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

(belonging to the higher hyperplane) and the S-duality conjecture predicates that the partition function Z_τ of our theory admits some modular properties under integral linear fractional transformations of τ. The function Z_τ is a power series of $q = e^{2\pi i\tau}$, so, really the question is about behavior of the partition function under $\tau \mapsto -\frac{1}{\tau}$. Assumption about this symmetry (for $\theta = 0$) was made by Montonen and Olive [MO] more than twenty years old. To check this pure mathematical fact we mimic Vafa and Witten's method [VW], which is applied unfortunately only to partial cases of 4-varieties (algebraic surfaces del Pezzo and K3) for which theorems of vanishing harmonic spinors in canonical and adjoint representations hold. We wouldn't use the vanishing arguments proposed in [VW] for the general case because of the problems with compactification. We hope to consider it in the subsequent articles.

The aim of this paper is to put mathematical problems that are related to testing S-duality conjecture and to show that exceptional bundles naturally appear in this subject.

In the first section we describe all configurations of fields such that the coefficient of β-function is zero or negative for small (1 and 2) ranks of the fiberwise gauge group and point out some series of such field configurations for an arbitrary rank of the fiberwise gauge group (the series a), b) and c) in item 3 of **1.4.** In particular, we show that the complete program of testing interesting for physicists S-duality conjecture in "high energies" for the gauge groups of rank 1 and 2 comes to five cases (in which mathematical research are being run now, see, for example, [GL], [Ku1], [Ku2]). For an arbitrary rank of the gauge group three series are given with two of which physicists are working (for the series a) see [VW] and for b) see [APS]).

In the second section we discuss SYM QFT over del Pezzo surfaces with the field configurations a) and b) given in Sec. 1. We use the interpretation of the spaces of solutions up to gauge of the field equations as a moduli spaces of stable bundles. Under this, the coefficients of the partition functions are interpreted as the degree of the top Chern class of some standard bundles over the moduli spaces (just like the top Chern class of the cotangent bundle sends us to the topological Euler class in [VW]).

In the third section we consider the arithmetical conditions on the topological invariants of the stable bundles that appear in Sec. 2. We show that the exceptional bundles with zero cohomology groups under some inequalities on the topological invariants are initial points of sequences of sheaf moduli spaces that give partition functions suitable for testing the S-duality conjecture.

In the last section we put an algebro-geometrical problem for c) series, but without physical motivation.

It is necessary to observe that there is a parallel program of testing S-duality, in "low energies", connected with pure geometric description for dynamics of Coulomb branch of vacua ([DW], [APS]). We don't discuss this approach here. In our cases non-abelian gauge symmetry is unbroken.

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The condition on the beta function coefficient gives the main selection for field configurations in QFT that are suitable for testing S-duality in high energies. We begin just with this.

1. Beta function coefficient.

Mathematical subject-matter of this section is certainly well known for specialists in representation theory. We give it here only for convenient transition to geometry in the sequel.

1.1. Let γ be an irreducible finite dimensional complex representation of $su(r)$ (the Lie algebra for the fiberwise gauge group of our theory) and let b_γ be the Casimir element of this representation. It is a central element of the universal enveloping algebra, so, all b_γ are proportional to each other.

Let can be the canonical representation of $su(r)$ in $\mathbb{C}^r$. Denote by $c_2(\gamma)$ the proportionality coefficient between b_{can} and b_γ:

$$b_{can} = c_2(\gamma) b_\gamma .$$

Any complex representation of $su(r)$ can be extended to its complexification $sl(r,\mathbb{C})$ without changing c_2, so we consider γ as an irreducible finite dimensional representation of $sl(r,\mathbb{C})$. Let $\{e_i\}$ be a basis in $sl(r,\mathbb{C})$. Consider the dual basis $\{f_i\}$ with respect to the bilinear form

$$B_\gamma(X;Y) = Tr(\gamma(X)\,\gamma(Y)) , \tag{1}$$

i.e. such that $B_\gamma(e_i; f_j) = \delta_{ij}$. Recall that by definition, $b_\gamma = \sum_i e_i f_i$. All bilinear forms (1) are also proportional to each other as invariant forms on simple Lie algebra. Therefore, we have:

$$B_\gamma = c_2(\gamma) B_{can} \,. \tag{2}$$

For the dual representation $B_{\gamma^*}(X;Y) = B_\gamma(X;Y)$, so the following equality holds:

$$c_2(\gamma^*) = c_2(\gamma) \,. \tag{3}$$

Remark. One can define c_2 for an irreducible $u(r)$ -representation $\tilde{\gamma}$ as $c_2(\gamma)$, where γ is the restriction of $\tilde{\gamma}$ on the subalgebra $su(r) \subset u(r)$. Clearly, all the results of this section hold also for $u(r)$ -representations.

Remark. Though notation $c_2(\gamma)$ is not quite conventional in the theory of Lie algebras, it justifies itself in subsequent transition to geometry. Namely, if the representation γ is considered as a tensor operation on vector bundles of rank r , then for such a bundle E $c_2(\gamma)$ is the coefficient at $c_2(E)$ in the expression of $c_2(\gamma(E))$ via the Chern classes of E .

Consider a finite collection of representations $\{\gamma_1, \dots, \gamma_{N_a}\}$, among which there are some isomorphic to each other. From physical viewpoint this collection describes configuration of fields for QFT under consideration (see Sec. 2). The coefficient of beta function takes the collection $\{\gamma_1, \dots, \gamma_{N_a}\}$ to the value

$$\beta(\{\gamma_i\}) = -c_2(ad) + \sum_i c_2(\gamma_i) \,.$$

The aim of this section is to describe all cases when $\beta(\{\gamma_i\}) \leqslant 0$ for algebras $su(2)$ and $su(3)$ and to point out some series of such cases for $su(r)$.

Remark. Since we deal with Lie algebras, freedom of choice of the gauge group is still remained for us. For instance, $su(2) \cong so(3)$, that corresponds to choosing the determinant of the bundle modulo its rank.

There is a general formula which expresses $c_2(\gamma)$ in terms of the higher weight of the representation γ .

