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LOCALLY NILPOTENT DERIVATIONS OF FREE ALGEBRA OF RANK TWO

VESSELIN DRENSKY AND LEONID MAKAR-LIMANOV

To the 80th anniversary of Dmitry Fuchs

Abstract. In commutative algebra, if δ is a locally nilpotent derivation of the polynomial algebra $K[x_1,\dots,x_d]$ over a field K of characteristic 0 and w is a nonzero element of the kernel of δ, then $\Delta = w\delta$ is also a locally nilpotent derivation with the same kernel as δ. In this paper we prove that the locally nilpotent derivation Δ of the free associative algebra $K\langle X,Y\rangle$ is determined up to a multiplicative constant by its kernel. We show also that the kernel of Δ is a free associative algebra and give an explicit set of its free generators.

1. Introduction

In this paper we study locally nilpotent derivations Δ of the free unitary associative algebra $K\langle X,Y\rangle$ over a field K of characteristic 0. By analogy with the commutative case we shall call the kernel of Δ the algebra of constants of Δ and shall denote it by $K\langle X,Y\rangle^{\Delta}$. It is easy to see that Δ is of the form $\Delta(U) = 0$, $\Delta(V) = f(U)$, with respect to a suitable system of generators U, V of $K\langle X,Y\rangle$. This follows from the description of Rentschler [R] of the locally nilpotent derivations of $K[x,y]$ and the isomorphism of the automorphism groups of $K[x,y]$ and $K\langle X,Y\rangle$ which is a consequence of the theorem of Jung–van der Kulk [J, K] and its analogue for the tameness of the automorphisms of $K\langle X,Y\rangle$ due to Czerniakiewicz [Cz] and Makar-Limanov [ML1]. This result is similar to the recent description of locally nilpotent derivations of the free Poisson algebra with two generators given by Makar-Limanov, Turusbekova and Umirbaev [MLTU]. Our main result is that the locally nilpotent derivations of $K\langle X,Y\rangle$ are determined up to a multiplicative constant by their algebras of constants. As a consequence of the result of Lane [L] and Kharchenko [Kh] the algebra of constants $K\langle X,Y\rangle^{\Delta}$ of the nontrivial Weitzenböck derivation Δ of $K\langle X,Y\rangle$ is a free associative algebra. A set of free generators of this algebra was given by Drensky and Gupta [DG]. We generalize this result and show that the algebra $K\langle X,Y\rangle^{\Delta}$ is free for any locally nilpotent derivation Δ of $K\langle X,Y\rangle$. As in [DG] we give an explicit set of free generators of $K\langle X,Y\rangle^{\Delta}$.

See also [Jo] where it is shown that $K\langle X,Y\rangle^{\Delta}$ is a free associative algebra for a nontrivial homogeneous derivation (and from which the freeness in our case can be deduced).

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2. Preliminaries

For an algebra R over a field K a linear operator $\delta : R \to R$ is called a derivation if it satisfies the Leibniz law $\delta(ab) = \delta(a)b + a\delta(b)$.

The kernel of a derivation δ is denoted by R^δ and the elements of the kernel are called δ-constants (or just constants when this is not confusing).

A derivation δ is called locally nilpotent if for any $r \in R$ there exists a natural number n (which depends on r) for which $\delta^n(r) = 0$. The function

$$\deg(r) = \max(d \mid \delta^d(r) \neq 0), \quad \deg(0) = -\infty,$$

is a degree function with familiar properties:

$$\deg(r_1r_2) = \deg(r_1) + \deg(r_2), \quad \deg(r_1 + r_2) \leq \max(\deg(r_1), \deg(r_2)),$$
$$\deg(r_1 + r_2) = \max(\deg(r_1), \deg(r_2) \text{ when } \deg(r_1) \neq \deg(r_2)),$$
$$\deg(\delta(r)) = \deg(r) - 1 \text{ if } \delta(r) \neq 0.$$

The set of all lnds (locally nilpotent derivations) of R is denoted by $\mathrm{LND}(R)$.

