

GEOMETRY OVER THE FIELD WITH ONE ELEMENT

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1. MOTIVATION

Two main sources have led to the development of several notions of F_1 -geometry in the recent five years. We will concentrate on one of these, which originated as remark in a paper by Jacques Tits ([10]). For a wide class of schemes X (including affine space \mathbb{A}^n , projective space \mathbb{P}^n , the Grassmannian $\text{Gr}(k, n)$, split reductive groups G), the function

$$N(q) = \#X(\mathbb{F}_q)$$

is described by a polynomial in q with integer coefficients, whenever q is a prime power. Taking the value $N(1)$ sometimes gives interesting outcomes, but has a 0 of order r in other cases. A more interesting number is the lowest non-vanishing coefficient of the development of $N(q)$ around $q - 1$, i.e. the number

$$\lim_{q \rightarrow 1} \frac{N(q)}{(q-1)^r},$$

which Tits took to be the number $\#X(\mathbb{F}_1)$ of “ \mathbb{F}_1 -points” of X . The task at hand is to extend the definition of the above mentioned schemes X to schemes that are “defined over \mathbb{F}_1 ” such that their set of \mathbb{F}_1 -points is a set of cardinality $\#X(\mathbb{F}_1)$. We describe some cases, and suggest an interpretation of the set of \mathbb{F}_1 -points:

- $\#\mathbb{P}^{n-1}(\mathbb{F}_1) = n = \#M_n$ with $M_n := \{1, \dots, n\}$.
- $\#\text{Gr}(k, n)(\mathbb{F}_1) = \binom{n}{k} = \#M_{k,n}$ with $M_{k,n} = \{\text{subsets of } M_n \text{ with } k \text{ elements}\}$.
- If G is a split reductive group of rank r , $T \simeq \mathbb{G}_m^r \subset G$ is a maximal torus, N its normalizer and $W = N(\mathbb{Z})/T(\mathbb{Z})$, then the Bruhat decomposition $G(\mathbb{F}_q) = \coprod_{w \in W} BwB(\mathbb{F}_q)$ (where B is a Borel subgroup containing T) implies that $N(q) = \sum_{w \in W} (q-1)^r q_w^d$ for certain $d_w \geq 0$. This means that $\#G(\mathbb{F}_1) = \#W$.

In particular, it is natural to ask whether the group law $m : G \times G \rightarrow G$ of a split reductive group may be defined as a “morphism over \mathbb{F}_1 ”. If so, one can define “group actions over \mathbb{F}_1 ”. The limit as $q \rightarrow 1$ of the action

$$\text{GL}(n, \mathbb{F}_q) \times \text{Gr}(k, n)(\mathbb{F}_q) \longrightarrow \text{Gr}(k, n)(\mathbb{F}_q)$$

induced by the action on $\mathbb{P}^{n-1}(\mathbb{F}_q)$ should be the action

$$S_n \times M_{k,n} \longrightarrow M_{k,n}$$

induced by the action on $M_n = \{1, \dots, n\}$.

The other, more lofty motivation for \mathbb{F}_1 -geometry stems from the search for a proof of the Riemann hypothesis. In the early 90s, Deninger gave criteria for a category of motives that would provide a geometric framework for translating Weil’s proof of the Riemann hypothesis for global fields of positive characteristic to number fields. In particular, the Riemann zeta function $\zeta(s)$ should have a cohomological interpretation, where an H^0 , an H^1 and an H^2 -term are involved. Manin proposed in [7] to interpret the H^0 -term as the zeta function of the “absolute point” $\text{Spec } \mathbb{F}_1$ and the H^2 -term as the zeta function of the “absolute Tate motive” or the “affine line over \mathbb{F}_1 ”.

2. OVERVIEW OVER RECENT APPROACHES

We give a rough description of the several approaches towards \mathbb{F}_1 -geometry, some of them looking for weaker structures than rings, e.g. monoids, others looking for a category of schemes with certain additional structures. In the following, a *monoid* always means an abelian multiplicative semi-group with 1. A *variety* is a scheme X that defines, via base extension, a variety X_k over any field k .