1.2. Proposition. *Let R be the root system for $sl(r, \mathbb{C})$, λ be the highest weight of γ with respect to a base of R , ρ be the half-sum of the positive roots, and $(\ ;\ )$ be the Killing form. Then*

$$c_2(\gamma) = \frac{(\lambda; \lambda + 2\rho)}{r^2 - 1} \dim \Gamma = \frac{(\lambda; \lambda + 2\rho)}{r^2 - 1} \prod_{\alpha \in R_+} \frac{(\lambda + \rho; \alpha)}{(\rho; \alpha)} \,. \tag{4}$$

Here Γ is the representation space for γ .

THE PROOF is based on two formulas of representation theory of semisimple Lie algebras. The former is the Freudenthal formula applied to the highest weight (see, for example, [FH, (25.14)]) which states that the Casimir element b_{can} acts on the space Γ of the representation γ as a homothety with the coefficient

$$(\lambda; \lambda + 2\rho) = \|\lambda + \rho\|^2 - \|\rho\|^2 \,. \tag{5}$$

From this formula it follows that

$$Tr(\gamma(b_{can})) = c_2(\gamma)\, Tr(\gamma(b_\gamma)) = (\lambda; \lambda + 2\rho) \dim \Gamma \,.$$

From the other hand, the definition of b_γ implies that

$$Tr(\gamma(b_\gamma)) = \sum_i Tr(\gamma(e_i)\gamma(f_i)) = \sum_i 1 = \dim sl(r,\mathbb{C}) = r^2 - 1 \ .$$

Therefore, we obtain:

$$c_2(\gamma) = \frac{(\lambda;\lambda+2\rho)}{r^2-1}\dim\Gamma = \frac{(\lambda;\lambda+2\rho)}{r^2-1}\prod_{\alpha\in R_+}\frac{(\lambda+\rho;\alpha)}{(\rho;\alpha)} \ .$$

The last equality holds because of the formula for the dimension of the representation space with given highest weight [FH, 24.6]. This is the latter formula of the representation theory on which the calculation of $c_2(\gamma)$ is based. The Proposition is proved.

In general case using (4) is hampered because the product by positive roots that equals $\dim\Gamma$ is cumbrous. However, for the exterior and symmetric powers of the canonical representation and for the adjoint representation c_2 's are computed easily.

1.3. Proposition.

$$c_2(\Lambda^k(can)) = \binom{r-2}{k-1}\ , \quad c_2(S^k(can)) = \binom{r+k}{k-1} \quad \textit{and} \quad c_2(ad) = 2r \ . \tag{6}$$

PROOF. Consider the Cartan subalgebra $\mathfrak{h} \subset sl(r,\mathbb{C})$ consisting of diagonal matrices with zero trace. We shall define a matrix $H \in \mathfrak{h}$ by its diagonal elements $(\lambda_1,\dots,\lambda_r)$.

The linear functions $\lambda_i - \lambda_j$, $i \neq j$, form the root system $R \subset \mathfrak{h}^*$. The set $S = \{\alpha_1,\dots,\alpha_{r-1}\}$, where $\alpha_i = \lambda_i - \lambda_{i+1}$, is a basis in this system. Matrices H_i that have only two nonzero elements $\lambda_i = 1$ and $\lambda_{i+1} = -1$, $i = 1,\dots,r-1$ form the dual basis in the dual system $R^* \subset \mathfrak{h}$. Observe that α_i is identified with H_i by isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$, which is defined by the Killing form $(X;Y) = Tr(X\,Y) = B_{can}|_{\mathfrak{h}}(X;Y)$. Herewith, the set R_+ of positive roots w.r.t. S, which consists of the elements $\alpha = \lambda_i - \lambda_j$, $i<j$, is identified with the set of matrices $H_\alpha = H_i + \ldots + H_j$, which also have two nonzero elements: $\lambda_i = 1$ and $\lambda_j = -1$.

As it is known [FH], there is one-to-one correspondence between irreducible $sl(r,\mathbb{C})$-modules and their highest weights. Herewith, if λ is the highest weight of an irreducible representation γ, then γ is finite dimensional iff all values $\lambda(H_i)$, $i = 1,\dots,r-1$, are nonnegative integers. This means that $\lambda = k_1\omega_1 + \cdots + k_{r-1}\omega_{r-1}$, where $k_i \in \mathbb{Z}$, $k_i \geqslant 0$ and ω_i are fundamental weights defined by the equalities $\omega_i(H_j) = \delta_{ij}$.

The fundamental weights have the form $\omega_i = \lambda_1 + \cdots + \lambda_i$, $1 \leqslant i \leqslant r-1$. We have: $(\omega_i;\omega_k) = \frac{i(r-k)}{r}$ for $i \leqslant k$. It is easily seen that the half-sum of the positive roots is $\rho = \sum_{i=1}^{r-1}\omega_i$. The highest weights of the representations $\Lambda^k(can)$, $S^k(can)$ and ad are equal to ω_k, $k\omega_1$ and $\omega_1 + \omega_{r-1}$ respectively. Calculating the homothety coefficients by formula (5), we get:

$$(\omega_k\,;\,\omega_k + 2\rho) = \frac{k(r-k)(r+1)}{r}\ , \qquad (k\omega_1\,;\,k\omega_1+2\rho) = \frac{k(r+k)(r-1)}{r}$$

$$(\omega_1+\omega_{r-1}\,;\,\omega_1+\omega_{r-1}+2\rho) = 2\,r\ .$$

By (4), multiplying these numbers respectively on $\dim\Lambda^k(can) = \binom{r}{k}$, $\dim S^k(can) = \binom{r+k-1}{k}$, $\dim(ad) = r^2-1$ and then dividing by $r^2 - 1 = \dim sl(r,\mathbb{C})$, we obtain the required formulae. This completes the proof.

Now we are ready to describe field configurations (i. e. collections of representations) such that the beta function coefficient is zero or negative. As an answer we obtain the following statement.