The intersection $\bigcap R^\delta$, $\delta \in \mathrm{LND}(R)$, of kernels of all locally nilpotent derivations of R is denoted by $\mathrm{AK}(R)$ (absolute Konstanten of R, sometimes denoted as $\mathrm{ML}(R)$).

If $\delta \in \mathrm{LND}(R)$ and characteristic of K is zero then the linear operator

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots .$$

is an automorphism of R.

In the sequel we fix a field K of characteristic 0 and consider the polynomial algebra $K[x, y]$ and the free associative algebra $K\langle X, Y\rangle$. Let

$$\pi : K\langle X, Y\rangle \to K[x, y]$$

be the natural homomorphism. We denote the elements U, V, etc. of $K\langle X, Y\rangle$ by upper case symbols and their images under π by the same lower case symbols u, v, etc. Let C be the commutator ideal of $K\langle X, Y\rangle$. It is generated by the commutator

$$Z_1 = [Y, X] = YX - XY.$$

By the theorem of Jung–van der Kulk [J, K], the automorphisms of $K[x, y]$ are tame, i.e. are compositions of affine automorphisms

$$x \to a_1x + a_2y + a_3,\ y \to b_1x + b_2y + b_3;\ a_1b_2 - a_2b_1 \neq 0,$$

and triangular automorphisms

$$x \to x, \quad y \to y + p(x), \quad p(x) \in K[x].$$

A similar theorem of Czerniakiewicz [Cz] and Makar-Limanov [ML1] states that the automorphisms of $K\langle X, Y\rangle$ are also tame. Therefore

$$\Psi(Z_1) = cZ_1, \quad c \in K^*,$$

for any automorphism Ψ of $K\langle X, Y\rangle$ (indeed, just check that this is true for affine and triangular automorphisms).

The structure of the automorphism groups of $K[x, y]$ and $K\langle X, Y\rangle$ is also known, it is a free product of the subgroups of affine and triangular automorphisms with amalgamation along their intersection [Se]. So we can think that there is a group G isomorphic to Aut $K[x, y]$ and Aut$K\langle X, Y\rangle$ which acts on $K[x, y]$ and $K\langle X, Y\rangle$.

Any automorphism of $K\langle X,Y\rangle$ induces an automorphism of $K[x,y]$ and, since the structure of the group G insures that this is one to one correspondence, any automorphism of $K[x,y]$ can be uniquely lifted to an automorphism of $K\langle X,Y\rangle$.

We shall use below a lexicographic ordering of monomials of $K\langle X,Y\rangle$ defined by $Y >> X > 1$ and denote by $\overline{S}$ the leading monomial of $S \in K\langle X,Y\rangle$.

3. Description of locally nilpotent derivations

Though the lnds of $K\langle X,Y\rangle$ are similar to the lnds of $K[x,y]$ there are also significant differences.

It is quite clear that $\mathrm{AK}(K[x,y]) = K$ (just observe that the partial derivatives $\dfrac{\delta}{\delta x}$ and $\dfrac{\delta}{\delta y}$ are locally nilpotent) but $\mathrm{AK}(K\langle X,Y\rangle) = K[Z_1]$.

Lemma 1. *$\delta(Z_1) = 0$ for any lnd of $K\langle X,Y\rangle$.*

Proof. If $\delta \in \mathrm{LND}(K\langle X,Y\rangle)$ then $\lambda\delta \in \mathrm{LND}(K\langle X,Y\rangle)$ for any $\lambda \in K$. Take $\Psi_\lambda = \exp(\lambda\delta)$; then $\Psi_\lambda([Y,X]) = c(\lambda)[Y,X]$, where $c(t) \in K[t]$ (recall that δ is an lnd). On the other hand $\Psi_\lambda\Psi_\mu = \Psi_{\lambda+\mu}$, i.e. $c(s)c(t) = c(s+t)$. Since $c(s) \neq 0$ this is possible only if $c(t) = 1$. Hence $\delta([Y,X]) = 0$. □

Of course this proves only that $\mathrm{AK}(K\langle X,Y\rangle) \supseteq K[Z_1]$. We shall see that $\mathrm{AK}(K\langle X,Y\rangle) = K[Z_1]$ later.