2.1. Soulé, 2004 ([9]). This is the first paper that suggests a candidate of a category of varieties over \mathbb{F}_1 . Soulé considers schemes together with a complex algebra, a functor on finite rings that are flat over \mathbb{Z} and certain natural transformations and a universal property that connects the scheme, the functor and the algebra. Soulé could prove that smooth toric varieties provide natural examples of \mathbb{F}_1 -varieties. In [6] the list of examples was broadened to contain models of all toric varieties over \mathbb{F}_1 , as well as split reductive groups. However, it seems unlikely that Grassmannians that are not projective spaces can be defined in this framework.

2.2. Connes-Consani, 2008 ([1]). The approach of Soulé was modified by Connes and Consani in the following way. They consider the category of schemes together with a functor on finite abelian groups, a complex variety, certain natural transformations and a universal property analogous to Soulé's idea. This category behaves only slightly different in some subtle details, but the class of established examples is the same (cf. [6]).

2.3. Deitmar, 2005 ([3]). A completely different approach was taken by Deitmar who uses the theory of prime ideals of monoids to define spectra of monoids. A \mathbb{F}_1 -scheme is a topological space together with a sheaf of monoids that is locally isomorphic to the spectrum of a ring. This theory has the advantage of having a very geometric flavour and one can mimic algebraic geometry to a large extent. However, Deitmar has shown himself in a subsequent paper that the \mathbb{F}_1 -schemes whose base extension to \mathbb{Z} are varieties are nothing more than toric varieties.

2.4. Toën-Vaquié, 2008 ([11]). Deitmar's approach is complemented by the work of Toën and Vaquié, which proposes locally representable functors on monoids as \mathbb{F}_1 -schemes. Marty shows in [8] that the Noetherian \mathbb{F}_1 -schemes in Deitmar's sense correspond to the Noetherian objects in Toën-Vaquié's sense. We raise the question: is the Noetherian condition necessary?

2.5. Borger, in progress. The category investigated by Borger are schemes X together with a family of morphism $\{\psi_p : X \rightarrow X\}_{p \text{ prime}}$, where the ψ_p 's are lifts of the Frobenius morphisms $\text{Frob}_p : X \otimes \mathbb{F}_p \rightarrow X \otimes \mathbb{F}_p$ and all ψ_p 's commute with each other.

There are further approaches by Durov ([4], 2007) and Haran ([5], 2007), which we do not describe here. In the following section we will examine more closely a new framework for \mathbb{F}_1 -geometry by Connes and Consani in spring 2009.

3. \mathbb{F}_1 -SCHEMES À LA CONNES-CONSANI AND TORIFIED VARIETIES

The new notion of an \mathbb{F}_1 -scheme due to Connes and Consani ([2]) combines the earlier approaches of Soulé and of themselves with Deitmar's theory of spectra of monoids and Toën-Vaquié's functorial viewpoint. First of all, Connes and Consani consider monoids with 0 and remark that the spaces that are locally isomorphic to spectra of monoids with 0, called *M_0 -schemes*, are the same as locally representable functors of monoids with 0. (Note that they do not make any Noetherian hypothesis). There is a natural notion of morphism in this setting. The base extension is locally given by taking the semi-group ring, i.e. if A is a monoid with zero 0_A and $X = \text{Spec } A$ is its spectrum, then

$$X_{\mathbb{Z}} := X \otimes_{\mathbb{F}_1} \mathbb{Z} := \text{Spec}(\mathbb{Z}[A]/(1 \cdot 0_A - 0_{\mathbb{Z}[A]})).$$

An \mathbb{F}_1 -scheme is a triple (\tilde{X}, X, e_X) , where \tilde{X} is an M_0 -scheme, X is a scheme and $e_X : \tilde{X}_{\mathbb{Z}} \rightarrow X$ is a morphism such that $e_X(k) : \tilde{X}_{\mathbb{Z}}(k) \xrightarrow{\sim} X(k)$ is a bijection for all fields k .