1.4. Theorem.
1. For $su(2)$ one has $\beta(\{\gamma_i\}) = 0$ in the following cases:
a) $N_a = 1$ and $\gamma_1 = ad$.
b) all $\gamma_i = can$ and $N_a = 4$;
The inequality $\beta(\{\gamma_i\}) < 0$ can be satisfied only in the case b) while $N_a < 4$.
2. For $su(3)$ one has $\beta(\{\gamma_i\}) = 0$ in the following cases:
a) $N_a = 1$ and $\gamma_1 = ad$.
b) all γ_i are isomorphic either to can or to $can^* \cong \Lambda^2(can)$, and $N_a = 6$;
c) $N_a = 2$, one of γ_i is isomorphic either to $can \cong \Lambda^2(can^*)$ or $can^* \cong \Lambda^2(can)$, and the other is either $\mathrm{S}^2(can)$ or $\mathrm{S}^2(can^*)$.
The inequality $\beta(\{\gamma_i\}) < 0$ can be satisfied either in the case b) while $N_a < 6$ or in a case of one representation isomorphic to $\mathrm{S}^2(can)$ or $\mathrm{S}^2(can^*)$.
3. For $su(r)$, $r \geqslant 2$, there is at least three series of cases in which $\beta(\{\gamma_i\}) = 0$:
a) $N_a = 1$ and $\gamma_1 = ad$;
b) all γ_i are isomorphic either to can or to $can^* \cong \Lambda^{r-1}(can)$, and $N_a = 2r$.
c) $N_a = 2$, one of γ_i is isomorphic either to $\Lambda^2(can)$, or to $\Lambda^2(can^*)$ and the other is either $\mathrm{S}^2(can)$ or $\mathrm{S}^2(can^*)$.

For $su(2)$ and $su(3)$ algebro-geometrical problem in some of these cases is beforehand divined and calculations are being performed using Vafa — Witten's method (geometric vanishing situation). For instance, the case 1a) is investigated in [VW], 1b) in [GL], algebro-geometrical problem for 2b) is considered in [Ku]. The case 2a) is geometrically analogous to 1a) (see below). At last, the most interesting case 2c) is a unique realization of the series 3c) on del Pezzo surfaces. The last section of this work is devoted to that.

Proof of Theorem 1.4.

The item 3 follows easily from Proposition 2. Suppose $r = 2$. It is known that any non-trivial irreducible representation of $sl(2, \mathbb{C})$ is isomorphic to $S^k(can)$ for $k \geqslant 1$. According to Proposition 2, we have: $c_2(S^k(can)) = \binom{k+2}{k-1} = \binom{k+2}{3}$. This sequence of numbers is increasing. Besides, $ad \cong S^2(can)$. This proves item 1.

To prove item 2, it suffices to estimate c_2 of $sl(3, \mathbb{C})$-representations. Suppose γ is an irreducible finite dimensional $sl(3, \mathbb{C})$-representation, P is the set of its weights, and Γ is the representation space. For any $\pi \in P$ denote by Γ^π the corresponding weight subspace. The number $m_\pi = \dim \Gamma^\pi$ is called the multiplicity of π. It is known that P is invariant under the action of the Weyl group W, which is generated by root reflections, and for any $\pi \in P$ and $w \in W$ the multiplicities of weights π and $w(\pi)$ coincide. Besides,

$$\Gamma = \bigoplus_{\pi \in P} \Gamma^\pi .$$

By definition, $c_2(can) = 1$ and it is evident that for the canonical representation $Tr(H_1^2) = 2$. Let λ be the highest weight of γ, then by (2) we have:

$$c_2(\gamma) = \frac{1}{2} Tr(\gamma(H_1)^2) = \frac{1}{2} \sum_{\pi \in P} \pi(H_1)^2 m_\pi \geqslant \frac{1}{2} \sum_{\pi \in \lambda^W} \pi(H_1)^2 .$$

Here λ^W is the orbit of the highest weight under the action of the Weyl group and m_λ always equals 1. If $\lambda = k_1\omega_1 + k_2\omega_2$, then the last half-sum can be easily calculated. It is equal to $2(k_1^2 + k_2^2)$ for $k_1, k_2 > 0$ and k^2 if one of k_1, k_2 equals 0 and the other is k. Suppose γ is contained in a collection of representations such that the beta function coefficient is non-positive; then $c_2(ad) = 6 \geqslant c_2(\gamma)$, whence in the former case $k_1^2 + k_2^2 \leqslant 3$ i.e. $k_1 = k_2 = 1$ and $\gamma = ad$ and in the latter case there are four possibilities:

(k_1, k_2)	$(1,0)$	$(2,0)$	$(0,1)$	$(0,2)$
γ	can	$\mathrm{S}^2(can)$	can^*	$\mathrm{S}^2(can^*)$
$c_2(\gamma)$	1	5	1	5

This concludes the proof.

2. From field equations to algebro-geometrical problems.

Theorem 1 shows that we have three series of field configurations: a), b) and c), which take place for all $su(r)$. We consider only the first two of them in this section.

2.1. For series a) our fields consist of multiplets in the adjoint representation. We write field equations for a special type of metrics, namely Kähler metrics, because our calculations will be related only to the case of algebraic surfaces, where necessary for us constants will have algebro-geometrical meaning and so can be computed within the framework of algebraic geometry. For simplicity we shall consider simply connected case. So, let S be simply connected Kähler surface and K be its canonical class. Configuration space is the direct product

$$\mathcal{A}(E) \times \Gamma(adE \otimes K)$$

where $\mathcal{A}(E)$ is affine space of $SU(r)$-connections in a complex vector bundle E on S of rank r with $c_1 = 0$ and $c_2 = k$. $\Gamma(adE \otimes K)$ is the space of Higgs — Hitchin fields. Then for a point $(a, \phi) \in \mathcal{A}(E) \times \Gamma(adE \otimes K)$ the field equations have the form

$$\begin{aligned} F_a^{2,0} = F_a^{0,2} &= 0 \\ \overline{\partial}_a(\phi) &= 0 \\ \omega \wedge F_a + [\phi, \overline{\phi}] &= 0 \end{aligned} \tag{7}$$

where $F_a^{i,j}$ are the Hodge decomposition components for the curvature tensor F_a of the connection a and ω is the Kähler form of our metric. Vafa and Witten in the work [VW] wrote generalizations of the equations above for a generic metric. Using linearization of these equations it is easy to show that the space of solutions is oriented and finite. Hence, for any $k = c_2(E)$ we obtain a number a_k of these solutions with provision for the orientation sign. Our partition function in this case has the form

$$Z_{(\mathrm{a})} = \sum_{k=0}^{\infty} a_k q^k$$

where as usual $q = e^{2\pi i k\tau}$ (see details in [VW]).