Now we shall prove that lnds of $K\langle X,\rangle$ are similar to those of $K[x,y]$.

Proposition 2. *Let Δ be a locally nilpotent derivation of $K\langle X,Y\rangle$. Then there is a system of generators U,V of $K\langle X,Y\rangle$ and a polynomial $f(U)$ depending on U only, such that $\Delta(U) = 0$, $\Delta(V) = f(U)$.*

Proof. Let Δ be a locally nilpotent derivation of $K\langle X,Y\rangle$. Clearly, Δ induces a locally nilpotent derivation δ of $K[x,y]$. By the theorem of Rentschler [R], $K[x,y]$ has a system of generators u,v such that $\delta(u) = 0$, $\delta(v) = f(u)$ for some $f(u) \in K[u]$.

As was mentioned above this pair of generators can be uniquely lifted to the pair U,V of generators of $K\langle X,Y\rangle$.

Let us consider the automorphisms

$$\Phi = \exp(\Delta) \in \mathrm{Aut}\ K\langle X,Y\rangle = \mathrm{Aut}\ K\langle U,V\rangle$$

and

$$\varphi = \exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots \in \mathrm{Aut}\ K[x,y] = \mathrm{Aut}\ K[u,v].$$

Then

$$\varphi : u \to u, \quad \varphi : v \to v + f(u).$$

From the uniqueness mentioned in Preliminaries

$$\varphi(u) = u, \quad \varphi(v) = v + f(u)$$

implies $\Phi(U) = U$, $\quad \Phi(V) = V + f(U)$. Since $\Phi = \exp(\Delta) = 1 + \Theta$, where

$$\Theta = \frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots$$

and $\Theta^n(S) = 0$ for $S \in K\langle X,Y\rangle$ and a sufficiently large n, we have that

$$\Delta = \log(1+\Theta) = \frac{\Theta}{1} - \frac{\Theta^2}{2} + \frac{\Theta^3}{3} - \cdots$$

and Φ determines uniquely the lnd Δ. Hence $\Delta(U)=0,\ \Delta(V)=f(U)$. □

Another difference between the locally nilpotent derivations of $K[x,y]$ and $K\langle X,Y\rangle$ is that in the latter case they can be distinguished by their algebras of constants.

Theorem 3. *Let Δ_1 and Δ_2 be two non-zero locally nilpotent derivations of $K\langle X,Y\rangle$. Then Δ_1 and Δ_2 have the same algebra of constants if and only if $\Delta_2=\alpha\Delta_1$ for a nonzero $\alpha\in K$.*

Proof. Changing the coordinates of $K\langle X,Y\rangle$, by Proposition 2 we may assume that $\Delta_1(X)=0,\quad \Delta_1(Y)=f(X)$ for some nonzero $f(X)\in K\langle X,Y\rangle$. By Lemma 1 $\Delta_1(Z_1)=0$ and this implies that the algebra of constants $K\langle X,Y\rangle^{\Delta_1}$ is generated by $f(X)$ and the commutator ideal C of $K\langle X,Y\rangle$. Since $K\langle X,Y\rangle^{\Delta_1}=K\langle X,Y\rangle^{\Delta_2}$ we have that $\Delta_2(X)=0$. Let us consider the automorphisms $\Phi_1=\exp(\Delta_1)$ and $\Phi_2=\exp(\Delta_2)$ of $K\langle X,Y\rangle$. Since Δ_1 and Δ_2 have the same algebras of constants we obtain that their images δ_1 and δ_2 in $K[x,y]$ also have the same algebras of constants. Hence $\delta_2(y)=g(x)$ and $\Delta_2(Y)=g(X)+W(X,Y)$ for some nonzero $g(X)\in K\langle X,Y\rangle$ and $W(X,Y)\in C$. Then