Note that an M_0 -scheme \tilde{X} defines the \mathbb{F}_1 -scheme $(\tilde{X}, \tilde{X}_{\mathbb{Z}}, \text{id}_{\tilde{X}_{\mathbb{Z}}})$. We give first examples of \mathbb{F}_1 -schemes of this kind. The affine line $\mathbb{A}_{\mathbb{F}_1}^1$ is the spectrum of the monoid $\{T^i\}_{i \in \mathbb{N}} \amalg \{0\}$ and, indeed, we have $\mathbb{A}_{\mathbb{F}_1}^1 \otimes_{\mathbb{F}_1} \mathbb{Z} \simeq \mathbb{A}^1$. The multiplicative group $\mathbb{G}_{m, \mathbb{F}_1}$ is the spectrum of the monoid $\{T^i\}_{i \in \mathbb{Z}} \amalg \{0\}$, which base extends to \mathbb{G}_m as desired. Both examples can be extended to define $\mathbb{A}_{\mathbb{F}_1}^n$ and $\mathbb{G}_{m, \mathbb{F}_1}^n$ by considering multiple variables T_1, \dots, T_n . More generally, all \mathbb{F}_1 -schemes in the sense of Deitmar deliver examples of M_0 and thus \mathbb{F}_1 -schemes in this new sense. In particular, toric varieties can be realized.

To obtain a richer class of examples, we recall the definition of a torified variety as given in a joint work with Javier López Peña ([6]). A *torified variety* is a variety X together with morphism $e_X : T \rightarrow X$ such that $T \simeq \coprod_{i \in I} \mathbb{G}_m^{d_i}$, where I is a finite index set and d_i are non-negative integers and such that for every field k , the morphism e_X induces a bijection $T(k) \xrightarrow{\sim} X(k)$. We call $e_X : T \rightarrow X$ a *torification* of X .

Note that T is isomorphic to the base extension $\tilde{X}_{\mathbb{Z}}$ of the M_0 -scheme $\tilde{X} = \coprod_{i \in I} \mathbb{G}_{m, \mathbb{F}_1}^{d_i}$. Thus every torified variety $e_X : T \rightarrow X$ defines an \mathbb{F}_1 -scheme (\tilde{X}, X, e_X) .

In [6], a variety of examples are given. Most important for our purpose are toric varieties, Grassmannians and split reductive groups. If X is a toric variety of dimension n with fan $\Delta = \{\text{cones } \tau \subset \mathbb{R}^n\}$, i.e. $X = \text{colim}_{\tau \in \Delta} \text{Spec } \mathbb{Z}[A_{\tau}]$, where $A_{\tau} = \tau^{\vee} \cap \mathbb{Z}^n$ is the intersection of the dual cone $\tau^{\vee} \subset \mathbb{R}^n$ with the dual lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Then the natural morphism $\coprod_{\tau \in \Delta} \text{Spec } \mathbb{Z}[A_{\tau}^{\times}] \rightarrow X$ is a torification of X .

The Schubert cell decomposition of $\text{Gr}(k, n)$ is a morphism $\coprod_{w \in M_{k, n}} \mathbb{A}^{d_w} \rightarrow \text{Gr}(k, n)$ that induces a bijection of k -points for all fields k . Since the affine spaces in this decomposition can be further decomposed into tori, we obtain a torification $e_X : T \rightarrow \text{Gr}(k, n)$. Note that the lowest-dimensional tori are 0-dimensional and the number of 0-dimensional tori is exactly $\#M_{k, n}$.

Let G be a split reductive group of rank r with maximal torus $T \simeq \mathbb{G}_m^r$, normalizer N and Weyl group $W = N(\mathbb{Z})/T(\mathbb{Z})$. Let B be a Borel subgroup containing T . The Bruhat decomposition $\coprod_{w \in W} BwB \rightarrow G$, where $BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{d_w}$ for some $d_w \geq 0$, yields a torification $e_G : T \rightarrow G$ analogously to the case of the Grassmannian. This defines a model $\mathcal{G} = (\tilde{G}, G, e_G)$ over \mathbb{F}_1 . Note that in this case the lowest-dimensional tori are r -dimensional and that the number of r -dimensional tori is exactly $\#W$.

Clearly, there is a close connection between torified varieties and the \mathbb{F}_1 -schemes in the sense of Connes and Consani with the idea that Tits had in mind. However, the natural choice of morphism in this category is a morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of M_0 -schemes together with a morphism $f : X \rightarrow Y$ of schemes such that

$$\begin{array}{ccc} \tilde{X}_{\mathbb{Z}} & \xrightarrow{\tilde{f}_{\mathbb{Z}}} & \tilde{Y}_{\mathbb{Z}} \\ \downarrow e_X & & \downarrow e_X \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Unfortunately, the only reductive groups G whose group law $m : G \times G \rightarrow G$ extends to a morphism $\mu : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ in this sense such that (\mathcal{G}, μ) becomes a group object in the category of \mathbb{F}_1 -schemes are algebraic groups of the form $G \simeq \mathbb{G}_m^r \times (\text{finite group})$. In the following section we will show how to modify the notion of morphism to realize Tits' idea.