Geometric meaning of equations (7) is as follows. The first equation is equivalent to statement that the curvature form has type (1,1), this is equivalent to defining a holomorphic structure on E. The second equation is equivalent to holomorphicity of the map $\phi : E \longrightarrow E \otimes K$ and the third equation has a sense of zero level of the moment map for the gauge group action, i.e. it means some stability conditions for the holomorphic pair $(E\,,\,\phi)$.

Now suppose that for our Kähler structure on S this stability condition implies vanishing of the holomorphic Higgs — Hitchin field ϕ. It will be so, for example, if the canonical class $K \leqslant 0$, i.e. if S is a del Pezzo surface or K3 surface. In this case our Kähler metric is non-general because the system (7) comes to anti-self-duality (ASD) equations on the connection a. We use here the following Donaldson's result [DK]: the set of gauge orbits of irreducible ASD connections on the bundle E is in one-to-one correspondence with the set of stable holomorphic structures on E, where the stability is by Mumford — Takemoto w.r.t. $[\omega]$. Therefore, the moduli space of solutions up to gauge of the field equations (7) is the whole space $\mathcal{M}_k(r)$ of $SU(r)$-instantons on S with the charge k. For example, for $r=2$ this variety has dimension $4k-3$ for del Pezzo surfaces and $4k-6$ for K3 surfaces. To regularize the problem we are to apply deformation to normal cone as described in chapter III of [PT]. In this case the obstruction bundle on $\mathcal{M}_k(r)$ has as a fiber the cokernel of the map $\overline{\partial}_a$, i.e. over a point $E \in \mathcal{M}_k(r)$ the fiber of the obstruction bundle is $H^1(adE \otimes K)$ by Dolbeault isomorphism. Thus, the obstruction bundle for our problem is $\Omega\mathcal{M}_k(r)$ by Kodaira — Spenser isomorphism. According to construction expounded in chapter III of [PT] the number of solutions of our problem is

$$a_k = c_{top}(\Omega\mathcal{M}_k(r)) \tag{8}$$

i.e. the top Chern class of the cotangent bundle.

Remark. Variety $\mathcal{M}_k(r)$ can be non compact (for instance, this takes place for $r=2$) and we should consider it as a variety "with ends". We are to estimate which number of solutions approaches infinity on these "ends" from general number of solutions defining by topological Euler characteristics of Fried — Uhlenbeck compactification of $\mathcal{M}_k(r)$ considering as orbifold (see [DK]). (Unfortunately, this analysis is absent in basic work [VW], and the reader is to make it himself using as a model analogous problem of expressing spin-polynomials in terms of Donaldson polynomials. A first step of solving this problem is described in [W-L]).

Thus, (after realization of the program expounded in the remark) the partition function for a) series takes the form

$$Z_{(\mathrm{a})} = \sum_{k=0}^{\infty} \chi(\overline{\mathcal{M}_k(r)})q^k\,, \tag{9}$$

where χ is the topological Euler characteristics of $\overline{\mathcal{M}_k(r)}$ as an orbifold.

Testing this sum for modularity performed in [VW] (for K3 surfaces and $\mathbb{CP}^2$) is based on a remarkable work [Kl] of Klyachko who has expressed coefficients of the series (9) in terms of arithmetical Hurwitz function.

2.2. We now describe a physical setup for the b) series.

Let M, g be smooth compact simply-connected Riemannian manifold. Consider the Levi — Civitá connection on the tangent bundle. A choice of $Spin^{\mathbb{C}}$-structure gives us a pair of complex Hermitian bundles $W^{\pm}$ of rank 2 with a class $c = c_1(\det W^{\pm})$.

The Levi — Civitá connection induces an $SO(3)$-connections on $W^{\pm}$. Consider a Hermitian bundle E of rank r with unitary connection a defined up to homothety. Besides, consider a $U(1)$-connection b on line bundle $\det(E \otimes W^{+})$.

Configuration space for our field equations is the set of collections

$$(a, b, \phi_1, \dots, \phi_{2r}) ,$$

where ϕ_j are sections of the tensor product $E \otimes W^+$.

Recall that Dirac operator $D_{a,b}$ is a differential operator defined by the following way. Covariant derivative determines a map

$$d_{a,b} : \Gamma(E \otimes W^+) \longrightarrow \Gamma(E \otimes W^+ \otimes T^*M), \tag{10}$$

and the last tensor product is contracted as

$$\Gamma(E \otimes W^+ \otimes T^*M) = \Gamma(E \otimes W^+ \otimes \mathrm{Hom}(W^-, W^+)^*) = \Gamma(E \otimes W^-)$$

by means of Clifford multiplication. Superposition of covariant derivation and contraction gives us the twisted Dirac operator

$$D_{a,b} : \Gamma(E \otimes W^+) \longrightarrow \Gamma(E \otimes W^-) . \tag{11}$$

Our field equations take the form:

$$\begin{aligned} &D_{a,b}(\phi_j) = 0 \\ &F_a^+ = i \sum_j (\phi_j \otimes \overline{\phi_j})_0 , \end{aligned} \tag{12}$$

where the index $_0$ means taking traceless part in $\mathrm{End}\,(W^+)$. Here we use the natural identification $ad\, W^+ \cong \Omega^+$ (see [T]).

Remark. One can see that these equations have the form of zero level for moment map of hyper-Kähler reduction w.r.t. the gauge group. In particular, the first equation contains an interpretation of zero level of moment map for holomorphic symplectic structure.

Now suppose that we have chosen a bundle E such that moduli space $\mathcal{M}_c(E)$ of orbits of solutions of (12) is finite. Let $\#(\mathcal{M}_c^g(E))$ be the number of solutions weighted with the orientation sign. Since $\#(\mathcal{M}_c^g(E))$ depends only on topological type of the bundle E, we can write

$$\#(\mathcal{M}_c^g(E)) = \#(\mathcal{M}_c^g(r, c_1, k)) ,$$

where as usual $k = c_2(E)$.