$$\Phi_2(X)=X,\quad \Phi_2(Y)=Y+g(X)+W(X,Y).$$

But the only automorphisms of $K\langle X,Y\rangle$ with $\Phi_2(X)=X$ have the property $\Phi(Y)=\varepsilon Y+h(X)$, $\varepsilon\in K^*$, $h(X)\in K\langle X,Y\rangle$. Hence $W(X,Y)=0$ and $\Delta_2(Y)=g(X)$.

A direct computation gives that $Z_2=YZ_1f(X)-f(X)Z_1Y\in K\langle X,Y\rangle^{\Delta_1}$. Hence $\Delta_2(Z_2)=g(X)Z_1f(X)-f(X)Z_1g(X)=0$ which implies that $g(x)=\alpha f(x)$ for some $\alpha\in K$. Therefore $\Delta_2=\alpha\Delta_1$. Since $\Delta_1,\Delta_2\neq 0$, we obtain that $\alpha\neq 0$. □

4. Algebras of constants of derivations of $K\langle X,Y\rangle$

By Proposition 2, up to a change of the free generators of $K\langle X,Y\rangle$ every nontrivial locally nilpotent derivation Δ of $K\langle X,Y\rangle$ is of the form

$$\Delta(X)=0,\quad \Delta(Y)=f(X),$$

where $0\neq f(x)\in K[x]$. In the sequel we shall fix $\deg(f)=m\geq 0$ and Δ as defined above.

Now we can check that $\mathrm{AK}(K\langle X,Y\rangle)=K[Z_1]$. Indeed, let us consider derivations

$$\delta_m:\delta_m(X)=0,\quad \delta_m(Y)=X^m.$$

Suppose $\delta_m(P)=0$ for all m. We may assume that P is homogeneous relative to X and Y. Write $P=XP_0+YP_1$, then

$$0=\delta_m(P)=X\delta_m(P_0)+X^mP_1+Y\delta_m(P_1).$$

Hence $\delta_m(P_1)=0$ and we can assume by induction on $\deg_Y$ that P_1 belongs to the subalgebra $K\langle X,Z_1\rangle$ of $K\langle X,Y\rangle$ generated by X and Z_1 and write $P_1=XP_{10}+Z_1P_{11}$. If $P_{11}\neq 0$ then $\overline{X^mZ_1P_{11}}$ cannot be canceled by any monomial of $X\delta_m(P_0)$ if m is sufficiently large. Hence $P_{11}=0$ and $P_{10}\in K\langle X,Z_1\rangle$. Therefore

$$P=XP_0+YXP_{10}=XP_0+Z_1P_{10}+XYP_{10}=X(P_0+YP_{10})+Z_1P_{10}.$$

Then $\delta_m(P_0+YP_{10})=0$ because $Z_1P_{10}\in K\langle X,Z_1\rangle$ and we can assume by induction on $\deg_X$ that $P_0+YP_{11}\in K\langle X,Z_1\rangle$, i.e. $P\in K\langle X,Z_1\rangle$. Of course

$$\mathrm{AK}(K\langle X,Y\rangle)\subseteq K\langle X,Z_1\rangle\bigcap K\langle Y,Z_1\rangle=K[Z_1]$$

since we can switch X an Y.