4. STRONG MORPHISMS

Let $\mathcal{X} = (\tilde{X}, X, e_X)$ and $\mathcal{Y} = (\tilde{Y}, Y, e_Y)$ be \mathbb{F}_1 -schemes. Then we define the *rank of a point* x in the underlying topological space \tilde{X} as $\text{rk } x := \text{rk } \mathcal{O}_{\tilde{X}, x}^{\times}$, where $\mathcal{O}_{\tilde{X}, x}$ is the stalk

(of monoids) at x and $\mathcal{O}_{X,x}^\times$ denotes its group of invertible elements. We define the *rank of X* as $\text{rk } X := \min_{x \in \tilde{X}} \{\text{rk } x\}$ and we let

$$\tilde{X}^{\text{rk}} := \coprod_{\text{rk } x = \text{rk } \tilde{X}} \text{Spec } \mathcal{O}_{X,x}^\times,$$

which is a sub- M_0 -scheme of \tilde{X} . A *strong morphism* $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ is a pair $\varphi = (\tilde{f}, f)$, where $\tilde{f} : \tilde{X}^{\text{rk}} \rightarrow \tilde{Y}^{\text{rk}}$ is a morphism of M_0 -schemes and $f : X \rightarrow Y$ is a morphism of schemes such that

$$\begin{array}{ccc} \tilde{X}_Z^{\text{rk}} & \xrightarrow{\tilde{f}_Z} & \tilde{Y}_Z^{\text{rk}} \\ \downarrow e_X & & \downarrow e_Y \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

This notion comes already close to achieving our goal. In the category of \mathbb{F}_1 -schemes together with strong morphisms, the object $(\text{Spec}\{0,1\}, \text{Spec } \mathbb{Z}, \text{id}_{\text{Spec } \mathbb{Z}})$ is the terminal object, which we should define as $\text{Spec } \mathbb{F}_1$. We define

$$\mathcal{X}(\mathbb{F}_1) := \text{Hom}_{\text{strong}}(\text{Spec } \mathbb{F}_1, \mathcal{X}),$$

which equals the set of points of \tilde{X}^{rk} as every strong morphism $\text{Spec } \mathbb{F}_1 \rightarrow \mathcal{X}$ is determined by the image of the unique point $\{0\}$ of $\text{Spec}\{0,1\}$ in \tilde{X}^{rk} . We see at once that $\#\mathcal{X}(\mathbb{F}_1) = \#M_{k,n}$ if \mathcal{X} is a model of the Grassmannian $\text{Gr}(k,n)$ as \mathbb{F}_1 -scheme and that $\#\mathcal{G}(\mathbb{F}_1) = \#W$ if $\mathcal{G} = (\tilde{G}, G, e_G)$ is a model of a split reductive group G with Weyl group W .

Furthermore, if the Weyl group can be lifted to $N(\mathbb{Z})$ as group, i.e. if the short exact sequence of groups

$$1 \longrightarrow T(\mathbb{Z}) \longrightarrow N(\mathbb{Z}) \longrightarrow W \longrightarrow 1$$

splits, then from the commutativity of

$$\begin{array}{ccc} \tilde{G}_Z^{\text{rk}} \times \tilde{G}_Z^{\text{rk}} & \xrightarrow{\tilde{m}_Z} & \tilde{G}_Z^{\text{rk}} \\ \downarrow (e_G, e_G) & & \downarrow e_G \\ G \times G & \xrightarrow{m} & G \end{array}$$

we obtain a morphism $\tilde{m} : \tilde{G}^{\text{rk}} \times \tilde{G}^{\text{rk}} \rightarrow \tilde{G}^{\text{rk}}$ of M_0 -schemes such that $\mu = (\tilde{m}, m) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is a strong morphism that makes \mathcal{G} into a group object.

However, $\text{SL}(n)$ provides an example where the Weyl group cannot be lifted. This leads us, in the following section, to introduce a second kind of morphisms.