Our partition function has the form

$$Z_{(\mathrm{b})} = \sum_{k=0}^{\infty} \#(\mathcal{M}_c^g(r, c_1, k)) q^k ,$$

where again $q = e^{2\pi i \tau}$ and the metric g is sufficiently general (for non-general metrics the moduli space can have parameters).

Remark. It is easy to see that our equations define a multiplet of non-abelian monopoles which are generalizations of abelian Seiberg — Witten monopole (see, for example, [T]).

Recall that we started with a Kähler metrics in describing a) series i.e. we reduced our problem to pure algebro-geometrical problem. To do this for b) series we have to act more accurately. Here we illustrate the general situation by the case $r = 2$ and $M = \mathbb{CP}^2$.

Consider the standard Fubini — Study metric g_{FS}. This metric has positive scalar curvature. The Weitzenbök formula shows that in this case the multiplet of twisted harmonic spinors $(\phi_1, \ldots, \phi_4)$ vanishes, and the field equations come to ASD equations. The moduli space of solutions $\mathcal{M}^{g_{FS}}_c(2, c_1, k)$ coincides with the moduli space of instantons (holomorphic stable bundles) of this topological type. Arithmetic calculations of the next section show that the unique possibilities for c_1 are -1 or -5 and

$$\dim \mathcal{M}^{g_{FS}}_c(2, -1, k) = 4(k-1)\,, \quad \dim \mathcal{M}^{g_{FS}}_c(2, -5, k) = 4(k-7)\,.$$

To regularize the problem we again have to apply the deformation to the normal cone as it is described in [PT, chapter III]. In this case the fiber of the obstruction bundle over a point in $\mathcal{M}^{g_{FS}}_c(2, c_1, k)$ is the direct sum of four cokernels of twisted Dirac operator (11). As usual, we interpret the an ASD connection as a holomorphic stable structure on E and Dirac operator as $\overline{\partial}_a \oplus \overline{\partial}_a{}^*$, where $Spin^{\mathbb{C}}$-structure equals $-K_S$ (see, for example, [DK]). So, the cokernel of Dirac operator is the coherent cohomology space $H^1(E)$.

Thus, our moduli spaces $\mathcal{M}^{g_{FS}}_c(2, c_1, k)$ are the moduli spaces of stable bundles $\mathcal{M}(2, c_1, k)$ with this topological type, where $c_1 \in \{-1, -5\}$. Hence, the fiber of the obstruction bundle over a point $E \in \mathcal{M}^{g_{FS}}_c(2, c_1, k)$ is $H^1(E)^{\oplus 4}$. According to the construction given in chapter III of [PT] the number of solutions of our problem is

$$\#(\mathcal{M}^{g_{FS}}_c(2, c_1, k)) = c_{top}((\mathcal{H}_k)^{\oplus 4})\,, \tag{13}$$

where $\mathcal{H}_k$ is a bundle over the moduli space with the fiber $H^1(E)$ and specially chosen normalization (see [GL]).

In fact the above reasoning hold when S is a del Pezzo surface, $Spin^{\mathbb{C}}$-structure is $-K_S$ and g is the Hodge metric. In this situation the space $\mathcal{M}^g_{-K_S}(r, c_1, k)$ of solutions up to gauge of the field equations is also interpreted as the moduli space $\mathcal{M}(r, c_1, k)$ of stable bundles with the given topological type. The fiber of the obstruction bundle over a point E is the direct sum of $2r$ cokernels of the Dirac operator i.e. $H^1(E)^{\oplus 2r}$. Hence, to have a finite number of solutions it is necessary to have the equality

$$\dim \mathcal{M}(r, c_1, k) = 2r \dim H^1(E)\,.$$

If this condition holds, then the partition function is

$$Z_{(\mathrm{b})} = \sum_{k=0}^{\infty} \#(\mathcal{M}^g_{-K_S}(r, c_1, k))q^k = \sum_{k=0}^{\infty} c_{top}((\mathcal{H}_k)^{\oplus 2r})q^k \tag{14}$$

where $\mathcal{H}_k$ is a bundle over $\mathcal{M}(r, c_1, k)$ with the fiber $H^1(E)$.

Observe that the moduli space $\mathcal{M}(r, c_1, k)$ is non-compact as a rule, so c_{top}'s in the last formula depend on cohomology theory which we are working with. A natural way to give a meaning to this quantities is to consider the Gieseker compactification $\overline{\mathcal{M}}(r, c_1, k)$ which is the coarse moduli space of semistable sheaves. We discuss this in the next section.

3. Exceptional bundles in realization of b) series.

3.1. Let S be a del Pezzo surface with the canonical class K, $\mathcal{M}(r, c_1, c_2)$ be the moduli space of stable bundles over S with the given topological invariants, $\overline{\mathcal{M}}(r, c_1, c_2)$ be the Gieseker compactification of $\mathcal{M}(r, c_1, c_2)$. The stability notion is by Mumford —

Takemoto w.r.t. $(-K)$. This means that a sheaf E is stable if it is torsion free and for any subsheaf F of E with $r(F) < r(E)$

$$\mu(F) < \mu(E), \quad \text{where} \quad \mu = \frac{c_1 \cdot (-K)}{r} \quad \text{is the slope.}$$

In this section we regard $\mathcal{M}(r, c_1, c_2)$ as the space of solutions of (12) up to gauge. We consider the partition function (14) under the following assumptions:

dimension condition $$\dim \mathcal{M}(r, c_1, k) = 2r \dim H^1(E) \tag{15}$$

slope inequality $$-K^2 < \mu(E) < 0 \tag{16}$$

The dimension condition is necessary for the coefficients of the partition function (14) to be really a numbers. From the slope inequality, the stability properties and Serre duality it follows that $H^0(E) = H^2(E) = 0$. Therefore, we have:

$$\dim H^1(E) = -\chi(E) = k - \frac{1}{2} c_1 \cdot (c_1 - K) - r . \tag{17}$$

Further, the space $\operatorname{Ext}^1(E, E)$ coincides with formal tangent space to $\mathcal{M}(r, c_1, k)$ at the point E. Using stability of E, we have $\operatorname{Hom}(E, E) \cong \mathbb{C}$. In addition, $(-K)$ is ample, so by Serre duality $\dim \operatorname{Ext}^2(E, E) < \dim \operatorname{Hom}(E, E)$ (see [KO]). This implies that $\operatorname{Ext}^2(E, E) = 0$, and we obtain

$$\dim \mathcal{M}(r, c_1, k) = 1 - \chi(E, E) = 1 - \chi(E^* \otimes E) = 2rk - (r - 1)c_1^2 - r^2 + 1 \tag{18}$$