Consider the operator $\boxdot$ on $K\langle X,Y\rangle$ defined by

$$\boxdot(a) = Yaf - faY.$$

We shall prove in this section that the algebra of constants of Δ is the minimal algebra A_f which contains $K\langle X, Z_1\rangle$ and is closed under this operator. Since $\boxdot\Delta = \Delta\boxdot$ it is clear that $A_f \subseteq K\langle X,Y\rangle^{\Delta}$. It is worth observing that the kernel of $\boxdot$ is $K[Y]$ if $\deg_X(f) = 0$ and 0 if $\deg_X(f) > 0$ and that $\deg(\boxdot(a)) = \deg(a)$ (where deg is the degree function induced by Δ) if $\deg_X(f) > 0$. We shall also denote $\boxdot(a)$ by $\{a\}$. This bracketing is a bit unusual since $\boxdot^n(a)$ will be recorded as $\{\{\ldots\{a\}\ldots\}\}$ with the same number n of the left and right brackets and there can be more than two terms inside of a pair of brackets, but as in the ordinary bracketing in a configuration of three brackets like this $\{a_1\{a_2\}$ the first bracket cannot match the third bracket, it should be matched by a bracket $\}$ to the right of the third bracket and second and third brackets are matched.

Theorem 4. *If $\Delta^n(F) = 0$ then F belongs to the linear span A_f^n of elements $a_1Ya_2Y\cdots Ya_k$, where $k \le n$ and each a_i, $1 \le i \le k$, is a monomial from A_f, endowed with an arbitrary number of matching pairs of brackets $\{\}$.*

Proof. We consider two cases separately.

(a) $m = 0$ (we can assume that $\Delta(Y) = 1$). Consider the sequence of elements $Z_1,\ Z_2,\ Z_i,\ \ldots$ defined by $Z_1 = YX - XY,\ Z_{i+1} = \boxdot^i(Z_1)$. In this case $\overline{Z_i} = Y^iX$ and any element $S \in K\langle X,Y\rangle$ can be written as $S = \sum_{j=0}^{k} S_jY^j$ where $S_j \in K\langle X, Z_1, \ldots, Z_i, \ldots\rangle$. Since $\Delta(S) = \sum_{j=0}^{k} jS_jY^{j-1}$, $\Delta^n(S) = 0$, and $\Delta^k(S) \neq 0$ if $S_k \neq 0$ it is clear that $k < n$.

(b) $m > 0$. Let us introduce a weight degree function on $K\langle X,Y\rangle$ by $w(X) = 1$, $w(Y) = m$. Then the space V_N spanned by monomials of the weight not exceeding N is mapped by the derivation onto itself. We proceed by induction on $w(S)$. If $w(S)$ is sufficiently small, say does not exceed m, the claim is obvious. Assume that for the weight less than N the claim is true.

Take an F for which $w(F) = N$ and $F^{(k)} = 0$ (here and further on $F^{(k)}$ denotes $\Delta^k(F)$). We can assume that $F(X,0) = 0$ and write

$$F = F_mf + \sum_{i=0}^{m-1} F_iYX^i.$$

Then

$$F_m^{(k)}f + k\sum_{i=0}^{m-1} F_i^{(k-1)}X^if + \sum_{i=0}^{m-1} F_i^{(k)}YX^i = 0.$$

Hence $F_i^{(k)} = 0$ for $i < m$ and

$$(F_m' + k\sum_{i=0}^{m-1} F_iX^i)^{(k-1)} = 0.$$

Therefore $\hat{F}^{(k)} = 0$ for $\hat{F} = F_mf + \sum_{i=0}^{m-1} F_iX^iY$.

It is sufficient to check the claim for $\hat{F}$ since $F - \hat{F} = \sum_{i=0}^{m-1} F_i[Y, X^i]$ satisfies the claim by induction ($w(F_i) < N$ and $[Y, X^i] \in A_f$).

Write $\hat{F} = F_m f + F_0 Y$. Then $F_0^{(k)} = 0$ and $(F_m' + kF_0)^{(k-1)} = 0$. Hence $F_m^{(k+1)} = 0$ and $\widetilde{F}^{(k)} = 0$ for $\widetilde{F} = kF_m f - F_m' Y$. It is sufficient to check the claim for $\widetilde{F}$ since $k\hat{F} - \widetilde{F} = (kF_0 + F_m')Y$ and $kF_0 + F_m'$ satisfy the claim by induction.