5. WEAK MORPHISMS

The morphism $\text{Spec } \mathcal{O}_{X,x}^\times \rightarrow *_{M_0}$ to the terminal object $*_{M_0} = \text{Spec}\{0,1\}$ in the category of M_0 -schemes induces a morphism

$$\tilde{X}^{\text{rk}} = \coprod_{x \in \tilde{X}^{\text{rk}}} \text{Spec } \mathcal{O}_{X,x}^\times \longrightarrow *_{\mathcal{X}} := \coprod_{x \in \tilde{X}^{\text{rk}}} *_{M_0}.$$

Given $\tilde{f} : \tilde{X}^{\text{rk}} \rightarrow \tilde{Y}^{\text{rk}}$, there is a unique morphism $*_{\mathcal{X}} \rightarrow *_{\mathcal{Y}}$ such that

$$\begin{array}{ccc} \tilde{X}^{\text{rk}} & \xrightarrow{\tilde{f}} & \tilde{Y}^{\text{rk}} \\ \downarrow & & \downarrow \\ *_{\mathcal{X}} & \longrightarrow & *_{\mathcal{Y}} \end{array}$$

commutes. Let X^{rk} denote the image of $e_X : \tilde{X}^{\text{rk}} \rightarrow X$. A *weak morphism* $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ is a pair $\varphi = (\tilde{f}, f)$, where $\tilde{f} : \tilde{X}^{\text{rk}} \rightarrow \tilde{Y}^{\text{rk}}$ is a morphism of M_0 -schemes and $f : X \rightarrow Y$

is a morphism of schemes such that

$$\begin{array}{ccc}
 \tilde{X}^{\text{rk}}_{\mathbb{Z}} & \xrightarrow{\tilde{f}_{\mathbb{Z}}} & \tilde{Y}^{\text{rk}}_{\mathbb{Z}} \\
 \searrow & & \searrow \\
 & (*\mathcal{X})_{\mathbb{Z}} & \xrightarrow{\quad} & (*\mathcal{Y})_{\mathbb{Z}} \\
 \nearrow & & \nearrow \\
 X^{\text{rk}} & \xrightarrow{f} & Y^{\text{rk}}
 \end{array}$$

commutes.

The key observation is that a weak morphism $\varphi = (\tilde{f}, f) : \mathcal{X} \rightarrow \mathcal{Y}$ has a base extension $f : X \rightarrow Y$ to \mathbb{Z} , but also induces a morphism $\tilde{f}_* : \mathcal{X}(\mathbb{F}_1) \rightarrow \mathcal{Y}(\mathbb{F}_1)$. With this in hand, we yield the following results.

6. ALGEBRAIC GROUPS OVER \mathbb{F}_1

The idea of Tits' paper is now realized in the following form.

Theorem 6.1. *Let G be a split reductive group with group law $m : G \times G \rightarrow G$ and Weyl group W . Let $\mathcal{G} = (\tilde{G}, G, e_G)$ be the model of G as described before as \mathbb{F}_1 -scheme. Then there is morphism $\tilde{m} : \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ of M_0 -schemes such that $\mu = (\tilde{m}, m)$ is a weak morphism that makes \mathcal{G} into a group object. In particular, $\mathcal{G}(\mathbb{F}_1)$ inherits the structure of a group that is isomorphic to W .*

We have already seen that $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ when \mathcal{X} is a model of $\text{Gr}(n, k)$ as \mathbb{F}_1 -scheme. Furthermore, we have the following.

Theorem 6.2. *Let \mathcal{G} be a model of $G = \text{GL}(n)$ as \mathbb{F}_1 -scheme and let \mathcal{X} be a model of $X = \text{Gr}(k, n)$ as \mathbb{F}_1 -scheme. Then the group action*

$$f : \text{GL}(n) \times \text{Gr}(k, n) \longrightarrow \text{Gr}(k, n),$$

induced by the action on \mathbb{P}^{n-1} , can be extended to a strong morphism $\varphi : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$ such that the group action

$$\varphi(\mathbb{F}_1) : S_n \times M_{k,n} \longrightarrow M_{k,n},$$

of $\mathcal{G}(\mathbb{F}_1) = S_n$ on $\mathcal{X}(\mathbb{F}_1) = M_{k,n}$ is induced by the action on $M_n = \{1, \dots, n\}$.

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