Remark. If the numbers r and $c_1 \cdot K$ are coprime, then any semistable sheaf in $\overline{\mathcal{M}}(r, c_1, c_2)$ is stable. This implies that the moduli space $\overline{\mathcal{M}}(r, c_1, c_2)$ is smooth, since $\dim \operatorname{Ext}^1(E, E)$ is independent of E.

By the equalities (17) and (18), the dimension condition (15) takes the form:

$$c_1^2 - r\, c_1 \cdot K + r^2 + 1 = 0 \tag{19}$$

The most interesting solutions of this equation are on the complex projective plane.

3.2. Theorem. *A stable sheaf E on $\mathbb{P}^2$ with invariants satisfying (16) obeys the equality (15) iff the triple of numbers $(r, -c_1, 1)$ satisfies the Markov equation*

$$x^2 + y^2 + z^2 = 3\, xyz \, . \tag{20}$$

The proof follows immediately from the fact that $c_1 \cdot K = -3\, c_1$ for $\mathbb{P}^2$.

The solution set of (20) is well-known [GR]. Namely, any solution can be obtained from $(1, 1, 1)$ by a finite number of mutations. A mutation is changing one variable to another root of the quadratic equation while the rest variables are fixed. So, for $z = 1$ we have a sequence of mutations that gives the set of all solutions (r, c_1) to (19). Here are several first solutions such that $r > -c_1$:

$$(2, -1)\, ,\ (5, -2)\, ,\ (13, -5)\, ,\ (34, -13)\, ,\ (89, -34)\, ,\ (233, -89)\, ,\ \ldots \tag{21}$$

The pair after (r, c_1) in this sequence is $(3r + c_1, -r)$.

It is known [GR] that for any exceptional collection of three bundles on $\mathbb{P}^2$ the ranks form a solution of Markov equation. Moreover, there are natural mutations of exceptional collections, which induce the above-mentioned mutations of rank collections. Let us enumerate the pairs in (21); then there is a unique sequence E_n of exceptional bundles such that $(r(E_n), c_1(E_n))$ is exactly the n-th term of (21). For example, $E_1 \cong \Omega_{\mathbb{P}^2}(1) \cong \mathrm{T}\mathbb{P}^2(-2)$. Moreover, for any n collection $(E_n, E_{n+1}, \mathcal{O}_{\mathbb{P}^2})$ is exceptional and the mutations preserving $\mathcal{O}_{\mathbb{P}^2}$ are given by the exact triples

$$0 \longrightarrow E_{n-1} \longrightarrow \mathrm{Hom}(E_n, E_{n+1}) \otimes E_n \longrightarrow E_{n+1} \longrightarrow 0\,, \tag{22}$$

where $\mathrm{Hom}(E_n, E_{n+1})$ is canonically identified with $\mathrm{Hom}(E_{n-1}, E_n)^*$. In fact, E_n's are defined also for $n \leqslant 0$ by the triple above. For example, $E_0 = \mathcal{O}_{\mathbb{P}^2}(-1)$, $E_{-1} = \mathcal{O}_{\mathbb{P}^2}(-2)$, and for $n = 0$ the exact triple (22) is the Euler exact sequence twisted by (-2). The bundles $E_n = \overline{\mathcal{M}}(r_n, -r_{n-1}, c_2(E_n))$ are the initial points for series of moduli spaces $\overline{\mathcal{M}}(r_n, -r_{n-1}, k)$, where $k \geqslant c_2(E_n)$. Any of these spaces is smooth because r_n and $3\,r_{n-1}$ are coprime. Thus, we have a diagram:

$$\begin{array}{ccccl}
\vdots & \vdots & \vdots & & \\
\mathcal{O}_{\mathbb{P}^2}(-2) & Hilb^1(\mathbb{P}^2) & Hilb^2(\mathbb{P}^2) & \ldots & Z_{-1} \text{ is a polynomial of degree } \leqslant 4 \\
 & || & & & \\
\mathcal{O}_{\mathbb{P}^2}(-1) & \mathbb{P}^2 & Hilb^2(\mathbb{P}^2) & \ldots & Z_0 = 1 + q \\
\Omega_{\mathbb{P}^2}(1) & \overline{\mathcal{M}}(2,-1,2) & \overline{\mathcal{M}}(2,-1,3) & \ldots & Z_1 = q + 13q^4 + 729q^5 + 85025q^6 + \\
 & & & & \qquad +15650066q^7 + \cdots \\
E_2 & \overline{\mathcal{M}}(5,-2,5) & \overline{\mathcal{M}}(5,-2,6) & \ldots & \\
E_3 & \overline{\mathcal{M}}(13,-5,19) & \overline{\mathcal{M}}(13,-5,20) & \ldots & \\
\vdots & \vdots & \vdots & \ddots &
\end{array}$$

Each row of this diagram produces a partition function Z_n. All what is known about these functions for the moment is written in the right hand side. On the other hand, the terms of the first column are connected by (22).

Question. Exist there a transformation between the whole rows of the diagram that generalizes the mutations (22)?

Conjecture. If such a transformation exists, then one can calculate all the partition functions knowing only two of them.

We would like to observe that the calculation of the first coefficients of Z_1 is contained in the paper [GL]. It is important that the moduli spaces $\overline{\mathcal{M}}(2, -1, k)$ are really fine i.e. there exist universal families which are needed to construct the bundles $\mathcal{H}_k$. The first part of the paper [GL] is description of $\overline{\mathcal{M}}(2, -1, k)$ as fine moduli spaces. A similar description of all $\overline{\mathcal{M}}(r_n, -r_{n-1}, k)$ in the diagram above and some moduli spaces of sheaves over $\mathbb{P}^1 \times \mathbb{P}^1$ is given in [Ku2]. The second part of [GL] explains the calculations of the coefficients via Bott's formula performed by MAPLE V on computer.