Since $F_m^{(k+1)} = 0$ and $w(F_m) < N$ we can write

$$F_m = \sum_{\mathbf{j}} \alpha_{j_0} Y \alpha_{j_1} Y \cdots Y \alpha_{j_k} + S,$$

where $\alpha_{j_i} \in A_f$, the summands are endowed with brackets $\{\}$, and S is the sum of terms in which Y appears less than k times. We can omit S since $kSf - S'Y \in A_f^k$.

Take one of the summands $\mu_{\mathbf{j}}$ and consider $\nu_{\mathbf{j}} = k\mu_{\mathbf{j}} f - \mu_{\mathbf{j}}' Y$. Since Δ and $\boxdot$ commute

$$\nu_{\mathbf{j}} = k\mu_{\mathbf{j}} f - \sum_{i=1}^{k} \alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} f \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y$$

where each term $\alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} f \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y$ has the same bracketing as μ.

Consider $P_{\mathbf{j},i} = \mu_{\mathbf{j}} f - \alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} f \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y$. It is clear that $P_{\mathbf{j},i}^{(k)} = 0$ so we should check that $P_{\mathbf{j},i}$ can be recorded as a sum of terms containing only $k-1$ entries of Y (we do not count Y's appearing in $\boxdot$).

Write $\mu = v_1 Y u_1$ where Y is the one which is replaced by f in $P_{\mathbf{j},i}$ and introduce two operations:

$$\bigtriangledown_{r,u}(v_1 Y u_1) = v_1 Y u_1 u f - v_1 f u_1 u Y \text{ and } \bigtriangledown_{l,u}(v_1 Y u_1) = f u v_1 Y u_1 - Y u v_1 f u_1.$$

We shall write $\bigtriangledown_r$ and $\bigtriangledown_l$ when $u = 1$, so $P_{\mathbf{j},i} = \bigtriangledown_r(v_1 Y u_1)$.

The operator $\boxdot$ is defined on all algebra while the operations $\bigtriangledown_{r,u}$, $\bigtriangledown_{l,u}$ are defined only on specially recorded elements and their extension does not seem to be canonical.

Assume that $v_1 Y u_1 = \boxdot(v_2 Y u_2)$. Then we need to simplify $\bigtriangledown_r(\boxdot(v_2 Y u_2))$. In order to do this let us compute $[\bigtriangledown_r, \boxdot](v_2 Y u_2)$.

This is a bit tedious but not difficult:

$$\bigtriangledown_r(\boxdot(v_2 Y u_2)) = [Y(v_2 Y u_2) f - f(v_2 Y u_2) Y] f - [Y(v_2 f u_2) f - f(v_2 f u_2) Y] Y,$$

$$\boxdot(\bigtriangledown_r(v_2 Y u_2)) = Y[(v_2 Y u_2) f - (v_2 f u_2) Y] f - f[(v_2 Y u_2) f - (v_2 f u_2) Y] Y$$

Hence

$$[\bigtriangledown_r, \boxdot](v_2 Y u_2) = -f(v_2 Y u_2) Y f + f(v_2 Y u_2) f Y - Y(v_2 f u_2) f Y + Y(v_2 f u_2) Y f$$
$$= [Y(v_2 f u_2) - f(v_2 Y u_2)][Y, f] = -\bigtriangledown_l (v_2 Y u_2)[Y, f].$$

Therefore

$$\bigtriangledown_r(\boxdot(v_2 Y u_2)) = \boxdot(\bigtriangledown_r(v_2 Y u_2)) - \bigtriangledown_l(v_2 Y u_2)[Y, f].$$

Since $w(v_2 Y u_2) < w(v_1 Y u_1)$ we can apply induction.