3.3. In fact, the situation described above for $\mathbb{P}^2$ admits a generalization for del Pezzo surfaces. The bundles E_n described in the previous subsection are exactly all exceptional bundles that belong to the subcategory

$$\mathcal{O}^\perp = \{F : \operatorname{Ext}^i(\mathcal{O}, F) = 0\}$$

of the category of coherent sheaves. We see that for $\mathbb{P}^2$ each exceptional bundle from $\mathcal{O}^\perp$ (i.e. with zero cohomology groups) produces a partition function. It is not hard to check that the slope inequality (16) holds for these bundles automatically. That is not so for other del Pezzo surfaces. For example, the bundles $\mathcal{O}(-1, m)$ over $\mathbb{P}^1 \times \mathbb{P}^1$ belong to $\mathcal{O}^\perp$ but have the slope $2(m-1)$.

Proposition. *If an exceptional bundle E over del Pezzo surface satisfies the slope inequality (16), then*

$$E \in \mathcal{O}^\perp \iff \textit{the moduli spaces } \overline{\mathcal{M}}(r(E), c_1(E), k) \textit{ obey the dimension condition.}$$

Proof. ($\Rightarrow$) Obviously, the dimension condition is independent of k and holds for $E = \overline{\mathcal{M}}(r(E), c_1(E), c_2(E))$.
($\Leftarrow$) It is known [KO] that exceptional bundles are stable by Mumford — Takemoto w.r.t. $(-K)$. Therefore, as in **3.1** (16) implies that $H^0(E) = H^2(E) = 0$. By the dimension condition applied to $E = \overline{\mathcal{M}}(r(E), c_1(E), c_2(E))$, we have: $H^1(E) = 0$. This concludes the proof.

Corollary. *Each exceptional bundle from $\mathcal{O}^\perp$ that satisfies the slope inequality gives a partition function for mathematical testing of S-duality conjecture.*

In particular, any system of bundles generated by an exceptional pair produces a diagram of moduli spaces which is similar to the diagram for $\mathbb{P}^2$. The question and the conjecture are the same.

4. An algebro-geometrical model for c) series.

Physical setup for the c) series in unknown for the moment. In this section we consider pure mathematical problem which is similar to one in the previous section.

Suppose that E is stable bundle with $r \geqslant 3$ over a del Pezzo surface and

$$-\frac{1}{2}K^2 < \mu(E) < 0\,. \tag{23}$$

The spaces $H^0(S^2E)$ and $H^0(\Lambda^2 E)$ are canonically identified with the spaces of symmetric and skew-symmetric maps of bundles $E^* \longrightarrow E$. From stability properties and the inequality $\mu(E) < 0 < \mu(E^*)$ it follows that $\operatorname{Hom}(E^*, E) = 0$. Therefore, we have: $H^0(S^2E) = H^0(\Lambda^2E) = 0$.

By Serre duality, $H^2(S^2E)^* = H^0(S^2E^* \otimes K)$. The last space identifies with the space of maps $\varphi : E \longrightarrow E^* \otimes K$ such that $\varphi^*(K) = \varphi$. Since $0 < \mu(E^*) < \frac{1}{2}K^2$, we have $\mu(E^* \otimes K) < -\frac{1}{2}K^2 < \mu(E)$. This implies that $\operatorname{Hom}(E, E^* \otimes K) = 0$ and so, $H^2(S^2E) = 0$. By similar reasoning, we have $H^2(\Lambda^2 E) = 0$. Thus, the dimensions of H^1 spaces for symmetric and exterior square of the bundle E are expressed via (r, c_1, k) by the Riemann — Roch formula:

$$\dim H^1(S^2E) = -\chi(S^2E) = (r+2)k - \frac{r+3}{2}c_1^2 + \frac{r+1}{2}c_1 \cdot K - \frac{r(r+1)}{2}\,,$$

$$\dim H^1(\Lambda^2 E) = -\chi(\Lambda^2 E) = (r-2)k - \frac{r-1}{2}c_1^2 + \frac{r-1}{2}c_1 \cdot K - \frac{r(r-1)}{2} \, .$$

Taking into account (18) we conclude that the dimension condition

$$\dim \mathcal{M}(r, c_1, k) = \dim H^1(S^2 E) + \dim H^1(\Lambda^2 E) \tag{24}$$

is equivalent to the equation on r and c_1 :

$$2\,c_1^2 - r\,c_1 \cdot K + 1 = 0 \, . \tag{25}$$

It is easy to see that for $\mathbb{P}^2$ this equation has no solutions. Indeed, in this case we have $c_1 \cdot K = -3c_1$ and discriminant of (25) w.r.t. c_1 equals $9r^2 - 8$. This can not be a square of an integer for $r \geqslant 3$ because the difference between neighbouring squares is $9r^2 - (3r-1)^2 = 6r - 1 > 16$.

For $\mathbb{P}^1 \times \mathbb{P}^1$ there are no solutions also because $c_1 \cdot K$ is even.

Now consider the plane with m blowed points in general position, where $1 \leqslant m \leqslant 8$. Denote this surface by X_m. We have:

$$\operatorname{Pic} X_m = \mathbb{Z}\ell_0 \oplus \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2 \oplus \cdots \oplus \mathbb{Z}\ell_m \, ,$$

where ℓ_0 is preimage of a line in $\mathbb{P}^2$ and ℓ_i, $1 \leqslant i \leqslant m$, are exceptional curves. The canonical class of X_m is

$$K = -3\ell_0 + \sum_{i=1}^{m} \ell_i \, ,$$

and the intersection form is defined by relations:

$$\ell_0^2 = 1 \, , \; \ell_i^2 = -1 \, , \; 1 \leqslant i \leqslant m \, ; \quad \ell_i \cdot \ell_j = 0 \, , \; i \neq j \, .$$

Consider $c_1 = a\ell_0 + \sum_{i=1}^{m} b_i \ell_i$. We have $c_1 \cdot K = -3a - \sum_{i=1}^{m} b_i$; $c_1^2 = a^2 - \sum_{i=1}^{m} b_i^2$. Suppose $c_1 \cdot K = A$; then from (25) we obtain: $c_1^2 = \frac{1}{2}(rA - 1)$. Together with (23) this gives the system of conditions:

$$\begin{cases} \sum_{i=1}^{m} b_i + 3a + A = 0 \\ \sum_{i=1}^{m} b_i^2 = a^2 - \frac{1}{2}(rA - 1) \\ 0 < A < \dfrac{r}{2}(9 - m) \, . \end{cases} \tag{26}$$

The first equation defines a hyperplane in the Euclidean space with coordinates $(b_1, \ldots, b_m)$ and the second one defines a sphere with center in the origin. Now we show that in all cases except only one the distance $d = \frac{|3a+A|}{\sqrt{m}}$ between the origin end the hyperplane is greater then the radius R of the sphere. Consider

$$m(d^2 - R^2) = (9 - m)a^2 + 6a + A^2 + \frac{m}{2}(rA - 1) \, .$$

This is a quadratic polynomial w.r.t. a with discriminant

$$D = \frac{m}{2}(rA - 1)\left(m - 9 + \frac{2A^2}{rA - 1}\right) \, .$$

For $0 < m < 9 - \frac{2A^2}{rA-1}$ we have $D < 0$, so $d^2 - R^2 > 0$, and there are no solutions of the system (26). Hence, if there are solutions, then the number of blowed points and $c_1 \cdot K$ are restricted with the inequality: $m \geqslant 9 - \frac{2A^2}{rA-1}$. From the other hand, the third inequality of (26) implies that $m < 9 - \frac{2A}{r}$, i.e. m belongs to the interval

$$I = \left[9 - \frac{2A^2}{rA-1} \ ; \ 9 - \frac{2A}{r} \right) \ .$$

Since A and r are integers, we have that the right bounds of I are either integers or distant from integers by the distance greater than or equal to $\frac{1}{r}$. At the same time the length of I is $\frac{2A^2}{rA-1} - \frac{2A}{r} = \frac{2}{r(r-\frac{1}{A})} \leqslant \frac{2}{r(r-1)}$, that is less then or equal to $\frac{2}{3r}$ for $r \geqslant 4$. Therefore, for $r \geqslant 4$ the interval I does not contain an integer and the system (26) has no solution. For $r = 3$ after a simple computation we see that there is a unique solution: $m = 8$, $A = 1$, $a = -3$, $b_1 = \cdots = b_8 = 1$. Thus, we have proved the following

Proposition. *A stable sheaf E over X_m satisfies to the restriction (23) and the dimension condition (24) iff $m = 8$, $r(E) = 3$ and $c_1(E) = K$.*

Observe that under conditions of the Proposition $c_1(E) \cdot K$ and $r(E)$ are coprime, so the moduli space of stable bundles in question has smooth Gieseker compactification. Therefore, we have an algebro-geometrical problem ("for testing S-duality conjecture"). Namely, let $\mathcal{S}_k$ and $\mathcal{L}_k$ be bundles over $\mathcal{M}(3, K, k)$ with fibers $H^1(S^2E)$ and $H^1(\Lambda^2 E)$. The problem is to test the function

$$Z = \sum_k c_{top}(\mathcal{S}_k \oplus \mathcal{L}_k)\, q^k \ .$$

for modular behaviour.

References.

[APS] Ph.C.Argyres, M.R.Plesser, N.Seiberg. The Moduli Space of Vacua of $N = 2$ SUSY QCD and Duality in $N = 2$ SUSY QCD //Nucl. Phys. B **471** (1996). P.159-194. (available also from hep-th/9603042)

[GR] A. L. Gorodentsev, A. N. Rudakov. Exceptional Vector Bundles on Projective Spaces// Duke Math. J., 54 (1987) 115–130.

[GL] A.L.Gorodentsev, M.I.Leenson. How to calculate the correlation function in twisted SYM $N = 2$, $N_f = 4$ QFT on projective plane// Preprint MPI 96-49

[DK] S. Donaldson and P. Kronheimer. The Geometry of Four-Manifolds// Clarendon Press, Oxford, 1990.

[DW] R.Donagi, E.Witten. Supersymmetric Yang-Mills theory and integrable systems// Nucl. Phys. B **460** (1996). P. 299-334.(available also from hep-th/9510101)

[FH] W. Fulton, J. Harris. Representation Theory. A First Course// Springer-Verlag (1990).

[Kl] A. A. Klyachko. Moduli of vector bundles and numbers of classes// Translation: Functional-Anal-Appl. **25** (1991), no. 1. P. 67–69.

[KO] S. A. Kuleshov and D. O. Orlov. Exceptional sheaves on del Pezzo surfaces// Russian Acad. Sci. Izv. Math. Vol. **44** (1995) No 1. P. 479–513.

[Ku1] S.A.Kuleshov. Moduli Space of Bundles on a Quadric// Warwick Preprints: 18/1996

[Ku2] S.A.Kuleshov. Moduli Spaces of sheaves necessary for testing S-duality conjecture// Preprint MPI 97-32.

[MO] C. Montonen, D. Olive, Phys. Lett 72B (1977), 117-132.

[PT] V. Ya. Pidstrigach and A. N. Tyurin. Invariants of the smooth structure of an algebraic surface arising from the Dirac operator// Russian Acad. Sci. Izv. Math. Vol. **40** (1993) No 2. P. 267–351.

[T] N. A. Tyurin. Necessary and sufficient condition for a deformation of a B-monopole into an instanton// Izvestiya: Mathematics, Vol. **60** (1996), no. 1. P. 217–231.

[VW] G. Vafa, E. Witten. A strong coupling test of S-duality// Nucl. Phys. B **431** (1995). P.3–77 (available also from hep-th/9408074).

[W] E.Witten. Monopoles and Four-Manifolds// Math. Res. Letts. 1 (1994), 769–796.

[W-L] W.-M. R. Leung. On $Spin^{\mathbb{C}}$-Invariants of Four-Manifolds// Thesis PhD, Magdalene College, Oxford 1995.