Assume now that either $\mu = v \boxdot (v_1 Y u_1)$ or $\mu = \boxdot(v_1 Y u_1) u$. If $\mu = v \boxdot (v_1 Y u_1)$ then $\bigtriangledown_r(v \boxdot (v_1 Y u_1)) = v \bigtriangledown_r (\boxdot(v_1 Y u_1))$. If $\mu = \boxdot(v_1 Y u_1) u$ then $\bigtriangledown_r(\mu) = \bigtriangledown_{r,u}(\boxdot(v_1 Y u_1))$. Now,

$$[\bigtriangledown_{r,u}, \boxdot](v_1 Y u_1) = \boxdot[\bigtriangledown_r(v_1 Y u_1) u - \bigtriangledown_{r,u}(v_1 Y u_1)] - \bigtriangledown_l(v_1 Y u_1) \boxdot (u)$$

and induction can be applied in these cases as well.

The last case is when Y does not belong to a bracketed subword. Then $\mu = v_1Yu_1$ and $\bigtriangledown_r(\mu) = v_1 \boxdot (u_1)$.

The proof is completed. □

Corollary 5. *The algebra of constants $K\langle X, Y\rangle^\Delta$ coincides with the algebra A_f.*

Proof. As we already mentioned $A_f \subseteq K\langle X, Y\rangle^\Delta$ and it is sufficient to show that if $\Delta(F) = 0$ for $F \in K\langle X, Y\rangle$, then F belongs to A_f. But this is a direct consequence of the case $n = 1$ in Theorem 4. □

Now we are able to establish one of the main properties of the algebra of constants $K\langle X, Y\rangle^\Delta$.

Theorem 6. *The algebra of constants $K\langle X, Y\rangle^\Delta$ is a free algebra.*

Proof. By Corollary 5 we may work with the algebra A_f instead with $K\langle X, Y\rangle^\Delta$. When $m = 0$ we saw (in the proof of Theorem 4) that A_1 is generated by $Z_1, Z_2, \ldots$. Since $\overline{Z_i} = Y^iX$ these elements freely generate A_1. For $m > 0$ producing a generating set is more involved but the freeness can be deduced from a theorem of Jooste [Jo]. It follows from his theorem that the kernel of the derivation $\overline{\Delta}(X) = 0$, $\overline{\Delta}(Y) = X^m$ is a free algebra. For this derivation any w-homogeneous component (recall that $w(X) = 1,\ w(Y) = m$) is also a constant, hence there is a homogeneous free generating set $F_1, F_2, \ldots$ of A_{X^m}. There is a bijection π between the elements of A_{X^m} and A_f obtained by replacing X^m in each bracket of an element of A_{X^m} by $f(X)$. Therefore $\pi(F_1), \pi(F_2), \ldots$ is a generating set of A_f which is free since $w(\pi(F_i) - F_i) < w(F_i)$. □

It remains to produce a homogeneous set freely generating A_{X^m}.

Lemma 7. *Algebra A_{X^m} is generated by X and bracketed words*

$$Z_1^{i_1}X^{j_1}\cdots X^{j_{k-1}}Z_1^{i_k},$$

where $i_1, i_2, \ldots, i_k > 0$ and $j_1, j_2, \ldots, j_{k-1} < m$.

Proof. Denote by B the subalgebra of A_{X^m} which is generated by words described in the Lemma. Any element of A_{X^m} can be written as a linear combination of bracketed words $\mu = X^{j_0}Z_1^{i_1}X^{j_1}\cdots Z_1^{i_k}X^{j_k}$. We shall find an element $b \in B$ with the same leading monomial $\overline{b}$ as the leading monomial $\overline{\mu}$ of μ in the lexicographic order defined by $Y >> X > 1$. Clearly this is sufficient for the proof of the Lemma.

To find the leading monomial $\overline{\mu}$ of a bracketed word μ we should replace all left brackets $\{$ by Y and all right brackets $\}$ by X^m.

If $\overline{\mu}$ starts with X then $\mu = X\mu_1$ (as an element of $K\langle X, Y\rangle$) where $\mu_1 \in A_{X^m}$ and we can use induction on weight to claim that there is an element $b_1 \in B$ such that $\overline{\mu_1} = \overline{b}$ (or even that $\mu_1 \in B$).

If μ cannot be written as $\boxdot(\nu)$ then $\mu = (\mu_1)(\mu_2)$ where brackets () separate elements of A_{X^m} and $w(\mu_i) < w(\mu)$. Hence we can use induction to claim that $\overline{\mu_1} = \overline{b_1},\ \overline{\mu_2} = \overline{b_2}$ where $b_i \in B$.

If $\mu = \boxdot(\nu)$ then $w(\mu) = w(\nu) + 2m$ and we may assume that $\overline{\nu} = \overline{b}$ where $b \in B$. Since $b \in B$ we can write $b = (X^{j_0})(v_1)(X^{j_1})\cdots(v_k)(X^{j_k})$ where $v_i \in B$ and $(X^j) = X^j$ and $\overline{\mu} = YX^{j_0}\overline{(v_1)(X^{j_1})\cdots(v_k)}X^{j_k+m}$. Inasmuch as $v_i \in B$ we may assume that the first and the last letters in all v_i (as bracketed words) are Z_1.

If $j_0 > 0$ then $\overline{Z_1}(X^{j_0-1})\overline{(v_1)(X^{j_1})\cdots(v_k)}(X^{j_k+m}) = \overline{\mu}$.

If $j_0 = 0,\ j_s \geq m$ where s is the smallest possible then

$$\overline{\{(v_1)(X^{j_1})\cdots(v_s)\}(X^{j_s-m})\cdots(v_k)}(X^{j_k+m}) = \overline{\mu}.$$

If all $j_s < m$ then $\mu \in B$. □

Theorem 8. *The algebra $B = A_{X^m}$, $m > 0$, is freely generated by X and words $\boxdot(Z_1^{i_1}X^{j_1}\cdots X^{jk-1}Z_1^{i_k})$, where $i_1, i_2, \dots, i_k > 0$ and $j_1, j_2, \dots, j_{k-1} < m$, and $Z_1^{i_1}X^{j_1}\cdots X^{jk-1}Z_1^{i_k}$ is a bracketed word (we shall refer to these words as permissible and to $Z_1^{i_1}X^{j_1}\cdots X^{jk-1}Z_1^{i_k}$ without brackets as the root of the corresponding word).*

Proof. It is sufficient to check that μ satisfying conditions of the Theorem cannot be presented as $(\mu_1)(\mu_2)$. To check this consider the leading monomial $\overline{\mu} = Y^{b_1}\cdots X^{a_{s-1}}Y^{b_s}X^{a_s}$ of μ. (Observe that $b_1a_s > 0$ since $\overline{\boxdot(v)} = Y\overline{v}X^m$.) The number of Z_1 in the bracketed representation of $\mu \in B$ must be equal to s since in any word from B a subword YX can appear in $\overline{\mu}$ only as $\overline{Z_1}$. So the number of brackets $\{$ in μ is $\deg_Y(\overline{\mu}) - s$. Of course the number of brackets $\}$ is the same. Since all permissible words have roots starting and ending with Z_1 a monomial Y^bX^a can appear in $\overline{\mu}$ only as $\{\dots\{Z_1\}\dots\}X^d$ where the number of left brackets is $a-1$, the number of right brackets is the integral part of $\frac{b-1}{m}$ and $0 \leq d < m$ is the remainder of the division of $b-1$ by m. Therefore the root and the bracketing of μ is uniquely determined by $\overline{\mu}$. The root and bracketing uniquely determine μ, i.e. the pairing of brackets is unique. Indeed if we have a configuration $\{v\}$ where v is a subword of the root then these two brackets must be paired and can be omitted. After that we can use induction on the number of brackets.

The Theorem is proved. □

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